Multipole Expansion Method in Micromechanics of Composites

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Introduction

In scientific literature, the Multipole Expansion Method is associated with a group of techniques and algorithms designed to study behavior of large scale collections of interacting objects of various nature, from atoms and molecules up to stars and galaxies. Analytical in nature, this method provides a theoretical basis of very efficient (e.g., [18]) computer codes and found numerous applications in cosmology, physics, chemistry, engineering, statistics, etc. This list includes also mechanics of heterogeneous solids and fluid suspensions, where a certain progress is observed in development of the multipole expansion based theories and applications.

The author's opinion is, however, that importance of this method for the micromechanics of composites is underestimated and its potential in the area is still not fully discovered. The contemporary studies on composites are still often based on the single inclusion model even if this is inappropriate in the problem under study. As known, the single inclusion-based theories provide $O(c)$ estimate of effective properties, c being the volume content of inclusions, so their application is justified to the composites with low c only. In order to get the next, $O(c^2)$ virial expansion term of the effective property, the pair interaction effect must be taken into account by means of the two-inclusion model (e.g., [26]). Further accuracy improvement requires the model with several interacting inclusions to be considered. The multipole expansion is an efficient tool for studying, from the multiple inclusion models, the effects caused by micro structure on the local fields and effective properties being the central problem of the science of composites.

It should be mentioned that some diversity exists in literature in using the words "multipole" and "multipole expansion". The idea of multipoles is traced back to

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Maxwell [57] who defined them as the complex point charges and studied the relationship between the potential fields of multipoles and spherical harmonics. To avoid confusing, the notions "multipole" (point source), "multipole field" (potential) and "multipole moment" (strength) should be clearly distinguished. Among several available in literature definitions, the most general one, probably, is [25]:

> "*...A multipole expansion is a series expansion of the effect produced by a given system in terms of an expansion parameter which becomes small as the distance away from the system increases*".

The basis functions and the expansion coefficients are referred as the potential fields and moments (strengths) of multipoles, respectively. What is important, this definition imposes no restrictions on the basis functions and, in what follows, we accept it.

By tradition, we call the method exposed in this Chapter the Multipole Expansion Method (MEM) despite the fact that multipole expansion is only a part of solution procedure. The basic idea of the method consists in reducing the boundary value problem stated on the piece-homogeneous domain to the ordinary system of linear algebraic equations. In so doing, a considerable analytical effort involving the mathematical physics and theory of special functions is required. This effort is quite rewarding, in view of the obtained this way remarkably simple and efficient computational algorithms.

MEM is essentially the series method, where the partial solutions of differential equation obtained by separation of variables in an appropriate coordinate system constitute a countable set of basis functions. The specific curvilinear coordinate system is dictated by the inclusion shape and introduced in a way that the matrix-inclusion interface coincides with the coordinate surface. An important for our study feature of the basis functions is that at this coordinate surface they form a full and orthogonal set of surface harmonics and thus provide an efficient way of fulfilling the interface boundary conditions.

In this Chapter, we review the work done for the scalar (conductivity) and vectorial (linear elasticity) problems. Two matrix type composites under study are (a) particulate composites with spherical and spheroidal inclusions and (b) unidirectional fibrous composite materials with circular and elliptic (in cross-section) fibers. The isotropic as well as anisotropic properties of constituents are considered. The review is structured as follows.

The homogenization problem, in particular, the rational way of introducing the macro parameters and effective properties of composite is briefly discussed in Sect. 1. The general formulas for the macroscopic flux vector and stress tensor are derived in terms of corresponding average gradient fields and dipole moments (stresslets) of the disturbance fields, i.e., in the form most appropriate for the multipole expansion approach.

In Sect. 2, we consider the Multipole Expansion Method in application to conductivity of composite with spherical inclusions as the most widely used and traditionally associated with multipoles geometry. This problem is well explored and we revisit it with aim to demonstrate the basic technique of the method and discuss the principal moments. In the subsequent Sections, the Multipole Expansion Method is applied to the elasticity problem as well as expanded on the composites with more complicated geometry of inclusions and properties of constituents.

All the Sections are structured uniformly, in accordance with the MEM solution flow. We begin with the problem for a single inclusion, immersed in non-uniform far field. These results, on the one hand, provide a necessary background for the subsequent study. On the other hand, they can be viewed as the *generalized Eshelby's model* expanded on the case of non-uniform far load—but still readily implanted in that or another self-consistent scheme.

Next, the Finite Cluster Model (FCM) is considered. To obtain an accurate solution of the multiple inclusion problem, the above solution for a single inclusion is combined with the superposition principle and the re-expansion formulas for a given geometry of inclusion. These results constitute the intermediate, second step of the method and will be further developed in order to obtain the full-featured model of composite. At the same time, this model can be viewed as the *generalized Maxwell's model*, where the particle-particle interactions are taken into account.

Then, the Representative Unit Cell (RUC) model of composite is studied. Here, the periodic solutions and corresponding lattice sums are introduced. A complete solution of the model provides a detailed analysis of the local fields, their analytical integration gives the exact, only dipole moments containing expressions of the effective conductivity and elasticity tensors.This model can be viewed as the *generalized Rayleigh's model* expanded on the general type geometry (both regular and random) of composite, with an adequate account for the interaction effects.

1 Homogenization Problem

The homogenization problem is in the focus of the composite mechanics for the last 50 years. The various aspects of this problem including (a) structure levels, (c) representative volume element (RVE) size and shape, (b) way of introducing the macro parameters and effective properties of composite, etc., were widely discussed in several books and thousands of papers. Our aim is more limited and specific: here, we will discuss how the multipole expansion solutions apply to the homogenization problem.

1.1 Conductivity

1.1.1 Definition of Macroscopic Quantities: Volume Versus Surface Averaging

The macroscopic, or effective, conductivity tensor $\mathbf{\Lambda}^* = \left\{\lambda_{ij}^*\right\}$ is defined by the Fourier law:

$$\langle\mathbf{q}\rangle = -\mathbf{\Lambda}^* \cdot \langle\nabla T\rangle\,. \tag{1.1}$$

In (1.1), $\langle\nabla T\rangle$ and $\langle\mathbf{q}\rangle$ are the macroscopic temperature gradient and heat flux vector, respectively. Their introduction is not as self-obvious and the researchers are not unanimous in this matter. In most publications, $\langle\nabla T\rangle$ and $\langle\mathbf{q}\rangle$ are taken as the volume-averaged values of corresponding local fields:

$$\langle\nabla T\rangle = \frac{1}{V}\int_V \nabla T dV; \quad \langle\mathbf{q}\rangle = \frac{1}{V}\int_V \mathbf{q} dV; \tag{1.2}$$

where V is a volume of the representative volume element (RVE) of composite solid. For the matrix type composite we consider, $V = \sum_{q=0}^N V_q$, V_q being the volume of qth inclusion and V_0 being the matrix volume inside RVE. An alternate, surface averaging-based definition of the macroscopic conductivity parameters is [90]:

$$\langle\nabla T\rangle = \frac{1}{V}\int_{S_0} T\mathbf{n} dS, \quad \langle\mathbf{q}\rangle = \frac{1}{V}\int_{S_0} (\mathbf{q}\cdot\mathbf{n})\,\mathbf{r} dS. \tag{1.3}$$

It is instructive to compare these two definitions. We employ the gradient theorem [64] to write

$$\frac{1}{V}\int_V \nabla T dV = \frac{1}{V}\int_{S_0} T^{(0)}\mathbf{n} dS + \frac{1}{V}\sum_{q=1}^N \int_{S_q} \left(T^{(q)} - T^{(0)}\right)\mathbf{n} dS, \tag{1.4}$$

where S_q is the surface of V_q, S_0 is the outer surface of RVE and $\mathbf{n}$ is the unit normal vector. As seen from (1.4), the compared formulas coincide only if temperature is continuous $\left(T^{(0)} = T^{(q)}\right)$ at the interface. Noteworthy, (1.3) holds true for composites with imperfect interfaces whereas (1.2) obviously not. On order to compare two definitions of $\langle\mathbf{q}\rangle$, we employ the identity $\mathbf{q} = \nabla\cdot\left(\mathbf{q}\otimes\mathbf{r}\right)$ and the divergence theorem [64] to get

$$\frac{1}{V}\int_V \mathbf{q} dV = \frac{1}{V}\int_{S_0} \left(\mathbf{q}^{(0)}\cdot\mathbf{n}\right)\mathbf{r} dS + \frac{1}{V}\sum_{q=1}^N \int_{S_q} \left[\left(\mathbf{q}^{(q)}\cdot\mathbf{n}\right) - \left(\mathbf{q}^{(0)}\cdot\mathbf{n}\right)\right]\mathbf{r} dS.$$

Again, two definitions coincide only if the normal flux $q_n = \mathbf{q} \cdot \mathbf{n}$ is continuous across the interface—and again Eq. (1.3) holds true for composites with imperfect interfaces.

Thus, (1.2) is valid only for the composites with perfect thermal contact between the constituents. The definition (1.3) is advantageous at least in the following aspects:

- It involves only the observable quantities—temperature and flux—at the surface of composite specimen. In essence, we consider RVE as a "black box" whose interior structure may affect numerical values of the macro parameters—but not *the way* they were introduced.
- This definition is valid for composites with arbitrary interior microstructure and arbitrary (not necessarily perfect) interface bonding degree as well as for porous and cracked solids.
- Numerical simulation becomes quite similar to (and reproducible in) the experimental tests where we apply the temperature drop (voltage, etc.) to the surface of specimen and measure the heat flux (current, etc.) passing the surface. Macroscopic conductivity of composite is then found as the output-to-input ratio. In so doing, we have no need to study interior microstructure of composite and/or perform volume averaging of the local fields.

1.1.2 Formula for Macroscopic Flux

Now, we derive the formula, particularly useful for the effective conductivity study by the Multipole Expansion Method. We start with the generalized Green's theorem

$$\int_V (uLv - vLu)dV = \int_S \left(u\frac{\partial v}{\partial M} - v\frac{\partial u}{\partial M} \right) dS, \tag{1.5}$$

where

$$Lu = \sum_{i,j=1}^{m} \lambda_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \frac{\partial u}{\partial M} = \sum_{i,j=1}^{m} \lambda_{ij} n_j \frac{\partial u}{\partial x_i}. \tag{1.6}$$

In our context, $m = 2$ or 3. Physical meaning of the differential operators (1.6) is clear from the formulas

$$Lu = \nabla \cdot (\mathbf{\Lambda} \cdot \nabla u) = -\nabla \cdot \mathbf{q}(u); \ \frac{\partial u}{\partial M} = (\mathbf{\Lambda} \cdot \nabla u) \cdot \mathbf{n} = -q_n(u). \tag{1.7}$$

We apply Eqs. (1.5) and (1.7) to the matrix part (V_0) of RVE: with no loss in generality, we assume the outer boundary of RVE S_0 entirely belonging to the matrix. In new notations,

$$\int_{V_0} \left[T^{(0)}\nabla \cdot \mathbf{q}\left(T'\right) - T'\nabla \cdot \mathbf{q}\left(T^{(0)}\right)\right]dV$$

$$= \sum_{q=0}^{N} \int_{S_q} \left[T^{(0)}q_n\left(T'\right) - T'q_n\left(T^{(0)}\right)\right]dS, \tag{1.8}$$

where $T^{(0)}$ is an actual temperature field in matrix phase of composite and T' is a trail temperature field obeying, as well as $T^{(0)}$, the energy conservation law $\nabla \cdot \mathbf{q}\,(T) = 0$ in every point of V_0. Therefore, the volume integral in the left hand side of (1.8) is identically zero.

