Chapter 2
Physics in External Gravitational Fields

I was made aware of these (works by Ricci and Levi-Civita) by my friend Grossmann in Zürich, when I put to him the problem to investigate generally covariant tensors, whose components depend only on the derivatives of the coefficients of the quadratic fundamental form.

—A. Einstein (1955)

We already emphasized in the introduction that the principle of equivalence is one of the foundation pillars of the general theory of relativity. It leads naturally to the kinematical framework of general relativity and determines, suitable interpreted, the coupling of physical systems to external gravitational fields. This will be discussed in detail in the present chapter.

2.1 Characteristic Properties of Gravitation

Among the known fundamental interactions only the electromagnetic and gravitational are of long range, thus permitting a classical description in the macroscopic limit. While there exists a highly successful quantum electrodynamics, a (unified) quantum description of gravity remains a fundamental theoretical task.

2.1.1 Strength of the Gravitational Interaction

Gravity is by far the weakest of the four fundamental interactions. If we compare for instance the gravitational and electrostatic force between two protons, we find in obvious notation

$$\frac{Gm_p^2}{r^2} = 0.8 \times 10^{-36} \frac{e^2}{r^2}. \tag{2.1}$$

N. Straumann, *General Relativity*, Graduate Texts in Physics,
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The tiny ratio of the two forces reflects the fact that the *Planck mass*

$$M_{Pl} = \left(\frac{\hbar c}{G}\right)^{1/2} = 1.2 \times 10^{19}\,\frac{\mathrm{GeV}}{c^2} \simeq 10^{-5}\ \mathrm{g} \tag{2.2}$$

is huge in comparison to known mass scales of particle physics. The numerical factor on the right in Eq. (2.1) is equal to $\alpha^{-1} m_p^2/M_{Pl}^2$, where $\alpha = e^2/\hbar c \simeq 1/137$ is the fine structure constant. Quite generally, gravitational effects in atomic physics are suppressed in comparison to electromagnetic ones by factors of the order $\alpha^n (m/M_{Pl})^2$, where $m = m_e, m_p, \ldots$ and $n = 0, \pm 1, \ldots$. There is thus no chance to measure gravitational effects on the atomic scale. Gravity only becomes important for astronomical bodies. For sufficiently large masses it sooner or later predominates over all other interactions and will lead to the catastrophic collapse to a black hole. One can show (see Chap. 7) that this is always the case for stars having a mass greater than about

$$\frac{M_{Pl}^3}{m_N^2} \simeq 2 M_\odot, \tag{2.3}$$

where m_N is the nucleon mass. Gravity wins because it is not only long range, but also universally attractive. (By comparison, the electromagnetic forces cancel to a large extent due to the alternating signs of the charges, and the exclusion principle for the electrons.) In addition, not only matter, but also antimatter, and every other form of energy acts as a source for gravitational fields. At the same time, gravity also acts on all forms of energy.

2.1.2 Universality of Free Fall

Since the time of Galilei, we learned with increasing precision that all test bodies fall at the same rate. This means that for an appropriate choice of units, the inertial mass is equal to the gravitational mass. Newton established that the "weight" of a body (its response to gravity) is proportional to the "quantity of matter" in it already to better than a part in 1000. He achieved this with two pendulums, each 11 feet long ending in a wooden box. One was a reference; into the other he put successively "gold, silver, lead, glass, common salt, wood, water and wheat". Careful observations showed that the times of swing are independent of the material. In Newton's words:

> *And by experiments made with the greatest accuracy, I have always found the quantity of matter in bodies to be proportional to their weight.*

In Newton's theory of gravity there is no explanation for this remarkable fact. A violation would not upset the conceptual basis of the theory. As we have seen in the introduction, Einstein was profoundly astonished by this fact.[1] The equality of the inertial and gravitational masses has been experimentally established with an accuracy of one part in 10^{12} (For a review, see [96]). This remarkable fact suggests the validity of the following *universality of free fall*, also called the *Weak Equivalence Principle*:

Weak Equivalence Principle (WEP) The motion of a test body in a gravitational field is independent of its mass and composition (at least when one neglects interactions of spin or of a quadrupole moment with field gradients).

For the Newtonian theory, universality is of course a consequence of the equality of inertial and gravitational masses. We postulate that it holds generally, in particular also for large velocities and strong fields.

2.1.3 Equivalence Principle

The equality of inertial and gravitational masses provides experimental support for a stronger version of the principle of equivalence.

Einstein's Equivalence Principle (EEP) In an arbitrary gravitational field no local non-gravitational experiment can distinguish a freely falling nonrotating system (local inertial system) from a uniformly moving system in the absence of a gravitational field.

Briefly, we may say that gravity can be locally transformed away.[2] Today of course, this is a well known fact to anyone who has watched space flight on television.

Remarks

1. The EEP implies (among other things) that inertia and gravity cannot be (uniquely) separated.

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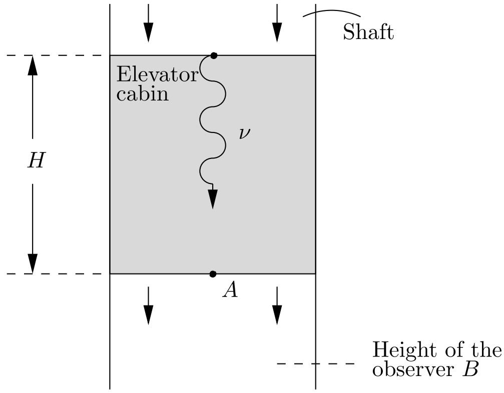
[1] In popular lectures which have only recently been published [70], H. Hertz said about inertial and gravitational mass:

> *And in reality we do have two properties before us, two most fundamental properties of matter, which must be thought as being completely independent of each other, but in our experience, and only in our experience, appear to be exactly equal. This correspondence must mean much more than being just a miracle .... We must clearly realize, that the proportionality between mass and inertia must have a deeper explanation and cannot be considered as of little importance, just as in the case of the equality of the velocities of electrical and optical waves.*

[2] This 'infinitesimal formulation' of the priniple of equivalence was first introduced by Pauli in 1921, [1, 2], p. 145. Einstein dealt only with the very simple case of homogeneous gravitational fields. For a detailed historical discussion, we refer to [81].

Fig. 2.1 EEP and blueshift



2. The formulation of the EEP is somewhat vague, since it is not entirely clear what is meant by a local experiment. At this point, the principle is thus of a heuristic nature. We shall soon translate it into a mathematical requirement.
3. The EEP is even for test bodies stronger than the universality of free fall, as can be seen from the following example. Consider a fictitious world in which, by a suitable choice of units, the electric charge is equal to the mass of the particles and in which there are no negative charges. In a classical framework, there are no objections to such a theory, and by definition, the universality property is satisfied. However, the principle of equivalence is not satisfied. Consider a homogeneous magnetic field. Since the radii and axes of the spiral motion are arbitrary, there is no transformation to an accelerated frame of reference which can remove the effect of the magnetic field on all particles at the same time.
4. We do not discuss here the so-called *strong equivalence principle (SEP)* which includes self-gravitating bodies and experiments involving gravitational forces (e.g., Cavendish experiments). Interested readers are referred to [8] and [96].

2.1.4 Gravitational Red- and Blueshifts

An almost immediate consequence of the EEP is the gravitational redshift (or blueshift) effect. (Originally, Einstein regarded this as a crucial test of GR.) Following Einstein, we consider an elevator cabin in a static gravitational field. For simplicity, we consider a homogeneous field of strength (acceleration) g, but the result (2.4) below also applies for inhomogeneous fields; this is obvious if the height H of the elevator is taken to be infinitesimal. Suppose the elevator cabin is dropped from rest at time $t = 0$, and that at the same time a photon of frequency ν is emitted from its ceiling toward the floor (see Fig. 2.1). The EEP implies two things:

(a) The light arrives at a point A of the floor at time $t = H/c$;
(b) no frequency shift is observed in the freely falling cabin.

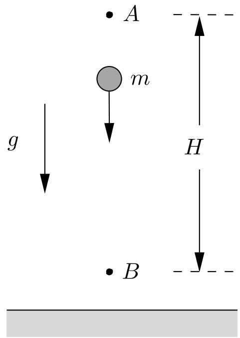


Fig. 2.2 Conservation of energy

Consider beside A an observer B at rest in the shaft at the same height as the point A of the floor when the photon arrives there. Clearly, B moves relative to A with velocity $v \simeq gt$ (neglecting higher order terms in t). Therefore, B sees the light Doppler shifted to the blue by the amount (in first order)[3]

$$z := \frac{\Delta\nu}{\nu} \simeq \frac{v}{c} \simeq \frac{gH}{c^2}.$$

If we write this as

$$z = \frac{\Delta\phi}{c^2}, \tag{2.4}$$

where $\Delta\phi$ is the difference in the Newtonian potential between the receiver and the emitter at rest at different heights, the formula also holds for inhomogeneous gravitational fields to first order in $\Delta\phi/c^2$. (The exact general relativistic formula will be derived in Sect. 2.9.)

Since the early 1960s the consequence (2.4) of the EEP has been tested with increasing accuracy. The most precise result so far was achieved with a rocket experiment that brought a hydrogen-maser clock to an altitude of about 10,000 km. The data confirmed the prediction (2.4) to an accuracy of 2×10^{-4}. Gravitational redshift effects are routinely taken into account for Earth-orbiting clocks, such as for the Global Positioning System (GPS). For further details see [96].

At the time when Einstein formulated his principle of equivalence in 1907, the prediction (2.4) could not be directly verified. Einstein was able to convince himself of its validity indirectly, since (2.4) is also a consequence of the conservation of energy. To see this, consider two points A and B, with separation H in a homogeneous gravitational field (see Fig. 2.2). Let a mass m fall with initial velocity zero from A to B. According to the Newtonian theory, it has the kinetic energy mgH at point B. Now let us assume that at B the entire energy of the falling body (rest energy plus kinetic energy) is annihilated to a photon, which subsequently returns to the point A. If the photon did not interact with the gravitational field, we could convert it back

[3] B is, of course, not an inertial observer. It is, however, reasonable to assume that B makes the same measurements as a freely falling (inertial) observer B' momentarily at rest relative to B.

to the mass m and gain the energy mgH in each cycle of such process. In order to preserve the conservation of energy, the photon must experience a redshift. Its energy must satisfy

$$E_{lower} = E_{upper} + mgH = mc^2 + mgH = E_{upper}\left(1 + \frac{gH}{c^2}\right).$$

For the wavelengths we then have (h is Planck's constant)

$$1 + z = \frac{\lambda_{upper}}{\lambda_{lower}} = \frac{h\nu_{lower}}{h\nu_{upper}} = \frac{E_{lower}}{E_{upper}} = 1 + \frac{gH}{c^2},$$

in perfect agreement with (2.4).

2.2 Special Relativity and Gravitation

> *That the special theory of relativity is only the first step of a necessary development became completely clear to me only in my efforts to represent gravitation in the framework of this theory.*
>
> —A. Einstein
> (Autobiographical Notes, 1949)

From several later recollections and other sources we know that Einstein recognized very early that gravity does not fit naturally into the framework of special relativity. In this section, we shall discuss some arguments which demonstrate that this is indeed the case.

2.2.1 Gravitational Redshift and Special Relativity

According to special relativity, a clock moving along the timelike world line $x^\mu(\lambda)$ measures the proper time interval

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda, \tag{2.5}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. In the presence of a gravitational field, (2.5) can no longer be valid, as is shown by the following argument.

Consider a redshift experiment in the Earth's gravitational field and assume that a special relativistic theory of gravity exists, which need not be further specified here. For such an experiment we may neglect all masses other than that of the Earth and regard the Earth as being at rest relative to some inertial system. In a spacetime diagram (height z above the Earth's surface versus time), the Earth's surface, the emitter and the absorber all move along world lines of constant z (see Fig. 2.3).

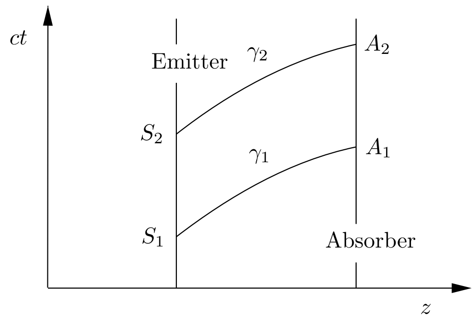


Fig. 2.3 Redshift in the Earth field

The transmitter is supposed to emit at a fixed frequency from S_1 to S_2. The photons registered by the absorber move along world lines γ_1 and γ_2, that are not necessarily straight lines at an angle of 45°, due to a possible interaction with the gravitational field, but must be *parallel*, since we are dealing with a static situation. Thus, if the flat Minkowski geometry holds and the time measurement is given by (2.5), it follows that the time difference between S_1 and S_2 must be equal to the time difference between A_1 and A_2. Thus, there would be no redshift. This shows that at the very least (2.5) is no longer valid. The argument does not exclude the possibility that the metric $g_{\mu\nu}$ might be proportional to $\eta_{\mu\nu}$. (This possibility will be rejected below.)

2.2.2 Global Inertial Systems Cannot Be Realized in the Presence of Gravitational Fields

In Newtonian–Galileian mechanics and in special relativity, the law of inertia distinguishes a special class of equivalent frames of reference (inertial systems). Due to the universality of gravitation, only the free fall of electrically neutral test bodies can be regarded as particularly distinguished motion in the presence of gravitational fields. Such bodies experience, however, relative accelerations. There is no operational procedure to uniquely separate inertia and gravitation. In spite of this, the fiction of a linear affine Galilei spacetime (with a flat affine connection) is maintained in the traditional presentation of Newton's theory,[4] and gravity is put on the side of the forces. But since the concept of an inertial system cannot be defined operationally, we are deprived of an essential foundation of the special theory of relativity.

We no longer have any reason to describe spacetime as a linear affine space. The absolute, integrable affine structure of the spacetime manifold in Newtonian–Galilean mechanics and in special relativity was, after all, suggested by the *law of*

[4] A more satisfactory formulation was given by E. Cartan, [67, 68] and K. Friederichs, [69] (see also Exercise 3.2).

inertia. A more satisfactory theory should account for inertia and gravity in terms of a single, indecomposable structure.

2.2.3 Gravitational Deflection of Light Rays

Consider again the famous Einstein elevator cabin in an elevator shaft attached to the Earth, and a light ray emitted perpendicular to the direction of motion of the freely falling cabin. According to the principle of equivalence, the light ray propagates along a straight line inside the cabin relative to the cabin. Since the elevator is accelerated relative to the Earth, one expects that the light ray propagates along a parabolic path relative to the Earth. This consequence of the EEP holds, a priori, only *locally*. It does not necessarily imply bending of light rays from a distant source traversing the gravitational field of a massive body and arriving at a distant observer. Indeed, we shall see later that it is possible to construct a theory which satisfies the principle of equivalence, but in which there is no deflection of light. (For a detailed discussion of how this comes about, see [95].) At any rate, the deflection of light is an experimental fact (the precise magnitude of the effect does not concern us at the moment).

It is therefore not possible to describe the gravitational field (as in the Einstein–Fokker theory, discussed in Sect. 3.2) in terms of a *conformally flat metric*, i.e., by a metric field proportional to the Minkowski metric: $g_{\mu\nu}(x) = \phi(x)\eta_{\mu\nu}$, where $\phi(x)$ plays the role of the gravitational potential. Indeed, for such a metric the light cones are the same as in the Minkowski spacetime; hence, there is no light deflection.

2.2.4 Theories of Gravity in Flat Spacetime

> *I see the most essential thing in the overcoming of the inertial system, a thing that acts upon all processes, but undergoes no reaction. This concept is, in principle, no better than that of the center of the universe in Aristotelian physics.*
>
> —A. Einstein (1954)

In spite of these arguments one may ask, how far one gets with a theory of gravity in Minkowski spacetime, following the pattern of well understood field theories, such as electrodynamics. Attempts along these lines have a long tradition, and are quite instructive. Readers with some background in special relativistic (classical) field theory should find the following illuminating. One may, however, jump directly to the conclusion at the end of this subsection (p. 18).

Scalar Theory

Let us first try a scalar theory. This simplest possibility was studied originally by Einstein, von Laue, and others, but was mainly developed by G. Nordstrøm, [62–64].

The field equation for the scalar field φ, generalizing the Newtonian potential, in the limit of *weak* fields (linear field equation) is unique:

$$\Box\varphi = -4\pi G T. \tag{2.6}$$

Here, T denotes the trace of the energy-momentum tensor $T^{\mu\nu}$ of matter. For a Newtonian situation this reduces to the Poisson equation.

We formulate the equation of motion of a test particle in terms of a Lagrangian. For weak fields this is again unique:

$$L\big(x^\mu, \dot{x}^\mu\big) = -\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}(1+\varphi), \tag{2.7}$$

because only for this the Newtonian limit for weak static fields and small velocities of the test bodies comes out right:

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) \approx \frac{1}{2}\dot{\boldsymbol{x}}^2 - \varphi + \text{const.}$$

The basic equations (2.6) and (2.7) imply a perihelion motion of the planets, but this comes out wrong, even the sign is incorrect. One finds $(-1/6)$ times the value of general relativity (see Exercise 2.3). In spite of this failure we add some further instructive remarks.

