Chapter 2
The Ramanujan τ-Function

1 Introduction

The 1916 memoir of Ramanujan, innocuously entitled "On certain arithmetic functions", introduced the τ-function. This is an integer-valued function on the natural numbers which, at first, manifested as part of an "error term" in counting the number of ways that a number could be written as a sum of 24 squares. However, Ramanujan realized that it was a function worthy of study in its own right. It would not be an overstatement to say that one of the significant themes of mathematics in the 20th century has emanated from this observation.

The τ-function is defined by the formal identity

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} \bigl(1 - q^n\bigr)^{24}. \tag{1}$$

It is in fact more than a formal identity. If we think of q as a complex number with $|q| < 1$, then taking the logarithm of the infinite product and expanding, we see that it is

$$\log q - 24 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q^{nm}}{m}.$$

Interchanging sums, we see that the double sum is

$$-24 \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1 - q^m},$$

and this converges for $|q| < 1$. For a complex number z with $\Im(z) > 0$, we see that

$$q = e^{2\pi i z}$$

satisfies $|q| < 1$, and so we may define a function $\Delta(z)$ by the right-hand side of (1). Moreover, we see that $\Delta(z) \neq 0$ as it is given as an absolutely convergent product of non-zero terms.

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The function $\Delta(z)$ was known to previous authors. If we consider the Eisenstein series

$$E_{2k}(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^{2k}}$$

where the sum ranges over pairs of integers (m,n) which are not both zero, then there is the identity

$$\Delta(z) = \frac{1}{1728}\big(E_4(z)^3 - E_6(z)^2\big).$$

Dedekind had studied $\Delta(z)$ (and a 24th root of it known as the η-function). The η-function occurs in the transformation properties of the Dedekind sums, and the Δ-function occurs in the theory of the moduli of elliptic curves. It also occurs in the limit formula of Kronecker. However, Ramanujan was the first to realize that the coefficients of the q-expansion give rise to an interesting arithmetic sequence.

2 The τ-Function and Partitions

Writing integers as sums of elements of a distinguished subset is a theme that occupied Ramanujan in many of his works. As we said, the τ-function itself arises in the problem of representing an integer as a sum of 24 squares. Ramanujan also gave considerable attention to the partition problem, namely the number of ways of writing a positive integer as a sum of positive integers. If we denote by $p(n)$ the number of such representations of a positive integer n, then we see that the first few values are given by $p(1)=1$, $p(2)=2$, $p(3)=3$, $p(4)=5$, $p(5)=7$, $p(6)=11$ and so on. If we consider the generating function

$$\sum_{n=1}^{\infty} p(n)q^n$$

then it is easily seen to be equal to

$$\prod_{m=1}^{\infty} \left(1-q^m\right)^{-1}.$$

This bears some superficial resemblance to (1). Indeed, such series had been studied classically by Euler, Jacobi and others. However, it was Ramanujan who began to observe arithmetical properties of the coefficients, such as congruences. For example, he observed that

$$p(5n+4) \equiv 0 \pmod 5$$

and

$$p(7n+5) \equiv 0 \pmod 7.$$

He continued this line of thought to the τ-function itself. We know now that such congruences are based on deep aspects of the theory of modular forms.

While the sequence $\{p(n)\}$ and $\{\tau(n)\}$ share properties such as congruence relations, they are, however, very different in terms of their growth properties. Ramanujan's circle method (described in another chapter) shows that $p(n)$ grows exponentially as a function of n, while as we shall see in this chapter, $\tau(n)$ has a polynomial growth in n.

3 Related Generating Functions

Euler had studied the function

$$\prod_{n=1}^{\infty}(1-q^n) \tag{2}$$

and proved that it is equal to

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{(3n^2+n)/2}.$$

Numbers of the form $n(3n+1)/2$ are called *pentagonal*. This and related q-expansion identities can be expressed in terms of the number of partitions of an integer into other integers satisfying various constraints. For example, Euler's identity can be interpreted as giving an expression for the number of partitions of a number into an even number of unequal parts minus the number of partitions of the same number into an odd number of unequal parts.