In the right hand side of (1.8), we take $T' = x_k$ and multiply by the Cartesian unit vector $\mathbf{i}_k$ to get

$$\sum_{q=0}^{N} \int_{S_q} \left[T^{(0)}\mathbf{\Lambda}_0 \cdot \mathbf{n} + q_n\left(T^{(0)}\right)\mathbf{r}\right]ds = 0,$$

where $\mathbf{r} = x_k\mathbf{i}_k$ is the radius-vector and $\mathbf{n} = n_k\mathbf{i}_k$ is the unit normal vector to the surface S_q. In view of $q_n\left(T^{(0)}\right) = \mathbf{q}\left(T^{(0)}\right) \cdot \mathbf{n}$ and (1.3), we come to the formula

$$\langle\mathbf{q}\rangle = -\mathbf{\Lambda}_0 \cdot \langle\nabla T\rangle + \frac{1}{V}\sum_{q=1}^{N} \int_{S_q} \left[T^{(0)}q_n\left(\mathbf{r}\right) - q_n(T^{(0)})\mathbf{r}\right]ds, \tag{1.9}$$

where we denote $q_n(\mathbf{r}) = q_n(x_k)\mathbf{i}_k$.

This formula is remarkable in several aspects.

- First, and most important, this equation together with (1.1) provide evaluation of the effective conductivity tensor of composite solid. Using RUC as the representative volume enables further simplification of Eq. (1.9).
- In derivation, no constraints were imposed on the shape of inclusions and interface conditions. Therefore, (1.9) is valid for the composite with anisotropic constituents and arbitrary matrix-to-inclusion interface shape, structure and bonding type.
- Integrals in (1.9) involve only the matrix phase temperature field, $T^{(0)}$. Moreover, these integrals are identically zero for all but dipole term in the $T^{(0)}$ multipole expansion in a vicinity of each inclusion and, in fact, represent contribution of these inclusions to the overall conductivity tensor.
- In the Multipole Expansion Method, where temperature in the matrix is initially taken in the form of multipole expansion, an analytical integration in (1.9) is straightforward and yields the exact, finite form expressions for the effective properties.

1.2 Elasticity

The fourth rank effective elastic stiffness tensor $\mathbf{C}^* = \left\{C^*_{ijkl}\right\}$ is defined by

$$\langle \sigma \rangle = \mathbf{C}^* : \langle \varepsilon \rangle\,, \tag{1.10}$$

where the macroscopic strain $\langle \varepsilon \rangle$ and stress $\langle \sigma \rangle$ tensors are conventionally defined as volume-averaged quantities:

$$\langle \varepsilon \rangle = \frac{1}{V}\int_V \varepsilon dV; \quad \langle \sigma \rangle = \frac{1}{V}\int_V \sigma dV. \tag{1.11}$$

This definition is valid for the composites with perfect mechanical bonding—and fails completely for the composites with imperfect interfaces. Also, this definition is "conditionally" correct for the porous and cracked solids.

Analogous to (1.3) surface averaging-based definition of the macroscopic strain and stress tensors [4]

$$\langle \varepsilon \rangle = \frac{1}{2V}\int_{S_0} (\mathbf{n} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{n})\, dS; \quad \langle \sigma \rangle = \frac{1}{V}\int_{S_0} \mathbf{r} \otimes (\sigma \cdot \mathbf{n})\, dS; \tag{1.12}$$

resolves the problem. In the case of perfect interfaces, this definition agrees with the conventional one, (1.11). This result is known in elastostatics as the mean strain theorem (e.g., [20]). Also, it follows from the mean stress theorem [20] that the volume averaged $\langle \sigma \rangle$ in (1.11) is consistent with (1.12) in the case of perfect mechanical contact between the matrix and inclusions. What is important for us, (1.12) holds true for the composites with imperfect interfaces (e.g., [11]).

1.2.1 Formula for Macroscopic Stress

The Betti's reciprocal theorem [20] written for the matrix domain V_0 of RVE states that the equality

$$\sum_{q=0}^{N} \int_{S_q} \left[\mathbf{T}_n(\mathbf{u}^{(0)}) \cdot \mathbf{u}' - \mathbf{T}_n(\mathbf{u}') \cdot \mathbf{u}^{(0)}\right] dS = 0$$

is valid for any displacement vector $\mathbf{u}'$ obeying the equilibrium equation $\nabla \cdot (\mathbf{C} : \nabla \mathbf{u}) = 0$. Following [44], we take it in the form $\mathbf{u}'_{ij} = \mathbf{i}_i x_j$. The dot product $\mathbf{T}_n(\mathbf{u}^{(0)}) \cdot \mathbf{u}'_{ij} = \sigma^{(0)}_{il} n_l x_j$ and, by definition (1.12),

$$\int_{S_0} \mathbf{T}_n(\mathbf{u}^{(0)}) \cdot \mathbf{u}'_{ij} dS = V \left\langle \sigma_{ij} \right\rangle.$$

On the other hand,

$$\mathbf{T}_n(\mathbf{u}'_{ij}) \cdot \mathbf{u}^{(0)} = \sigma_{kl}(\mathbf{u}'_{ij}) n_l u_k^- = C_{ijkl}^{(0)} n_l u_k^-;$$

comparison with (1.12) gives us

$$\frac{1}{V}\int_{S_0} \mathbf{T}_n(\mathbf{u}'_{ij}) \cdot \mathbf{u}^{(0)} dS = C_{ijkl}^{(0)} \left\langle \varepsilon_{kl} \right\rangle .$$

Thus, we come out with the formula

$$\langle \sigma_{ij} \rangle = C_{ijkl}^{(0)} \langle \varepsilon_{kl} \rangle + \frac{1}{V} \sum_{q=1}^{N} \int_{S_q} \left[\mathbf{T}_n(\mathbf{u}^{(0)}) \cdot \mathbf{u}'_{ij} - \mathbf{T}_n(\mathbf{u}'_{ij}) \cdot \mathbf{u}^{(0)} \right] dS \tag{1.13}$$

consistent with [74].

The Eq. (1.13) is the counterpart of (1.9) in the elasticity theory and everything said above with regard to (1.9) holds true for (1.13).

- This formula is valid for the composite with *arbitrary* (a) shape of disperse phase, (b) anisotropy of elastic properties of constituents and (c) interface bonding type.
- Together with (1.10), (1.13) enables evaluation of the effective stiffness tensor of composite provided the local displacement field $\mathbf{u}^{(0)}$ is known/found in some way.
- An yet another remarkable property of this formula consists in that the integral it involves (stresslet, in [74] terminology) is non-zero only for the dipole term in the vector multipole expansion of $\mathbf{u}^{(0)}$.

2 Composite with Spherical Inclusions: Conductivity Problem

The multipoles are usually associated with the spherical geometry, and the most work in the multipoles theory have been done for this case. In particular, the conductivity problem for a composite with spherical inclusions has received much attention starting from the pioneering works of Maxwell [57] and Rayleigh [73]. Now, this problem has been thoroughly studied and we revisit it to illustrate the basic technique of the method. To be specific, we consider thermal conductivity of composite. These results apply also to the mathematically equivalent physical phenomena (electric conductivity, diffusion, magnetic permeability, etc.).

2.1 Background Theory

2.1.1 Spherical Harmonics

The spherical coordinates (r, θ, φ) relate the Cartesian coordinates (x_1, x_2, x_3) by

$$x_1 + \mathrm{i}x_2 = r \sin\theta \exp(\mathrm{i}\varphi), \quad x_3 = r\cos\theta \quad (r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi < 2\pi). \tag{2.1}$$

Separation of variables in Laplace equation

$$\Delta T(\mathbf{r}) = 0 \tag{2.2}$$

in spherical coordinates gives us a set of partial ("normal", in Hobson's [23] terminology) solutions of the form

$$r^t P_t^s(\cos\theta)\exp(\mathrm{i}s\varphi) \quad (-\infty < t < \infty, \ -t \leq s \leq t) \tag{2.3}$$

referred [57] as scalar *solid* spherical harmonics of degree t and order s. Here, P_t^s are the associate Legendre's functions of first kind [23]. With regard to the asymptotic behavior, the whole set (2.3) is divided into two subsets: regular (infinitely growing with $r \to \infty$) and singular (tending to zero with $r \to \infty$) functions. We denote them separately as

$$y_t^s(\mathbf{r}) = \frac{r^t}{(t+s)!}\chi_t^s(\theta,\varphi); \quad Y_t^s(\mathbf{r}) = \frac{(t-s)!}{r^{t+1}}\chi_t^s(\theta,\varphi) \quad (t \geq 0, |s| \leq t), \tag{2.4}$$

respectively. In (2.4), χ_t^s are the scalar *surface* spherical harmonics

$$\chi_t^s(\theta,\varphi) = P_t^s(\cos\theta)\exp(\mathrm{i}s\varphi). \tag{2.5}$$

They possess the orthogonality property

$$\frac{1}{S}\int_S \chi_t^s\,\overline{\chi_k^l}\,dS = \alpha_{ts}\delta_{tk}\delta_{sl}, \quad \alpha_{ts} = \frac{1}{2t+1}\frac{(t+s)!}{(t-s)!}, \tag{2.6}$$

where integral is taken over the spherical surface S; over bar means complex conjugate and δ_{ij} is the Kronecker's delta. Adopted in (2.4) normalization is aimed to simplify the algebra [29, 71]: so, we have

$$y_t^{-s}(\mathbf{r}) = (-1)^s\overline{y_t^s(\mathbf{r})}, \quad Y_t^{-s}(\mathbf{r}) = (-1)^s\overline{Y_t^s(\mathbf{r})}. \tag{2.7}$$

We mention also the differentiation rule [23]:

$$D_1 y_t^s = y_{t-1}^{s-1}, \quad D_2 y_t^s = -y_{t-1}^{s+1}, \quad D_3 y_t^s = y_{t-1}^s;$$
$$D_1 Y_t^s = Y_{t+1}^{s-1}, \quad D_2 Y_t^s = -Y_{t+1}^{s+1}, \quad D_3 Y_t^s = -Y_{t+1}^s; \tag{2.8}$$

or, in the compact form,

$$(D_2)^s (D_3)^{t-s} \left(\frac{1}{r}\right) = (-1)^t Y_t^s(\mathbf{r}) \tag{2.9}$$

where D_i are the differential operators

$$D_1 = \left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right), \quad D_2 = \overline{D_1} = \left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right), \quad D_3 = \frac{\partial}{\partial x_3}. \tag{2.10}$$

These operators can be viewed as the directional derivatives along the complex Cartesian unit vectors $\mathbf{e}_i$ defined as

$$\mathbf{e}_1 = (\mathbf{i}_1 + \mathrm{i}\mathbf{i}_1)/2, \mathbf{e}_2 = \overline{\mathbf{e}_1}, \mathbf{e}_3 = \mathbf{i}_3. \tag{2.11}$$

2.1.2 Spherical Harmonics Versus Multipole Potentials

Maxwell [57] has discovered the relationship between the solid spherical harmonics (2.7) and the potential fields of multipoles. So, the potential surrounding a point charge (being a singular point of zeroth order, or *monopole*) is $1/r = Y_0^0(\mathbf{r})$. The first order singular point, or *dipole*, is obtained by pushing two monopoles of equal strength—but with opposite signs—toward each other. The potential of the dipole to be given (up to rescaling) by the directional derivative $\nabla_{\mathbf{u}_1}(1/r)$, where $\mathbf{u}_1$ is the direction along which the two monopoles approach one another. Similarly, pushing together two dipoles with opposite signs gives (up to rescaling) a *quadrupole* with potential $\nabla_{\mathbf{u}_1}\nabla_{\mathbf{u}_2}(1/r)$, where $\mathbf{u}_2$ s the direction along which the dipoles approach, and so on. In general, the multipole of order t is constructed with aid of 2^t point charges and has the potential proportional to $\nabla_{\mathbf{u}_1}\nabla_{\mathbf{u}_2}\dots\nabla_{\mathbf{u}_t}(1/r)$. The latter can be expanded into a weighted sum of $2t+1$ spherical harmonics of order t, i.e., $Y_t^s(\mathbf{r}),\ -t \le s \le t$. And, *vise versa*, $Y_t^s(\mathbf{r})$ can be written as $\nabla_{\mathbf{u}_1}\nabla_{\mathbf{u}_2}\dots\nabla_{\mathbf{u}_t}(1/r)$ provided the directions $\mathbf{u}_i$ are taken in accordance with the formula (2.9). This is why the series expansion in terms of solid spherical harmonics (2.7) is often referred as the *multipole expansion*.

