

Chapter 2

Lorentz Connections and Inertia

In Special Relativity, Lorentz connections represent inertial effects present in non-inertial frames. In these frames, any relativistic equation acquires a manifestly Lorentz covariant form due to the presence of the inertial connection. By virtue of the local equivalence between gravitation and inertia, these notions are essential for the study of the gravitational interaction.

2.1 Purely Inertial Connection

In Special Relativity, Lorentz connections represent inertial effects present in a given frame. In the class of inertial frames, for example, where these effects are absent, the Lorentz connection vanishes identically. Since this is the class most used in Field Theory, Lorentz connections usually do not show up in relativistic physics. Of course, as long as Physics is frame independent, it can be described in any class of frames. For the sake of simplicity, however, one always uses the class of inertial frames.

To see how an inertial Lorentz connection shows up, let us denote by $e^a{}_\mu$ a generic frame in Minkowski spacetime. The class of inertial (or holonomic) frames, defined by all frames for which $f'^c{}_{ab} = 0$, will be denoted by $e'^a{}_\mu$ [see Eq. (1.14) and around]. In a general coordinate system, the frames belonging to this class have the holonomic form

$$e'^a{}_\mu = \partial_\mu x'^a, \quad (2.1)$$

with x'^a a spacetime-dependent Lorentz vector: $x'^a = x'^a(x)$. The spacetime metric

$$\eta'_{\mu\nu} = e'^a{}_\mu e'^b{}_\nu \eta_{ab} \quad (2.2)$$

still represents the Minkowski metric, but in a general coordinate system. In the specific case of cartesian coordinates, the inertial frame assumes the form

$$e'^a{}_\mu = \delta^a_\mu \quad (2.3)$$

and the spacetime metric $\eta'_{\mu\nu}$ is given by (1.16). Under a local Lorentz transformation,

$$x^a = \Lambda^a_b(x)x'^b, \quad (2.4)$$

the holonomic frame (2.1) transforms according to

$$e^a_\mu = \Lambda^a_b(x)e'^b_\mu. \quad (2.5)$$

As a simple computation shows, it has the explicit form

$$e^a_\mu = \partial_\mu x^a + \dot{A}^a_{b\mu} x^b \equiv \dot{\mathcal{D}}_\mu x^a, \quad (2.6)$$

where

$$\dot{A}^a_{b\mu} = \Lambda^a_e(x) \partial_\mu \Lambda_b^e(x) \quad (2.7)$$

is a Lorentz connection that represents the inertial effects present in the new frame. It is just the connection obtained from a Lorentz transformation of the vanishing spin connection $\dot{A}'^e_{d\mu} = 0$, as can be seen from Eq. (1.78):

$$\dot{A}^a_{b\mu} = \Lambda^a_e(x) \dot{A}'^e_{d\mu} \Lambda_b^d(x) + \Lambda^a_e(x) \partial_\mu \Lambda_b^e(x). \quad (2.8)$$

Starting from an inertial frame, in which $\dot{A}'^a_{b\mu} = 0$, different classes of frames are obtained by performing *local* (point-dependent) Lorentz transformations $\Lambda^a_b(x^\mu)$. Inside each class, the infinitely many frames are related through *global* (point-independent) Lorentz transformations, $\Lambda^a_b = \text{constant}$.

The inertial connection (2.7) is sometimes referred to as the Ricci coefficient of rotation [1]. Due to its presence, the transformed frame e^a_μ is no longer holonomic. In fact, its coefficient of anholonomy is given by

$$f^c_{ab} = -(\dot{A}^c_{ab} - \dot{A}^c_{ba}), \quad (2.9)$$

where we have used the identity $\dot{A}^a_{bc} = \dot{A}^a_{b\mu} e^c{}^\mu$. The inverse relation is

$$\dot{A}^a_{bc} = \frac{1}{2}(f^a_b{}_c + f^a_c{}_b - f^a_{bc}). \quad (2.10)$$

Of course, as a purely inertial connection, $\dot{A}^a_{b\mu}$ has vanishing curvature and torsion:

$$\dot{R}^a_{b\nu\mu} \equiv \partial_\nu \dot{A}^a_{b\mu} - \partial_\mu \dot{A}^a_{b\nu} + \dot{A}^a_{e\nu} \dot{A}^e_{b\mu} - \dot{A}^a_{e\mu} \dot{A}^e_{b\nu} = 0 \quad (2.11)$$

and

$$\dot{T}^a_{\nu\mu} \equiv \partial_\nu e^a_\mu - \partial_\mu e^a_\nu + \dot{A}^a_{e\nu} e^e_\mu - \dot{A}^a_{e\mu} e^e_\nu = 0. \quad (2.12)$$

