Preface

I don't see any problem with the math, but this is not a dissertation in economics. We can't give you a Ph.D. in economics for a dissertation that isn't about economics. It's not economics. It's not mathematics. It's not even business administration.

Milton Friedman, about H. Markowitz's manuscript

Balance laws appear in many areas of application, ranging from fluid mechanics modeling, or semi-classical WKB approximations of linear quantum models, to discrete-ordinate reduction of multi-dimensional kinetic equations. These are partial differential equations describing the evolution in time of intensive (or bulk) quantities which are submitted to a physical process involving both convection and another mechanism (reaction, relaxation, or even diffusion). In many situations, such a system of equations stabilizes onto a large-time behavior which is characterized by an accurate balancing between the transport terms and the other ones. Another interesting configuration is the one in which the system contains an independent parameter which variation deeply affects the qualitative behavior of the solutions. We shall therefore speak about qualitatively correct numerical approximations when either (or both) aforementioned distinguished behaviors can be reproduced algorithmically without salient restrictions on the computational grid. Such accurate computations usually result from the use of sophisticate numerical flux functions, which display consistency not only with the convection terms, but with other parts of the equation. Perceiving simultaneously several (if not all) the terms appearing in the partial differential equation helps in preserving at the numerical level desirable qualitative properties, like dissipation of certain norms, respect of positively invariant domains, entropy inequalities or Lyapunov functionals in a robust manner. The objective of the present book is to raise the reader's awareness of how such elaborate flux functions can be built, mainly in a one-dimensional context for hyperbolic systems admitting shock-type solutions and for kinetic equations in the discrete-ordinate approximation as well. An effort will be dedicated to rigorous mathematical derivations and to the analysis of the net gain retrieved from this approach.

In particular, one should often keep in mind that an equilibrium has to be sought between the three edges of the *golden triangle*[1]: observations, modeling and analysis, numerical simulation. While observations are imposed by our surrounding world, modeling can be instead achieved at several levels of complexity. A more intricate

[1] I learnt this nice expression from Prof. Vincent Courtillot.

model can lead to bigger difficulties in terms of mathematical analysis, even if the development of powerful tools in the field of non-linear analysis allowed to successfully resolve delicate problems in terms of existence, uniqueness and stability of appropriate weak solutions (arousing some reflexions[2] about what is called *solving*). Impressive achievements in theoretical analysis don't yield automatically powerful algorithms to simulate efficiently these weak solutions on a computer: concerning balance laws, only Tai-Ping Liu's extension of James Glimm's theorem was actually based on an astute numerical algorithm. One insight in that work was a seemingly simple finite-difference scheme which building block contains a complete time-asymptotic wave pattern, including both convection and source terms. Slightly later, Gary Sod developed a similar processing for convection-diffusion systems, involving again a solver consistent with all the terms. There is an unpleasant fact about increasing the complexity of a physical model: even if mathematical issues can be overcome by means of an elegant theory, usually the level of noise produced by standard approximation algorithms increases too. Second-order accurate numerical schemes which behave nicely on smooth classical solutions can display spurious oscillations when asked to compute discontinuous waves emanating from models endowed with degenerate or vanishing viscosity: the case of the Lax-Wendroff scheme is quite revealing of this type of drawback. Shock solutions are a visual expression of the mathematical fact that no strong dissipation has been kept at the Sobolev level: however dissipation helps when designing algorithms because it smears off part of the numerical truncation errors. *The gain in accuracy when reproducing real-life observations that one obtains by increasing the complexity of a mathematical model must always be vastly superior to the increase of numerical noise resulting from dissipation processes being removed*. There's little doubt that homogeneous systems of conservation laws are somewhat limited when it comes to rendering certain situations: when thinking about large-scale gas dynamics, gravity is an external force which can hardly be bypassed thus leading to the inclusion of source terms on the right-hand side of both momentum and total energy equations. Such terms make the system "less dissipative", therefore more sensitive to truncation errors as new mechanisms appear likely to amplify them. Solvers involving a whole non-interacting, time-asymptotic wave pattern sometimes can help.

Bari, L'Aquila and Rome, August 2012 *Laurent Gosse*

[2] Clément Mouhot, *Que signifie* résoudre *les équations de la physique pour un mathématicien?*

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