2.2 *General Solution for a Single Inclusion*

Let consider the regular, non-uniform temperature far field T_{far} in unbounded solid (matrix) of conductivity λ_0. We insert a spherical inclusion of radius R and conductivity λ_1 inclusion assuming that its presence does not alter the incident field. The inclusion causes local disturbance T_{dis} of temperature field vanishing at infinity and depending, besides T_{far}, on the shape and size of inclusion, conductivities of the matrix and inclusion materials and the matrix-inclusion bonding type. The temperature field T ($T = T^{(0)}$ in the matrix, $T = T^{(1)}$ in the inclusion) satisfies (2.2). At the interface S ($r = R$), the perfect thermal contact is supposed:

$$[[T]] = 0; \quad [[q_r]] = 0; \tag{2.12}$$

where $[[f]] = (f^{(0)} - f^{(1)})|_{r=R}$ is a jump of quantity f through the interface S and $q_n = -\lambda \nabla T \cdot \mathbf{n}$ is the normal heat flux. Our aim is to determine the temperature in and outside the inclusion.

2.2.1 Multipole Expansion Solution

The temperature field in the inclusion $T^{(1)}$ is finite and hence its series expansion contains the regular solutions $y_t^s(\mathbf{r})$ (2.4) only:

$$T^{(1)}(\mathbf{r}) = \sum_{t=0}^{\infty} \sum_{s=-t}^{t} d_{ts} y_t^s(\mathbf{r}) \tag{2.13}$$

where d_{ts} are the unknown coefficients (complex, in general). Temperature is a real quantity, so (2.7) leads to analogous relation between the series expansion coefficients: $d_{t,-s} = (-1)^s \overline{d_{ts}}$. In accordance with physics of the problem, temperature $T^{(0)}$ in the matrix domain is written as $T^{(0)} = T_{far} + T_{dis}$, where $T_{dis}(\mathbf{r}) \to 0$ with $\|\mathbf{r}\| \to \infty$. It means that T_{dis} series expansion contains the singular solutions Y_t^s only. So, we have

$$T^{(0)}(\mathbf{r}) = T_{far}(\mathbf{r}) + \sum_{t=1}^{\infty} \sum_{s=-t}^{t} A_{ts} Y_t^s(\mathbf{r}), \tag{2.14}$$

where A_{ts} are the unknown coefficients. Again, $A_{t,-s} = (-1)^s \overline{A_{ts}}$. The second, series term in (2.14) is the *multipole expansion* of the disturbance field T_{dis}.

2.2.2 Far Field Expansion

We consider T_{far} as the governing parameter. It can be prescribed either analytically or in tabular form (e.g., obtained from numerical analysis). In fact, it suffices to know T_{far} values in the integration points at the interface S defined by $r = R$. Due to regularity of T_{far} in a vicinity of inclusion, its series expansion is analogous to (2.13), with the another set of coefficients c_{ts}. In view of (2.6), they are equal to

$$c_{ts} = \frac{(t+s)!}{4\pi R^2 \alpha_{ts}} \int_S T_{far} \, \overline{\chi_t^s} \, dS. \tag{2.15}$$

For a given T_{far}, we can consider c_{ts} as the known values. Integration in (2.15) can be done either analytically or numerically: in the latter case, the appropriate scheme [42] comprises uniform distribution of integration points in azimuthal direction φ with Gauss-Legendre formula [1] for integration with respect to θ.

2.2.3 Resolving Equations

The last step consists is substituting $T^{(0)}$ (2.14) and $T^{(1)}$ (2.13) into the bonding conditions (2.12). From the first, temperature continuity condition we get for $r = R$

$$\sum_{t=0}^{\infty} \sum_{s=-t}^{t} c_{ts} \frac{R^t}{(t+s)!} \chi_t^s(\theta, \varphi) + \sum_{t=1}^{\infty} \sum_{s=-t}^{t} A_{ts} \frac{(t-s)!}{R^{t+1}} \chi_t^s(\theta, \varphi) \tag{2.16}$$
$$= \sum_{t=0}^{\infty} \sum_{s=-t}^{t} d_{ts} \frac{R^t}{(t+s)!} \chi_t^s(\theta, \varphi).$$

From here, for $t = 0$ ($\chi_0^0 \equiv 1$) we get immediately $d_{00} = c_{00}$. For $t \neq 0$, in view of χ_t^s orthogonality property (2.6), we come to a set of linear algebraic equations

$$\frac{(t-s)!\,(t+s)!}{R^{2t+1}} A_{ts} + c_{ts} = d_{ts}. \tag{2.17}$$

The second, normal flux continuity condition gives us also

$$-\frac{(t+1)}{t} \frac{(t-s)!\,(t+s)!}{R^{2t+1}} A_{ts} + c_{ts} = \omega d_{ts}, \tag{2.18}$$

where $\omega = \lambda_1/\lambda_0$. By eliminating d_{ts} from (2.17) to (2.18), we get the coefficients A_{ts}:

$$\frac{(\omega + 1 + 1/t)}{(\omega - 1)} \frac{(t-s)!\,(t+s)!}{R^{2t+1}} A_{ts} = -c_{ts}; \tag{2.19}$$

then, the d_{ts} coefficients can be found from (2.17). The obtained **general solution** is exact and, in the case of polynomial far field, finite one.

2.3 *Finite Cluster Model (FCM)*

Now, we consider an unbounded solid containing a finite array of N spherical inclusions of radius R_q and conductivity λ_q centered in the points O_q. In the (arbitrarily introduced) global Cartesian coordinate system $Ox_1x_2x_3$, position of qth inclusion is defined by the vector $\mathbf{R}_q = X_{1q}\mathbf{e}_{x_1} + X_{2q}\mathbf{e}_{x_2} + X_{3q}\mathbf{e}_{x3}$, $q = 1, 2, \ldots, N$. Their non-overlapping condition is $\left|\mathbf{R}_{pq}\right| > R_p + R_q$, where the vector $\mathbf{R}_{pq} = \mathbf{R}_q - \mathbf{R}_p$ gives relative position of pth and qth inclusions. We introduce the local, inclusion-associated coordinate systems $Ox_{1q}x_{2q}x_{3q}$ with origins in O_q. The local variables $\mathbf{r}_p = \mathbf{r} - \mathbf{R}_p$ of different coordinate systems relate each other by $\mathbf{r}_q = \mathbf{r}_p - \mathbf{R}_{pq}$.

2.3.1 Superposition Principle

A new feature of this problem consists in the following. Now, a given inclusion undergoes a joint action of incident far field and the disturbance fields caused by all other inclusions. In turn, this inclusion affects the field around other inclusions. This means that the problem must be solved for all the inclusions *simultaneously*. For this purpose, we apply the **superposition principle** widely used for tailoring the solution of linear problem in the multiple-connected domain. This principle [81] states that

> *a general solution for the multiple-connected domain can be written as a superposition sum of general solutions for the single-connected domains whose intersection gives the considered multiple-connected domain.*

The derived above general solution for a single-connected domain allows to write a formal solution of the multiple inclusion problem. Moreover, the above exposed integration based expansion procedure (2.15) provides a complete solution of the problem. An alternate way consists in using the **re-expansion formulas** (referred also as the addition theorems) for the partial solutions. This way does not involve integration and appears more computationally efficient. The re-expansion formulas is the second component added to the solution procedure at this stage.

2.3.2 Re-Expansion Formulas

In notations (2.4), the re-expansion formulas for the scalar solid harmonics take the simplest possible form. Three kinds of re-expansion formulas are: singular-to-regular (S2R)

$$Y_t^s(\mathbf{r} + \mathbf{R}) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} (-1)^{k+l} Y_{t+k}^{s-l}(\mathbf{R}) y_k^l(\mathbf{r}), \quad \|\mathbf{r}\| < \|\mathbf{R}\|; \tag{2.20}$$

regular-to-regular (R2R)

$$y_t^s(\mathbf{r} + \mathbf{R}) = \sum_{k=0}^{t} \sum_{l=-k}^{k} y_{t-k}^{s-l}(\mathbf{R}) y_k^l(\mathbf{r}); \tag{2.21}$$

and singular-to-singular (S2S)

$$Y_t^s(\mathbf{r}+\mathbf{R}) = \sum_{k=t}^{\infty}\sum_{l=-k}^{k}(-1)^{t+k+s+l}y_{k-t}^{s-l}(\mathbf{R})Y_k^l(\mathbf{r}), \quad \|\mathbf{r}\| > \|\mathbf{R}\|\,. \tag{2.22}$$

The formula (2.21) is finite and hence exact and valid for any $\mathbf{r}$ and $\mathbf{R}$. In [77], (2.21) and (2.22) are regarded as translation of regular and singular solid harmonics, respectively. In [19], they are called translation of local and multipole expansions whereas (2.20) is referred as conversion of a multipole expansion into a local one. The formulas (2.20)–(2.22) can be derived in several ways, one of them based on using the formula (2.15). Noteworthy, these formulas constitute a theoretical background of the Fast Multipole Method [19].

2.3.3 Multipole Expansion Theorem

To illustrate the introduced concepts and formulas, we consider a standard problem of the multipoles theory. Let N monopoles of strength q_p are located at the points $\mathbf{R}_p$. We need to find the multipole expansion of the total potential field in the point $\mathbf{r}$ where $\|\mathbf{r}\| > R_s$ and $R_s = \max_p \left\|\mathbf{R}_p\right\|$. In other words, we are looking for the multipole expansion outside the sphere of radius R_s containing all the point sources.