We can ask for which integers m there is an $n \in \mathbb{Z}$ such that

$$m = \frac{1}{2}n(3n+1)? \tag{3}$$

We need to have $1 + 24m$ to be a perfect square, say r^2. Moreover, we need $-1 \pm r$ divisible by 6, and in particular, 6 should not divide r. When these conditions are satisfied, we have

$$n = \frac{1}{6}(-1 \pm r).$$

In particular, given m, there are at most two values of n satisfying (3). Thus, when (2) is written as a power series in q, the coefficients are bounded. Moreover, the number of $m \le x$ for which (3) has a solution is $\le \sqrt{2x/3}$. In particular, most of the coefficients are zero, and the series is "lacunary". We can say a little more about which coefficients are nonzero. Indeed, for $1 + 8m = r^2$ to be satisfied, we need r odd, and

$$r+1 = a, \qquad r-1 = b$$

for some factorization $8m = ab$. This implies that $r = \frac{1}{2}(a+b)$ and $a = b+2$. In particular, both a and b are even, and $r = b+1$ and $8m = b(b+2)$.

Jacobi studied the third power of (2) and proved that

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(2n+1)q^{n(n+1)/2}. \tag{4}$$

Numbers of the form $n(n+1)/2$ are called *triangular*. Again, it is seen that this is a lacunary series, in the sense that the set of n for which the coefficient of q^n is nonzero has density zero. Note that we have

$$\sum_{n=1}^{\infty}\tau(n)q^n = q\left(\sum_{m\in\mathbb{Z}}(-1)^m q^{(3m^2+m)/2}\right)^{24}$$

and

$$\sum_{n=1}^{\infty}\tau(n)q^n = q\left(\sum_{m=0}^{\infty}(2m+1)q^{m(m+1)/2}\right)^{8}.$$

In particular, one can derive the following formulas for $\tau(n)$:

$$(n-1)\tau(n) = \sum_{1\le|m|\le a_n}\left(n-1-\frac{25m}{2}(3m+1)\right)\tau\left(n-\frac{m}{2}(3m+1)\right)$$

where

$$a_n = \frac{1}{6}\left(1+(1+24n)^{\frac{1}{2}}\right);$$

$$(n-1)\tau(n) = \sum_{1\le m\le b_n}(-1)^{2m+1}(2m+1)\left(n-1-\frac{9m}{2}(m+1)\right)\tau\left(n-\frac{m}{2}(m+1)\right)$$

where

$$b_n = \frac{1}{2}\left((1+8n)^{\frac{1}{2}}-1\right).$$

The first of these is due to Lehmer, and the second to Ramanujan. However, these formulas do not seem to be useful in giving an expression for $\tau(n)$ in terms of elementary functions. They cannot be viewed as 'closed-form' expressions since the range of summation is a function of the argument. However, they do suffice to show that $\tau(n)$ has at most polynomial growth in n.

4 Values of the τ-Function

It is easy to compute the first few values: $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252$. The following table is copied from [116].

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	11	534,612	21	−4,219,488
2	−24	12	−370,944	22	−12,830,688
3	252	13	−577,738	23	18,643,272
4	−1472	14	401,856	24	21,288,960
5	4830	15	1,217,160	25	−25,499,225
6	−6048	16	987,136	26	13,865,712
7	−16744	17	−6,905,934	27	−73,279,080
8	84480	18	2,727,432	28	24,647,168
9	−113,643	19	10,661,420	29	128,406,630
10	−115,920	20	−7,109,760	30	−29,211,840

Looking at this table and others that give more values, many natural questions come to mind. Firstly, we see that the numbers are growing fairly rapidly. However, the growth is not exponential since it was shown by Ramanujan that

$$\left|\tau(n)\right| \ll n^7.$$

He conjectured that

$$\left|\tau(n)\right| \le d(n)n^{11/2}$$

where $d(n)$ denotes the number of positive divisors of n. This is known as the Ramanujan conjecture (actually Hardy called it the Ramanujan hypothesis), and it is now a theorem as we shall explain in another chapter. The only proof of this relies on "reducing" it to a special case of the Weil conjectures and appealing to the proof of these conjectures by Deligne.

