Chapter 2
Ramanujan: The Second Century

When European mathematicians first came to know of Ramanujan's spectacular results during the early part of this century, they perceived him as a singular genius who produced numerous beautiful but mysterious identities. To a mathematician a result is mysterious if he is not able to understand it in terms of well-known theorems or see it as part of a general theory. Lacking formal education, Ramanujan was in no position to motivate his results or supply rigorous proofs. Even Professor G.H. Hardy could not fully understand many of these identities on infinite series and products. Although Hardy compared Ramanujan to Euler and Jacobi for sheer manipulative ability, he expressed the opinion that Ramanujan's results lacked the simplicity of the very greatest works. But during the last half a century, many of Ramanujan's identities have been studied in detail and put in proper perspective with respect to contemporary theories. Hence his results do not appear now to be quite that mysterious, and in fact by the time his centenary was celebrated, it became clear that his work compared well with those of the very greatest mathematicians. But the study of Ramanujan's formulae is by no means over. As Professor Atle Selberg of The Institute for Advanced Study, Princeton, remarked during the Ramanujan Centenary, it will take many more decades, possibly even more than a century, to completely understand Ramanujan's contributions. Mathematicians know well that Selberg is not given to hyperbole, and so this is very high praise! I will now describe some features of Ramanujan's work which continue to excite researchers today and will engage them in the near future.

Mock Theta Functions Ramanujan's work on mock theta functions is considered to be one of his deepest contributions. These results were discovered by him just before he died, and he communicated them to Hardy in his last letter dated January 1920. A major portion of The Lost Notebook is devoted to mock theta functions. In his letter Ramanujan listed several mock theta functions of orders three, five, and

This article appeared in *The Hindu*, India's national newspaper, on December 22, 1991, on Ramanujan's 104-th birth anniversary.

K. Alladi, *Ramanujan's Place in the World of Mathematics*,
DOI 10.1007/978-81-322-0767-2_2, © Springer India 2013

seven. Hardy passed on to Professor G.N. Watson the task of analysing Ramanujan's mock theta identities. Watson wrote two papers on this topic, the first of which was his presidential address to The London Mathematical Society entitled "The Final Problem: An Account of the Mock Theta Functions." Watson explained the choice of the title as follows: "I doubt whether a more suitable title could be found for it than used by John H. Watson, M.D., for what he imagined to be his final memoir on Sherlock Holmes." Watson's first paper (1936) dealt with mock theta functions of third order, and the second (1937) with those of fifth order. Watson did not consider the seventh-order functions, but these were investigated by Selberg in 1938. In the last two decades Professor George Andrews has analysed and explained combinatorially many of Ramanujan's mock theta identities. In collaboration with his former student Frank Garvan, Andrews was led to conjecture that some of Ramanujan's mock theta identities were equivalent to certain results on partitions (a partition of a positive integer n is a representation of n as a sum of positive integers not exceeding n). These were called the "Mock Theta Conjectures." These conjectures were settled by Dean Hickerson in 1989, after the Ramanujan Centenary. Just this year Andrews and Hickerson have completed the study of eleven identities of Ramanujan on sixth-order mock theta functions in The Lost Notebook. Another recent advance is the work of Henri Cohen who explained certain mock theta identities in the context of Algebraic Number Theory.

In spite of these breakthroughs, several fundamental questions remain. For instance, no one knows what Ramanujan meant by the "order" of a mock theta function. Ramanujan divided his list of functions into those of third, fifth, and seventh orders. Known identities indicate that these are related to the numbers 3, 5, and 7, but a precise definition of order is yet to be given. So for now, the order of a mock theta function is a convenient label which may or may not have deeper significance. Ramanujan had defined mock theta functions to be those satisfying two conditions. But no one has rigorously shown yet that any of these mock theta functions actually satisfy the second of Ramanujan's conditions. Also, in dealing with mock theta functions, special techniques have been used based on the specific function being discussed. There are attempts to find a unified approach to deal with mock theta functions like the theory of modular forms that is used in the study of theta functions.