Since the monopoles possess the fixed strength and do not interact, the total potential is equal to

$$T(\mathbf{r}) = \sum_{p=1}^{N}\frac{q_p}{\left\|\mathbf{r}-\mathbf{R}_p\right\|} \tag{2.23}$$

being a trivial case of the superposition sum. Next, by applying the formula (2.22) for $t = s = 0$, namely,

$$\frac{1}{\left\|\mathbf{r}-\mathbf{R}_p\right\|} = Y_0^0(\mathbf{r}-\mathbf{R}_p) = \sum_{t=0}^{\infty}\sum_{s=-t}^{t}\overline{y_t^s(\mathbf{R}_p)}Y_t^s(\mathbf{r}), \tag{2.24}$$

valid at $\|\mathbf{r}\| > R_s$ for all p, one finds easily

$$T(\mathbf{r}) = \sum_{t=0}^{\infty}\sum_{s=-t}^{t}A_{ts}Y_t^s(\mathbf{r}), \quad A_{ts} = \sum_{p=1}^{N}q_p\overline{y_t^s(\mathbf{R}_p)}. \tag{2.25}$$

For the truncated ($t \le t_{\max}$) series (2.25), the following error estimate exists:

$$\left|T(\mathbf{r}) - \sum_{t=0}^{t_{\max}}\sum_{s=-t}^{t}A_{ts}Y_t^s(\mathbf{r})\right| \le A\frac{(R_s/r)^{t_{\max}+1}}{r-R_s}, \quad A = \sum_{p=1}^{N}|q_p|. \tag{2.26}$$

These results constitute the multipole expansion theorem [19]. The more involved problem for the multiple finite-size, interacting inclusions is considered below.

2.4 FCM Boundary Value Problem

Let temperature field $T = T^{(0)}$ in a matrix, $T = T^{(p)}$ in the pth inclusion of radius R_p and conductivity $\lambda = \lambda_p$. On the interfaces $r_p = R_p$, perfect thermal contact (2.12) is supposed. Here, $\left(r_p, \theta_p, \varphi_p\right)$ are the local spherical coordinates with the origin O_p in the center of the pth inclusion.

2.4.1 Direct (Superposition) Sum

In accordance with the superposition principle,

$$T^{(0)}(\mathbf{r}) = T_{far}(\mathbf{r}) + \sum_{p=1}^{N} T_{dis}^{(p)}(\mathbf{r}_p) \tag{2.27}$$

where $T_{far} = \mathbf{G} \cdot \mathbf{r} = G_i x_i$ is the linear far field, and

$$T_{dis}^{(p)}(\mathbf{r}_p) = \sum_{t=1}^{\infty} \sum_{s=-t}^{t} A_{ts}^{(p)} Y_t^s(\mathbf{r}_p) \tag{2.28}$$

is a disturbance field caused by pth inclusion centered in O_p: $T_{dis}^{(p)}(\mathbf{r}_p) \to 0$ with $\left\|\mathbf{r}_p\right\| \to \infty$.

2.4.2 Local Series Expansion

In a vicinity of O_q, the following expansions are valid:

$$T_{far}(\mathbf{r}_q) = \sum_{t=0}^{\infty} \sum_{s=-t}^{t} c_{ts}^{(q)} y_t^s(\mathbf{r}_q), \tag{2.29}$$

where $c_{00}^{(q)} = \mathbf{G} \cdot \mathbf{R}_q$, $c_{10}^{(q)} = G_3$, $c_{11}^{(q)} = G_1 - iG_2$, $c_{1,-1}^{(q)} = -\overline{c_{11}^{(q)}}$ and $c_{ts}^{(q)} = 0$ otherwise. In (2.27), $T_{dis}^{(q)}$ is already written in qth basis. For $p \neq q$, we apply the re-expansion formula (2.20) to get

$$T_{dis}^{(p)}(\mathbf{r}_q) = \sum_{t=0}^{\infty}\sum_{s=-t}^{t} a_{ts}^{(q)} y_t^s(\mathbf{r}_q),\ a_{ts}^{(q)} = (-1)^{t+s}\sum_{p\neq q}^{N}\sum_{k=1}^{\infty}\sum_{l=-k}^{k} A_{kl}^{(p)} Y_{k+t}^{l-s}(\mathbf{R}_{pq}). \tag{2.30}$$

By putting all the parts together, we get

$$T^{(0)}(\mathbf{r}_q) = \sum_{t=1}^{\infty}\sum_{s=-t}^{t} A_{ts}^{(q)} Y_t^s(\mathbf{r}_q) + \sum_{t=0}^{\infty}\sum_{s=-t}^{t}\left(a_{ts}^{(q)} + c_{ts}^{(q)}\right) y_t^s(\mathbf{r}_q) \tag{2.31}$$

and the problem is reduced to the considered above single inclusion study.

2.4.3 Infinite Linear System

By substituting $T^{(0)}$ (2.31) and $T^{(q)}$ (2.13) written in local coordinates into (2.12), we come to the set of equations with unknowns $A_{ts}^{(q)}$, quite analogous to (2.17). Namely,

$$\frac{(\omega_q + 1 + 1/t)}{(\omega_q - 1)}\frac{(t-s)!\,(t+s)!}{\left(R_q\right)^{2t+1}} A_{ts}^{(q)} + a_{ts}^{(q)} = -c_{ts}^{(q)} \quad (\omega_q = \lambda_q/\lambda_0); \tag{2.32}$$

or, in an explicit form,

$$\begin{aligned}&\frac{(\omega_q + 1 + 1/t)}{(\omega_q - 1)}\frac{(t-s)!\,(t+s)!}{\left(R_q\right)^{2t+1}} A_{ts}^{(q)} \\ &\quad + (-1)^{t+s}\sum_{p\neq q}^{N}\sum_{k=1}^{\infty}\sum_{l=-k}^{k} A_{kl}^{(p)} Y_{k+t}^{l-s}(\mathbf{R}_{pq}) = -c_{ts}^{(q)}.\end{aligned} \tag{2.33}$$

A total number of unknowns in (2.33) can be reduced by a factor two by taking account of $A_{k,-l}^{(p)} = (-1)^l\overline{A_{kl}^{(p)}}$.

The theoretical solution (2.33) we found is formally exact—but, in contrast to (2.17), leads to the infinite system of linear algebraic equations. The latter can be solved, with any desirable accuracy, by the truncation method provided a sufficient number of harmonics (with $t \le t_{\max}$) is retained in solution [17, 27]. Hence, the numerical solution of the truncated linear system can be regarded as an asymptotically exact, because any accuracy can be achieved by the appropriate choice of $t_{\max}$. The smaller distance between the inhomogeneities is, the higher harmonics must be retained in the numerical solution to ensure the same accuracy of computations.

2.4.4 Modified Maxwell's Method for Effective Conductivity

Since we have $A_{ts}^{(q)}$ found from (2.33), one can evaluate the temperature field in any point in and around the inclusions. Moreover, this model allows to evaluate an effective conductivity of composite with geometry represented by FCM. In fact, it was Maxwell [57] who first suggested this model and derived his famous formula by equating "*the potential at a great distance from the sphere*" (in fact, total **dipole moment**) of an array of inclusions to that of the equivalent inclusion with unknown effective conductivity. In so doing, Maxwell neglected interaction between the inclusions—but wrote "*...when the distance between the spheres is not great compared with their radii..., then other terms enter into the result which we shall not now consider.*" Our solution contains *all* the terms and hence one can expect better accuracy of the Maxwell's formula. In our notations, it takes the form

$$\frac{\lambda_{eff}}{\lambda_0} = \frac{1 - 2c\,\langle A_{10}\rangle}{1 + c\,\langle A_{10}\rangle}, \tag{2.34}$$

where λ_{eff} is the effective conductivity of composite with volume content c of spherical inclusions, the mean dipole moment $\langle A_{10}\rangle = \frac{1}{N}\sum_{p=1}^{N} A_{10}^{(p)}$ and $A_{10}^{(p)}$ are calculated from (2.33) for $G_3 = 1$.

Recently, this approach was explored in [63]. The reported there numerical data reveal that taking the interaction effects into account substantially improves an accuracy of (2.34). Among other FCM-related publications, we mention [26] and similar works where an effective conductivity was estimated up to $O(c^2)$ from the "pair-of-inclusions" model.

2.5 *Representative Unit Cell Model*

The third model we consider in our review is so-called "unit cell" model of composite. Its basic idea consists in modeling an actual micro geometry of composite by some periodic structure with a unit cell containing several inclusions. It is known in literature as "generalized periodic", or "quasi-random" model: in what follows, we call it the *Representative Unit Cell* (RUC) model. This model is advantageous in that it allows to simulate the micro structure of composite and, at the same time, take the interactions of inhomogeneities over entire composite space into account accurately. This makes the cell approach appropriate for studying the local fields and effective properties of high-filled and strongly heterogeneous composites where the arrangement and interactions between the inclusions substantially affects the overall material behavior. RUC model can be applied to a wide class of composite structures and physical phenomena and, with a rapid progress in the computing technologies, is gaining more and more popularity.

Noteworthy, this model stems from the famous work by Lord Rayleigh [73] who considered "*a medium interrupted by spherical obstacles arranged in rectangular order*" and has evaluated its effective conductivity by taking into account the dipole, quadrupole and octupole moments of all inclusions. Almost a century later, the complete multipole-type analytical solutions are obtained for three cubic arrays of identical spheres [59, 60, 75, 82, 90]. In a series of more recent papers ([5, 29, 79, 88], among others), the conductivity problem for the random structure composite was treated as a triple-periodic problem with random arrangement of particles in the cubic unit cell.

2.5.1 RUC Geometry

The RUC model is essentially the above considered FCM model, periodically continued (replicated) in three orthogonal directions with period a, without overlapping of any two inclusions. In fact, we consider an unbounded solid containing a number N of periodic, equally oriented simple cubic (SC) arrays of inclusions. For a given geometry, any arbitrarily placed, oriented along the principal axes of lattice cube with side length a can be taken as RUC. It contains the *centers* of exactly N inclusions, randomly (but without overlapping) placed within a cell. The inclusions may partially lie outside the cube and, *vise versa*, a certain part of cube may be occupied by the inclusions which do not belong to the cell, Fig. 1a. Equally, one can take the unit cell as a cuboid with curvilinear boundary (but parallel opposite faces): for convenience, we assume with no loss in generality that the cell boundary S_0 entirely belongs to the matrix, see Fig. 1b.

It should be noted that the model problem is formulated and solved for a whole composite bulk rather than for the cube with plane faces. The RUC concept is nothing more than convenient "gadget" for introducing the model geometry and averaging the periodic strain and stress fields—and we use it for this purpose. We define geometry of the cell by its side length a and position $\mathbf{R}_q = X_{iq}\mathbf{i}_i$ of qth inclusion center

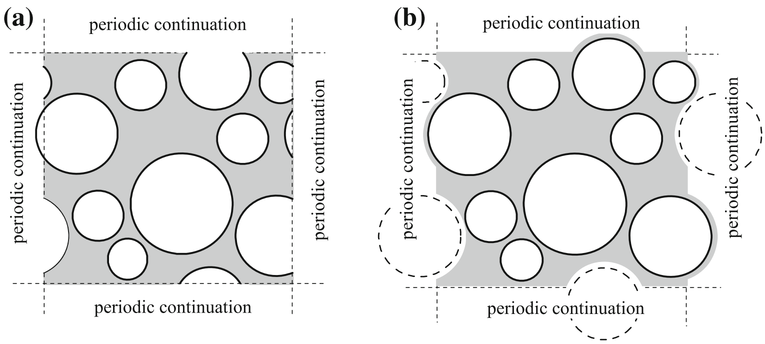


Fig. 1 RUC model of composite

($q = 1, 2, \ldots, N$) where $\mathbf{i}_i$ are the unit axis vectors of the global Cartesian coordinate system $Ox_1x_2x_3$. Number N can be taken large sufficiently to simulate arrangement of disperse phase in composite. We assume the inclusions equally sized, of radius R, and made from the same material, their volume content is $c = N\frac{4}{3}\pi R^3/a^3$. In what follows, the parameter $\mathbf{R}_{pq} = \mathbf{R}_q - \mathbf{R}_p$ is understood as the minimal distance between the inclusions of pth and qth SC arrays.