First, we want to emphasize that the interaction is necessarily *attractive*, independent of the matter content. To show this, we start from the general form of the Lagrangian density for the scalar theory

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + gT\cdot\phi + \mathcal{L}_{mat}, \tag{2.8}$$

where ϕ is proportional to φ and g is a coupling constant. Note first that only g^2 is significant: Setting $\tilde{\phi} = g\phi$, we have

$$\mathcal{L} = -\frac{1}{2g^2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + T\cdot\tilde{\phi} + \mathcal{L}_{mat},$$

involving only g^2. Next, it has to be emphasized that it is not allowed to replace g^2 by $-g^2$, otherwise the field energy of the gravitational field would be negative. (This "solution" of the energy problem does not work.) Finally, we consider the field energy for *static* sources.

The total (canonical) energy-momentum tensor

$$T^\mu_\nu = -\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\phi_{,\nu} + \cdots + \delta^\mu_\nu\mathcal{L}$$

gives for the ϕ-contribution

$$(T_\phi)_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi + \eta_{\mu\nu}gT\phi.$$

For the corresponding total energy we find

$$E = \int (T_\phi)_{00}\, d^3x = \frac{1}{2}\int \left((\nabla\phi)^2 - 2gT\phi\right) d^3x$$
$$= \frac{1}{2}\int \left(\phi(-\Delta\phi) - 2gT\phi\right) d^3x = -\frac{1}{2}g\int T\phi\, d^3x. \tag{2.9}$$

Since $\Delta\phi = -gT$, we have

$$\phi(\boldsymbol{x}) = \frac{g}{4\pi}\int \frac{T(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'.$$

Inserting this in (2.9) gives finally

$$E = -\frac{g^2}{4\pi}\frac{1}{2}\int \frac{T(\boldsymbol{x})T(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x\, d^3x',$$

showing that indeed the interaction is attractive.

This can also be worked out in quantum field theory by computing the effective potential corresponding to the one-particle exchange diagram with the interaction Lagrangian $\mathcal{L}_{int} = g\bar{\psi}\psi\phi_{m=0}$. One finds

$$V_{eff} = -\frac{g^2}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

both for fermion-fermion and fermion-antifermion interactions. The same result is found for the exchange of massless spin-2 particles, while for spin-1 we obtain *repulsion* between particles, and attraction between particles and antiparticles (see Exercise 2.4).

The scalar theory predicted that there is *no* light deflection, simply because the trace of the electromagnetic energy-momentum tensor vanishes. For this reason Einstein urged in 1913 astronomers (E. Freundlich in Potsdam) to measure the light deflection during the solar eclipse the coming year in the Crimea. Shortly before the event the first world war broke out. Over night Freundlich and his German colleagues were captured as prisoners of war and it took another five years before the light deflection was observed. For further discussion of the scalar theory we refer to [97], and references therein.

Tensor (spin-2) Theory

We are led to study the spin-2 option. (There are no consistent higher spin equations with interaction.) This means that we try to describe the gravitational field by a symmetric tensor field $h_{\mu\nu}$. Such a field has 10 components. On the other hand, we learned from Wigner that in the massless case there are only *two* degrees of freedom. How do we achieve the truncation from 10 to 2?

Recall first the situation in the *massive case*. There we can require that the trace $h = h^\mu_\mu$ vanishes, and then the field $h_{\mu\nu}$ transforms with respect to the homogeneous Lorentz group irreducibly as $D^{(1,1)}$ (in standard notation). With respect to the subgroup of rotations this reduces to the reducible representation

$$D^1 \otimes D^1 = D^2 \oplus D^1 \oplus D^0.$$

The corresponding unwanted spin-1 and spin-0 components are then eliminated by imposing 4 subsidiary conditions

$$\partial_\mu h^\mu_{\ \nu} = 0.$$

The remaining 5 degrees of freedom describe (after quantization) massive spin-2 particles (W. Pauli and M. Fierz, [98, 99]; see, e.g., the classical book [22]).

In the *massless case* we have to declare certain classes of fields as physically equivalent, by imposing—as in electrodynamics—a *gauge invariance*. The gauge transformations are

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{2.10}$$

where ξ_μ is an arbitrary vector field.

Let us first consider the *free* spin-2 theory which is unique (W. Pauli and M. Fierz)

$$\mathcal{L} = -\frac{1}{4} h_{\mu\nu,\sigma} h^{\mu\nu,\sigma} + \frac{1}{2} h_{\mu\nu,\sigma} h^{\sigma\nu,\mu} + \frac{1}{4} h_{,\sigma} h^{,\sigma} - \frac{1}{2} h_{,\sigma} h^{\nu\sigma}_{\ \ ,\nu}. \tag{2.11}$$

Let $G_{\mu\nu}$ denote the Euler–Lagrange derivative of $\mathcal{L}$,

$$\begin{aligned} G_{\mu\nu} = {} & \frac{1}{2} \partial^\sigma \partial_\sigma h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\nu \partial^\sigma h_{\mu\sigma} - \partial_\mu \partial^\sigma h_{\sigma\nu} \\ & + \eta_{\mu\nu} \big(\partial^\alpha \partial^\beta h_{\alpha\beta} - \partial^\sigma \partial_\sigma h \big). \end{aligned} \tag{2.12}$$

The free field equations

$$G_{\mu\nu} = 0 \tag{2.13}$$

are identical to the linearized Einstein equations (as shown in Sect. 5.1) and describe, for instance, the propagation of weak gravitational fields.

The gauge invariance of $\mathcal{L}$ (modulo a divergence) implies the identity

$$\partial_\nu G^{\mu\nu} \equiv 0, \quad \text{“linearized Bianchi identity”}. \tag{2.14}$$

This should be regarded in analogy to the identity $\partial_\mu(\Box A^\mu - \partial^\mu \partial_\nu A^\nu) \equiv 0$ for the left-hand side of Maxwell's equations.

Let us now introduce couplings to matter. The simplest possibility is the linear coupling

$$\mathcal{L}_{int} = -\frac{1}{2} \kappa h_{\mu\nu} T^{\mu\nu}, \tag{2.15}$$

leading to the field equation

$$G^{\mu\nu} = \frac{\kappa}{2} T^{\mu\nu}. \tag{2.16}$$

This can, however, not yet be the final equation, but only an approximation for weak fields. Indeed, the identity (2.14) implies $\partial_\nu T^{\mu\nu} = 0$ which is unacceptable (in contrast to the charge conservation of electrodynamics). For instance, the motion of a fluid would then not at all be affected by the gravitational field. Clearly, we must introduce a *back-reaction* on matter. Why not just add to $T^{\mu\nu}$ in (2.16) the energy-momentum tensor ${}^{(2)}t^{\mu\nu}$ which corresponds to the *Pauli–Fierz Lagrangian* (2.11)? But this modified equation cannot be derived from a Lagrangian and is still not consistent, but only the second step of an iteration process

$$\mathcal{L}_{free} \longrightarrow {}^{(2)}t^{\mu\nu} \longrightarrow \mathcal{L}_{cubic} \longrightarrow {}^{(3)}t^{\mu\nu} \longrightarrow \cdots ?$$

The sequence of arrows has the following meaning: A Lagrangian which gives the quadratic terms ${}^{(2)}t^{\mu\nu}$ in

$$G^{\mu\nu} = \frac{\kappa}{2}\left(T^{\mu\nu} + {}^{(2)}t^{\mu\nu} + {}^{(3)}t^{\mu\nu} + \cdots\right) \tag{2.17}$$

must be cubic in $h_{\mu\nu}$, and in turn leads to cubic terms ${}^{(3)}t^{\mu\nu}$ of the gravitational energy-momentum tensor. To produce these in the field equation (2.17), we need quartic terms in $h_{\mu\nu}$, etc. This is an infinite process. By a clever reorganization it stops already after the second step, and one arrives at field equations which are equivalent to Einstein's equations (see [100]). The physical metric of GR is given in terms of $\phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$ by

$$\sqrt{-g}\,g^{\mu\nu} = \eta^{\mu\nu} - \phi^{\mu\nu}, \tag{2.18}$$

where $g := \det(g_{\mu\nu})$.

At this point one can reinterpret the theory geometrically. Thereby the flat metric disappears completely and one arrives in a pedestrian way at GR. This approach is further discussed in [97]. There it is also shown that $g_{\mu\nu}$ is really the *physical* metric.

Conclusion The consequent development of the theory shows that it is possible to eliminate the flat Minkowski metric, leading to a description in terms of a curved metric which has a direct physical meaning. The originally postulated Lorentz invariance turns out to be physically meaningless and plays no useful role. The flat Minkowski spacetime becomes a kind of unobservable ether. The conclusion is inevitable that spacetime is a pseudo-Riemannian (Lorentzian) manifold, whereby the metric is a dynamical field, subjected to field equations.

2.2.5 Exercises

Exercise 2.1 Consider a homogeneous electric field in the z-direction and a charged particle with $e = m$. Show that a particle which, originally at rest, moves faster in the vertical direction than a particle which was originally moving horizontally.

Exercise 2.2 Consider a self gravitating body (star) moving freely in the neighborhood of a black hole. Estimate at which distance D the star is disrupted by relative forces due to inhomogeneities of the gravitational field (*tidal forces*).

Solution Relative gravitational accelerations in Newtonian theory are determined by the second derivative of the Newtonian potential. A satellite with mass M and radius R at distance r from a compact body (neutron star, black hole) of mass M_c experiences a tidal force at the surface (relative to the center) of magnitude

$$\left|\frac{d}{dr}\left(\frac{GM_c}{r^2}\right)R\right|.$$

Once this becomes larger than the gravitational acceleration of its own field at the surface, the satellite will be disrupted. The critical distance D is thus estimated to be

$$D \simeq \left(\frac{2M_c}{M}\right)^{1/3} R.$$

Let us introduce the average mass density $\bar{\rho}$ of the satellite by $M = \frac{4\pi}{3}R^3\bar{\rho}$, then

$$D \simeq \left(\frac{3}{2\pi}\right)^{1/3}\left(\frac{M_c}{\bar{\rho}}\right)^{1/3}$$

Put in the numbers for $M_c \simeq 10^8 M_\odot$ and the parameters of the sun for the satellite. Compare D with the *Schwarzschild radius* $R_s = 2GM_c/c^2$ for M_c.

Exercise 2.3 Determine the perihelion motion for Nordstrøm's theory of gravity (basic equations (2.6) and (2.7)). Compare the result with that of GR, derived in Sect. 4.3. Even the sign turns out to be wrong.

Exercise 2.4 Show that a vector theory of gravity, similar to electrodynamics, leads necessarily to *repulsion*.

2.3 Spacetime as a Lorentzian Manifold

Either, therefore, the reality which underlies space must form a discrete manifold, or we must seek the ground of its metric relations (measure conditions) outside it, in binding forces which act upon it.

—B. Riemann (1854)

The discussion of Sect. 2.2 has shown that in the presence of gravitational fields the spacetime description of SR has to be generalized. According to the EEP, special relativity remains, however, valid in "infinitesimal" regions. This suggests that the metric properties of spacetime have to be described by a symmetric tensor field $g_{\mu\nu}(p)$ for which it is not possible to find coordinate systems such that

$g_{\mu\nu}(p) = \mathrm{diag}(-1, 1, 1, 1)$ in finite regions of spacetime. This should only be possible when no true gravitational fields are present. We therefore postulate: *The mathematical model for* spacetime *(i.e., the set of all elementary events) in the presence of gravitational fields is a pseudo-Riemannian manifold M, whose metric g has the same signature as the Minkowski metric. The pair (M, g) is called a* Lorentz manifold *and g is called a* Lorentzian metric.

Remark At this point, readers who are not yet familiar with (pseudo-) Riemannian geometry should study the following sections of the differential geometric part at the end of the book: All of Chaps. 11 and 12, and Sects. 15.1–15.6. These form a self-contained subset and suffice for most of the basic material covered in the first two chapters (at least for a first reading). References to the differential geometric Part III will be indicated by DG.

As for Minkowski spacetime, the metric g determines, beside the metric properties, also the causal relationships, as we shall see soon. At the same time, we also interpret the metric field as the *gravitational potential*. In the present chapter our goal is to describe how it influences non-gravitational systems and processes.

Among the metric properties, the generalization of (2.5) of the *proper time* interval for a timelike curve $x^\mu(\lambda)$ (i.e., a curve with timelike tangent vectors) is

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda. \tag{2.19}$$

A good (atomic) clock, moving along $x^\mu(\lambda)$, measures this proper time.

The coupling of the metric to non-gravitational systems should satisfy two principles. First, the basic equations must have *intrinsic* meaning in (M, g). In other words, they should be expressible in terms of the intrinsic calculus on Lorentz manifolds, developed in DG. Equivalently, the basic laws should not distinguish any coordinate system. All charts of any atlas, belonging to the differential structure, are on the same footing. One also says that the physical laws have to be *covariant* with respect to smooth coordinate transformations (or are *generally covariant*). Let us formulate this property more precisely:

Definition A system of equations is *covariant with respect to the group $\mathcal{G}(M)$* of (germs of) smooth coordinate transformations, provided that for any element of $\mathcal{G}(M)$ the quantities appearing in the equations can be transformed to new quantities in such a way that

(i) the assignment preserves the group structure of $\mathcal{G}(M)$;
(ii) both the original and the transformed quantities satisfy the same system of equations.

Only generally covariant laws have an intrinsic meaning in the Lorentz manifold. If a suitable calculus is used, these can be formulated in a coordinate-free manner. The general covariance is at this point a matter of course. It should, however, not be confused with general *invariance*. The difference of the two concepts will be clarified later when we shall consider the coupled dynamical system of metric plus matter variables. It will turn out that general invariance is, like gauge invariance, a powerful symmetry principle (see Sect. 3.5).

From DG, Sect. 15.3, we know that in a neighborhood of every point p a coordinate system exists, such that

$$g_{\mu\nu}(p) = \eta_{\mu\nu} \quad \text{and} \quad g_{\mu\nu,\lambda}(p) = 0, \tag{2.20}$$

where $(\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$. Such coordinates are said to be inertial or normal at p, and are interpreted as *locally inertial systems*. We also say that such a coordinate system is *locally inertial with origin* p. The metric g describes the behavior of clocks and measuring sticks in such locally inertial systems, exactly as in special relativity. Relative to such a system, the usual laws of electrodynamics, mechanics, etc. in the special relativistic form are locally valid. The form of these laws for an arbitrary system is to a large extent determined by the following two requirements (we shall discuss possible ambiguities in Sect. 2.4.6):

(a) Aside from the metric and its derivatives, the laws should contain only quantities which are also present in the special theory of relativity.[5]
(b) The laws must be generally covariant and reduce to the special relativistic form at the origin of a locally inertial coordinate system.

These requirements provide a *mathematical* formulation of Einstein's Equivalence Principle. We shall soon arrive at a more handy prescription.

2.4 Non-gravitational Laws in External Gravitational Fields

We now apply this mathematical formulation of the EEP and discuss possible ambiguities at the end of this section. Familiarity with the concept of covariant differentiation (DG, Sects. 15.1–15.6) will be assumed. We add here some remarks about notation.

Remarks (Coordinate-free versus abstract index notation) Modern mathematical texts on differential geometry usually make use of indices for vectors, tensors, etc. only for their components relative to a local coordinate system or a *frame*, i.e. a local basis of vector fields. If indices for the objects themselves are totally avoided, computations can, however, quickly become very cumbersome, especially when higher rank tensors with all sorts of contractions are involved. For this reason, relativists usually prefer what they call the *abstract index notation*. This has *nothing* to do with coordinates or frames. For instance, instead of saying: "… let *Ric* be the Ricci tensor, u a vector field, and consider $Ric(u, u)$ …", one says: "… let $R_{\mu\nu}$ be the Ricci

[5] It is not permitted to introduce in addition to $g_{\mu\nu}$ other "external" (absolute) elements such as a flat metric which is independent of g.

tensor, u^μ a vector field, and consider $R_{\mu\nu}u^\mu u^\nu, \ldots$." In particular, the number and positions of indices specify which type of tensor is considered, and repeated indices indicate the type of contraction that is performed.

Some authors (e.g., R. Wald in his text book [9]) distinguish this abstract meaning of indices from the usual component indices by using different alphabets. We do not want to adopt this convention, since it should be clear from the context whether the indices can be interpreted abstractly, or whether they refer to special coordinates or frames which are adapted to a particular spacetime (with certain symmetries, distinguished submanifolds, ...).

When dealing, for example, with differential forms we often avoid abstract indices and follow the habits of mathematicians. We hope that the reader will not be disturbed by our notational flexibility. After having studied the present subsection, he should be familiar with our habits.