Ramanujan's Congruences Some of the most surprising observations by Ramanujan concern congruences or divisibility properties for the partition function. Hardy had asked MacMahon to prepare a table of first two hundred values of the partition function using a certain formula of Euler. As soon as Ramanujan saw this table, he pointed out three congruences involving the primes 5, 7, and 11. The first congruence states that the number of partitions of an integer of the form $5n + 4$ is divisible by 5. For example, there are 30 partitions of 9, and 30 is divisible by 5. Hardy was simply stunned, because partitions represent an additive process, and so he did not expect such divisibility properties. MacMahon had prepared the table, and Hardy had checked it, but neither observed such a relation! Ramanujan had the eye for such connections, and this is an example of the element of surprise that is present throughout Ramanujan's work. Ramanujan generalised his congruences to the powers of 5, 7, and 11. Watson (1938) proved the congruences for the powers of 5 and (in a slightly modified form) for the powers of 7; the congruences involving the powers of 11 were established later by Atkin.

In 1944, Freeman Dyson, then a young student at Cambridge University, conjectured a combinatorial explanation for the congruences involving the primes 5 and 7 using the concept of "rank" for partitions. Dyson published his conjecture in "Eureka", a Cambridge student journal. The Dyson rank conjectures were proved in 1954 by A.O.L. Atkin and H.P.F. Swinnerton-Dyer using the theory of modular forms. Dyson had pointed out that the rank does not explain the third (and deeper) congruence involving the prime 11, but he conjectured the existence of a statistic, which he called the "crank", that would explain the third congruence combinatorially. But he had no idea of what the crank would be. Freeman Dyson has humorously remarked that this was the only instance in mathematics when an object had been named before it had been found! The crank sought by Dyson was found in 1987, one day after the Ramanujan centenary conference in Urbana, Illinois, by Andrews and Garvan. The solution was based on Garvan's Ph.D. thesis at Pennsylvania State University.

Whenever Ramanujan pointed out a relation, it was usually one of many that existed, and often the most striking among those. It has been shown that the coefficients in the expansions of various modular forms satisfy such congruences. During the last two years, Garvan, who is now at the University of Florida, has developed the idea of the crank to combinatorially prove and explain many such congruences. Also, Garvan, Kim and Stanton have applied ideas from Group Theory to explain deeper congruences. Thus the study of Ramanujan-type congruences will continue to be an active line of research in the future.

Rogers–Ramanujan Identities This pair of identities (discovered independently by Rogers in 1894 and Ramanujan around 1910) are considered among the most beautiful in mathematics. The combinatorial description of the first identity is that the number of partitions of an integer n into parts differing by at least two equals the number of partitions of n into parts which when divided by 5 leave remainder 1 or 4. The second identity has a similar description. The simplicity of the identities belies their depth. Several proofs have been given, but none can be considered simple or straightforward. In an attempt to understand these identities, a rich theory has developed, concerning, on the one hand, partitions whose parts satisfy gap conditions and, on the other, partitions whose parts satisfy congruence conditions. For a quarter century beginning around 1960, considerable work has been done in this direction, especially by Professor Basil Gordon of the University of California, Los Angeles, and by Professors George Andrews and David Bressoud of Pennsylvania State University.

Rogers actually found several elegant companions to the Rogers–Ramanujan identities. Fundamental discoveries always find applications eventually. In 1979, the Australian mathematical physicist Rodney Baxter showed that the Rogers–Ramanujan identities and these companions are the solutions to the Hard Hexagon

Model in Statistical Mechanics. For this work, Professor Baxter was awarded the Boltzman medal of the American Physical Society. In recent years, identities of the Rogers–Ramanujan type have found more applications to problems in mathematical physics.

Ramanujan considered the Rogers–Ramanujan identities as arising out of a continued fraction possessing a product representation. It was Ramanujan's insight to have realised the importance of this continued fraction in the theory of modular forms. This is only one of many continued fractions studied by Ramanujan, but perhaps the most appealing. Professor Bruce Berndt of The University of Illinois has analysed several continued fractions of Ramanujan. These continued fractions can be approached in various ways and offer a wide range of problems for exploration.

Products of the Ramanujan type established for this continued fraction are of interest in themselves. In 1980, Andrews and Bressoud showed that there was a pattern among the coefficients of certain Rogers–Ramanujan-type products that had value zero. Professor Gordon and I have recently extended these results to general Rogers–Ramanujan-type products, and there is scope for more work in this area.