Several methods ([79, 83, 89], among others) have been developed to generate the random structure RUC statistically close to that of actual disordered composite. The generated arrangement of particles, likewise the real composite micro structure, can be characterized by several parameters: packing density, coordination number, radial distribution function, nearest neighbor distance, etc. [7, 83]. Another parameter, often introduced in this type models, is the *minimum allowable spacing* $\delta_{\min} = \min_{p,q}\left(\left\|\mathbf{R}_{pq}\right\|/2R-1\right)$ [9]. It is also known as impenetrability parameter, in terms of the cherry-pit model [83]. A small positive value is usually assigned to this parameter in order to separate inclusions and thus alleviate analysis of the model problem.

2.5.2 RUC Model Problem

The macroscopically uniform temperature field in the composite bulk is considered. This means constancy of the volume-averaged, or macroscopic, temperature gradient $\langle\nabla T\rangle$ and heat flux $\langle\mathbf{q}\rangle$ vectors. Here and below, $\langle f\rangle = V^{-1}\int_V f\,dV$ and $V = a^3$ is the cell volume; $\langle\nabla T\rangle$ is taken as the load governing parameter. In this case, periodicity of geometry results in quasi-periodicity of the temperature field and periodicity of the temperature gradient and heat flux:

$$\begin{aligned} T\left(\mathbf{r} + a\mathbf{i}_i\right) &= T\left(\mathbf{r}\right) + a\left\langle\nabla T\right\rangle\cdot\mathbf{i}_i; \\ \nabla T\left(\mathbf{r} + a\mathbf{i}_i\right) &= \nabla T\left(\mathbf{r}\right); \quad \mathbf{q}\left(\mathbf{r} + a\mathbf{i}_i\right) = \mathbf{q}\left(\mathbf{r}\right). \end{aligned} \tag{2.35}$$

2.5.3 Temperature Filed

The conditions (2.35) are satisfied by taking T in the form $T\left(\mathbf{r}\right) = \left\langle\nabla T\right\rangle\cdot\mathbf{r} + T_{dis}\left(\mathbf{r}\right)$, T_{dis} being now the periodic disturbance field. In the matrix domain, we write it as a sum of linear mean field and disturbances form the infinite, periodic arrays of particles:

$$T^{(0)}(\mathbf{r}) = \mathbf{G}\cdot\mathbf{r} + \sum_{p=1}^{N} T_{dis}^{*(p)}(\mathbf{r}_p), \tag{2.36}$$

The Eq. (2.36) is similar to (2.27), where the single inclusion disturbance terms $T_{dis}^{(p)}$ are replaced with their periodic counterparts given by the sums over all the lattice nodes $\mathbf{k} = k_i\mathbf{i}_i$ ($-\infty < k_i < \infty$):

$$T_{dis}^{*(p)}(\mathbf{r}_p) = \sum_{\mathbf{k}} T_{dis}^{(p)}\left(\mathbf{r}_p + a\mathbf{k}\right), \tag{2.37}$$

In view of (2.37) and (2.27), $T_{dis}^{*(p)}$ can be expressed in terms of the periodic harmonic potentials Y_{ts}^*:

$$T_{dis}^{*(p)}(\mathbf{r}_p) = \sum_{t=1}^{\infty}\sum_{s=-t}^{t} A_{ts}^{(p)} Y_{ts}^*(\mathbf{r}_p), \quad \text{where} \quad Y_{ts}^*(\mathbf{r}_p) = \sum_{\mathbf{k}} Y_t^s(\mathbf{r}_p + a\mathbf{k}). \tag{2.38}$$

In fact, this is a direct formal extension of the FCM model when a number of particles becomes infinitely large—and, at the same time, direct extension of the Rayleigh's [73] approach. For almost a century, the Rayleigh's solution was questioned due to conditional convergence of (2.38) for $t = 1$. The limiting process has been legitimized by [59, 82] who resolved the convergence issue of dipole lattice sums. An alternate, the generalized periodic functions based approach has been applied by [75, 90]. In [15], the solution has been found in terms of doubly periodic functions, in [88] the RUC problem was solved by the boundary integral method. Not surprisingly, all the mentioned methods give the resulting sets of linear equations consistent with

$$\frac{(\omega_q + 1 + \frac{1}{t})}{(\omega_q - 1)} \frac{(t-s)!(t+s)!}{\left(R_q\right)^{2t+1}} A_{ts}^{(q)} + (-1)^{t+s} \sum_{p=1}^{N}\sum_{k=1}^{\infty}\sum_{l=-k}^{k} A_{kl}^{(p)} Y_{k+t,l-s}^*(\mathbf{R}_{pq}) \tag{2.39}$$
$$= -\delta_{t1}\left[\delta_{s0}G_3 + \delta_{s1}\left(G_1 - iG_2\right) - \delta_{s,-1}\left(G_1 + iG_2\right)\right].$$

Among them, the multipole expansion method provides, probably, the most straightforward and transparent solution procedure. In fact, the system (2.39) is obtained from (2.33) by replacing the matrix coefficients $Y_{k+t}^{l-s}(\mathbf{R}_{pq})$ with the corresponding lattice sums

$$Y_{k+t,l-s}^*(\mathbf{R}_{pq}) = \sum_{\mathbf{k}} Y_{k+t}^{l-s}\left(\mathbf{R}_{pq} + a\mathbf{k}\right). \tag{2.40}$$

Convergence of the series (2.40) was widely discussed in literature and several fast summation techniques have been developed for them ([17, 59, 88], among others) and we refer there for the details.

As would be expected, the Y_{ts}^* definition (2.38) for $t = 1$ also suffers the above-mentioned convergence problem. Indeed, obtained with aid of (2.20) the multipole expansion

$$Y_{1s}^*(\mathbf{r}) = Y_1^s(\mathbf{r}) + \sum_{k=0}^{\infty}\sum_{l=-k}^{k} (-1)^{k+l} Y_{k+1,s-l}^*(\mathbf{R}) y_k^l(\mathbf{r}) \tag{2.41}$$

also involves the conditionally convergent ("shape-dependent") dipole lattice sum $Y_{20}^*(\mathbf{0})$. McPhedran et al. [59] have argued that the Rayleigh's [73] result $Y_{20}^*(\mathbf{0}) = 4\pi/3a^3$ is consistent with the physics of the problem. Alternatively, $Y_{1s}^*(\mathbf{r})$ can be defined by means of periodic fundamental solution S_1 [21] given by the triple Fourier series. Specifically, we define

$$Y_{10}^* = -D_3(S_1); \quad Y_{11}^* = -\overline{Y_{1,-1}^*} = -D_2(S_1); \tag{2.42}$$

where D_i are the differential operators (2.10). It appears that this definition is equivalent to (2.41): differentiation of S_1 multipole expansion [21] gives, in our notations,

$$D_3(S_1) = -Y_1^0(\mathbf{r}) + \frac{4\pi}{3a^3}x_3 - \sum_{k=2}^{\infty}\sum_{l=-k}^{k}(-1)^{k+l}Y_{k+1,-l}^*(\mathbf{0})y_k^l(\mathbf{r}). \tag{2.43}$$

As expected, exactly the same result follows from the formula (2.41) where $y_1^0(\mathbf{r})$ is replaced with x_3 and $Y_{20}^*(\mathbf{0})$—with its numerical value, $4\pi/3a^3$.

2.5.4 Effective Conductivity

The second rank effective conductivity tensor $\Lambda^* = \{\lambda_{ij}^*\}$ is defined by (1.1). In order to evaluate λ_{ij}^* for a given composite, one has to conduct a series of numerical tests, with three different $\langle\nabla T\rangle$, and evaluate the macroscopic heat flux it causes. Specifically, $\lambda_{ij}^* = -\langle q_i\rangle$ for $\langle\nabla T\rangle = \mathbf{i}_j$, so we need an explicit expression of the macroscopic temperature gradient and heat flux corresponding to our temperature solution (2.36).

Evaluation of the macroscopic gradient, $\langle\nabla T\rangle$ is ready. First, we recall that we take the unit cell of RUC with $S_0 \in V_0$ and, hence, $T = T^{(0)}$ in (1.3). Next, we observe that for the periodic part of solution (2.36) in the boundary points $\mathbf{r}_a \in S_0$ and $\mathbf{r}_b = \mathbf{r}_a + a\mathbf{i}_j \in S_0$ belonging to the opposite cell faces we have $T_{dis}(\mathbf{r}_b) = T_{dis}(\mathbf{r}_a)$ whereas the normal unit vector changes the sign: $\mathbf{n}(\mathbf{r}_b) = -\mathbf{n}(\mathbf{r}_a)$. Hence, the integrals of T_{dis} over the opposite faces cancel each other and the total integral over S_0 equals to zero. Integration of the linear part of $T^{(0)}$ is elementary: the gradient theorem yields

$$\frac{1}{V}\int_{S_0}(\mathbf{G}\cdot\mathbf{r})\,\mathbf{n}\,dS = \frac{1}{V}\int_V \nabla(\mathbf{G}\cdot\mathbf{r})\,dV = \mathbf{G}. \tag{2.44}$$

Comparison with (1.3) gives the expected $\langle\nabla T\rangle = \mathbf{G}$ provided T obeys (2.35).

This is general result: the derived formula is invariant of the shape, properties and arrangement of inclusions, interface bonding type and shape of the unit cell. In the subsequent Sections, this equality will be applied without derivation.