2.4.1 Motion of a Test Body in a Gravitational Field

What is the equation of motion of a freely falling test particle? Let $\gamma(\tau)$ be its timelike world line, parameterized by the proper time τ. According to (2.19) its tangent vector $\dot\gamma$ (four-velocity) satisfies

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1 \quad \text{or} \quad g(\dot\gamma,\dot\gamma) =: \langle\dot\gamma,\dot\gamma\rangle = -1. \tag{2.21}$$

Consider some arbitrary point p along the orbit $\gamma(\tau)$, and introduce coordinates which are locally inertial at p. The weak equivalence principle implies

$$\left.\frac{d^2x^\mu}{d\tau^2}\right|_p = 0. \tag{2.22}$$

Since the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ vanish at p, we can write this as

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{2.23}$$

at p. This is just the geodesic equation (DG, Sect. 15.3), which is generally covariant. Therefore, Eq. (2.23) holds in any coordinate system. Moreover, since the point p is arbitrary, it is valid along the entire orbit of the test body. In coordinate-free notation it is equivalent to the statement that $\dot\gamma$ is autoparallel along γ

$$\nabla_{\dot\gamma}\dot\gamma = 0. \tag{2.24}$$

Note that (2.21) and (2.23) are compatible. This follows with the Ricci identity from

$$\frac{d}{d\tau}\langle\dot\gamma,\dot\gamma\rangle = \nabla_{\dot\gamma}\langle\dot\gamma,\dot\gamma\rangle = 2\langle\nabla_{\dot\gamma}\dot\gamma,\dot\gamma\rangle = 0.$$

The geodesic equation (2.23) is the Euler–Lagrange equation for the variational principle

$$\delta \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda = 0 \tag{2.25}$$

(see the solution of Exercise 2.5).

The basic equation (2.23) can be regarded as the generalization of the Galilean law of inertia in the presence of a gravitational field. It is a great triumph that the universality of the inertial/gravitational mass ratio is automatic. It is natural to regard the connection coefficients $\Gamma^\mu_{\alpha\beta}$ as the gravitational-inertial *field strength* relative to the coordinates $\{x^\mu\}$.

2.4.2 World Lines of Light Rays

Using the same arguments, the following equations for the world line $\gamma(\lambda)$ of a light ray, parameterized by an affine parameter λ, are obtained

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \tag{2.26a}$$

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{2.26b}$$

In other words, the world lines of light rays are null geodesics. (See also Exercise 2.7.) Later (in Sect. 2.8) we shall derive these equations from Maxwell's equations in the eikonal approximation.

At each spacetime point $p \in M$ we can consider, as in Minkowski spacetime, the past and future *null cones* in the tangent space T_pM. These are tangent to past and future *light cones*, generated by light rays ending up in p, respectively emanating from p. The set of all these light cones describes the *causal structure* of spacetime.

Relativists are used to draw spacetime diagrams. A typical example that illustrates some of the basic concepts we have introduced so far is shown in Fig. 2.4.

2.4.3 Exercises

Exercise 2.5 Let

$$\begin{aligned} \gamma : [a,b] &\longrightarrow M \\ \tau &\longmapsto \gamma(\tau), \end{aligned}$$

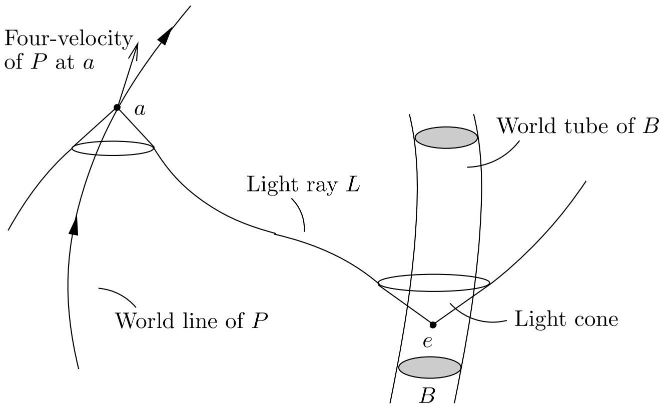


Fig. 2.4 Spacetime diagram representing a particle P, a body B and a light ray L emitted at $e \in B$ and absorbed at a point of P's worldline

be a smooth timelike curve (at least of class C^2). Show that when γ minimizes the *distance*

$$L(\gamma) = \int_a^b \sqrt{-\langle \dot{\gamma}, \dot{\gamma} \rangle}\, d\tau \tag{2.27}$$

for fixed endpoints p and q, then γ is a geodesic if τ is the proper time.

Solution Note first that $L(\gamma)$ is independent of the parametrization. For simplicity, we assume that the minimizing curve γ lies in the domain U of a chart with associated coordinates $\{x^\mu\}$. Consider a family $\{\gamma_\varepsilon\}$, $-\alpha < \varepsilon < \alpha$, of smooth ($C^2$) curves from p to q ($\gamma_\varepsilon \subset U$, $\varepsilon \in (-\alpha, \alpha)$), defined by the coordinates $x^\mu(\tau, \varepsilon) = x^\mu(\tau) + \varepsilon \xi^\mu(\tau)$, with $\xi^\mu(a) = \xi^\mu(b) = 0$. We use the notation $\cdot = \partial/\partial\tau$ and $' = \partial/\partial\varepsilon$. Since the quantity

$$L(\gamma_\varepsilon) = \int_a^b \big[-g_{\mu\nu}\big(x(\tau,\varepsilon)\big)\dot{x}^\mu(\tau,\varepsilon)\dot{x}^\nu(\tau,\varepsilon)\big]^{1/2} d\tau$$

attains a minimum at $\varepsilon = 0$, we have

$$\begin{aligned} L'(\gamma_\varepsilon)|_{\varepsilon=0} &= -\frac{1}{2}\int_a^b \big[\partial_\lambda g_{\mu\nu}\big(\gamma(\tau)\big)\xi^\lambda \dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + 2g_{\mu\nu}\big(\gamma(\tau)\big)\dot{\xi}^\mu(\tau)\dot{x}^\nu(\tau)\big]\, d\tau \\ &= 0. \end{aligned}$$

Integration by part of the second term gives

$$\int_a^b \big[(\partial_\lambda g_{\mu\nu} - 2\partial_\mu g_{\lambda\nu})\dot{x}^\mu \dot{x}^\nu - 2g_{\lambda\nu}\ddot{x}^\nu\big]\xi^\lambda\, d\tau = 0.$$

By a standard argument, the curly bracket must vanish. Renaming some indices and using the symmetry of $g_{\mu\nu}$, one finds that $x^\mu(\tau)$ satisfies the geodesic equation (2.23).

Remark We show in DG, Sect. 16.4 how this variational calculation can be done in a coordinate-free manner. For the second derivative, see, e.g., [46], Chap. 10.

Exercise 2.6 Beside $L(\gamma)$ one can consider the *energy functional*

$$E(\gamma) = \frac{1}{2}\int \langle\dot{\gamma}, \dot{\gamma}\rangle \, d\tau, \tag{2.28}$$

which depends on the parametrization. From the solution of Exercise 2.5 it should be obvious that minimization of $E(\gamma)$ again leads to the geodesic equation (we shall often use this for practical calculations). Show by a direct calculation that the geodesic equation implies

$$\frac{d}{d\tau}\big(g_{\mu\nu}\big(x(\tau)\big)\dot{x}^\mu \dot{x}^\nu\big) = 0,$$

whence the parametrization is proportional to proper time. It should be clear that the variational principle for the energy functional also applies to null geodesics.

Exercise 2.7 Consider a conformal change of the metric

$$g \longmapsto \tilde{g} = \mathrm{e}^{2\phi} g. \tag{2.29}$$

Show that a null geodesic for g is also a null geodesic for $\tilde{g}$.

Hints In transforming the geodesic equation one has to carry out a re-parametrization $\lambda \longmapsto \tilde{\lambda}$ such that $d\tilde{\lambda}/d\lambda = \mathrm{e}^{2\phi}$. The relation between the Christoffel symbols for the two metrics is

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \delta^\mu_\alpha \phi_{,\beta} + \delta^\mu_\beta \phi_{,\alpha} - g_{\alpha\beta} g^{\mu\nu} \phi_{,\nu}. \tag{2.30}$$

2.4.4 Energy and Momentum "Conservation" in the Presence of an External Gravitational Field

According to the special theory of relativity, the energy-momentum tensor $T^{\mu\nu}$ of a closed system satisfies, as a result of translation invariance, the conservation law

$$T^{\mu\nu}{}_{,\nu} = 0.$$

In the presence of a gravitational field, we define a corresponding tensor field on (M, g) such that it reduces to the special relativistic form at the origin of a locally inertial system.

Example (Energy-momentum tensor for an ideal fluid) In SR the form of the energy-momentum tensor is established as follows. A fluid is by definition *ideal* if a comoving observer sees the fluid around him as isotropic. So let us consider a *local rest frame*, that is an inertial system such that the fluid is at rest at some particular spacetime point. Relative to this system (indicated by a tilde) the energy-momentum tensor has at that point the form

$$\tilde{T}^{00} = \rho, \qquad \tilde{T}^{0i} = \tilde{T}^{i0} = 0, \qquad \tilde{T}^{ij} = p\delta_{ij}, \tag{2.31}$$

where ρ is the *proper energy density* and p is the *pressure* of the fluid ($c = 1$). The four-velocity u^μ of the fluid has in the local rest frame the value $\tilde{u}^0 = 1$, $\tilde{u}^i = 0$, and hence we can write (2.31) as

$$\tilde{T}^{\mu\nu} = (\rho + p)\tilde{u}^\mu \tilde{u}^\nu + p\eta^{\mu\nu}.$$

Since this is a tensor equation it holds in any inertial system.

In the presence of gravitational fields our general prescription leads uniquely to

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{2.32}$$

with the normalization

$$g_{\mu\nu}u^\mu u^\nu = -1. \tag{2.33}$$

of the four-velocity field. For an introduction to special relativistic fluid dynamics, see for instance [21].

Remark We shall discuss in Sect. 3.3.4 a general method of constructing the energy-momentum tensor in the framework of the Lagrangian formalism.

At the origin $p \in M$ of a locally inertial system we have, by the EEP, $T^{\mu\nu}{}_{,\nu} = 0$ at p. We may just as well write $T^{\mu\nu}{}_{;\nu} = 0$ at p, where the semicolon denotes the covariant derivative of the tensor field. This equation is generally covariant, and hence is valid in any coordinate system. We thus arrive at

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{2.34}$$

Conclusion From this consideration, we conclude quite generally that the physical laws of special relativity are changed in the presence of a gravitational field simply by the substitution of covariant derivatives for ordinary derivatives, often called the principle of minimal coupling (or comma $\longrightarrow$ semicolon rule). This is an expression of the principle of equivalence. (Possible ambiguities for higher order derivatives are discussed at the end of this section.)

In this manner the coupling of the gravitational field to physical systems is determined in an extremely simple manner.

We may write (2.34) as follows: General calculational rules give (DG, Eq. (15.23))

$$T^{\mu\nu}{}_{;\sigma} = T^{\mu\nu}{}_{,\sigma} + \Gamma^{\mu}_{\sigma\lambda} T^{\lambda\nu} + \Gamma^{\nu}_{\sigma\lambda} T^{\mu\lambda}.$$

Hence,

$$T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\nu\lambda} T^{\lambda\nu} + \Gamma^{\nu}_{\nu\lambda} T^{\mu\lambda}.$$

Now we have[6]

$$\Gamma^{\nu}_{\nu\lambda} = \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g}), \tag{2.35}$$

where g is the determinant of $(g_{\mu\nu})$. Hence, (2.34) is equivalent to

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} T^{\mu\nu}\right) + \Gamma^{\mu}_{\nu\lambda} T^{\lambda\nu} = 0. \tag{2.36}$$

Because of the second term in (2.36), this is *no longer* a conservation law. We cannot form any constants of the motion from (2.36). This should also not be expected, since the system under consideration can exchange energy and momentum with the gravitational field.

Equations (2.34) (or (2.36)) and (2.32) provide the basic hydrodynamic equations for an ideal fluid in the presence of a gravitational field (see the exercises below).

Show that (2.36) is for a symmetric $T^{\mu\nu}$ equivalent to

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} T_\mu{}^\nu\right) - \frac{1}{2} g_{\alpha\beta,\mu} T^{\alpha\beta} = 0. \tag{2.37}$$

Remark In the derivation of the field equations for the gravitational field, (2.34) will play an important role.

2.4.5 Exercises

Exercise 2.8 Contract Eq. (2.34) with u^μ and show that the stress-energy tensor (2.32) for a perfect fluid leads to

$$\nabla_u \rho = -(\rho + p) \nabla \cdot u. \tag{2.38}$$

[6] From linear algebra we know (Cramer's rule) that $g g^{\mu\nu}$ is the cofactor (minor) of $g_{\mu\nu}$, hence $\partial_\alpha g = \frac{\partial g}{\partial g_{\mu\nu}} \partial_\alpha g_{\mu\nu} = g g^{\mu\nu} \partial_\alpha g_{\mu\nu}$. This gives

$$\begin{aligned} \Gamma^{\nu}_{\nu\alpha} &= g^{\mu\nu} \frac{1}{2} (\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\nu\alpha}) = \frac{1}{2} g^{\mu\nu} \partial_\alpha g_{\mu\nu} \\ &= \frac{1}{2g} \partial_\alpha g = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g}). \end{aligned}$$

Exercise 2.9 Contract Eq. (2.34) with the "projection tensor"

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \tag{2.39}$$

and derive the following general relativistic *Euler equation* for a perfect fluid:

$$(\rho + p)\nabla_u u = -\operatorname{grad} p - (\nabla_u p)u. \tag{2.40}$$

The *gradient* of a function f is the vector field $\operatorname{grad} f := (df)^\sharp$.

2.4.6 Electrodynamics

We assume that the reader is familiar with the four-dimensional tensor formulation of electrodynamics in SR. The basic dynamical object is the antisymmetric electromagnetic field tensor $F_{\mu\nu}$, which unifies the electric and magnetic fields as follows:

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

In the language of differential forms (DG, Chap. 14) $F_{\mu\nu}$ can be regarded as the components of the 2-form

$$\begin{aligned} F &= \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \\ &= \left(E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3\right) \wedge dx^0 \\ &\quad + B_1\, dx^2 \wedge dx^3 + B_2\, dx^3 \wedge dx^1 + B_3\, dx^1 \wedge dx^2, \end{aligned} \tag{2.41}$$

sometimes called the *Faraday form*. The homogeneous Maxwell equations are

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{2.42}$$

expressing that the Faraday 2-form is closed:

$$dF = 0. \tag{2.43}$$

Obviously, this law makes no use of a metric.

If $j^\mu = (\rho, \boldsymbol{J})$ denotes the current four-vector, the inhomogeneous Maxwell equations are ($c = 1$),

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu. \tag{2.44}$$

With the calculus of differential forms this can be written as

$$\delta F = -4\pi J, \tag{2.45}$$

where δ is the codifferential[7] and J denotes the current 1-form

$$J = j_\mu \, dx^\mu. \tag{2.46}$$

The generalization of these fundamental equations to Einstein's gravity theory is simple: We have to define $F_{\mu\nu}$ and j^μ such that they transform as tensor fields, and have the same meaning as in SR in locally inertial systems. Secondly, we must apply the $\partial_\mu \longrightarrow \nabla_\mu$ rule. Maxwell's equations in GR are thus

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0, \tag{2.47a}$$

$$\nabla_\nu F^{\mu\nu} = 4\pi j^\mu, \tag{2.47b}$$

with

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}. \tag{2.48}$$

Because of (2.43) the metric should drop out in (2.47a). The reader may verify explicitly that the following identity

$$\nabla_\lambda F_{\mu\nu} + \cdots = \partial_\lambda F_{\mu\nu} + \cdots$$

holds for any antisymmetric tensor field $F_{\mu\nu}$. A more general statement is derived in DG, Sect. 15.4 (Eq. (15.25)).

As expected, the inhomogeneous equations (2.47b) imply (covariant) current conservation

$$\nabla_\mu j^\mu = 0. \tag{2.49}$$

This follows from the identity $\nabla_\mu \nabla_\nu F^{\mu\nu} \equiv 0$. A simple way to show this is to note that for an antisymmetric tensor field $F^{\mu\nu}$ and a vector field j^μ the following identities hold (see Exercise 2.10)

$$\nabla_\nu F^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right), \tag{2.50}$$

$$\nabla_\mu j^\mu \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} j^\mu \right). \tag{2.51}$$

These can also be used to rewrite (2.47b) and (2.49) as

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right) = 4\pi j^\mu, \tag{2.52}$$

$$\partial_\mu \left(\sqrt{-g} j^\mu \right) = 0. \tag{2.53}$$

In terms of differential forms, things are again much more concise. Due to the identity $\delta \circ \delta = 0$, Eq. (2.45) implies immediately $\delta J = 0$, and this is equivalent to

[7]Note the sign convention for δ adopted in DG, Sect. 14.6.4, which is not universally used.

(2.49) (see DG, Exercise 15.8). Because of Gauss' Theorem (DG, Theorem 14.12), the vanishing of the divergence of J implies an integral conservation law (conservation of electric charge).