Special Functions Ramanujan wrote down several beautiful formulae involving various special functions (like the Beta and Gamma functions). For the past two decades, Professor Richard Askey of the University of Wisconsin, with his students and co-workers, systematically studied q-analogues of various special functions. In the course of this study, many of Ramanujan's identities found in his original notebooks and in the Lost Notebook were extremely useful. Many of the q-analogues found by Askey and others are now finding important applications in Physics, through the idea of Quantum Groups.

The Notebooks When Bruce Berndt began editing the notebooks of Ramanujan, he envisaged publishing three volumes. Springer-Verlag has brought out three volumes, but the work is not over yet. Professor Berndt has almost completed work on the fourth volume, and there will be a fifth! This clearly demonstrates the depth and scope of Ramanujan's contributions. It is now possible to offer courses on Ramanujan's work since much of his work has been edited and books available on the subject. Thus a greater number of bright students will take to a study of Ramanujan's formulae in the decades to follow. In this connection, Robert Kanigel's recent book "The man who knew infinity" will open the eyes of the general public to the wonder that Ramanujan was.

The Lost Notebook It was George Andrews who discovered the Lost Notebook in 1976 at the Wren Library in Cambridge University. Since then, he has analysed hundreds of incredible identities contained in it and published several papers on them, most notably in the journal Advances in Mathematics. On 22 December 1987, Ramanujan's hundredth birthday, the printed version of the Lost Notebook was released. Professors Andrews and Berndt are planning an edited version of the Lost Notebook, much like Berndt's edited version of the original notebooks. This project will have great impact in the coming decades.

The Undying Magic In closing we emphasise certain features about Ramanujan's mathematics.

Ramanujan had the knack of spotting seemingly unexpected relations. Thus there is always an element of surprise for someone who studies his work. Quite often, a closer analysis reveals that there are many more such relations and that Ramanujan was pointing out only the most striking cases. So, one begins to suspect whether Ramanujan had a method to generate such relations. In an effort to find such methods, interesting theories emerge, sometimes leading to connections between different areas of mathematics. Some of the most intriguing connections recently found are between root systems of Lie algebras and the theory of q-series and modular forms. This is the work of Professors V.G. Kac, I.G. MacDonald, and D.H. Peterson. Such connections not only enrich the two areas but also offer several fruitful research projects.

Ramanujan's mathematics remains youthful even in the modern world of the computer. His modular equations were used by Canadians Jonathan and Peter Borwein to calculate π (the ratio of the circumference of a circle to its diameter) to several million decimal places. The Borweins showed that these modular equations produce efficient algorithms to obtain approximations to π and other numbers. More recently, Ramanujan's transformations for elliptic functions were used by David and Gregory Chudnovsky to produce very rapidly convergent algorithms to compute π; in fact the Chudnovskys have now calculated π to the order of about a billion digits!

Finally there is a lasting quality about Ramanujan's mathematics and about fundamental research in general. In the mid-eighteenth century, the British mathematician Stirling did calculations to produce the table of logarithms. With the advent of modern computers, such tables are among the least useful possessions of a library. But the mathematics that went into the construction of such tables never loses its lustre. Indeed, Stirling had developed methods for the acceleration of convergence of series, and this has been the basis for William Gosper's recent program to generate identities using computer algebra packages like MACSYMA. Motivated by Gosper's ideas, Ira Gessel and Dennis Stanton have used q-Lagrangian inversion to generate many identities of the Rogers–Ramanujan type.

Thus Ramanujan has left behind enough ideas to keep mathematicians busy well into the twenty-first century. Professor Dyson has remarked that we should be grateful to Ramanujan not only for discovering so much but also for providing others plenty to discover! What sets Ramanujan apart from the rest of the mathematical giants is that feeling of astonishment he creates with his stunningly beautiful identities. Ramanujan is like a gem with many faces. His identities can be studied from different viewpoints. Each face of this gem dazzles the beholder with its array of colours!

http://www.springer.com/978-81-322-0766-5

Ramanujan's Place in the World of Mathematics
Essays Providing a Comparative Study
Alladi, K.
2013, XVIII, 178 p., Hardcover
ISBN: 978-81-322-0766-5