For $\langle \mathbf{q} \rangle$ evaluation, we employ the formula (1.9). In the considered here isotropic case, $q_n(\mathbf{r}) = q_n(x_k)\,\mathbf{i}_k = -\lambda_0 \mathbf{n}$, it takes

$$\langle \mathbf{q} \rangle = -\boldsymbol{\Lambda}_0 \cdot \langle \nabla T \rangle - \frac{\lambda_0}{V} \sum_{q=1}^{N} \int_{S_q} \left(R_q \frac{\partial T^{(0)}}{\partial r_q} - T^{(0)} \right) \mathbf{n} dS. \tag{2.45}$$

The unit vector $\mathbf{n}$ is expressed in terms of surface spherical harmonics as $\mathbf{n} = \chi_1^1 \mathbf{e}_2 - 2\chi_1^{-1} \mathbf{e}_1 + \chi_1^0 \mathbf{e}_3$. whereas the local expansion of the integrand in right hand side of (2.45) is given by

$$R_q \frac{\partial T^{(0)}}{\partial r_q} - T^{(0)} = \sum_{t=0}^{\infty} \sum_{s=-t}^{t} \left[-(2t+1) A_{ts}^{(q)} \frac{(t-s)!}{R_q^{t+1}} \right.$$
$$\left. + (t-1)\left(a_{ts}^{(q)} + c_{ts}^{(q)}\right) \frac{R_q^t}{(t+s)!} \right] \chi_t^s(\theta_q, \varphi_q). \tag{2.46}$$

Due to orthogonality of the surface spherical harmonics (2.6), the surface integral in (2.45) equals to zero for all terms in Eq. (2.31) with $t \neq 1$. Moreover, $r \frac{\partial y_1^s}{\partial r} = y_1^s$ and hence only the dipole potentials Y_1^s contribute to (2.45). From here, we get the exact finite formula involving only the dipole moments of the disturbance field, $A_{1s}^{(q)}$:

$$\frac{\langle \mathbf{q} \rangle}{\lambda_0} = -\mathbf{G} + \frac{4\pi}{a^3} \sum_{q=1}^{N} \mathrm{Re} \left(2A_{11}^{(q)} \mathbf{e}_1 + A_{10}^{(q)} \mathbf{e}_3 \right). \tag{2.47}$$

In view of (2.39), the coefficients $A_{ts}^{(q)}$ are linearly proportional to $\mathbf{G}$:

$$\lambda_0 \frac{4\pi}{a^3} \sum_{q=1}^{N} \mathrm{Re} \left(2A_{11}^{(q)} \mathbf{e}_1 + A_{10}^{(q)} \mathbf{e}_3 \right) = \delta \boldsymbol{\Lambda} \cdot \mathbf{G}. \tag{2.48}$$

The components of the $\delta\boldsymbol{\Lambda}$ tensor are found by solving Eq. (2.39) for $\mathbf{G} = \mathbf{i}_j$. These equations, together with Eq. (1.1), provide evaluation of the effective conductivity tensor as

$$\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}_0 + \delta \boldsymbol{\Lambda}. \tag{2.49}$$

3 Composite with Spherical Inclusions: Elasticity Problem

3.1 Background Theory

3.1.1 Vectorial Spherical Harmonics

Vectorial spherical harmonics $\mathbf{S}_{ts}^{(i)} = \mathbf{S}_{ts}^{(i)}(\theta, \varphi)$ are defined as [64]

$$\begin{aligned}
\mathbf{S}_{ts}^{(1)} &= \mathbf{e}_\theta \frac{\partial}{\partial\theta}\chi_t^s + \frac{\mathbf{e}_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\chi_t^s; \\
\mathbf{S}_{ts}^{(2)} &= \frac{\mathbf{e}_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\chi_t^s - \mathbf{e}_\varphi \frac{\partial}{\partial\theta}\chi_t^s; \\
\mathbf{S}_{ts}^{(3)} &= \mathbf{e}_r \chi_t^s \quad (t \ge 0, |s| \le t).
\end{aligned} \tag{3.1}$$

The functions (3.1) constitute a complete and orthogonal on sphere set of vectorial harmonics with the orthogonality properties given by

$$\frac{1}{S}\int_S \mathbf{S}_{ts}^{(i)} \cdot \overline{\mathbf{S}_{kl}^{(j)}}\, dS = \alpha_{ts}^{(i)} \delta_{tk}\delta_{sl}\delta_{ij}, \tag{3.2}$$

where $\alpha_{ts}^{(1)} = \alpha_{ts}^{(2)} = t(t+1)\,\alpha_{ts}$ and $\alpha_{ts}^{(3)} = \alpha_{ts}$ given by (2.6). Also, these functions possess remarkable differential

$$\begin{aligned}
r\nabla \cdot \mathbf{S}_{ts}^{(1)} &= -t(t+1)\chi_t^s; \quad \nabla \cdot \mathbf{S}_{ts}^{(2)} = 0; \quad r\nabla \cdot \mathbf{S}_{ts}^{(1)} = 2\chi_t^s; \\
r\nabla \times \mathbf{S}_{ts}^{(1)} &= -\mathbf{S}_{ts}^{(2)}; \quad r\nabla \times \mathbf{S}_{ts}^{(2)} = \mathbf{S}_{ts}^{(1)} + t(t+1)\mathbf{S}_{ts}^{(3)}; \quad r\nabla \times \mathbf{S}_{ts}^{(3)} = \mathbf{S}_{ts}^{(2)};
\end{aligned} \tag{3.3}$$

and algebraic

$$\mathbf{e}_r \times \mathbf{S}_{ts}^{(1)} = -\mathbf{S}_{ts}^{(2)}; \quad \mathbf{e}_r \times \mathbf{S}_{ts}^{(2)} = \mathbf{S}_{ts}^{(1)}; \quad \mathbf{e}_r \times \mathbf{S}_{ts}^{(3)} = 0; \tag{3.4}$$

properties. In the vectorial—including elasticity—problems, the functions (3.1) play the same role as the surface harmonics (2.5) in the scalar problems. With aid of (3.3), (3.4), separation of variables in the vectorial harmonic $\Delta\mathbf{f} = 0$ and biharmonic $\Delta\Delta\mathbf{g} = 0$ equations is straightforward [17, 30] and yields the corresponding countable sets of partial solutions—vectorial solid harmonics and biharmonics, respectively.

3.1.2 Partial Solutions of Lame Equation

The said above holds true for the Lame equation

$$\frac{2(1-\nu)}{(1-2\nu)}\nabla(\nabla\cdot\mathbf{u})-\nabla\times\nabla\times\mathbf{u}=\mathbf{0} \tag{3.5}$$

being the particular case of vectorial biharmonic equation. In (3.5), $\mathbf{u}$ is the displacement vector and ν is the Poisson ratio. The regular partial solutions of (3.5) $\mathbf{u}_{ts}^{(i)}=\mathbf{u}_{ts}^{(i)}(\mathbf{r})$ are defined as [30]

$$\mathbf{u}_{ts}^{(1)}=\frac{r^{t-1}}{(t+s)!}\left(\mathbf{S}_{ts}^{(1)}+t\mathbf{S}_{ts}^{(3)}\right),\quad \mathbf{u}_{ts}^{(2)}=-\frac{r^{t}}{(t+1)(t+s)!}\mathbf{S}_{ts}^{(2)}, \tag{3.6}$$
$$\mathbf{u}_{ts}^{(3)}=\frac{r^{t+1}}{(t+s)!}\left(\beta_t\mathbf{S}_{ts}^{(1)}+\gamma_t\mathbf{S}_{ts}^{(3)}\right),\quad \beta_t=\frac{t+5-4\nu}{(t+1)(2t+3)},\quad \gamma_t=\frac{t-2+4\nu}{(2t+3)}.$$

The singular (infinitely growing at $r\to 0$ and vanishing at infinity) functions $\mathbf{U}_{ts}^{(i)}$ are given by $\mathbf{U}_{ts}^{(i)}=\mathbf{u}_{-(t+1),\,s}^{(i)}$, with the relations $\mathbf{S}_{-t-1,s}^{(i)}=(t-s)!(-t-1+s)!\mathbf{S}_{ts}^{(i)}$ taken into account. The functions $\mathbf{u}_{ts}^{(i)}(\mathbf{r})$ and $\mathbf{U}_{ts}^{(i)}(\mathbf{r})$ are the vectorial counterparts of scalar solid harmonics (2.4).

At the spherical surface $r=R$, the traction vector $\mathbf{T}_n=\sigma\cdot\mathbf{n}$ can be written as

$$\frac{1}{2\mu}\mathbf{T}_r(\mathbf{u})=\frac{\nu}{1-2\nu}\mathbf{e}_r\left(\nabla\cdot\mathbf{u}\right)+\frac{\partial}{\partial r}\mathbf{u}+\frac{1}{2}\mathbf{e}_r\times\left(\nabla\times\mathbf{u}\right), \tag{3.7}$$

μ being the shear modulus. For the vectorial partial solutions (3.6), it yields [30]

$$\frac{1}{2\mu}\mathbf{T}_r(\mathbf{u}_{ts}^{(1)})=\frac{(t-1)}{r}\mathbf{u}_{ts}^{(1)};\quad \frac{1}{2\mu}\mathbf{T}_r(\mathbf{u}_{ts}^{(2)})=\frac{(t-1)}{2r}\mathbf{u}_{ts}^{(2)}; \tag{3.8}$$
$$\frac{1}{2\mu}\mathbf{T}_r(\mathbf{u}_{ts}^{(3)})=\frac{r^{t}}{(t+s)!}\left(b_t\mathbf{S}_{ts}^{(1)}+g_t\mathbf{S}_{ts}^{(3)}\right);$$

where $b_t=(t+1)\beta_t-2(1-\nu)/(t+1)$ and $g_t=(t+1)\gamma_t-2\nu$. In view of (3.6), representation of $\mathbf{T}_r(\mathbf{u}_{ts}^{(i)})$ in terms of vectorial spherical harmonics (3.1) is obvious.

The total force $\mathbf{T}$ and moment $\mathbf{M}$ acting on the spherical surface S of radius R are given by the formulas

$$\mathbf{T}=\int_{S_r}\mathbf{T}_r dS,\mathbf{M}=\int_{S_r}\mathbf{r}\times\mathbf{T}_r dS, \tag{3.9}$$

It is straightforward to show that $\mathbf{T}=\mathbf{M}=0$ for all the regular functions $\mathbf{u}_{ts}^{(i)}$. Among the singular solutions $\mathbf{U}_{ts}^{(i)}$, we have exactly three functions with non-zero resultant force $\mathbf{T}$:

$$\mathbf{T}\left(\mathbf{U}_{10}^{(3)}\right)=16\mu\pi(\nu-1)\mathbf{e}_3;\,\mathbf{T}\left(\mathbf{U}_{11}^{(3)}\right)=-\overline{\mathbf{T}\left(\mathbf{U}_{1,-1}^{(3)}\right)}=32\mu\pi(\nu-1)\mathbf{e}_1. \tag{3.10}$$

By analogy with Y_0^0, $\mathbf{U}_{1s}^{(3)}$ can be regarded as the vectorial monopoles. The resultant moment is zero for all the partial solutions but $\mathbf{U}_{1s}^{(2)}$ for which we get

$$\mathbf{M}\left(\mathbf{U}_{10}^{(2)}\right) = -8\mu\pi\mathbf{e}_3; \mathbf{M}\left(\mathbf{U}_{11}^{(2)}\right) = \overline{\mathbf{M}\left(\mathbf{U}_{1,-1}^{(2)}\right)} = -16\mu\pi\mathbf{e}_1. \tag{3.11}$$

3.2 Single Inclusion Problem

In the elasticity problem, we deal with the vectorial displacement field $\mathbf{u}$ ($\mathbf{u} = \mathbf{u}^{(0)}$ in a matrix, $\mathbf{u} = \mathbf{u}^{(1)}$ in the spherical inclusion of radius R) satisfying (3.5). On the interface S, perfect mechanical contact is assumed:

$$[[\mathbf{u}]] = 0; \quad [[\mathbf{T}_r(\mathbf{u})]] = 0; \tag{3.12}$$

where $\mathbf{T}_r$ is given by (3.7). The elastic moduli are (μ_0, ν_0) for matrix material and (μ_1, ν_1) for inclusion, the non-uniform displacement far field $\mathbf{u}_{far}$ is taken as the load governing parameter.