The energy-momentum tensor of the electromagnetic field can be read off from the expression in SR

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha}F^{\nu}_{\ \alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]. \tag{2.54}$$

Note that its trace vanishes.

The Lorentz equation of motion for a charged test mass becomes in GR

$$m\left(\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right) = eF^\mu_{\ \nu}\frac{dx^\nu}{d\tau}. \tag{2.55}$$

The homogeneous Maxwell equation (2.43) allows us also in GR to introduce vector potentials, at least locally. By Poincaré's Lemma (DG, Sect. 14.4), F is locally exact

$$F = dA. \tag{2.56}$$

In components, with $A = A_\mu\, dx^\mu$, we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (\equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu). \tag{2.57}$$

As in SR there is a *gauge freedom*

$$A \longrightarrow A + d\chi \quad \text{or} \quad A_\mu \longrightarrow A_\mu + \partial_\mu\chi, \tag{2.58}$$

where χ is any smooth function. This can be used to impose gauge conditions, for instance the *Lorentz condition*

$$\nabla_\mu A^\mu = 0 \quad (\text{or } \delta A = 0). \tag{2.59}$$

We stay in this class if χ in (2.58) is restricted to satisfy

$$\Box\chi := \nabla_\mu\nabla^\mu\chi = 0. \tag{2.60}$$

In terms of the four-potential A^μ we can write the inhomogeneous Maxwell equations (2.47b) as

$$\nabla_\nu\nabla^\nu A^\mu - \nabla_\nu\nabla^\mu A^\nu = -4\pi j^\mu. \tag{2.61}$$

Let us impose the Lorentz condition. We can use this in the second term with the help of the Ricci identity for the commutator of two covariant derivatives (DG, Eq. (15.92))

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A^\alpha = R^\alpha_{\ \beta\mu\nu}A^\beta. \tag{2.62}$$

This leads to

$$\nabla_\nu \nabla^\nu A^\mu - R^\mu_\nu A^\nu = -4\pi j^\mu . \tag{2.63}$$

Note that in SR (2.63) reduces to the inhomogeneous wave equation $\partial_\nu \partial^\nu A^\mu = -4\pi j^\mu$. If we would substitute here covariant derivatives, we would miss the curvature term in (2.63). This example illustrates possible *ambiguities* in applying the $\partial \longrightarrow \nabla$ rule to second order differential equations, because covariant derivatives do not commute. In passing, we mention that without the curvature term we would, however, lose gauge invariance (see Exercise 2.11).

Let us finally derive a wave equation for F in vacuum ($J = 0$). With the calculus of exterior forms this is extremely simple: From $dF = 0$ and $\delta F = 0$ we deduce

$$\Box F = 0, \tag{2.64}$$

where

$$\Box = \delta \circ d + d \circ \delta . \tag{2.65}$$

In Exercise 2.13 the reader is asked to write this in terms of covariant derivatives, with the result (2.67).

2.4.7 Exercises

Exercise 2.10 Derive the identities (2.50) and (2.51).

Exercise 2.11 Show that the curvature term in (2.63) is needed in order to maintain gauge invariance within the Lorentz gauge class.

Exercise 2.12 Use the Ricci identity (2.60), as well as $\nabla_\mu \nabla_\nu f = \nabla_\mu \nabla_\nu f$ for functions f, to derive the following Ricci identity for covariant vector fields

$$\omega_{\alpha;\mu\nu} - \omega_{\alpha;\nu\mu} = R^\lambda{}_{\alpha\mu\nu}\omega_\lambda , \tag{2.66}$$

and its generalization for arbitrary tensor fields.

Exercise 2.13 As an application of the last exercise apply ∇^λ on the homogeneous Maxwell equations in the form (2.47a) and use the vacuum Maxwell equations $\nabla_\nu F^{\mu\nu} = 0$ to show that

$$F_{\mu\nu;\lambda}{}^{;\lambda} + \left(R^\sigma_\mu F_{\nu\sigma} - R^\sigma_\nu F_{\mu\sigma}\right) + R_{\alpha\beta\mu\nu} F^{\alpha\beta} = 0. \tag{2.67}$$

Exercise 2.14 Show that Maxwell's vacuum equations are invariant under conformal changes $g \longrightarrow e^{2\phi} g$ of the metric.

Exercise 2.15 Show that Maxwell's equations (2.47a), (2.47b) imply for the energy-momentum tensor of the electromagnetic field

$$T^{\mu\nu}{}_{;\nu} = -F^{\mu\nu} J_\nu. \tag{2.68}$$

Exercise 2.16 Show that the equation

$$\Box\psi - \frac{1}{6}R\psi = 0 \tag{2.69}$$

for a scalar field ψ is invariant under conformal changes $g \longrightarrow \mathrm{e}^{2\phi} g$ of the metric and the transformation law $\psi \longrightarrow e^{-\phi}\psi$.

Hints Use the formula

$$\Box\psi = \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}\partial^\mu\psi\big),$$

and the transformation law for R in Eq. (3.268).

2.5 The Newtonian Limit

For any generalization of a successful physical theory it is crucial to guarantee that the old theory is preserved within certain limits. The Newtonian theory should be an excellent approximation for slowly varying weak gravitational fields and small velocities of material bodies. At this point we can check only part of this requirement, because the dynamical equation for the metric field is not yet known to us.

We consider a test particle moving slowly in a quasi-stationary weak gravitational field. For weak fields, there are coordinate systems which are nearly Lorentzian. This means that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{2.70}$$

For a slowly moving particle (in comparison with the speed of light) we have $dx^0/d\tau \simeq 1$ and we neglect $dx^i/d\tau$ $(i = 1, 2, 3)$ in comparison to $dx^0/d\tau$ in the geodesic equation (2.23). We then obtain

$$\frac{d^2x^i}{dt^2} \simeq \frac{d^2x^i}{d\tau^2} = -\Gamma^i{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \simeq -\Gamma^i{}_{00}. \tag{2.71}$$

Thus only the components $\Gamma^i{}_{00}$ appear in the equation of motion. To first order in $h_{\mu\nu}$ these are given by

$$\Gamma^i{}_{00} \simeq -\frac{1}{2}h_{00,i} + h_{0i,0}. \tag{2.72}$$

Table 2.1 Numerical illustration of Eq. (2.75)

ϕ/c^2	On the surface of
10^{-9}	the Earth
10^{-6}	the Sun
10^{-4}	a white dwarf
10^{-1}	a neutron star
10^{-39}	a proton

For quasi-stationary fields we can neglect the last term, $\Gamma^i{}_{00} \simeq -\frac{1}{2}h_{00,i}$, obtaining

$$\frac{d^2x^i}{dt^2} \simeq \frac{1}{2}\partial_i h_{00}. \tag{2.73}$$

This agrees with the *Newtonian equation of motion*

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\boldsymbol{\nabla}\phi, \tag{2.74}$$

where ϕ is the Newtonian potential, if we set $h_{00} \simeq -2\phi + \text{const}$. For an isolated system ϕ and h_{00} should vanish at infinity. So we arrive at the important relation

$$g_{00} \simeq -1 - 2\phi. \tag{2.75}$$

Note that we only obtain information on the component g_{00} for a Newtonian situation. However, this does not mean that the other components of $h_{\mu\nu}$ must be small in comparison to h_{00}. The almost Newtonian approximation of the other components will be determined in Sect. 5.2. (In this connection an interesting remark is made in Exercise 2.18.) Table 2.1 shows that for most situations the correction in (2.75) is indeed very small.

Remark In the Newtonian limit, the Poisson equation for ϕ will follow from Einstein's field equation.

2.5.1 Exercises

Exercise 2.17 Use (2.75) to derive the Newtonian limit of the basic equation for an ideal fluid.

Exercise 2.18 The result (2.75) might suggest that the metric for a Newtonian situation is approximately

$$g = -(1 + 2\phi)\,dt^2 + dx^2 + dy^2 + dz^2.$$

Compute for this metric the deflection of light by the sun.

Remark It turns out that the deflection angle is only *half*[8] of the value of GR that will be derived in Sect. 4.4. The reason for this famous factor 2 is that the correct Newtonian approximation will be found to be

$$g = -(1+2\phi)\,dt^2 + (1-2\phi)\bigl(dx^2 + dy^2 + dz^2\bigr). \tag{2.76}$$

Thus, the spatial part of the metric is non-Euclidean. This result will be derived in Sect. 5.2.

2.6 The Redshift in a Stationary Gravitational Field

The derivation of the gravitational redshift in this section is a bit pedestrian, but instructive. A more elegant treatment will be given in Sect. 2.9.

We consider a clock in an arbitrary gravitational field which moves along an arbitrary timelike world line (not necessarily in free fall). According to the principle of equivalence, the clock rate is unaffected by the gravitational field when one observes it from a locally inertial system. Let Δt be the time between "ticks" of clocks *at rest* in some inertial system in the absence of a gravitational field. In the locally inertial system $\{\xi^\mu\}$ under consideration, we then have for the coordinate intervals $d\xi^\mu$ between two ticks

$$\Delta t = \sqrt{-\eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu}.$$

In an arbitrary coordinate system $\{x^\mu\}$ we obviously have

$$\Delta t = \sqrt{-g_{\mu\nu}\, dx^\mu\, dx^\nu}.$$

Hence,

$$\frac{dt}{\Delta t} = \left(-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{-1/2}, \tag{2.77}$$

where $dt = dx^0$ denotes the time interval between two ticks relative to the system $\{x^\mu\}$. If the clock is at rest relative to this system, i.e. $dx^i/dt = 0$, we have in particular

$$\frac{dt}{\Delta t} = \frac{1}{\sqrt{-g_{00}}}. \tag{2.78}$$

This is true for any clock. For this reason, we cannot verify (2.77) or (2.78) locally. However, we can compare the time dilations at two different points with each other. For this purpose, we specialize the discussion to the case of a *stationary* field. By this we mean that we can choose the coordinates x^μ such that the $g_{\mu\nu}$ are independent of t. Now consider Fig. 2.5 with two clocks at rest at the points 1 and 2. (One can convince oneself that the clocks are at rest in any other coordinate system in which

[8] Einstein got this result in 1911 during his time in Prague. He obtained the correct value only after he found his final vacuum equation in November 1915.

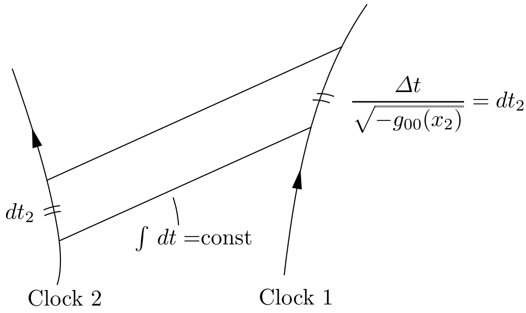


Fig. 2.5 Gravitational redshift in a stationary field

the $g_{\mu\nu}$ are independent of time. The concept "at rest" has an intrinsic meaning for stationary fields. See Sect. 2.9 for a geometrical discussion.)

Let a periodic wave be emitted at point 2. Since the field is stationary, the time (relative to our chosen coordinate system) which a wave crest needs to move from point 2 to point 1 is constant.[9] The time between the arrival of successive crests (or troughs) at point 1 is thus equal to the time dt_2 between their emission at point 2, which is, according to (2.78)

$$dt_2 = \Delta t \frac{1}{\sqrt{-g_{00}(x_2)}}.$$

If on the other hand, we consider the same atomic transition at point 1, then, according to (2.78) the time dt_1 between two wave crests, as observed at point 1 is

$$dt_1 = \Delta t \frac{1}{\sqrt{-g_{00}(x_1)}}.$$

For a given atomic transition, the ratio of frequencies observed at point 1 for light emitted at the points 2 and 1, respectively, is equal to

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}}. \tag{2.79}$$

For weak fields, $g_{00} \simeq -1 - 2\phi$ with $|\phi| \ll 1$, we have

$$\frac{\Delta\nu}{\nu} = \frac{\nu_2}{\nu_1} - 1 \simeq \phi(x_2) - \phi(x_1), \tag{2.80}$$

in agreement with our previous result in Sect. 2.1. The experimental situation was already discussed there.

[9] From $g_{\mu\nu}\, dx^\mu\, dx^\nu = 0$ along the light rays, we have

$$dt = \frac{1}{g_{00}} \cdot \left(-g_{i0}\, dx^i - \sqrt{(g_{i0}g_{j0} - g_{ij}g_{00})\, dx^i\, dx^j}\right).$$

The time interval being discussed is equal to the integral of the right hand side from 2 to 1, and is thus constant.

2.7 Fermat's Principle for Static Gravitational Fields

In the following we shall study in more detail light rays in a *static* gravitational field. A characteristic property of a static field is that in suitable coordinates the metric splits as

$$ds^2 = g_{00}(\boldsymbol{x})\,dt^2 + g_{ik}(\boldsymbol{x})\,dx^i\,dx^k. \tag{2.81}$$

Thus there are no off-diagonal elements g_{0i} and the $g_{\mu\nu}$ are independent of time. We shall give an intrinsic definition of a static field in Sect. 2.9.

If λ is an affine parameter, the paths $x^\mu(\lambda)$ of light rays can be characterized by the variational principle (using standard notation)

$$\delta\int_{\lambda_1}^{\lambda_2} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,d\lambda = 0, \tag{2.82}$$

where the endpoints of the path are held fixed. In addition, we have (see Exercise 2.6)

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{2.83}$$

Consider now a static spacetime with a metric of the form (2.81). If we vary only $t(\lambda)$, we have

$$\begin{aligned}\delta\int_{\lambda_1}^{\lambda_2} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,d\lambda &= \int_{\lambda_1}^{\lambda_2} 2g_{00}\frac{dt}{d\lambda}\delta\left(\frac{dt}{d\lambda}\right)d\lambda \\ &= \int_{\lambda_1}^{\lambda_2} 2g_{00}\frac{dt}{d\lambda}\frac{d}{d\lambda}(\delta t)\,d\lambda \\ &= 2g_{00}\frac{dt}{d\lambda}\delta t\bigg|_{\lambda_1}^{\lambda_2} - 2\int_{\lambda_1}^{\lambda_2}\frac{d}{d\lambda}\left(g_{00}\frac{dt}{d\lambda}\right)\delta t\,d\lambda, \end{aligned} \tag{2.84}$$

where δ denotes the derivative $\partial/\partial\varepsilon|_{\varepsilon=0}$, introduced in the solution of Exercise 2.5. The variational principle (2.82) thus implies ($\delta t = 0$ at the end points)

$$g_{00}\frac{dt}{d\lambda} = \text{const.}$$

We normalize λ such that

$$g_{00}\frac{dt}{d\lambda} = -1. \tag{2.85}$$

Now consider a general variation of the path $x^\mu(\lambda)$, for which only the *spatial* endpoints $x^i(\lambda)$ are held fixed, while the condition $\delta t = 0$ at the endpoints is dropped. If we require that the varied paths also satisfy the normalization condition

(2.85) for the parameter λ, the variational formula (2.84) reduces to

$$\delta \int_{\lambda_1}^{\lambda_2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \, d\lambda = -2\delta t|_{\lambda_1}^{\lambda_2} = -2\delta \int_{\lambda_1}^{\lambda_2} dt. \tag{2.86}$$

The time lapse on the right is a functional of the spatial path. If the varied orbit is also traversed at the speed of light (just as the original path), the left-hand side of (2.86) is equal to zero and for the varied light-like curves the relation

$$\sqrt{-g_{00}} \, dt = d\sigma \tag{2.87}$$

holds, where $d\sigma^2 = g_{ik} \, dx^i \, dx^k$ is the 3-dimensional Riemannian metric of the spatial sections. We thus have

$$\delta \int_{\lambda_1}^{\lambda_2} dt = 0 = \delta \int \frac{1}{\sqrt{-g_{00}}} \, d\sigma. \tag{2.88}$$

This is *Fermat's principle of least time*. The second equality in (2.88) determines the spatial path of the light ray. Note that the spatial path integral is parametrization invariant. The time has been completely eliminated in this formulation: The second equation in (2.88) is valid for an arbitrary portion of the spatial path of the light ray, for any variation such that the ends are held fixed. A comparison with Fermat's principle in optics shows that the role of the index of refraction has been taken over by $(g_{00})^{-1/2}$.

With this classical argument, that goes back to Weyl and Levi-Civita, we have arrived at the interesting result that the path of a light ray is a geodesic in the spatial sections for what is often called the *Fermat metric*

$$g_F = g^F_{ik} \, dx^i \, dx^k, \tag{2.89}$$

where $g^F_{ik} = g_{ik}/(-g_{00})$. We thus have the variational principle for the spatial path γ of a light ray

$$\delta \int \sqrt{g_F(\dot{\gamma}, \dot{\gamma})} \, d\lambda = 0, \tag{2.90}$$

where the spatial endpoints are kept fixed. Instead of the energy functional for g_F we can, of course, also use the length functional. This result is useful for calculating the propagation of light rays in gravitational fields. In many situations it suffices to use the almost Newtonian approximation (2.76) for the metric. The Fermat metric is then

$$g_F = \frac{1 - 2\phi}{1 + 2\phi} \, d\boldsymbol{x}^2, \tag{2.91}$$

with $d\boldsymbol{x}^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. Fermat's principle becomes

$$\delta \int (1 - 2\phi) \big| \dot{\boldsymbol{x}}(\lambda) \big| \, d\lambda = 0,$$

where $|\dot{\boldsymbol{x}}|$ denotes the Euclidean norm of $d\boldsymbol{x}/d\lambda$. This agrees with the Fermat principle of geometrical optics

$$\delta \int n\big(\boldsymbol{x}(\lambda)\big)\big|\dot{\boldsymbol{x}}(\lambda)\big|\,d\lambda = 0$$

for the refraction index

$$n = 1 - 2\phi. \tag{2.92}$$

This can be used as the starting point for much of gravitational lensing theory, an important branch of present day astronomy. Section 5.8 will be devoted to this topic.