It is a classical result that $d(n) = \mathbf{O}(n^\epsilon)$ for any $\epsilon > 0$, and so a weaker version of the Ramanujan conjecture is that for any $\epsilon > 0$,

$$\tau(n) \ll_\epsilon n^{11/2+\epsilon}$$

where the subscript indicates that the implied constant may depend on ϵ.

We might also notice from the table a fact that Ramanujan stated as a conjecture, namely that the τ-function is multiplicative:

$$\tau(mn) = \tau(m)\tau(n) \tag{5}$$

for positive integers m and n that are relatively prime. Moreover, Ramanujan observed that for a fixed prime p, the values of $\tau(p^m)$ satisfy a second-order recurrence relation: for $m \ge 1$, we have

$$\tau\left(p^{m+1}\right) = \tau(p)\tau\left(p^m\right) - p^{11}\tau\left(p^{m-1}\right). \tag{6}$$

p	$\tau(p)$	p	$\tau(p)$	p	$\tau(p)$
2	−24	31	−52,843,168	73	1,463,791,322
3	252	37	−182,213,314	79	38,116,845,680
5	4830	41	308,120,442	83	−29,335,099,668
7	−16744	43	−17,125,708	89	−24,992,917,110
11	534,612	47	2,687,48,496	97	75,013,568,546
13	−577,738	53	−1,596,055,698	101	81,742,959,102
17	−6,905,934	59	−5,189,203,740	103	−225,755,128,648
19	10,661,420	61	6,956,478,662	107	90,241,258,356
23	18,643,272	67	−15,481,826,884	109	73,482,676,310
29	128,406,630	71	9,791,485,272	113	−85,146,862,638

Both of these properties were proved by Mordell within a year of the publication of Ramanujan's paper. These relations imply that all of the values of the τ-function can be determined once they are known at prime arguments. Above is a table of $\tau(p)$ for primes $p \le 113$ (the first 30 primes).

Consider again the values of the τ-function on powers of a prime. Denote by α_p and β_p the complex numbers which are roots of the equation

$$T^2 - \tau(p)T + p^{11} = 0.$$

Then $\tau(p) = \alpha_p + \beta_p$ and $\alpha_p\beta_p = p^{11}$. Let us write

$$\alpha_p = p^{11/2}e^{i\theta_p}.$$

Here θ_p is a complex number, and the Ramanujan conjecture is the assertion that in fact θ_p is real. In any case, we deduce that

$$\tau(p) = 2p^{11/2}\cos(\theta_p).$$

This implies that

$$\tau(p)^2 = 4p^{11}\bigl(1 - \sin^2(\theta_p)\bigr).$$

We see from this that we cannot have $\sin(\theta_p) = 0$ as $\tau(p)$ is an integer. Moreover, relation (6) shows that

$$\tau\bigl(p^2\bigr) = \tau(p)^2 - p^{11} = \alpha_p^2 + \alpha_p\beta_p + \beta_p^2.$$

More generally, by induction, we can show that

$$\tau\bigl(p^a\bigr) = \frac{\alpha_p^{a+1} - \beta_p^{a+1}}{\alpha_p - \beta_p}.$$

Equivalently,

$$\tau\left(p^a\right) = p^{11a/2}\frac{\sin(a+1)\theta_p}{\sin\theta_p}. \tag{7}$$

5 Parity of the τ-Function

A few more calculations will show that $\tau(p)$ seems to be even for all primes p. This is in fact true and can be proved as follows. We have the congruence

$$\left(1-q^n\right)^{24} \equiv \left(1+q^{8n}\right)^3 \pmod 2.$$

Now, by a q-series identity of Jacobi, we have

$$\prod_{n=1}^{\infty}\left(1+q^{8n}\right)^3 = \sum_{m=0}^{\infty} q^{4m^2+4m}.$$

Thus, we deduce that

$$\sum_{n=1}^{\infty}\tau(n)q^n \equiv q\sum_{m=0}^{\infty} q^{4m^2+4m} \equiv \sum_{m=0}^{\infty} q^{(2m+1)^2} \pmod 2.$$

In particular, $\tau(n)$ is odd if and only if $n = (2m+1)^2$, in other words, if and only if n is an odd square. In particular, $\tau(p)$ is even for every prime p.