3.2.1 Series Solution

The displacement in the inclusion $\mathbf{u}^{(1)}$ is finite and so allows expansion into a series over the regular solutions $\mathbf{u}_{ts}^{(i)}(\mathbf{r})$ (3.6):

$$\mathbf{u}^{(1)}(\mathbf{r}) = \sum_{i,t,s} d_{ts}^{(i)}\mathbf{u}_{ts}^{(i)}(\mathbf{r}) \quad \left(\sum_{i,t,s} = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t}\right), \tag{3.13}$$

where $d_{ts}^{(i)}$ are the unknown constants. The components of the displacement vector are real quantities, so the property $\mathbf{u}_{t,-s}^{(i)} = (-1)^s\overline{\mathbf{u}_{ts}^{(i)}}$ gives $d_{t,-s}^{(i)} = (-1)^s\overline{d_{ts}^{(i)}}$. In the matrix domain, we write $\mathbf{u}^{(0)} = \mathbf{u}_{far} + \mathbf{u}_{dis}$, where the disturbance part $\mathbf{u}_{dis}(\mathbf{r}) \to 0$ with $\|\mathbf{r}\| \to \infty$ and hence can be written in terms of the singular solutions $\mathbf{U}_{ts}^{(i)}$ only:

$$\mathbf{u}^{(0)}(\mathbf{r}) = \mathbf{u}_{far}(\mathbf{r}) + \sum_{i,t,s} A_{ts}^{(i)}\mathbf{U}_{ts}^{(i)}(\mathbf{r}) \tag{3.14}$$

where $A_{ts}^{(i)}$ are the unknown coefficients. By analogy with (2.14), the series term is thought as the multipole expansion of $\mathbf{u}_{dis}$.

3.2.2 Far Field Expansion

Due to regularity of $\mathbf{u}_{far}$, we can expand it into a series (3.13) with coefficients $c_{ts}^{(j)}$. With aid of (3.6), we express $\mathbf{u}_{far}$ at $r = R$ in terms of vectorial spherical harmonics (3.1)

$$\mathbf{u}_{far}(\mathbf{r}) = \sum_{j,t,s} c_{ts}^{(i)} \mathbf{u}_{ts}^{(i)}(\mathbf{r}) = \sum_{j,t,s} \frac{c_{ts}^{(i)}}{(t+s)!} \sum_{i=1}^{3} UM_t^{ij}(R, \nu_0) \mathbf{S}_{ts}^{(i)}(\theta, \varphi) \tag{3.15}$$

where $\mathbf{UM}_t$ is a (3×3) matrix of the form

$$\mathbf{UM}_t(r, \nu) = \left\{UM_t^{ij}(r, \nu)\right\} = r^{t-1} \begin{Bmatrix} 1 & 0 & r^2\beta_t(\nu) \\ 0 & -\frac{r}{t+1} & 0 \\ t & 0 & r^2\gamma_t(\nu) \end{Bmatrix}. \tag{3.16}$$

Now, we multiply (3.15) by $\overline{\mathbf{S}_{ts}^{(i)}}$ and integrate the left-hand side (either analytically or numerically) over the interface S. In view of (3.2), analytical integration of the right-hand side of (3.15) is trivial and yields

$$J_{ts}^{(i)} = \frac{(t+s)!}{4\pi R^2 \alpha_{ts}^{(i)}} \int_S \mathbf{u}_{far} \cdot \overline{\mathbf{S}_{ts}^{(i)}}\, dS = \sum_{j=1}^{3} UM_t^{ij}(R, \nu_0)\, c_{ts}^{(j)} \tag{3.17}$$

From the above equation, we get the expansion coefficients in matrix-vector form as

$$\mathbf{c}_{ts} = \mathbf{UM}_t(R, \nu_0)^{-1} \mathbf{J}_{ts} \tag{3.18}$$

where $\mathbf{c}_{ts} = \left\{c_{ts}^{(i)}\right\}^T$ and $\mathbf{J}_{ts} = \left\{J_{ts}^{(i)}\right\}^T$. In the particular case of linear $\mathbf{u}_{far} = \mathbf{E} \cdot \mathbf{r}$, where $\mathbf{E} = \left\{E_{ij}\right\}$ is the uniform far-field strain tensor, the explicit analytical expressions for the expansion coefficients are

$$\begin{aligned} c_{00}^{(3)} &= \frac{(E_{11} + E_{22}^{+} E_{33})}{3\gamma_0(\nu_0)}, \quad c_{20}^{(1)} = \frac{(2E_{33} - E_{11} - E_{22})}{3}, \\ c_{21}^{(1)} &= E_{13} - iE_{23}, \quad c_{22}^{(1)} = E_{11} - E_{22} - 2iE_{12}; \end{aligned} \tag{3.19}$$

$c_{2,-s}^{(i)} = (-1)^s \overline{c_{2s}^{(i)}}$ and all other $c_{ts}^{(i)}$ are equal to zero.

3.2.3 Resolving Equations

Now, we substitute (3.13) and (3.14) into the first condition of (3.12) and use the orthogonality of $\mathbf{S}_{ts}^{(i)}$ to reduce it to a set of linear algebraic equations, written in matrix form as ($\mathbf{UG}_t = \mathbf{UM}_{-(t+1)}$)

$$(t-s)!(t+s)!\mathbf{UG}_t(R,\nu_0)\cdot\mathbf{A}_{ts}+\mathbf{UM}_t(R,\nu_0)\cdot\mathbf{c}_{ts}=\mathbf{UM}_t(R,\nu_1)\cdot\mathbf{d}_{ts}. \quad (3.20)$$

The second, traction continuity condition gives us another set of equations:

$$(t-s)!(t+s)!\mathbf{TG}_t(R,\nu_0)\cdot\mathbf{A}_{ts}+\mathbf{TM}_t(R,\nu_0)\cdot\mathbf{c}_{ts}=\omega\mathbf{TM}_t(R,\nu_1)\cdot\mathbf{d}_{ts}, \quad (3.21)$$

where $\omega=\mu_1/\mu_0$. In (3.21), the **TM** is (3×3) matrix

$$TM_t(r,\nu)=\left\{TM_t^{ij}(r,\nu)\right\}=r^{t-2}\left\{\begin{array}{ccc} t-1 & 0 & r^2 b_t(\nu)\\ 0 & -\frac{r(t-1)}{2(t+1)} & 0\\ t(t-1) & 0 & r^2 g_t(\nu)\end{array}\right\}. \quad (3.22)$$

where g_t and b_t are defined by (3.8); $\mathbf{TG}_t=\mathbf{TM}_{-(t+1)}$, $\mathbf{A}_{ts}=\left\{A_{ts}^{(i)}\right\}^T$ and $\mathbf{d}_{ts}=\left\{d_{ts}^{(i)}\right\}^T$.

For all indices $t\geq 0$ and $|s|\leq t$, the coefficients $\mathbf{A}_{ts}$ and $\mathbf{d}_{ts}$ can be found from linear system (3.20), (3.21). For computational purposes, it is advisable to eliminate $\mathbf{d}_{ts}$ from there and obtain the equations containing the unknowns $\mathbf{A}_{ts}$ only:

$$(t-s)!(t+s)!(\mathbf{RM}_t)^{-1}\mathbf{RG}_t\cdot\mathbf{A}_{ts}=-\mathbf{c}_{ts}, \quad (3.23)$$

where

$$\begin{aligned}\mathbf{RG}_t&=\omega\left[\mathbf{UM}_t(R,\nu_1)\right]^{-1}\mathbf{UG}_t(R,\nu_0)-\left[\mathbf{TM}_t(R,\nu_1)\right]^{-1}\mathbf{TG}_t(R,\nu_0),\\ \mathbf{RM}_t&=\omega\left[\mathbf{UM}_t(R,\nu_1)\right]^{-1}\mathbf{UM}_t(R,\nu_0)-\left[\mathbf{TM}_t(R,\nu_1)\right]^{-1}\mathbf{TM}_t(R,\nu_0). \quad (3.24)\end{aligned}$$

This transformation is optional for a single inclusion problem but can be rather useful for the multiple inclusion problems where the total number of unknowns becomes very large.

The equations with $t=0$ and $t=1$ deserve extra attention. First, we note that $\mathbf{U}_{00}^{(2)}=\mathbf{U}_{00}^{(3)}\equiv 0$ and so we have only one equation (instead of three) in (3.23). It can be resolved easily to get

$$\frac{1}{R^3}\frac{4\mu_0+3k_1}{3k_0-3k_1}a_{00}^{(1)}=-c_{00}^{(3)} \quad (3.25)$$

where the relation $\frac{a_0}{\gamma_0}=\frac{3k}{2\mu}$ (k being the bulk modulus) is taken into consideration. Noteworthy, Eq. (3.25) gives solution of the single inclusion problem in the case of equiaxial far tension: $E_{11}=E_{22}=E_{33}$. For $t=1$, the first two columns of the matrix $\mathbf{TM}_t$ (3.22) become zero: also, $a_1+2b_1=0$. From (3.23), we get immediately $A_{1s}^{(2)}=A_{1s}^{(3)}=0$.

The solution we obtain is complete and valid for any non-uniform far field. For any polynomial far field of order $t_{\max}$, this solution is exact and *conservative*, i.e., is given by the finite number of terms with $t\leq t_{\max}$. For example, in the

Eshelby-type problem, the expansion coefficients with $t \leq 2$ are nonzero only. The solution procedure is straightforward and remarkably simple as compared with the scalar harmonics-based approach (see, e.g., [72]). In fact, use of the vectorial spherical harmonics makes the effort of solving the vectorial boundary value problems comparable to that of solving scalar boundary value problems. Also, solution is written in the compact matrix-vector form, readily implemented by means of standard computer algebra.

3.3 Re-Expansion Formulas

The re-expansions of the vectorial solutions $\mathbf{U}_{ts}^{(i)}$ and $\mathbf{u}_{ts}^{(i)}$ (3.5) are [30]: singular-to-regular (S2R)

$$\mathbf{U}_{ts}^{(i)}(\mathbf{r}+\mathbf{R}) = \sum_{j=1}^{3}\sum_{k=0}^{\infty}\sum_{l=-k}^{k}(-1)^{k+l}\eta_{tksl}^{(i)(j)}(\mathbf{R})\mathbf{u}_{kl}^{(j)}(\mathbf{r}), \quad \|\mathbf{r}\| < \|\mathbf{R}\|\,; \tag{3.26}$$

where

$$\begin{aligned}
\eta_{tksl}^{(i)(j)} &= 0, \quad j > l;\ \eta_{tksl}^{(1)(1)} = \eta_{tksl}^{(2)(2)} = \eta_{tksl}^{(3)(3)} = Y_{t+k}^{s-l}; \\
\eta_{tksl}^{(2)(1)} &= \mathrm{i}\left(\frac{s}{t}+\frac{l}{k}\right)Y_{t+k-1}^{s-l};\ \eta_{tksl}^{(3)(2)} = -4\,(1-\nu)\,\eta_{tksl}^{(2)(1)},\ k \geq 1; \\
\eta_{tksl}^{(3)(1)} &= \frac{l}{k}\eta_{t,k-1,sl}^{(3)(2)} - Z_{t+k}^{s-l} - Y_{t+k-2}^{s-l} \\
&\quad\times\left[\frac{(t+k-1)^2-(s-l)^2}{2t+2k-1} + C_{-(t+1),s} + C_{k-2,l}\right];
\end{aligned} \tag{3.27}$$

regular-to-regular (S2R):

$$\mathbf{u}_{ts}^{(i)}(\mathbf{r}+\mathbf{R}) = \sum_{j=1}^{i}\sum_{k=0}^{t+i-j}\sum_{l=-k}^{k}(-1)^{k+l}\nu_{tksl}^{(i)(j)}(\mathbf{R})\mathbf{u}_{kl}^{(j)}(\mathbf{r}); \tag{3.28}$$