Exercise 2.19 Consider a stationary source-free electromagnetic field $F_{\mu\nu}$ in a static gravitational field g with metric (2.81). Show that the time independent scalar potential φ satisfies the Laplace equation

$$\Delta(g_F)\varphi = 0, \tag{2.93}$$

where $\Delta(g_F)$ is the 3-dimensional Laplace operator for the Fermat metric g_F.

Hints Use the conformal invariance of Maxwell's equations and work with the metric g/g_{00}.

2.8 Geometric Optics in Gravitational Fields

In most instances gravitational fields vary even over macroscopic distances so little that the propagation of light and radio waves can be described in the geometric optics limit (*ray optics*). We shall derive in this section the laws of geometric optics in the presence of gravitational fields from Maxwell's equations (see also the corresponding discussion in books on optics). In addition to the geodesic equation for light rays, we shall find a simple propagation law for the polarization vector.

The following characteristic lengths are important for our analysis:

1. The wavelength λ.
2. A typical length L over which the amplitude, polarization and wavelength of the wave vary significantly (for example the radius of curvature of a wave front).
3. A typical "radius of curvature" for the geometry; more precisely, take

$$R = \left|\begin{array}{c}\text{typical component of the Riemannian tensor}\\ \text{in a typical local inertial system}\end{array}\right|^{-1/2}.$$

The region of validity for geometric optics is

$$\lambda \ll L \quad \text{and} \quad \lambda \ll R. \tag{2.94}$$

Consider a wave which is highly monochromatic in regions having a size smaller than L (more general cases can be treated via Fourier analysis). Now separate the four-vector potential A_μ into a rapidly varying real phase ψ and a slowly varying complex amplitude $\mathcal{A}_\mu$ (eikonal ansatz)

$$A_\mu = \mathrm{Re}\{\mathcal{A}_\mu \mathrm{e}^{\mathrm{i}\psi}\}.$$

It is convenient to introduce the small parameter $\varepsilon = \lambda/\min(L, R)$. We may expand $\mathcal{A}_\mu = a_\mu + \varepsilon b_\mu + \cdots$, where $a_\mu, b_\mu, \ldots$ are independent of λ. Since $\psi \propto \lambda^{-1}$, we replace ψ by ψ/ε. We thus seek solutions of the form

$$A_\mu = \mathrm{Re}\{(a_\mu + \varepsilon b_\mu + \cdots)\mathrm{e}^{\mathrm{i}\psi/\varepsilon}\}. \tag{2.95}$$

In the following let $k_\mu = \partial_\mu \psi$ be the wave number, $a = \sqrt{(a_\mu \bar{a}^\mu)}$ the scalar amplitude and $f_\mu = a_\mu/a$ the polarization vector, where f_μ is a complex unit vector. By definition, *light rays* are integral curves of the vector field k^μ and are thus perpendicular to the surfaces of constant phase ψ, in other words perpendicular to the *wave fronts*.

Now insert the geometric-optics ansatz (2.95) into Maxwell's equations. In vacuum, these are given (see Sect. 2.4.6)

$$A^{\nu;\mu}{}_{;\nu} - A^{\mu;\nu}{}_{;\nu} = 0. \tag{2.96}$$

We use the Ricci identity

$$A^{\nu;\mu}{}_{;\nu} = A^{\mu;\nu}{}_{;\nu} + R^\mu_\nu A^\nu \tag{2.97}$$

and impose the Lorentz gauge condition

$$A^\nu{}_{;\nu} = 0. \tag{2.98}$$

Equation (2.96) then takes the form

$$A^{\mu;\nu}{}_{;\nu} - R^\mu_\nu A^\nu = 0. \tag{2.99}$$

If we now insert (2.95) into the Lorentz condition, we obtain

$$0 = A^\nu{}_{;\nu} = \mathrm{Re}\left\{\left(\mathrm{i}\frac{k_\mu}{\varepsilon}\left(a^\mu + \varepsilon b^\mu + \cdots\right) + \left(a^\mu + \varepsilon b^\mu + \cdots\right)_{;\mu}\right)\mathrm{e}^{\mathrm{i}\psi/\varepsilon}\right\}. \tag{2.100}$$

From the leading term, it follows that $k_\mu a^\mu = 0$, or equivalently

$$k_\mu f^\mu = 0. \tag{2.101}$$

Thus, the polarization vector is perpendicular to the wave vector. The next order in (2.100) leads to $k_\mu b^\mu = \mathrm{i}a^\mu{}_{;\mu}$. Now substitute (2.95) in (2.99) to obtain

$$
\begin{aligned}
0 &= -A^{\mu;\nu}{}_{;\nu} + R^{\mu}_{\nu} A^{\nu} \\
&= \mathrm{Re}\bigg\{ \bigg(\frac{1}{\varepsilon^2} k^{\nu} k_{\nu} \big(a^{\mu} + \varepsilon b^{\mu} + \cdots \big) - 2\frac{\mathrm{i}}{\varepsilon} k^{\nu} \big(a^{\mu} + \varepsilon b^{\mu} + \cdots \big)_{;\nu} \\
&\quad - \frac{\mathrm{i}}{\varepsilon} k^{\nu}_{\ ;\nu} \big(a^{\mu} + \varepsilon b^{\mu} + \cdots \big) - \big(a^{\mu} + \cdots \big)_{;\nu}^{\ \nu} + R^{\mu}_{\nu} \big(a^{\nu} + \cdots \big) \bigg) \mathrm{e}^{\mathrm{i}\psi/\varepsilon} \bigg\}. \qquad (2.102)
\end{aligned}
$$

This gives, in order ε^{-2}, $k^{\nu} k_{\nu} a^{\mu} = 0$, which is equivalent to

$$k^{\nu} k_{\nu} = 0, \tag{2.103}$$

telling us that the wave vector is null. Using $k_{\mu} = \partial_{\mu} \psi$ we obtain the general relativistic *eikonal equation*

$$g^{\mu\nu} \partial_{\mu} \psi \partial_{\nu} \psi = 0. \tag{2.104}$$

The terms of order ε^{-1} give

$$k^{\nu} k_{\nu} b^{\mu} - 2\mathrm{i} \bigg(k^{\nu} a^{\mu}_{\ ;\nu} + \frac{1}{2} k^{\nu}_{\ ;\nu} a^{\mu} \bigg) = 0.$$

With (2.103), this implies

$$k^{\nu} a^{\mu}_{\ ;\nu} = -\frac{1}{2} k^{\nu}_{\ ;\nu} a^{\mu}. \tag{2.105}$$

As a consequence of these equations, we obtain the geodesic law for the propagation of light rays: Eq. (2.103) implies

$$0 = \big(k^{\nu} k_{\nu}\big)_{;\mu} = 2 k^{\nu} k_{\nu;\mu}.$$

Now $k_{\nu} = \psi_{,\nu}$ and since $\psi_{;\nu;\mu} = \psi_{;\mu;\nu}$ we obtain, after interchanging indices,

$$k^{\nu} k_{\mu;\nu} = 0, \quad (\nabla_k k = 0). \tag{2.106}$$

We have thus demonstrated that, as a consequence of Maxwell's equations, the paths of light rays are *null geodesics*.

Now consider the amplitude $a^{\mu} = a f^{\mu}$. From (2.105) we have

$$
\begin{aligned}
2 a k^{\nu} a_{,\nu} &= 2 a k^{\nu} a_{;\nu} = k^{\nu} \big(a^2\big)_{;\nu} = k^{\nu} \big(a_{\mu} \bar{a}^{\mu}\big)_{;\nu} \\
&= \bar{a}^{\mu} k^{\nu} a_{\mu;\nu} + a_{\mu} k^{\nu} \bar{a}^{\mu}_{\ ;\nu} \overset{(2.105)}{=} -\frac{1}{2} k^{\nu}_{\ ;\nu} \big(\bar{a}^{\mu} a_{\mu} + a_{\mu} \bar{a}^{\mu}\big),
\end{aligned}
$$

so that

$$k^{\nu} a_{,\nu} = -\frac{1}{2} k^{\nu}_{\ ;\nu} a. \tag{2.107}$$

This can be regarded as a propagation law for the scalar amplitude. If we now insert $a^\mu = af^\mu$ into (2.105) we obtain

$$
\begin{aligned}
0 &= k^\nu\left(af^\mu\right)_{;\nu} + \frac{1}{2}k^\nu_{\ ;\nu}af^\mu \\
&= ak^\nu f^\mu_{\ ;\nu} + f^\mu\left(k^\nu a_{;\nu} + \frac{1}{2}k^\nu_{\ ;\nu}a\right) \overset{(2.107)}{=} ak^\nu f^\mu_{\ ;\nu}
\end{aligned}
$$

or

$$k^\nu f^\mu_{\ ;\nu} = 0, \quad (\nabla_k f = 0). \tag{2.108}$$

We thus see that the polarization vector f^μ is perpendicular to the light rays and is parallel-propagated along them.

Remark The gauge condition (2.101) is consistent with the other equations: Since the vectors k^μ and f^μ are parallel transported along the rays, one must specify the condition $k_\mu f^\mu = 0$ at only one point. For the same reason, the equations $f_\mu \bar{f}^\mu = 1$ and $k_\mu k^\mu = 0$ are preserved.

Equation (2.107) can be rewritten as follows. After multiplying by a, we have

$$\left(k^\nu \nabla_\nu\right)a^2 + a^2\nabla_\nu k^\nu = 0$$

or

$$\left(a^2 k^\mu\right)_{;\mu} = 0, \tag{2.109}$$

thus $a^2 k^\mu$ is a conserved "current".

Quantum mechanically this has the meaning of a conservation law for the number of photons. Of course, the photon number is not in general conserved; it is an *adiabatic invariant*, in other words, a quantity which varies very slowly for $R \gg \lambda$, in comparison to the photon frequency.

Let us consider the eikonal equation (2.104) for the almost Newtonian metric (2.76)

$$-(1-2\phi)(\partial_t\psi)^2 + (1+2\phi)(\boldsymbol{\nabla}\psi)^2 = 0.$$

Since ϕ is time independent we set

$$\psi(\boldsymbol{x}, t) = S(\boldsymbol{x}) - \omega t \tag{2.110}$$

and obtain (up to higher orders in ϕ)

$$(\boldsymbol{\nabla} S)^2 = n^2\omega^2, \quad n = 1 - 2\phi. \tag{2.111}$$

This has the standard form of the eikonal equation in ray optics with refraction index n. The connection between n and the Newtonian potential ϕ was already found earlier with the help of Fermat's principle.

2.8.1 Exercises

Exercise 2.20 Consider light rays, i.e., integral curves $x^\mu(\lambda)$ of $\nabla^\mu\psi$. Derive from the eikonal equation (2.104) that $\dot{x}^\mu$ is an autoparallel null vector.

Exercise 2.21 Show that the energy-momentum tensor, averaged over a wavelength, is (for $\varepsilon = 1$)

$$\langle T^{\mu\nu}\rangle = \frac{1}{8\pi}a^2 k^\mu k^\nu.$$

In particular, the energy flux is

$$\langle T^{0j}\rangle = \langle T^{00}\rangle n^j,$$

where $n^j = k^j/k^0$. The Eqs. (2.106) and (2.109) imply

$$8\pi\,\nabla_\nu\langle T^{\mu\nu}\rangle = \nabla_\nu\big(a^2k^\mu k^\nu\big) = \nabla_\nu\big(a^2k^\nu\big)k^\mu + a^2k^\nu\nabla_\nu k^\mu = 0.$$

2.9 Stationary and Static Spacetimes

For this section the reader should be familiar with parts of DG covered in Chaps. 12 and 13.

In Sect. 2.5 we defined somewhat naively a gravitational field to be stationary if there exist coordinates $\{x^\mu\}$ for which the components $g_{\mu\nu}$ of the metric tensor are independent of $t = x^0$. We translate this definition into an intrinsic property of the spacetime (M, g). Let $K = \partial/\partial x^0$, i.e., $K^\mu = \delta^\mu_0$. The Lie derivative $L_K g$ of the metric tensor is then

$$\begin{aligned}(L_K g)_{\mu\nu} &= K^\lambda g_{\mu\nu,\lambda} + g_{\lambda\nu}K^\lambda_{\ ,\mu} + g_{\mu\lambda}K^\lambda_{\ ,\nu}\\ &= 0+0+0,\end{aligned} \tag{2.112}$$

so that

$$L_K g = 0. \tag{2.113}$$

A vector field K which satisfies (2.113) is a *Killing field* or an *infinitesimal isometry*. This leads us to the

Definition 2.1 A spacetime (M, g) is *stationary* if there exists a timelike Killing field K.

This means that observers moving with the flow of the Killing field K recognize no changes (see DG, Theorem 13.11). It may be, as in the case for black holes, that a Killing field is only timelike in some open region of M. We then say that this part of spacetime is stationary.

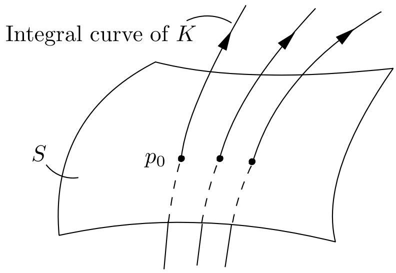


Fig. 2.6 Adapted (stationary) coordinates

Let us conversely show that Definition 2.1 implies the existence of local coordinates for which the $g_{\mu\nu}$ are independent of time. Choose a spacelike hypersurface S of M and consider the integral curves of K passing through S (see Fig. 2.6). In S we choose arbitrary coordinates and introduce local coordinates of M as follows: If $p = \phi_t(p_0)$, where $p_0 \in S$ and ϕ_t is the flow of K, then the (Lagrange-) coordinates of p are $(t, x^1(p_0), x^2(p_0), x^3(p_0))$. In terms of these coordinates, we have $K = \partial/\partial x^0$, and $L_K g = 0$ implies (using (2.112))

$$g_{\mu\nu,0} + 0 + 0 = 0.$$

We call such coordinates to be *adapted* to the Killing field.

Static fields are special cases of stationary fields. The following heuristic consideration will lead us to their proper definition. We choose adapted coordinates and assume that $g_{0i} = 0$ for $i = 1, 2, 3$. Then the Killing field is orthogonal to the spatial sections $\{t = \text{const.}\}$. The 1-form ω corresponding to K ($\omega = K^\flat$, $\omega_\mu = K_\mu = g_{\mu\nu}K^\nu$) is then

$$\omega = g_{00}\,dt = \langle K, K\rangle\,dt. \tag{2.114}$$

This implies trivially the *Frobenius condition*

$$\omega \wedge d\omega = 0. \tag{2.115}$$

Conversely, let us assume that the Frobenius condition holds for a stationary spacetime with Killing field K. We apply the interior product i_K to (2.115) and use Cartan's formula $L_K = d \circ i_K + i_K \circ d$:

$$0 = i_K(\omega \wedge d\omega) = \underbrace{(i_K\omega)}_{\omega(K)=\langle K,K\rangle}\, d\omega - \omega \wedge \underbrace{i_K\,d\omega}_{L_K\omega - d\langle K,K\rangle}.$$

We expect that $L_K\omega = 0$; indeed, for any vector field X we have

$$(L_K\omega)(X) = K\big(\omega(X)\big) - \omega\big([K, X]\big) = K\langle K, X\rangle - \big\langle K, [K, X]\big\rangle.$$

On the other hand

$$0 = (L_K g)(K, X) = K\langle K, X\rangle - \big\langle [K, K], X\big\rangle - \big\langle K, [K, X]\big\rangle,$$

and the right-hand sides of both equations agree.

Using the abbreviation $V := \langle K, K\rangle \neq 0$ we thus arrive at

$$V\,d\omega + \omega \wedge dV = 0 \quad \text{or} \quad d(\omega/V) = 0. \tag{2.116}$$

Together with the Poincaré Lemma we see that locally $\omega = V\,df$ for a function f. We use this function as our time coordinate t,

$$\omega = \langle K, K\rangle\,dt. \tag{2.117}$$

K is perpendicular to the spacelike sections $\{t = \text{const.}\}$. Indeed, for a tangential vector field X to such a section, $\langle K, X\rangle = \omega(X) = V\,dt(X) = V(Xt) = 0$. In adapted coordinates we have therefore $K = \partial_t$ and $g_{0i} = \langle \partial_t, \partial_i\rangle = \langle K, \partial_i\rangle = 0$.

