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★**Computing qualitatively correct approximations of balance laws.**
Exponential-fit, well-balanced and asymptotic-preserving.
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This book is devoted to the numerical analysis of hyperbolic systems of balance laws. These are partial differential equations of the general form

$$\partial_t u + \partial_x f(u) = g(x,u),$$

where the function g takes various possible sources into account. Such systems occur in a wide variety of physical phenomena, many of which are studied or mentioned in the book. Let us quote, among others: compressible flow in a duct of variable cross-section and/or in the presence of gravity, some models of chemotaxis, radiative transfer, semiconductors or other kinetic equations for which the solution is approximated by a finite number of its moments. In some situations, the underlying system of balance laws includes a small parameter, for instance the Knudsen number or the Debye length. One is then interested in capturing the asymptotic behavior of solutions as this small parameter goes to zero.

When the source g vanishes, one is left with a system of conservation laws for which a large number of numerical approximations have been developed. A first attempt at the numerical simulation of the balance law could therefore rely on an operator-splitting technique where one alternatively solves a conservation law and an ordinary differential equation. Despite its broad use in the approximation of systems with (possibly stiff) relaxation, this splitting strategy may not be satisfactory in a number of situations. For instance, splitting schemes can fail to capture stationary equilibria or exponential growth of the exact solution. They can also produce numerical errors that grow exponentially in time. The overall goal of the book is therefore to explain how to derive accurate numerical approximations of solutions to balance laws. Accuracy may refer either to the large-time behavior of the numerical solution, which should be compatible with the large-time behavior of the solution to the partial differential equation, or to the asymptotic behavior of the numerical solution as a small parameter in the model tends to zero. These are very active fields of research, and the book covers a rather large spectrum. Each chapter includes comments and historical notes, as well as a list of references. The book is divided into two parts, the first of which (from chapter 2 to chapter 7) deals with general quasilinear hyperbolic balance laws. Part two of the book (from chapter 8 to chapter 15) deals with various problems in kinetic theory that give rise to balance laws through some kind of approximation. A final chapter gathers some concluding remarks and gives some possible directions for future research.

I shall now try to summarize each chapter. Chapter 1 makes a quick comparison between a standard finite difference approximation to the drift-diffusion equation and the Scharfetter-Gummel scheme. Special attention is paid to the asymptotic regime where the diffusion coefficient

tends to zero. This is in some sense an introduction to the issue of designing asymptotic-preserving schemes. Eventually, this first chapter includes a general presentation of the book. Chapter 2 deals with scalar balance laws. For source terms in the form $k(x)\,g(u)$, one introduces the anti-derivative a of k and rewrites artificially the balance law as a first-order system for (u, a). The lifted system is strictly hyperbolic under the non-resonant assumption $f'(u) \neq 0$. A Godunov-type scheme is constructed by approximating a by a piecewise constant function, which amounts to approximating $k \in L^1$ by Dirac masses. An error bound $O(\Delta x\,(\mathrm{e}^{C\,t} - 1))^{1/2} + O(\mathrm{e}^{C\,t}\,\Delta x)$ is then proved. Chapter 3 presents an improvement of this error bound for a modified Godunov wave-front tracking scheme. The new error bound reads $O(\Delta x + t)$. Numerical simulations illustrate this linear growth of the error with respect to time, and comparisons are made with time-splitting techniques. Chapter 4 presents a numerical scheme for compressible flow in a duct of variable cross-section, an extension of the celebrated random choice method for reaction-diffusion systems, and a numerical scheme for compressible flow with gravity. Chapter 5 gives an overview of the well-posedness theory for systems of balance laws in the non-resonant case: the underlying hyperbolic system of conservation laws has nonzero characteristic speeds. Waves thus propagate towards infinity where the effect of the source is small. The semigroup is constructed with the help of a suitable wave-front tracking algorithm whose stability relies on interaction estimates "à la Glimm". Results on the large-time behavior of solutions are also presented. Chapter 6 starts with a numerical discretization of the Vlasov equation. Simulations are presented for the Vlasov-Poisson system. The chapter also presents numerical approximations of highly oscillating solutions to the Schrödinger equation. A convenient framework for simulations beyond caustics is proposed and illustrated by numerous computations. Chapter 7 deals with the porous media equation. Using a convenient formulation of the equation, a stable and consistent scheme is studied. A contraction property is also proved. The chapter then moves to the gravitational Navier-Stokes-Poisson system, which is reformulated in Lagrangian variables.

Chapter 8 gives a detailed account of numerical approximations for a kinetic model with two velocities. A numerical scheme is proposed by simplifying the collisional term into a combination of Dirac masses. An asymptotic-preserving modification is studied in the diffusive regime. Similar techniques apply to a hyperbolic-parabolic model of chemotaxis. Chapter 9 is devoted to a simplified model of radiative transfer. Velocities are discretized with a Gaussian quadrature rule. Numerical approximations are proposed, even for a variable opacity and in the diffusive regime. Coupling with the temperature is also considered. Chapter 10 presents various models of chemotaxis and numerical procedures to approximate them. The cell density evolves according to a kinetic equation, and its moments are again approximated by a Gaussian quadrature rule. Special attention is paid to aggregation phenomena. Chapter 11 studies kinetic models for semiconductors. These are a coupling of a kinetic equation for the charged particles with the Poisson equation for the electric potential. Various simulations, including Landau damping, illustrate the efficiency of the well-balanced approximation. Chapter 12 is devoted to the Fokker-Planck equation. The kinetic equation thus involves a diffusive term rather than a relaxation operator. Coupling with an exterior electric field is considered for testing the validity of the numerical approximation. A model of spray is also discussed, which couples the Fokker-Planck equation for the particles with a Burgers equation for the fluid in which they evolve. The short chapter 13 is devoted to

a variant of the Fokker-Planck equation, which is sometimes called the Vlasov-Lorentz model. Chapter 14 first reviews the Boltzmann equation and its (linearized) BGK approximation, and proposes a numerical approximation for the latter model. Numerical results are presented for a heat transfer problem and the propagation of a sound wave. Eventually, chapter 15 goes back to some kinetic models and proposes genuinely two-dimensional numerical schemes. The main issue in this approach is to understand where the source term should be localized.

The book requires a broad knowledge either in hyperbolic equations, kinetic theory or numerical analysis, but should be of interest to researchers willing to learn well-balanced techniques.

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