2.2 Particle Equation of Motion

In the class of inertial frames e'^a_μ , a free particle is described by the equation of motion

$$\frac{du'^a}{d\sigma} = 0, \quad (2.13)$$

with u'^a the particle four-velocity, and

$$d\sigma^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad (2.14)$$

the quadratic Minkowski invariant interval, with σ the proper time. In an anholonomic frame $e^a{}_\mu$, related to $e'^a{}_\mu$ by the local Lorentz transformation (2.5), the equation of motion assumes the manifestly covariant form

$$\frac{du^a}{d\sigma} + \dot{A}^a{}_{b\mu} u^b u^\mu = 0, \quad (2.15)$$

where

$$u^a = \Lambda^a{}_b(x) u'^b \quad (2.16)$$

is the Lorentz transformed four-velocity, and

$$u^\mu = u^a e_a{}^\mu \quad (2.17)$$

is the spacetime-indexed four-velocity, which has the usual holonomic form

$$u^\mu = \frac{dx^\mu}{d\sigma}. \quad (2.18)$$

It is important to remark that, since the anholonomy of the new frame $e^a{}_\mu$ is related to inertial effects only, the spacetime metric

$$\eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}, \quad (2.19)$$

still represents the Minkowski metric, though in a general x -dependent form.

Comment 2.1 This allows a better characterization of the metrics defined by (1.27). Anholonomic basis fields related by Lorentz transformations define one same metric. On the other hand, anholonomic basis fields not related by Lorentz transformations define different metrics. This second case is what happens in the presence of gravitation. In fact, a tetrad whose anholonomy represents a gravitational field cannot be obtained through a Lorentz transformation from a tetrad whose anholonomy represents inertial effects only.

In terms of the holonomic four-velocity, the equation of motion (2.15) assumes the form

$$\frac{du^\rho}{d\sigma} + \dot{\gamma}^\rho{}_{\nu\mu} u^\nu u^\mu = 0, \quad (2.20)$$

where

$$\dot{\gamma}^\rho{}_{\nu\mu} = e_c{}^\rho \partial_\mu e^c{}_\nu + e_c{}^\rho \dot{A}^c{}_{b\mu} e^b{}_\nu \equiv e_c{}^\rho \dot{\mathcal{D}}_\mu e^c{}_\nu \quad (2.21)$$

is the spacetime-indexed version of the inertial spin connection $\dot{A}^a{}_{b\mu}$, obtained through contractions with the trivial tetrad $e^a{}_\mu$. Of course, since it has vanishing torsion, it is symmetric in the last two indices:

$$\dot{\gamma}^\rho{}_{\nu\mu} = \dot{\gamma}^\rho{}_{\mu\nu}. \quad (2.22)$$

The inverse relation is

$$\dot{A}^a{}_{b\mu} = e^a{}_\rho \partial_\mu e_b{}^\rho + e^a{}_\rho \dot{\gamma}^\rho{}_{\nu\mu} e_b{}^\nu \equiv e^a{}_\rho \dot{\nabla}_\mu e_b{}^\rho. \quad (2.23)$$

In an inertial frame $e'^a{}_\mu$, where $\dot{A}'^a{}_{b\mu} = 0$, we see from Eq. (2.21) that

$$\dot{\gamma}'^\rho{}_{v\mu} = e'^\rho{}_c \partial_\mu e'^c{}_v. \quad (2.24)$$

In cartesian coordinates, where $e'^a{}_\mu = \delta^a_\mu$, the connection $\dot{\gamma}'^\rho{}_{v\mu}$ vanishes and the equation of motion (2.20) assumes the usual form

$$\frac{du'^\rho}{d\sigma} = 0 \quad (2.25)$$

with $u'^\rho = e'^a{}_\mu u'^a$.

We have shown above how inertial and coordinate effects show up in the equation of motion of a free particle. Actually, this can be done for any relativistic equation. For example, in an inertial frame, and using cartesian coordinates, the sourceless Maxwell's equation reads

$$\partial_\mu F^{\mu\nu} = 0, \quad (2.26)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (2.27)$$

is the field strength, with A^μ the electromagnetic potential. In a non-inertial frame, and considering a general coordinate system, it assumes the manifestly covariant form

$$\dot{\nabla}_\mu F^{\mu\nu} = 0, \quad (2.28)$$

with the field strength given now by

$$F^{\mu\nu} = \dot{\nabla}^\mu A^\nu - \dot{\nabla}^\nu A^\mu = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (2.29)$$

the last equality coming from the symmetry of $\dot{\gamma}'^\rho{}_{v\mu}$ in the last two indices. And so on for any relativistic equation of Physics. Of course, for the sake of simplicity, one always uses inertial frames when dealing with field theory, where these effects are absent and the inertial connection (2.7) vanishes.