6 Congruences Satisfied by the τ-Function

The result of the previous paragraph is that

$$\tau(p) \equiv 0 \pmod 2$$

for all primes p. There are many other congruence relations discovered by Ramanujan. Here is a partial list:

(1) $\tau(p) \equiv 1 + p^3 \pmod{2^5}$
(2) $\tau(p) \equiv 1 + p \pmod 3$
(3) $\tau(p) \equiv p + p^{10} \pmod{5^2}$
(4) $\tau(p) \equiv p + p^4 \pmod 7$
(5) $\tau(p) \equiv 1 + p^{11} \pmod{691}$

More congruences were discovered later by other authors including Bambah, Chowla, Gandhi, Swinnerton-Dyer and Wilton.

As mentioned briefly earlier, Ramanujan was also the first to find congruences satisfied by the partition function. Some of these are

(1) $p(5m+4) \equiv 0 \pmod{5}$
(2) $p(7m+5) \equiv 0 \pmod{7}$
(3) $p(11m+6) \equiv 0 \pmod{11}$

In Chap. 7, we will indicate how such congruences can be proved.

7 Vanishing of the τ-Function

Of the many problems that are open with respect to the τ-function, there is a conjecture of Lehmer [117] that asserts that

$$\tau(p) \neq 0$$

where p is a prime. Equivalently,

$$\tau(n) \neq 0$$

for any $n \geq 1$. In fact, we have the following elementary result of Lehmer.

Proposition 7.1 *Let n_0 denote the least value of n for which $\tau(n) = 0$ (if it exists). Then n_0 is prime.*

Proof The multiplicativity of the τ-function (5) shows that n_0 is a prime power, say $n_0 = p^a$. Suppose that $a > 1$ (in other words, $\tau(p) \neq 0$). Then from (7) we deduce that

$$\sin(a+1)\theta_p = 0$$

and so

$$\theta_p = k\pi/(a+1) \tag{8}$$

for some integer k. The number

$$4(\cos\theta_p)^2 = \tau(p)^2/p^{11}$$

is rational. On the other hand, by (8), it is an algebraic integer. Thus, it is in fact an integer, say m, and it is also clear that $m > 0$. Again, by (8), θ_p is real, and so $|\cos\theta_p| \leq 1$ and so $m \leq 4$. Then

$$\tau(p)^2 = mp^{11}$$

and as $\tau(p)$ is an integer, m must be divisible by p. Putting all of these constraints together, we see that $p = m = 2$ or $p = m = 3$. This implies that

$$\tau(2) = \pm 2^6 \quad \text{or} \quad \tau(3) = \pm 3^6.$$

But neither of these can hold as we see from the tables above: $\tau(2) = -24$ and $\tau(3) = 252$. This completes the proof. □

We also have the following curious result.

Proposition 7.2 *Suppose that the set* $\{n : \tau(n) = 0\}$ *has density zero. Then it is, in fact, empty.*

Proof Suppose that $\tau(n) = 0$ for some n. Then by the previous result, the least such n is a prime number, p (say). It follows that if $n = mp$ with $p \nmid m$, then $\tau(n) = 0$. Thus, the set

$$\{n : \tau(n) = 0\}$$

has density $\geq (p-1)/p^2$, contradicting our hypothesis. □

Using the congruences satisfied by the τ-function, it is possible to show that n_0 (if it exists) must be quite large. Indeed, Lehmer himself observed that if $\tau(p) = 0$, then the congruences imply that

$$p^3 \equiv -1 \pmod{2^5}.$$

Since $p \neq 2$, we also have that

$$p^{16} \equiv 1 \pmod{2^5}.$$

These two congruences together imply that

$$p^2 \equiv 1 \pmod{2^5}$$

from which it follows that

$$p \equiv -1 \pmod{2^5}.$$

Similarly, one finds $p \equiv -1 \pmod{2^5 \cdot 3 \cdot 5^2 \cdot 691}$. This immediately implies that

$$p \geq 2^5 \cdot 3 \cdot 5^2 \cdot 691 - 1 = 1{,}658{,}399.$$

In fact, one can do much better, and it is currently known that $\tau(p) \neq 0$ for $p < 10^{15}$. Of course. Lehmer's conjecture is that it should be nonzero for all p. Recently, this conjecture has been related to the irrationality of coefficients of certain mock modular forms.