Summarizing, if the Frobenius condition (2.115) for the timelike Killing field is satisfied, the metric splits locally as

$$g = g_{00}(\boldsymbol{x})\,dt^2 + g_{ik}(\boldsymbol{x})\,dx^i\,dx^k, \tag{2.118}$$

and

$$g_{00} = \langle K, K\rangle, \tag{2.119}$$

where $K = \partial/\partial t$. This leads us to the

Definition 2.2 A stationary spacetime (M, g) with timelike Killing field is *static*, if $\omega = K^\flat$ satisfies the Frobenius condition $\omega \wedge d\omega = 0$, whence locally $\omega = \langle K, K\rangle\,dt$ for an adapted time coordinate t, which is unique up to an additive constant.

The flow of K maps the hypersurfaces $\{t = \text{const.}\}$ isometrically onto each other. An *observer at rest* moves along integral curves of K.

2.9.1 Killing Equation

According to DG, Eq. (15.104) we have for any vector field X and its associated 1-form $\alpha = X^\flat$ the identity

$$\nabla\alpha = \frac{1}{2}(L_X g - d\alpha). \tag{2.120}$$

For the special case $X = K$ and $\alpha = \omega$ this gives

$$\nabla\omega = -\frac{1}{2}\,d\omega. \tag{2.121}$$

In components, this is equivalent to the *Killing equation*

$$K_{\mu;\nu} + K_{\nu;\mu} = 0. \tag{2.122}$$

A shorter derivation in terms of local coordinates goes as follows. From (2.112) we obtain for a Killing field

$$K^\lambda g_{\mu\nu,\lambda} + g_{\lambda\nu} K^\lambda_{\ ,\mu} + g_{\mu\lambda} K^\lambda_{\ ,\nu} = 0.$$

Now introduce, for a given point p, normal coordinates with origin p. At this point the last equation reduces to $K_{\mu,\nu} + K_{\nu,\mu} = 0$ or, equivalently, to (2.122). But (2.122) is generally invariant and so holds in any coordinate system.

The reader may wonder, how one might obtain (2.116) in terms of local coordinates. We want to demonstrate that such a derivation can be faster. We write the Frobenius condition (2.115) in components

$$K_\mu K_{\nu,\lambda} + K_\nu K_{\lambda,\mu} + K_\lambda K_{\mu,\nu} = 0.$$

The left hand side does not change if partial derivatives are replaced by covariant derivatives. If then multiply the resulting equation by K^λ and use the Killing equation (2.122), we obtain

$$-K_\mu \big(K^\lambda K_\lambda\big)_{;\nu} + K_\nu \big(K^\lambda K_\lambda\big)_{;\mu} + K^\lambda K_\lambda (K_{\mu;\nu} - K_{\nu;\mu}) = 0.$$

This implies

$$\big[K_\nu / \langle K, K\rangle\big]_{,\mu} - \big[K_\mu / \langle K, K\rangle\big]_{,\nu} = 0,$$

and this is equivalent to (2.116).

2.9.2 *The Redshift Revisited*

The discussion of the gravitational redshift in Sect. 2.6 was mathematically a bit ugly. Below we give two derivations which are more satisfactory, mathematically.

First Derivation

In the eikonal approximation (see Sect. 2.8) we have for the electromagnetic field tensor

$$F_{\mu\nu} = \mathrm{Re}\big(f_{\mu\nu} \mathrm{e}^{\mathrm{i}\psi}\big),$$

where $f_{\mu\nu}$ is a slowly varying amplitude. Light rays are integral curves of the vector field $k^\mu = \psi^{,\mu}$ and are null geodesics (see Sect. 2.8). Since $k^\mu \psi_{,\mu} = 0$, the light rays propagate along surfaces of constant phase, i.e. wave fronts.

Consider now the world lines, parameterized by proper time, of a transmitter and an observer, as well as two light rays which connect the two (see Fig. 2.7). Let the corresponding phases be $\psi = \psi_0$ and $\psi = \psi_0 + \Delta\psi$. We denote the interval of proper time between the events at which the two light rays intersect the world line i

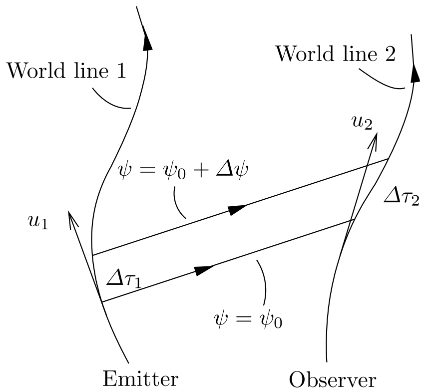


Fig. 2.7 Redshift (first derivation)

$(i = 1, 2)$ by $\Delta\tau_i$. The four-velocities of the emitter and observer are denoted by u_1^μ and u_2^μ, respectively. Obviously,

$$u_1^\mu(\partial_\mu\psi)_1\Delta\tau_1 = \Delta\psi = u_2^\mu(\partial_\mu\psi)_2\Delta\tau_2. \tag{2.123}$$

If ν_1 and ν_2 are the frequencies assigned to the light by 1 and 2, respectively, then (2.123) gives

$$\frac{\nu_1}{\nu_2} = \frac{\Delta\tau_2}{\Delta\tau_1} = \frac{\langle k, u_1\rangle}{\langle k, u_2\rangle}. \tag{2.124}$$

This equation gives the combined effects of Doppler and gravitational redshifts (and is also useful in SR).

Now, we specialize (2.124) to a *stationary* spacetime with Killing field K. For an observer at rest (along an integral curve of K) with four-velocity u, we have

$$K = \big(-\langle K, K\rangle\big)^{1/2}u. \tag{2.125}$$

Furthermore, we note that $\langle k, K\rangle$ is *constant* along a light ray, since

$$\nabla_k\langle k, K\rangle = \langle\nabla_k k, K\rangle + \langle k, \nabla_k K\rangle = 0.$$

Note that the last term is equal to $k^\alpha k^\beta K_{\alpha;\beta}$ and vanishes as a result of the Killing equation; alternatively, due to (2.121) it is proportional to $d\omega(k, k) = 0$.

If both emitter and observer are at rest, we obtain from (2.124) and (2.125)

$$\frac{\nu_1}{\nu_2} = \left(\frac{\langle K, K\rangle_2}{\langle K, K\rangle_1}\right)^{1/2}. \tag{2.126}$$

In adapted coordinates, $K = \partial/\partial t$ and $\langle K, K\rangle = g_{00}$, we can write (2.126) as

$$\frac{\nu_1}{\nu_2} = \left(\frac{g_{00}|_2}{g_{00}|_1}\right)^{1/2}. \tag{2.127}$$

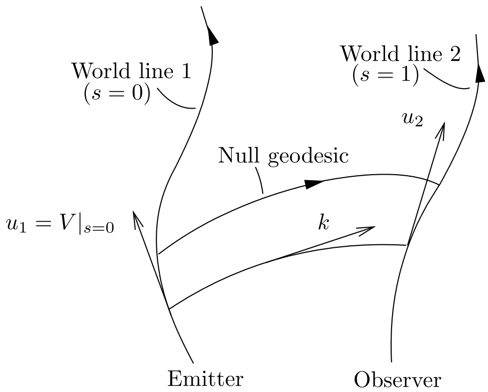


Fig. 2.8 Redshift (second derivation)

Remark At first sight this appears to be inconsistent with (2.79). However, the frequencies are defined there differently. In Sect. 2.6, ν_1 and ν_2 are both measured at 1, but ν_2 refers to a definite atomic transition at 2, while ν_1 is the frequency of the same transition of an atom at the observer's position 1. In (2.127) the meaning of ν_1 and ν_2 is different: ν_1 and ν_2 are the frequencies assigned to the light by 1 and 2, respectively.

Second Derivation

We again work in the limit of geometric optics, and consider the same situation as before. Emitter and observer, with four-velocities u_1 and u_2, can be connected to each other by null geodesics with tangent vectors k (see Fig. 2.8). We assume that for a finite τ_1-interval, null geodesics exist which are received by the observer. This family of null geodesics can be parameterized by the emission time τ_1 or by the observer time τ_2, and defines a function $\tau_2(\tau_1)$. The frequency ratio r is clearly the derivative of this function,

$$r = \frac{d\tau_2}{d\tau_1}. \tag{2.128}$$

We can parameterize the null geodesics by an affine parameters s, such that $s = 0$ along the world line 1 and $s = 1$ along 2. In what follows we parameterize the 1-parameter family of null geodesics by $(s, \tau_1) \longmapsto H(s, \tau_1)$. With this set up, we are in a situation that has been studied generally in DG, Sect. 16.4. We use the concepts and results which have been developed there (including Sect. 16.2 and Sect. 16.3).

k is a tangential vector field along the map H

$$k = TH \circ \frac{\partial}{\partial s}. \tag{2.129}$$

Beside this we also use the field of tangent vectors for curves of constant s

$$V = TH \circ \frac{\partial}{\partial \tau_1}. \tag{2.130}$$

Note that

$$V|_{s=0} = u_1, \qquad V|_{s=1} = ru_2, \tag{2.131}$$

because $\tau_1 \longmapsto H(s=1, \tau_1)$ is the world line 2, parameterized by τ_1.

We shall show below that $\langle V, k\rangle$ is constant along a null geodesic. Using this, the ratio r can easily be computed. We obtain the previous result (2.124) from

$$\langle V, k_1\rangle_1 = \langle u_1, k\rangle = \langle V, k\rangle_2 = r\langle u_2, k\rangle.$$

The rest is as before.

It remains to prove that $\langle V, k\rangle$ is constant. The null geodesics satisfy

$$\langle k, k\rangle = 0, \quad \nabla_k k = 0. \tag{2.132}$$

(We denote the Levi-Civita connection and the induced covariant derivatives for vector fields along the map H by the same letter.) For the tangential vector fields along H, such as k and V, we have (DG, Proposition 16.4 and 16.5)

$$\nabla_A B - \nabla_B A = [A, B], \tag{2.133}$$

where $A, B \in \mathcal{X}(H)^T$ and

$$A'\langle X, Y\rangle = \langle \nabla_A X, Y\rangle + \langle X, \nabla_A Y\rangle \tag{2.134}$$

for $A = TH \circ A'$ and $X, Y \in \mathcal{X}(H)$. Using this we get (for $A' = \partial/\partial s$)

$$\frac{\partial}{\partial s}\langle V, k\rangle = \langle \nabla_k V, k\rangle + \langle V, \nabla_k k\rangle = \langle \nabla_k V, k\rangle.$$

From (2.129) and (2.130) we see that $[V, k] = 0$ (see DG, Eq. (16.9)). Hence,

$$\frac{\partial}{\partial s}\langle V, k\rangle = \langle \nabla_k V, k\rangle = \langle \nabla_V k, k\rangle = \frac{1}{2}\frac{\partial}{\partial \tau_1}\langle k, k\rangle = 0.$$

2.10 Spin Precession and Fermi Transport

Suppose that an observer moves along a timelike world line in a gravitational field (not necessarily in free fall). One might, for example, consider an astronaut in a space capsule. For practical reasons he will choose a coordinate system in which all apparatus attached to his capsule is at rest. What is the equation of motion of a freely falling test body in this coordinate system? More specifically, the following questions arise:

1. How should the observer orient his space ship so that "Coriolis forces" do not appear?

2. How does one describe the motion of a gyroscope? One might expect that it will not rotate relative to the frame of references, provided the latter is chosen such that Coriolis forces are absent.
3. It is possible to find a spatial frame of reference for an observer at rest in a stationary field, which one might call Copernican? What is the equation of motion of a spinning top in such a frame of reference? Under what conditions will it not rotate relative to the Copernican frame?

2.10.1 Spin Precession in a Gravitational Field

By *spin* we mean either the polarization vector of a particle (i.e., the expectation value of the spin operator for a particle in a particular quantum mechanical state) or the intrinsic angular momentum of a rigid body, such as a gyroscope.

In both cases this is initially defined only relative to a local inertial system in which the body is at rest (its *local rest system*). In this system the spin is described by a three vector $\boldsymbol{S}$. For a gyroscope or for an elementary particle, the equivalence principle implies that in the local rest system, in the absence of external forces,

$$\frac{d}{dt}\boldsymbol{S}(t) = 0. \tag{2.135}$$

(We assume that the interaction of the gyroscope's quadrupole moment with inhomogeneities of the gravitational field can be neglected; this effect is studied in Exercise 2.23 at the end of this section.) We now define a four-vector S which reduces to $(0, \boldsymbol{S})$ in the local rest system. This last requirement can be expressed invariantly as

$$\langle S, u\rangle = 0, \tag{2.136}$$

where u is the four-velocity.

We shall now rewrite (2.135) in a covariant form. For this we consider $\nabla_u S$. In the local rest system (indicated by R) we have

$$(\nabla_u S)_R = \left(\frac{dS^0}{dt}, \frac{d}{dt}\boldsymbol{S}\right) = \left(\frac{dS^0}{dt}, \boldsymbol{0}\right). \tag{2.137}$$

It follows from Eq. (2.136) that

$$\langle \nabla_u S, u\rangle = -\langle S, \nabla_u u\rangle = -\langle S, a\rangle, \tag{2.138}$$

where $a = \nabla_u u$ is the acceleration. Hence,

$$\langle \nabla_u S, u\rangle = -\left.\frac{dS^0}{dt}\right|_R = -\langle S, a\rangle. \tag{2.139}$$

From (2.137) and (2.139) we then have

$$(\nabla_u S)_R = \big(\langle S, a\rangle, \boldsymbol{0}\big) = \big(\langle S, a\rangle u\big)_R. \tag{2.140}$$

The desired covariant equation is thus

$$\nabla_u S = \langle S, a \rangle u. \tag{2.141}$$

Equation (2.136) is consistent with (2.141). Indeed from (2.141) we find

$$\langle u, \nabla_u S \rangle = \langle S, a \rangle \langle u, u \rangle = -\langle S, a \rangle = -\langle S, \nabla_u u \rangle,$$

so that $\nabla_u \langle u, S \rangle = 0$.

2.10.2 Thomas Precession

For Minkowski spacetime (2.141) reduces to

$$\dot{S} = \langle S, \dot{u} \rangle u, \tag{2.142}$$

where the dot means differentiation with respect to proper time. One can easily derive the Thomas precession from this equation.

Let $x(\tau)$ denote the path of a particle. The instantaneous rest system (at time τ) is obtained form the laboratory system via the special Lorentz transformation $\Lambda(\boldsymbol{\beta})$, where $\boldsymbol{\beta} = \boldsymbol{v}/c$ and $\boldsymbol{v}$ the 3-velocity. With respect to this family of instantaneous rest systems S has the form $S = (0, \boldsymbol{S}(t))$, where t is the time in the laboratory frame. We obtain the equation of motion for $S(t)$ easily from (2.142). Since S is a four vector, we have, in the laboratory frame, with standard notation of SR

$$S = \left(\gamma \boldsymbol{\beta} \cdot \boldsymbol{S}, \boldsymbol{S} + \boldsymbol{\beta} \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} \cdot \boldsymbol{S} \right). \tag{2.143}$$

In addition,

$$u = (\gamma, \gamma \boldsymbol{\beta}), \qquad \dot{u} = (\dot{\gamma}, \dot{\gamma} \boldsymbol{\beta} + \gamma \dot{\boldsymbol{\beta}}). \tag{2.144}$$

Hence,

$$\begin{aligned} \langle S, \dot{u} \rangle &= -\dot{\gamma} \gamma \boldsymbol{\beta} \cdot \boldsymbol{S} + (\dot{\gamma} \boldsymbol{\beta} + \gamma \dot{\boldsymbol{\beta}}) \cdot \left(\boldsymbol{S} + \boldsymbol{\beta} \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} \cdot \boldsymbol{S} \right) \\ &= \gamma \left(\dot{\boldsymbol{\beta}} \cdot \boldsymbol{S} + \frac{\gamma^2}{\gamma + 1} \dot{\boldsymbol{\beta}} \cdot \boldsymbol{\beta} \boldsymbol{\beta} \cdot \boldsymbol{S} \right). \end{aligned} \tag{2.145}$$

From (2.142) and (2.145) we then obtain

$$(\gamma \boldsymbol{\beta} \cdot \boldsymbol{S})^{\cdot} = \gamma^2 \left(\dot{\boldsymbol{\beta}} \cdot \boldsymbol{S} + \frac{\gamma^2}{\gamma + 1} \dot{\boldsymbol{\beta}} \cdot \boldsymbol{\beta} \boldsymbol{\beta} \cdot \boldsymbol{S} \right),$$

$$\left(\boldsymbol{S} + \boldsymbol{\beta} \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} \cdot \boldsymbol{S} \right)^{\cdot} = \boldsymbol{\beta} \gamma^2 \left(\dot{\boldsymbol{\beta}} \cdot \boldsymbol{S} + \frac{\gamma^2}{\gamma + 1} \dot{\boldsymbol{\beta}} \cdot \boldsymbol{\beta} \boldsymbol{\beta} \cdot \boldsymbol{S} \right).$$

After some rearrangements, one finds

$$\dot{\boldsymbol{S}} = \boldsymbol{S} \times \boldsymbol{\omega}_T, \tag{2.146}$$

where $\boldsymbol{\omega}_T = \frac{\gamma-1}{\beta^2}\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}$. This is the well-known expression of the Thomas precession.