Comment 2.2 Inertial frames can only be defined in absence of gravitation. In the presence of gravitation, however, it is possible to define some generalizations. For example, one can introduce the notion of inertia-free frames. In these *global* frames, the inertial connection $\dot{A}^a{}_{bv}$ vanishes, and consequently their coefficients of anholonomy represent gravitation only, not inertia. This kind of frame can only be defined in the context of Teleparallel Gravity, and will be studied in more detail in Chap. 6. On the other hand, in the context of General Relativity, it is possible to define what is usually called a *locally inertial frame* [2]. It is a local frame in which the general relativistic spin connection $\dot{A}^a{}_{bv}$ vanishes. In such local frame gravitation is exactly compensated by inertial effects, so that gravitation becomes undetectable at a given point. Of course, in order to produce an inertial effect that exactly compensates gravitation, this frame must be accelerated, and consequently cannot be inertial. Its name comes from the fact that, in this local frame the laws of Physics reduce to that of Special Relativity *as described from an inertial frame*.

2.3 Four-Acceleration and Parallel Transport

Let us consider now a general riemannian spacetime with metric

$$g_{\mu\nu} = \eta_{ab} h^a{}_{\mu} h^b{}_{\nu}. \quad (2.30)$$

A curve $\gamma(s)$ on this spacetime, parametrized by proper time s , with

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.31)$$

will have as four-velocity the vector of components

$$u^\mu = \frac{dx^\mu}{ds}. \quad (2.32)$$

The corresponding four-acceleration cannot be given a covariant meaning without a connection—and each different connection $\Gamma^\rho{}_{\mu\nu}$ will define a different four-acceleration

$$a^\rho \equiv u^\nu \nabla_\nu u^\rho = \frac{du^\rho}{ds} + \Gamma^\rho{}_{\mu\nu} u^\mu u^\nu. \quad (2.33)$$

But acceleration must remain a measure of the velocity variation with *time*, and time appears in that formula as the proper time defined by metric $g_{\mu\nu}$. If acceleration is to keep a meaning, it is necessary that the same metric be considered all along the curve. In other words, the acceleration-defining connection must parallel-transport $g_{\mu\nu}$, satisfying the metricity condition (1.46).

Comment 2.3 Observe that, differently from the four-acceleration, the definition of the four-velocity does not require a connection. In fact, even defined with an ordinary derivative, the four-velocity u^μ turns out to be a four-vector. The reason is that x^μ is not a four-vector, but a set of four scalar functions $\gamma^\mu(s)$ parameterizing the curve γ . As such, its ordinary derivative turns out to be covariant. This is similar to what happens with a scalar field, whose ordinary four-derivative is (in this case) covariant.

As a^ρ is orthogonal to u^ρ , its vanishing means that the u^ρ keeps parallel to itself along the curve. This leads to the notion of parallel transport: we say that u^ρ is parallel-transported along γ when $a^\rho = 0$. Further, as every vector field is locally tangent to a curve (its local “integral curve”), a condition like

$$z^\mu \nabla_\mu u^\rho \equiv \nabla_{z^\rho} u^\rho = 0 \quad (2.34)$$

says that u^ρ is parallel-transported along the integral curve of z^ρ . The metric compatibility condition (1.46) implies that

$$z^\rho \nabla_\rho g_{\mu\nu} = 0 \quad (2.35)$$

for any vector field z^ρ , which is equivalent to say that the metric $g_{\mu\nu}$ is parallel-transported everywhere on spacetime. This is true, in particular, for the Levi-Civita connection (1.66), in which case we get

$$z^\rho \overset{\circ}{\nabla}_\rho g_{\mu\nu} = 0. \quad (2.36)$$

2.4 Inertial Effects

Let us consider now an observer attached to a particle moving along curve γ . An observer is abstractly conceived as a timelike worldline [3, 4]. More: notice that the four members of a tetrad are (pseudo-)orthogonal to each other. This means that one of them is timelike, and the other three are spacelike. As

$$\eta_{ab} = h_{av} h_b^v, \quad (2.37)$$

then

$$h_{0v} h_0^v = \eta_{00} = +1,$$

so that, in our convention with $\eta = \text{diag}(+1, -1, -1, -1)$, the member h_0 is timelike and has unit modulus. The remaining h_k ($k = 1, 2, 3$) are spacelike. We then “attach” h_0 to the observer by identifying

$$u = h_0 = \frac{d}{ds}, \quad (2.38)$$

with components $u^\mu = h_0^\mu$. Of course, h_0 will be the observer velocity. The tetrad field, in this way, is made into a *reference frame*, with an observer attached to it.