There is an even stronger conjecture than that of Lehmer, which has been suggested by Atkin and Serre. For every $\epsilon > 0$, they ask whether

$$\left|\tau(p)\right| \gg_\epsilon p^{\frac{9}{2}-\epsilon}?$$

Nothing is known about this conjecture. In terms of the angle θ_p introduced earlier, the above conjecture is equivalent to the assertion that

$$\left|\theta_p - \frac{\pi}{2}\right| \gg_\epsilon \frac{1}{p^{1+\epsilon}}.$$

If we consider $\tau(n)$ rather than $\tau(p)$, it is possible to prove a kind of lower bound. It is shown in [143] that there is an effectively computable absolute constant $c > 0$ such that for all positive integers n for which $\tau(n)$ is *odd*, we have the lower bound

$$\left|\tau(n)\right| \geq (\log n)^c.$$

The condition on the parity of $\tau(n)$ ensures that it is not divisible by the first power of any prime. Thus, the set of n for which the result applies is the so-called squarefull numbers (that is, numbers for which every prime divisor occurs to at least the second power).

It is also interesting to ask for lower bounds that hold infinitely often. Hardy showed that

$$\tau(n) > n^{11/2}$$

holds infinitely often. The best result in this regard is to due to R. Murty [136], who showed that there is an absolute and effective constant $c > 0$ such that

$$\left|\tau(n)\right| > n^{11/2} \exp\{c \log n / \log\log n\}.$$

This result is essentially best possible since we know that

$$d(n) < \exp\left\{c' \log n / \log\log n\right\}$$

and by Ramanujan's conjecture (Deligne's theorem), we have

$$\left|\tau(n)\right| \leq d(n) n^{11/2}.$$

With recent developments on the Sato–Tate conjecture (see Chaps. 10 and 12), these results are valid for any $c < \log 2$. □

8 Divisibility of $\tau(p)$ by p

The many congruences satisfied by the τ-function all have a fixed modulus and varying argument. A different kind of congruence is the one in the title of this section, namely whether

$$\tau(p) \equiv 0 \pmod{p}. \tag{9}$$

Indeed, this does occur, and the first example is $p = 2$ as we have already seen above. The only primes known to satisfy (9) are $p = 2, 3, 5, 7, 2411, 7758337633$.

(This last prime was discovered quite recently [122].) It is not known whether there are infinitely many such primes. Neither is it known that the complement is infinite! In this regard, the situation is similar to that of Wieferich primes.

Heuristic reasoning would suggest that

$$\#\{p \le x : \tau(p) \equiv 0 \pmod{p}\} \sim \log\log x. \tag{10}$$

However, the function $\log\log x$ grows so slowly that it is computationally difficult to distinguish it from a constant. (For example, $\log\log 10^{15}$ is approximately 3.54.) If (10) were true, we would expect that

$$\sum_{\tau(p)\equiv 0 \pmod{p}} \frac{1}{\log p}$$

converges. However, we do not even know whether

$$\sum_{\tau(p)\equiv 0 \pmod{p}} \frac{1}{p}$$

converges.

Suppose that p is a prime for which (9) holds. Then from the relation

$$\tau\left(p^{a+1}\right) = \tau(p)\tau\left(p^a\right) - p^{11}\tau\left(p^{a-1}\right)$$

we see that for all $a \ge 1$,

$$\tau\left(p^a\right) \equiv 0 \pmod{p^a}.$$

We also note that if $\tau(p) \equiv 0 \pmod{p}$ and $n = pm$ where $(p, m) = 1$, then

$$\left(\tau(n), n\right) \ne 1. \tag{11}$$

If the set of primes for which (9) holds has positive density, then it follows by an elementary sieve argument that the set of n for which (11) holds has density 1. This latter statement can be proved unconditionally. In fact, one knows that

$$\#\{n \le x : \left(\tau(n), n\right) = 1\} \ll x/\log\log\log x.$$

It is interesting to note that if there are infinitely many primes p such that $\tau(p) = 0$, then for any value of k, there exist infinitely many values of n such that $\tau(n) = \tau(n+1) = \tau(n+2) = \cdots = \tau(n+k) = 0$. This is an easy exercise using the Chinese remainder theorem.