2.10.3 Fermi Transport

Definition Let $\gamma(s)$, with s the proper time, be a timelike curve with tangent vector $u = \dot{\gamma}$ satisfying $\langle u, u\rangle = -1$. The *Fermi derivative* $\boldsymbol{F}_u$ of a vector field X along γ is defined by

$$\boldsymbol{F}_u X = \nabla_u X - \langle X, a\rangle u + \langle X, u\rangle a, \tag{2.147}$$

where $a = \nabla_u u$.

Since $\langle S, u\rangle = 0$ we may write (2.141) in the form

$$\boldsymbol{F}_u S = 0. \tag{2.148}$$

It is easy to show that the Fermi derivative (2.147) has the following important properties:

1. $\boldsymbol{F}_u = \nabla_u$ if γ is a geodesic;
2. $\boldsymbol{F}_u u = 0$;
3. If $\boldsymbol{F}_u X = \boldsymbol{F}_u Y = 0$ for vector fields X, Y along γ, then $\langle X, Y\rangle$ is constant along γ;
4. If $\langle X, u\rangle = 0$ along γ, then

$$\boldsymbol{F}_u X = (\nabla_u X)_\perp. \tag{2.149}$$

Here $\perp$ denotes the projection perpendicular to u.

These properties show that the Fermi derivative is a natural generalization of ∇_u.

We say that a vector field X is *Fermi transported* along γ if $\boldsymbol{F}_{\dot{\gamma}} X = 0$. Since this equation is linear in X, Fermi transport defines (analogously to parallel transport) a two parameter family of isomorphisms

$$\tau^{\boldsymbol{F}}_{t,s} : T_{\gamma(s)}(M) \longrightarrow T_{\gamma(t)}(M).$$

One can show that

$$\boldsymbol{F}_{\dot{\gamma}} X\big(\gamma(t)\big) = \frac{d}{ds}\bigg|_{s=t} \tau^{\boldsymbol{F}}_{t,s} X\big(\gamma(s)\big).$$

The proof is similar to that of Theorem 15.1 in DG.

As in the case of the covariant derivative (see DG, Sect. 15.4), the Fermi derivative can be extended to arbitrary tensor fields such that the following properties hold:

1. $\boldsymbol{F}_u$ transforms a tensor field of type (r,s) into another tensor field of the same type;
2. $\boldsymbol{F}_u$ commutes with contractions;
3. $\boldsymbol{F}_u(S\otimes T)=(\boldsymbol{F}_u S)\otimes T+S\otimes(\boldsymbol{F}_u T)$;
4. $\boldsymbol{F}_u f=df/ds$, when f is a function;
5. $\tau^{\boldsymbol{F}}_{t,s}$ induces linear isomorphisms

$$T_{\gamma(s)}(M)^r_s\longrightarrow T_{\gamma(t)}(M)^r_s.$$

We now consider the world line $\gamma(\tau)$ of an accelerated observer (τ is the proper time). Let $u=\dot\gamma$ and let $\{e_i\}$, with $i=1,2,3$, be an arbitrary orthonormal frame along γ perpendicular to $e_0:=\dot\gamma=u$. We then have

$$\langle e_\mu,e_\nu\rangle=\eta_{\mu\nu},$$

where $(\eta_{\mu\nu})=\mathrm{diag}(-1,1,1,1)$. For the acceleration $a:=\nabla_u u$ it follows from $\langle u,u\rangle=-1$ that $\langle a,u\rangle=0$. We set

$$\omega_{ij}=\langle\nabla_u e_i,e_j\rangle=-\omega_{ji}. \tag{2.150}$$

If $e^\mu=\eta^{\mu\nu}e_\nu$ and $\omega_i^{\ j}:=\langle\nabla_u e_i,e^j\rangle$, we have

$$\begin{aligned}\nabla_u e_i&=\big\langle\nabla_u e_i,e^\alpha\big\rangle e_\alpha\\&=-\langle\nabla_u e_i,u\rangle u+\big\langle\nabla_u e_i,e^j\big\rangle e_j\\&=\langle e_i,\nabla_u u\rangle u+\omega_i^{\ j}e_j,\end{aligned}$$

so that

$$\nabla_u e_i=\langle e_i,a\rangle u+\omega_i^{\ j}e_j. \tag{2.151}$$

Adding a vanishing term, this can be rewritten as $\nabla_u e_i=-\langle e_i,u\rangle a+\langle e_i,a\rangle u+\omega_i^{\ j}e_j$.

Let

$$\big(\omega_\alpha^{\ \beta}\big)=\begin{pmatrix}0&0\\0&\omega_i^{\ j}\end{pmatrix}$$

Then $\nabla_u e_\alpha=-\langle e_\alpha,u\rangle a+\langle e_\alpha,a\rangle u+\omega_\alpha^{\ \beta}e_\beta$, since for $\alpha=0$ the right-hand side is equal to $-\langle u,u\rangle a+\langle u,a\rangle u=a=\nabla_u u$. Using (2.147) this can be written in the form

$$\boldsymbol{F}_u e_\alpha=\omega_\alpha^{\ \beta}e_\beta. \tag{2.152}$$

${\omega_\alpha}^\beta$ thus describes the deviation from Fermi transport. For a spinning top we have $\boldsymbol{F}_u S = 0$ and $\langle S, u\rangle = 0$. If we write $S = S^i e_i$, then

$$0 = \boldsymbol{F}_u S = \frac{dS^i}{d\tau} e_i + S^j \boldsymbol{F}_u e_j = \frac{dS^i}{d\tau} e_i + S^j \omega^{\ i}_{j} e_i,$$

hence,

$$\frac{dS^i}{d\tau} = \omega^{\ i}_{j} S^j. \tag{2.153}$$

Thus the top precesses relative to the frame $\{e_i\}$ with angular velocity $\boldsymbol{\Omega}$, where

$$\omega_{ij} = \varepsilon_{ijk} \Omega^k. \tag{2.154}$$

We may write (2.153) in three-dimensional vector notation

$$\frac{d\boldsymbol{S}}{d\tau} = \boldsymbol{S} \times \boldsymbol{\Omega}. \tag{2.155}$$

If the frame $\{e_i\}$ is Fermi transported along γ, then clearly $\boldsymbol{\Omega} = 0$. We shall evaluate (2.150) for the angular velocity in a number of instances. A first example is given in Sect. 2.10.5.

2.10.4 The Physical Difference Between Static and Stationary Fields

We consider now an observer at rest in a stationary spacetime with timelike Killing field K. The observer thus moves along an integral curve $\gamma(\tau)$ of K. His four-velocity u is

$$u = \bigl(-\langle K, K\rangle\bigr)^{-1/2} K. \tag{2.156}$$

We now choose an orthonormal triad $\{e_i\}$ along γ which is *Lie-transported*

$$L_K e_i = 0, \tag{2.157}$$

for $i = 1, 2, 3$. Note that the e_i remain perpendicular to K and hence to u. Indeed, it follows from

$$0 = L_K g(X, Y) = K\langle X, Y\rangle - \langle L_K X, Y\rangle - \langle X, L_K Y\rangle$$

that the orthogonality of the vector fields X and Y is preserved when $L_K X = L_K Y = 0$. Also note that K itself is Lie-transported: $L_K K = [K, K] = 0$.

The $\{e_i\}$ can then be interpreted as "axes at rest" and define what one may call a "Copernican system" (see Fig. 2.9). We are interested in the change of the spin relative to this system. Our starting point is (2.150) or, making use of (2.156)

$$\omega_{ij} = \bigl(-\langle K, K\rangle\bigr)^{-1/2} \langle e_j, \nabla_K e_i\rangle. \tag{2.158}$$

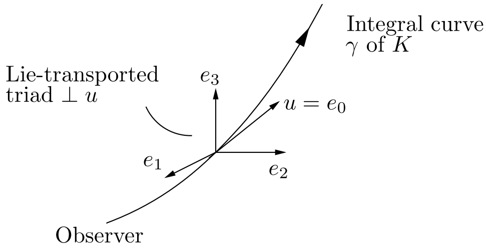


Fig. 2.9 Spin precession in stationary fields relative to a "Copernican system"

Now

$$0 = T(K, e_i) = \nabla_K e_i - \nabla_{e_i} K - [K, e_i],$$

and $[K, e_i] = L_K e_i = 0$. Hence (2.158) implies

$$\omega_{ij} = \big(-\langle K, K\rangle\big)^{-1/2} \langle e_j, \nabla_{e_i} K\rangle = \big(-\langle K, K\rangle\big)^{-1/2} \nabla \boldsymbol{K}(e_j, e_i),$$

where $\boldsymbol{K} = K^\flat$. Since ω_{ij} is antisymmetric

$$\omega_{ij} = -\big(-\langle K, K\rangle\big)^{-1/2} \frac{1}{2} \big(\nabla \boldsymbol{K}(e_i, e_j) - \nabla \boldsymbol{K}(e_j, e_i)\big)$$

or, since any one-form φ satisfies $\nabla\varphi(X, Y) - \nabla\varphi(Y, X) = -d\varphi(X, Y)$, we have also

$$\omega_{ij} = \frac{1}{2} \big(-\langle K, K\rangle\big)^{-1/2} d\boldsymbol{K}(e_i, e_j). \tag{2.159}$$

We shall show below that

$$\omega_{ij} = 0, \quad \text{if and only if} \quad \boldsymbol{K} \wedge d\boldsymbol{K} = 0. \tag{2.160}$$

From this it follows that a *Copernican system does not rotate if and only if the stationary field is static*.

The one-form $*(\boldsymbol{K} \wedge d\boldsymbol{K})$ can be regarded as a measure of the "absolute" rotation, because the vector $\varOmega = \varOmega^k e_k$ can be expressed in the form

$$\boldsymbol{\varOmega} = -\frac{1}{2} \langle K, K\rangle^{-1} * (\boldsymbol{K} \wedge d\boldsymbol{K}), \tag{2.161}$$

where $\boldsymbol{\varOmega}$ denotes the one-form corresponding to $\varOmega$.

It remains to derive Eq. (2.161). Let $\{\theta^\mu\}$ denote the dual basis of $\{e_\mu\}$, $e_0 = u$. From well-known properties of the $*$-operation (see DG, Sect. 14.6.2), we have

$$\theta^\mu \wedge (\boldsymbol{K} \wedge d\boldsymbol{K}) = \eta \big\langle \theta^\mu, *(\boldsymbol{K} \wedge d\boldsymbol{K})\big\rangle. \tag{2.162}$$

Since $\boldsymbol{K} = -(-\langle K, K\rangle)^{1/2} \theta^0$, the left hand side of (2.162) is equal to $-(-\langle K, K\rangle)^{1/2} \theta^\mu \wedge \theta^0 \wedge d\boldsymbol{K}$ and vanishes for $\mu = 0$. From (2.159) we conclude that

$$d\boldsymbol{K} = \big(-\langle K, K\rangle\big)^{1/2} \omega_{ij} \theta^i \wedge \theta^j + \text{terms containing } \theta^0.$$

Hence we have

$$\theta^\mu \wedge (\boldsymbol{K} \wedge d\boldsymbol{K}) = \begin{cases} 0 & \text{if } \mu = 0, \\ \langle K, K\rangle \varepsilon_{ijl} \Omega^l \theta^k \wedge \theta^0 \wedge \theta^i \wedge \theta^j = -2\langle K, K\rangle \eta \Omega^k & \text{if } \mu = k, \end{cases}$$

where we used $\theta^k \wedge \theta^0 \wedge \theta^i \wedge \theta^j = -\varepsilon_{ijk}\eta$. From this and (2.162) we get

$$\big(\theta^\mu, *(\boldsymbol{K} \wedge d\boldsymbol{K})\big) = \begin{cases} 0 & \text{if } \mu = 0, \\ -2\langle K, K\rangle \Omega^k & \text{if } \mu = k. \end{cases}$$

The left-hand side of this expression is equal to the contravariant components of $*(\boldsymbol{K} \wedge d\boldsymbol{K})$ and hence (2.161) follows. Obviously (2.161) implies (2.160).

2.10.5 Spin Rotation in a Stationary Field

The spin rotation relative to the Copernican system is given by (2.161). We now write this in terms of adapted coordinates, with $K = \partial/\partial t$ and $g_{\mu\nu}$ independent of $t = x^0$. Then

$$\begin{aligned} \boldsymbol{K} &= g_{00}\, dt + g_{0i}\, dx^i, \\ d\boldsymbol{K} &= g_{00,k}\, dx^k \wedge dt + g_{0i,k}\, dx^k \wedge dx^i, \\ \boldsymbol{K} \wedge d\boldsymbol{K} &= (g_{00} g_{0i,j} - g_{0i} g_{00,j})\, dt \wedge dx^j \wedge dx^i + g_{0k} g_{0i,j}\, dx^k \wedge dx^j \wedge dx^i. \end{aligned}$$

From DG, Exercise 14.7, we then have

$$\begin{aligned} *(\boldsymbol{K} \wedge d\boldsymbol{K}) &= g_{00}^2 \left(\frac{g_{0i}}{g_{00}}\right)_{,j} * \big(dt \wedge dx^j \wedge dx^i\big) + g_{0k} g_{0i,j} * \big(dx^k \wedge dx^j \wedge dx^i\big) \\ &= \frac{g_{00}^2}{\sqrt{-g}} \varepsilon_{ijl} \left(\frac{g_{0i}}{g_{00}}\right)_{,j} \\ &\quad \times \left[\big(g_{lk}\, dx^k + g_{l0}\, dx^0\big) - \left(g_{l0}\, dx^0 + \frac{g_{l0} g_{k0}}{g_{00}}\, dx^k\right)\right], \end{aligned}$$

where we used $*(dt \wedge dx^j \wedge dx^i) = \eta^{0jil} g_{l\mu}\, dx^\mu = -\frac{1}{\sqrt{-g}} \varepsilon_{jil} g_{l\mu}\, dx^\mu$ in the last step. We thus obtain

$$\boldsymbol{\Omega} = \frac{-g_{00}}{2\sqrt{-g}} \varepsilon_{ijl} \left(\frac{g_{0i}}{g_{00}}\right)_{,j} \left(g_{lk} - \frac{g_{l0} g_{k0}}{g_{00}}\right) dx^k \tag{2.163}$$

and it follows immediately that

$$\Omega = \frac{-g_{00}}{2\sqrt{-g}} \varepsilon_{ijl} \left(\frac{g_{0i}}{g_{00}}\right)_{,j} \left(\partial_k - \frac{g_{k0}}{g_{00}} \partial_0\right). \tag{2.164}$$

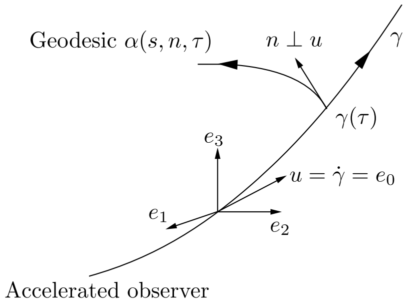


Fig. 2.10 Adapted coordinates for accelerated observer

We shall later apply this equation to the field outside of a rotating star or black hole. At sufficiently large distances we have

$$g_{00} \simeq -1, \qquad g_{ij} \simeq \delta_{ij}, \qquad \frac{g_{0k}}{g_{00}} \ll 1,$$

and a good approximation to (2.164) is given by

$$\Omega \simeq \frac{1}{2}\varepsilon_{ijk} g_{0i,j} \partial_k. \tag{2.165}$$

Since $e_k \cong \partial_k$, the gyroscope rotates relative to the Copernican frame with angular velocity (in three dimensional notation)

$$\boldsymbol{\Omega} \simeq -\frac{1}{2}\nabla \times \boldsymbol{g}, \tag{2.166}$$

where $\boldsymbol{g} := (g_{01}, g_{02}, g_{03})$.

We have shown that in a stationary (but not static) field, a gyroscope rotates relative to the Copernican system (relative to the "fixed stars") with angular velocity (2.164). In a weak field this can be approximated by (2.166). This means that *the rotation of a star drags along the local inertial system* (*Lense–Thirring effect*). We shall discuss possible experimental tests of this effect later.