Take now a general connection Γ and examine the corresponding frame acceleration

$$a_{(f)}^a \equiv h^a{}_\rho a_{(f)}^\rho = h^a{}_\rho \Gamma^\rho{}_{\mu\nu} h_0^\mu h_0^\nu + h^a{}_\rho h_0 (h_0^\rho). \quad (2.39)$$

Comparing with the spin connection components,

$$A^a{}_{bc} \equiv h^a{}_\rho \nabla_{h_c} h_b^\rho = h^a{}_\rho \Gamma^\rho{}_{\mu\nu} h_b^\mu h_c^\nu + h^a{}_\rho h_c (h_b^\rho), \quad (2.40)$$

we see that

$$a_{(f)}^a = A^a{}_{00} \quad (2.41)$$

for whatever connection. As $A^a{}_{bc}$ is antisymmetric in the first two indices, only $a_{(f)}^k$ is different from zero. The definition

$$A^a{}_{bc} \equiv h^a{}_\rho \nabla_{h_c} h_b^\rho, \quad (2.42)$$

which in words is the covariant derivative of h_b along h_c , projected along h_a , provides a general interpretation for $A^a{}_{bc}$: it is a *generalized frame proper acceleration*.

These considerations give a new perception of the acceleration

$$a_{(f)}^k = \frac{du^k}{ds} + A^k{}_{bc} u^b u^c, \quad (2.43)$$

as seen from an accelerated frame. Besides the first, kinetic term, it includes contributions from the frame itself. It can be decomposed in the form

$$a_{(f)}^k = \frac{du^k}{ds} + a_{(f)}^k u^0 u^0 + 2[u \times \omega_{(f)}]^k u^0 + A^k{}_{ij} u^i u^j, \quad (2.44)$$

where

$$\omega_{(f)}^k = \frac{1}{2} \varepsilon^{kij} A_{ij0} \quad (2.45)$$

is the space tetrad members angular velocity. Some of the terms turning up in (2.44) can be easily recognized: piece

$$a_{lin}^k = a_{(f)}^k u^0 u^0 \quad (2.46)$$

represents the frame linear acceleration, whereas piece

$$a_{Cor}^k = 2[u \times \omega_{(f)}]^k u^0 \quad (2.47)$$

represents the Coriolis force. The last piece represents additional inertial effects present in the frame [5].

Comment 2.4 Another transport, distinct from parallel transport, can be introduced which absorbs the inertial effects. Applied on a four-vector z^ρ , it is given by the Fermi-Walker derivative:

$$\nabla_u^{FW} z^\rho = \nabla_u z^\rho + a_\nu u^\rho z^\nu - a^\rho u_\nu z^\nu.$$

In the specific case of the Levi-Civita connection of General Relativity, it assumes the form

$$\overset{\circ}{\nabla}_u^{FW} z^\rho = \overset{\circ}{\nabla}_u z^\rho + \overset{\circ}{a}_\nu u^\rho z^\nu - \overset{\circ}{a}^\rho u_\nu z^\nu.$$

Applied to the four-velocity, it reads

$$\overset{\circ}{\nabla}_u^{FW} u^\rho = \overset{\circ}{\nabla}_u u^\rho + \overset{\circ}{a}_\nu u^\rho u^\nu - \overset{\circ}{a}^\rho u_\nu u^\nu.$$

Using the identities $u_\nu u^\nu = 1$ and $\overset{\circ}{a}_\nu u^\nu = 0$, it reduces to

$$\overset{\circ}{\nabla}_u^{FW} u^\rho = \overset{\circ}{\nabla}_u u^\rho - \overset{\circ}{a}^\rho.$$

Since $\overset{\circ}{\nabla}_u u^\rho = \overset{\circ}{a}^\rho$, we see that it vanishes identically:

$$\overset{\circ}{\nabla}_u^{FW} u^\rho = 0.$$

In particular,

$$\overset{\circ}{\nabla}_{h_0}^{FW} h_0^\rho \equiv \overset{\circ}{\nabla}_{h_0} h_0^\rho - \overset{\circ}{a}^\rho = 0$$

implies that h_0 , by the Fermi-Walker transport, is kept tangent along its own integral curve.

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