9 Lehmer's Conjecture and Harmonic Weak Maass Forms

Finally, we report on some recent work of Bruinier, Ono and Rhoades [28] that opens another line of investigation for Lehmer's conjecture. We give a brief (and slightly technical) explanation of this new development.

After the study of classical modular forms, it is natural to study the space of weakly holomorphic modular forms which are just meromorphic modular forms whose singularities may only occur at the cusps. These spaces are contained in the larger space of harmonic weak Maass forms which need not be holomorphic but are annihilated by a second-order differential operator. At the cusps, we allow for exponential growth of a special type. To be precise, let

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Let χ be a Dirichlet character modulo N. A *harmonic weak Maass form of weight k on $\Gamma_0(N)$ with Nebentypus χ* is a smooth function on the upper half-plane satisfying the following three conditions:

(1) $$f\left(\frac{az+b}{cz+d}\right)=\chi(d)(cz+d)^k f(z) \quad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N);$$

(2) $\Delta_k f = 0$;
(3) there is a polynomial $P_f = \sum_{n\le 0} c_f^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that for some fixed $\epsilon > 0$, $f(z) - P_f(z) = O(e^{-\epsilon y})$ as y tends to infinity, with analogous conditions at the other cusps. The polynomial P_f is called the *principal part* of f at the corresponding cusp.

This vector space of harmonic weak Maass forms is denoted $H_k(\Gamma_0(N),\chi)$. One can show that every weight $2-k$ harmonic weak Maass form $f(z)$ has a Fourier expansion at each cusp of the following form:

$$f(z)=\sum_{n\gg-\infty} c_f^+(n)q^n+\sum_{n<0} c_f^-(n)\Gamma\big(k-1,4\pi|n|y\big)q^n,$$

where $\Gamma(a,x)$ is the incomplete Gamma function given by

$$\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}\,dt.$$

The differential operator

$$\xi_w := 2iy^w\overline{\frac{\partial}{\partial\overline{z}}},$$

has the property that

$$\xi_{2-k}: H_{2-k}\big(\Gamma_0(N),\chi\big)\to S_k\big(\Gamma_0(N),\overline{\chi}\big).$$

A straightforward calculation shows that

$$\xi_{2-k}(f)=-(4\pi)^{k-1}\sum_{n=1}^\infty \overline{c_f^-(-n)}n^{k-1}q^n.$$

In other words, the coefficients $c_f^-(n)$ are really coefficients of classical cusp forms. Now let $g \in S_k(\Gamma_0(N), \chi)$ be a normalized newform. We say that a harmonic weak Maass form f is *good for* g if the following conditions are satisfied:

(1) The principal part of f at the cusp ∞ belongs to $F_g[q^{-1}]$, where F_g is the number field obtained by adjoining the Fourier coefficients of g to $\mathbb{Q}$;
(2) The principal parts of f at the other cusps of $\Gamma_0(N)$ are constant;
(3) $\xi_{2-k}(f) = g/(g, g)$, where $(\cdot,\cdot)$ denotes the Petersson inner product.

The main theorem of [28] is that if

$$g = \sum_{n=1}^{\infty} c_g(n) q^n \in S_k\big(\Gamma_0(N), \overline{\chi}\big)$$

is a normalized newform and $f \in H_{2-k}(\Gamma_0(N), \chi)$ is good for g, then for any p coprime to N for which $c_g(p) = 0$, we have $c_f^+(n)$ algebraic for any n with $\mathrm{ord}_p(n)$ odd. In other words, the vanishing of the Fourier coefficients of a Hecke eigenform implies the algebraicity of the Fourier coefficients of the corresponding harmonic weak Maass form. The authors in [28] discuss the case for Lehmer's conjecture in this context.

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