2.10.6 Adapted Coordinate Systems for Accelerated Observers

We again consider the world line $\gamma(\tau)$ of an (accelerated) observer. Let $u = \dot{\gamma}$ and let $\{e_i\}$ be an arbitrary orthonormal frame along γ which is perpendicular to $e_0 = u$. As before, $a = \nabla_u u$. Now construct a local coordinate system as follows: at every point on $\gamma(\tau)$ consider spacelike geodesics $\alpha(s)$ perpendicular to u, with proper length s as affine parameter. Thus $\alpha(0) = \gamma(\tau)$; let $n = \dot{\alpha}(0) \perp u$. In order to distinguish the various geodesics, we denote the geodesic through $\gamma(\tau)$ in the direction n with affine parameter s by $\alpha(s, n, \tau)$ (see Fig. 2.10). We have

$$n = \left(\frac{\partial}{\partial s}\right)_{\alpha(0,n,\tau)}, \qquad \langle n, n\rangle = 1. \tag{2.167}$$

Each point $p \in M$ in the vicinity of the observer's world line lies on precisely one of these geodesics. If $p = \alpha(s, n, \tau)$ and $n = n^j e_j$ we assign the following coordinates to p

$$\big(x^0(p), \dots, x^3(p)\big) = \big(\tau, sn^1, sn^2, sn^3\big). \tag{2.168}$$

This means

$$\begin{aligned} x^0\big(\alpha(s, n, \tau)\big) &= \tau \\ x^j\big(\alpha(s, n, \tau)\big) &= sn^j = sn_j = s\langle n, e_j\rangle. \end{aligned} \tag{2.169}$$

Calculation of the Christoffel Symbols Along $\gamma(\tau)$

Along the observer's world line, we have by construction

$$\left.\frac{\partial}{\partial x^\alpha}\right|_\gamma = e_\alpha, \tag{2.170}$$

and hence $g_{\alpha\beta} = \langle\partial_\alpha, \partial_\beta\rangle = \eta_{\alpha\beta}$ along $\gamma(\tau)$. If $\Gamma^\alpha_{\beta\gamma}$ denote the Christoffel symbols relative to the tetrad $\{e_\alpha\}$ (see DG, Sect. 15.7), then $\nabla_u e_\alpha = \nabla_{e_0} e_\alpha = \Gamma^\beta_{0\alpha} e_\beta$ and thus

$$\langle e_\beta, \nabla_u e_\alpha\rangle = \eta_{\beta\gamma}\Gamma^\gamma_{0\alpha}. \tag{2.171}$$

In particular,

$$\begin{aligned} \Gamma^0_{00} &= -\langle u, \nabla_u u\rangle = 0, \\ \Gamma^j_{00} &= \langle e_j, \nabla_u u\rangle = \langle e_j, a\rangle = a^j, \\ \Gamma^0_{0j} &= -\langle u, \nabla_u e_j\rangle = \langle e_j, a\rangle = a^j. \end{aligned}$$

If we make use of $\omega_{ij} = \langle\nabla_u e_i, e_j\rangle = -\omega_{ji}$, introduced in (2.150), and the angular velocity Ω^i,

$$\omega_{jk} = \varepsilon_{ijk}\Omega^i, \tag{2.172}$$

then

$$\Gamma^j_{0k} = -\varepsilon_{ijk}\Omega^i. \tag{2.173}$$

The remaining Christoffel symbols can be read off from the equation for the geodesics $s \longmapsto \alpha(s, n, \tau)$. According to (2.169) the coordinates x^μ for these geodesics are $x^0(s) = \text{const.}$ and $x^j(s) = sn^j$, hence $d^2x^\alpha/ds^2 = 0$. On the other hand, the geodesics satisfy the equation

$$0 = \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = \Gamma^\alpha_{jk}n^j n^k.$$

Hence, along the observer's world line γ we have

$$\Gamma^{\alpha}_{jk} = 0. \tag{2.174}$$

The partial derivatives of the metric coefficients can be determined from the Christoffel symbols. The general relation is

$$0 = g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^{\mu}_{\alpha\gamma} g_{\mu\beta} - \Gamma^{\mu}_{\beta\gamma} g_{\alpha\mu}.$$

If we substitute $g_{\alpha\beta} = \eta_{\alpha\beta}$ along $\gamma(\tau)$ and our previously derived results for the Christoffel symbols, we find

$$g_{\alpha\beta,0} = 0, \qquad g_{ik,l} = 0,$$
$$g_{00,j} = -2a^j, \qquad g_{0j,k} = -\varepsilon_{jkl}\Omega^l.$$

These relations, together with $g_{\alpha\beta} = \eta_{\alpha\beta}$ along $\gamma(\tau)$, imply that the metric near γ is given by

$$\begin{aligned} g = {} & -(1 + 2\boldsymbol{a}\cdot\boldsymbol{x})\big(dx^0\big)^2 - 2\varepsilon_{jkl}x^k\Omega^l\, dx^0\, dx^j \\ & + \delta_{jk}\, dx^j\, dx^k + O\big(|\boldsymbol{x}|^2\big)\, dx^\alpha\, dx^\beta. \end{aligned} \tag{2.175}$$

From this we see that

1. The acceleration leads to the additional term
$$\delta g_{00} = -2\boldsymbol{a}\cdot\boldsymbol{x}. \tag{2.176}$$
2. Since the observer's coordinates axes rotate ($\Omega^i \neq 0$), the metric has the "non-diagonal" term
$$g_{0j} = -\varepsilon_{jkl}x^k\Omega^l = -(\boldsymbol{x}\times\boldsymbol{\Omega})^j. \tag{2.177}$$
3. The lowest order corrections are not affected by the curvature. The curvature shows itself in second order.
4. If $a = \nabla_u u = 0$ and $\boldsymbol{\Omega} = 0$ (no acceleration and no rotation) we have a local inertial system along $\gamma(\tau)$.

2.10.7 Motion of a Test Body

Suppose that the observer, whose world line is $\gamma(\tau)$ (an astronaut in a capsule, for example), observes a nearby freely falling body. This obeys the equation of motion

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0.$$

We now replace the proper time τ of the test body by the coordinate time t, using

$$\frac{d}{d\tau} = \left(\frac{dt}{d\tau}\right)\frac{d}{dt} =: \gamma\frac{d}{dt}.$$

Since $dx^\alpha/d\tau = \gamma(1, dx^k/dt)$, we obtain

$$\frac{d^2x^\alpha}{dt^2} + \frac{1}{\gamma}\frac{d\gamma}{dt}\frac{dx^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0. \tag{2.178}$$

For $\alpha = 0$, this becomes

$$\frac{1}{\gamma}\frac{d\gamma}{dt} + \Gamma^0_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0.$$

Substitution of this into (2.178) for $\alpha = j$ gives

$$\frac{d^2x^j}{dt^2} + \left(-\frac{dx^j}{dt}\Gamma^0_{\beta\gamma} + \Gamma^j_{\beta\gamma}\right)\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0. \tag{2.179}$$

We work this out relative to the coordinate system introduced above for small velocities of the test body. To first order in $v^j = dx^j/dt$ the equation of motion is

$$\frac{dv^j}{dt} - v^j\Gamma^0_{00} + \Gamma^j_{00} + 2\Gamma^j_{k0}v^k = 0. \tag{2.180}$$

Since the particle is falling in the vicinity of the observer, the spatial coordinates (2.169) are small. To first order in x^k and v^k we have

$$\frac{dv^j}{dt} = v^j\Gamma^0_{00}|_{\boldsymbol{x}=0} - \Gamma^j_{00}|_{\boldsymbol{x}=0} - x^k\Gamma^j_{00,k}|_{\boldsymbol{x}=0} - 2v^k\Gamma^j_{k0}|_{\boldsymbol{x}=0}.$$

The Christoffel symbols along γ have already been determined. If we insert these, we find

$$\frac{dv^j}{dt} = -a^j - 2\varepsilon_{jik}\Omega^i v^k - x^k\Gamma^j_{00,k}|_{\boldsymbol{x}=0}.$$

The quantity $\Gamma^j_{00,k}$ is obtained from the Riemann tensor (see DG, Eq. (15.30))

$$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\gamma\mu}\Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\delta\mu}\Gamma^\mu_{\beta\gamma},$$

so that

$$\Gamma^j_{00,k} = R^j_{0k0} + \Gamma^j_{0k,0} - \Gamma^j_{k\mu}\Gamma^\mu_{00} + \Gamma^j_{0\mu}\Gamma^\mu_{0k}.$$

For $\boldsymbol{x} = 0$, we have

$$\Gamma^j_{0k,0} = -\varepsilon_{jkm}\Omega^m_{,0},$$
$$\Gamma^j_{k\mu}\Gamma^\mu_{00} = 0,$$
$$\Gamma^j_{0\mu}\Gamma^\mu_{0k} = \Gamma^j_{00}\Gamma^0_{0k} + \Gamma^j_{0m}\Gamma^m_{0k} = a^j a^k + \varepsilon_{mjn}\Omega^n\varepsilon_{kml}\Omega^l.$$

Hence,

$$x^k\Gamma^j_{00,k} = x^k\big(R^j_{0k0} - \varepsilon_{jkm}\Omega^m_{,0} + a^j a^k + \varepsilon_{mjn}\Omega^n\varepsilon_{kml}\Omega^l\big).$$

Expressed in three-dimensional vector notation, we end up with

$$\dot{\boldsymbol{v}} = -\boldsymbol{a}(1+\boldsymbol{a}\cdot\boldsymbol{x}) - 2\boldsymbol{\Omega}\times\boldsymbol{v} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}) - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} + \boldsymbol{f}, \tag{2.181}$$

where

$$f^j := R^j_{00k}x^k. \tag{2.182}$$

The first term in (2.181) is the usual "inertial acceleration", including the relativistic correction $(1+\boldsymbol{a}\cdot\boldsymbol{x})$, which is a result of (2.176). The terms containing $\boldsymbol{\Omega}$ are well-known from classical mechanics. The force $R^j_{00k}x^k$ is a consequence of the inhomogeneity of the gravitational field.

If the frame $\{e_i\}$ is Fermi transported ($\boldsymbol{\Omega}=0$), the Coriolis force (third term) vanishes. If the observer is freely falling, then also $\boldsymbol{a}=0$ and only the *tidal force* f^j remains. This cannot be transformed away. The equation of motion for the test body then becomes

$$\frac{d^2x^j}{dt^2} = -R^j{}_{0k0}x^k. \tag{2.183}$$

We shall discuss this *equation of geodesic deviation* in more detail at the beginning of the next chapter.

2.10.8 Exercises

Exercise 2.22 Carry out the rearrangements leading to the Thomas precession (2.146).

Exercise 2.23 A non spherical body of mass density ρ in an inhomogeneous gravitational field experiences a torque which results in a time dependence of the spin four vector. Suppose that the center of mass is freely falling along a geodesic with four-velocity u. Show that S satisfies the equation of motion

$$\nabla_u S^\rho = \eta^{\rho\beta\alpha\mu}u_\mu u^\sigma u^\lambda t_{\beta\nu}R^\nu_{\sigma\alpha\lambda},$$

where $t_{\beta\nu}$ is the "reduced quadrupole moment tensor"

$$t_{ij} = \int \rho \left(x^i x^j - \frac{1}{3}\delta_{ij} \right) d^3x$$

in the rest frame of the center of mass and $t^{\alpha\beta} u_\beta = 0$. It is assumed that the Riemann tensor is determined by an external field which is nearly constant over distances comparable to the size of the test body.

Solution Perform the computation in the local comoving Lorentz frame of the center of mass of the body. The center of mass is taken as the reference point of the equation of geodesic deviation (2.183). The relative acceleration of a mass element at position x^i due to the tidal force is thus $-R^j{}_{0k0} x^k$. Therefore, the i^{th} component of the torque per unit volume is equal to $-\varepsilon_{ilj} x^l \rho R^j{}_{0k0} x^k$. The total torque, which determines the time derivative of the intrinsic angular momentum, is the right hand side of the equation

$$\frac{dS_i}{dt} = -\varepsilon_{ilj} R^j{}_{0k0} \int \rho x^l x^k \, d^3x,$$

if the variation of $R^j{}_{0k0}$ over the body is neglected. Because of the symmetry properties of the expression in front of the integral, this equation is equivalent to

$$\frac{dS_i}{dt} = -\varepsilon_{ilj} t_{lk} R^j{}_{0k0},$$

where t_{lk} is the "reduced quadrupole tensor" in the exercise. The invariant tensor version of this equation is just the equation to be derived.

Exercise 2.24 Show that

$$H = \frac{1}{2} g^{\mu\nu} (\pi_\mu - eA_\mu)(\pi_\nu - eA_\nu)$$

is the Hamiltonian which describes the motion of a charged particle with charge e in a gravitational field (π_μ is the *canonical momentum*).

Exercise 2.25 (Maxwell equations in a static spacetime) Let g be a static spacetime. In adapted coordinates

$$g = -\alpha^2 \, dt^2 + g_{ik} \, dx^i \, dx^k, \tag{2.184}$$

where α (the *lapse*) and g_{ik} are independent of t. We introduce an orthonormal tetrad of 1-forms: $\theta^0 = \alpha \, dt$, $\{\theta^i\}$ an orthonormal triad for the spatial metric. Relative to this basis we decompose the Faraday 2-form as in SR

$$F = E \wedge \theta^0 + B, \tag{2.185}$$

where E is the electric 1-form $E = E_i \theta^i$ and B the magnetic 2-form $B = \frac{1}{2} B_{ij} \theta^i \wedge \theta^j$.

1. Show that the homogeneous Maxwell equation $dF = 0$ splits as

$$\boldsymbol{d}B = 0, \qquad \boldsymbol{d}(\alpha E) + \partial_t B = 0, \tag{2.186}$$

where $\boldsymbol{d}$ is the 3-dimensional (spatial) exterior derivative.

2. Show that

$$*F = -H \wedge \theta^0 + D, \tag{2.187}$$

where

$$H = \ast B, \qquad D = \ast E. \tag{2.188}$$

The symbol $\ast$ denotes the Hodge dual for the spatial metric $g_{ik}\, dx^i\, dx^k$.

3. Decompose the current 1-form J as

$$J = \rho\theta^0 + j_k\theta^k \equiv \rho\theta^0 + j$$

and show that the inhomogeneous Maxwell equation $\delta F = -J$ splits as

$$\boldsymbol{\delta} E = 4\pi\rho, \qquad \boldsymbol{\delta}(\alpha B) = \boldsymbol{\partial}_t E + 4\pi\alpha j, \tag{2.189}$$

where $\boldsymbol{\delta}$ is the spatial codifferential.

4. Deduce the continuity equation

$$\partial_t \rho + \boldsymbol{\delta}(\alpha j) = 0. \tag{2.190}$$

2.11 General Relativistic Ideal Magnetohydrodynamics

General relativistic extensions of magnetohydrodynamics (MHD) play an important role in relativistic astrophysics, for instance in accretion processes on black holes. In this section we present the basic equations in the limit of infinite conductivity (ideal MHD).

We consider a relativistic fluid with rest-mass density ρ_0, energy-mass density ρ, 4-velocity u^μ, and isotropic pressure p. The basic equations of ideal MHD are easy to write down. First, we have the baryon conservation

$$\nabla_\mu \left(\rho_0 u^\mu\right) = 0. \tag{2.191}$$

For a magnetized plasma the equations of motion are

$$\nabla_\nu T^{\mu\nu} = 0, \tag{2.192}$$

where the energy-stress tensor $T^{\mu\nu}$ is the sum of the matter (M) and the electromagnetic (EM) parts:

$$T_M^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}, \tag{2.193}$$

$$T_{EM}^{\mu\nu} = \frac{1}{4\pi}\left(F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right). \tag{2.194}$$

In addition we have Maxwell's equations

$$dF = 0, \qquad \nabla_\nu F^{\mu\nu} = 4\pi J^\mu. \tag{2.195}$$

We adopt the *ideal MHD approximation*

$$i_u F = 0, \tag{2.196}$$

which expresses that the electric field vanishes in the rest frame of the fluid (infinite conductivity). Then the inhomogeneous Maxwell equation provides the current 4-vector J^μ.

As a consequence of (2.195) and (2.196) we obtain, with the help of Cartan's identity $L_u = i_u \circ d + d \circ i_u$,

$$L_u F = 0,$$

i.e., that F is invariant under the plasma flow, implying flux conservation. The basic equations imply that

$$*F = B \wedge u, \tag{2.197}$$

where $B = i_u * F$ is the magnetic induction in the rest frame of the fluid (seen by a comoving observer); see the solution of Exercise 2.26. Note that $i_u B = 0$. Furthermore, one can show (Exercise 2.26) that the electromagnetic part of the energy-momentum tensor may be written in the form

$$T_{EM}^{\mu\nu} = \frac{1}{4\pi}\left[\frac{1}{2}\|B\|^2 g^{\mu\nu} + \|B\|^2 U^\mu U^\nu - B^\mu B^\nu\right], \tag{2.198}$$

with $\|B\|^2 := B_\alpha B^\alpha$.

2.11.1 Exercises

Exercise 2.26 Derive Eq. (2.197). Consider, more generally, for an arbitrary Faraday form F, the electric and magnetic fields measured by a comoving observer

$$E = -i_u F, \qquad B = i_u * F, \tag{2.199}$$

and show that

$$F = u \wedge E + *(u \wedge B). \tag{2.200}$$

Solution The right hand side of (2.200) can be written with the help of the identity (14.34) as

$$u \wedge E + *(u \wedge i_u * F) = u \wedge E - i_u(F \wedge u) = F. \tag{2.201}$$

Exercise 2.27 Derive Eq. (2.198).

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