where

$$\begin{aligned}
\nu_{tksl}^{(i)(j)} &= 0, \quad j > l; \quad \nu_{tksl}^{(1)(1)} = \nu_{tksl}^{(2)(2)} = \nu_{tksl}^{(3)(3)} = y_{t-k}^{s-l}; \\
\nu_{tksl}^{(2)(1)} &= \mathrm{i}\left(\frac{s}{t+1}-\frac{l}{k}\right)y_{t-k+1}^{s-l}; \quad \nu_{tksl}^{(3)(2)} = -4\,(1-\nu)\,\nu_{tksl}^{(2)(1)};
\end{aligned} \tag{3.29}$$

$$\nu_{tksl}^{(3)(1)} = \frac{l}{k}\nu_{t,k-1,sl}^{(3)(2)} - z_{t-k}^{s-l} - y_{t-k+2}^{s-l}\left[\frac{(t-k+2)^2-(s-l)^2}{2t-2k+3} - C_{ts} + C_{k-2,l}\right];$$

and singular-to-singular (S2S):

$$\mathbf{U}_{ts}^{(i)}(\mathbf{r}+\mathbf{R}) = \sum_{j=1}^{3}\sum_{k=t-i+j}^{\infty}\sum_{l=-k}^{k}(-1)^{t+k+s+l}\mu_{tksl}^{(i)(j)}(\mathbf{R})\mathbf{U}_{kl}^{(j)}(\mathbf{r}), \quad \|\mathbf{r}\| > \|\mathbf{R}\|; \tag{3.30}$$

where

$$\begin{aligned}
\mu_{tksl}^{(i)(j)} &= 0, \quad j > l; \quad \mu_{tksl}^{(1)(1)} = \mu_{tksl}^{(2)(2)} = \mu_{tksl}^{(3)(3)} = y_{k-t}^{s-l}; \\
\mu_{tksl}^{(2)(1)} &= \mathrm{i}\left(\frac{s}{t} - \frac{l}{k+1}\right)y_{k-t+1}^{s-l}; \quad \mu_{tksl}^{(3)(2)} = -4\,(1-\nu)\,\mu_{tksl}^{(2)(1)}; \\
\mu_{tksl}^{(3)(1)} &= -\frac{l}{k+1}\mu_{t,k+1,sl}^{(3)(2)} + z_{k-t}^{s-l} \\
&\quad - y_{k-t+2}^{s-l}\left[\frac{(t+k-2)^2-(s-l)^2}{2k-2t+3} + C_{-(t+1),s} + C_{-(k+3),l}\right].
\end{aligned} \tag{3.31}$$

In these formulas, $Z_t^s = \frac{r^2}{2t-1}Y_t^s$ and $z_t^s = \frac{r^2}{2t+3}y_t^s$ are the singular and regular, respectively, scalar solid biharmonics and $C_{ts} = \left[(t+1)^2 - s^2\right]\beta_t$.

> These formulas are the vectorial counterparts of (2.20)–(2.22): being combined with FMM [19] scheme, they provide the fast multipole solution algorithm for elastic interactions in the multiple inclusion problem.

For the Stokes interactions in suspension of spherical particles, similar work is done in [77].

3.4 FCM

Analysis of the FCM elasticity problem is analogous to that of conductivity, so we outline the procedure and formulas. For simplicity sake, we assume the far displacement field to be linear: $\mathbf{u}_{far} = \mathbf{E}\cdot\mathbf{r}$.

3.4.1 Direct (Superposition) Sum

We use the superposition principle to write

$$\mathbf{u}^{(0)}(\mathbf{r}) = \mathbf{u}_{far}(\mathbf{r}) + \sum_{p=1}^{N} \mathbf{u}_{dis}^{(p)}(\mathbf{r}_p), \tag{3.32}$$

where

$$\mathbf{u}_{dis}^{(p)}(\mathbf{r}) = \sum_{i,t,s} A_{ts}^{(i)(p)} \mathbf{U}_{ts}^{(i)}(\mathbf{r}) \tag{3.33}$$

is the displacement disturbance field caused by pth inclusion: $\mathbf{u}_{dis}^{(p)}(\mathbf{r}_p) \to 0$ as $\|\mathbf{r}_p\| \to \infty$.

3.4.2 Local Expansion Sum

In a vicinity of O_q, the following expansions are valid:

$$\mathbf{u}_{far}(\mathbf{r}) = \mathbf{u}_{far}(\mathbf{R}_q) + \mathbf{u}_{far}(\mathbf{r}_q) = \varepsilon^{\infty} \cdot \mathbf{R}_q + \sum_{t=0}^{\infty} \sum_{s=-t}^{t} c_{ts}^{(i)} \mathbf{u}_{ts}^{(i)}(\mathbf{r}_q), \tag{3.34}$$

where $c_{tst}^{(i)}$ are given by (3.19). Displacement $\mathbf{u}_{dis}^{(q)}$ in (3.32) is written in the local coordinate system of the qth inhomogeneity and ready for use. For $p \neq q$, we apply the re-expansion formula (3.26) to get

$$\mathbf{u}^{(0)}(\mathbf{r}) = \varepsilon^{\infty} \cdot \mathbf{R}_q + \sum_{i,t,s} \left[A_{ts}^{(i)(q)} \mathbf{U}_{ts}^{(i)}(\mathbf{r}_q) + a_{ts}^{(i)(q)} \mathbf{u}_{ts}^{(i)}(\mathbf{r}_q) \right], \tag{3.35}$$

where

$$a_{ts}^{(i)(q)} = \sum_{p \neq q}^{N} \sum_{j,k,l} (-1)^{k+l} A_{kl}^{(j)(p)} \eta_{ktls}^{(j)(i)}(\mathbf{R}_{pq}). \tag{3.36}$$

In matrix form,

$$\mathbf{a}_{ts}^{(q)} = \sum_{p \neq q}^{N} \sum_{k,l} \left[\eta_{ktls}(\mathbf{R}_{pq}) \right]^T \cdot \mathbf{A}_{kl}^{(p)} \tag{3.37}$$

where $\mathbf{A}_{ts}^{(q)} = \left\{ A_{ts}^{(i)(q)} \right\}^T$, $\mathbf{a}_{ts}^{(q)} = \left\{ a_{ts}^{(i)(q)} \right\}^T$ and $\eta_{tksl} = (-1)^{k+l} \left\{ \eta_{tksl}^{(i)(j)} \right\}$.

3.4.3 Infinite System of Linear Equations

With the displacement within qth inhomogeneity represented by the series (3.13) with coefficients $\mathbf{d}_{ts}^{(q)} \left\{ d_{ts}^{(i)(q)} \right\}^T$, we come to the above considered single inclusion problem. The resulting infinite system of linear equations has the form

$$(t-s)!(t+s)!(\mathbf{RM}_t^{(q)})^{-1}\mathbf{RG}_t^{(q)}\cdot\mathbf{A}_{ts}^{(q)}+\sum_{p\neq q}^{N}\sum_{k,l}\left[\eta_{ktls}(\mathbf{R}_{pq})\right]^T\cdot\mathbf{A}_{kl}^{(p)}=-\mathbf{c}_{ts}, \quad (3.38)$$

where $\mathbf{RM}_t^{(q)} = \mathbf{RM}_t(R_q, \nu_0, \nu_q)$ and $\mathbf{RG}_t^{(q)} = \mathbf{RG}_t(R_q, \nu_0, \nu_q)$ are given by (3.24). Likewise (2.33), this system can be solved by the truncation method.

Again, the obtained solution is straightforward and remarkably simple: see, for comparison, solution of two-sphere problem [8]. This approach enables an efficient analytical solution to a wide class of 3D elasticity problems for multiple-connected domains with spherical boundaries, in particular, study of elastic interactions between the spherical nano inclusions with Gurtin-Murdoch type interfaces [54].

3.4.4 FCM and Effective Elastic Moduli

We define, by analogy with FCM conductivity problem, an equivalent inclusion radius as $R_{eff}^3 = N/c$, c being a volume fraction of inclusions, and compare, in spirit of Maxwell's approach, an asymptotic behavior of disturbances caused by the finite cluster of inclusions and "equivalent" inclusion. To this end, we apply the (S2S) re-expansion (3.30), giving us for $\|\mathbf{r}\| > \max_p \|\mathbf{R}_p\|$

$$\sum_{p=1}^{N}\mathbf{u}_{dis}^{(p)}(\mathbf{r}) = \sum_{i,t,s} e_{ts}^{(i)}\mathbf{U}_{ts}^{(i)}(\mathbf{r}), \quad (3.39)$$

where

$$\mathbf{e}_{ts} = \sum_{p=1}^{N}\sum_{k,l}\left[\mu_{ktls}(-\mathbf{R}_p)\right]^T\cdot\mathbf{A}_{kl}^{(p)} \quad (3.40)$$

and $\mu_{tksl} = \left\{\mu_{tksl}^{(i)(j)}\right\}$.

Now, we equate $\mathbf{e}_{ts}$ given by (3.40) to $\mathbf{a}_{ts}$ in (3.23) to determine the effective moduli of composite. Considering the equiaxial far tension $\varepsilon_{11}^{\infty} = \varepsilon_{22}^{\infty} = \varepsilon_{33}^{\infty}$ gives us, in view of (3.25), an expression for the effective bulk modulus k_{eff}

$$k_{eff} = \frac{3k_0 + 4\mu_0\, c\left\langle A_{00}^{(1)}\right\rangle}{3\left(1 - c\left\langle A_{00}^{(1)}\right\rangle\right)}, \quad (3.41)$$

where $\left\langle A_{00}^{(1)}\right\rangle = \frac{1}{N}\sum_{p=1}^{N} A_{00}^{(1)(p)}$ is the mean dipole moment. In the case we neglect interactions between the inclusions, (3.41) reduces to the mechanical counterpart of the original Maxwell's formula (e.g., [58]).

3.5 RUC

The stress field in the composite bulk is assumed to be macroscopically uniform, which means constancy of the macroscopic strain $\langle\varepsilon\rangle$ and stress $\langle\sigma\rangle$ tensors. We take $\langle\varepsilon\rangle = \mathbf{E}$ as input (governing) load parameter. Alike the conductivity problem, periodic geometry of composite results in quasi-periodicity of the displacement field and periodicity of the strain and stress fields:

$$\begin{aligned}&\mathbf{u}\left(\mathbf{r}+a\mathbf{i}_i\right)=\mathbf{u}\left(\mathbf{r}\right)+a\mathbf{E}\cdot\mathbf{i}_i;\\&\varepsilon\left(\mathbf{r}+a\mathbf{i}_i\right)=\varepsilon\left(\mathbf{r}\right);\;\;\sigma\left(\mathbf{r}+a\mathbf{i}_i\right)=\sigma\left(\mathbf{r}\right).\end{aligned} \tag{3.42}$$

3.5.1 Displacement Field

The conditions (3.42) are fulfilled by taking $\mathbf{u}\left(\mathbf{r}\right) = \langle\varepsilon\rangle \cdot \mathbf{r} + \mathbf{u}_{dis}\left(\mathbf{r}\right)$, where $\mathbf{u}_{dis}$ is the periodic disturbance displacement field. In the matrix domain, we write $\mathbf{u}$ in the form analogous to (2.36):

$$\mathbf{u}^{(0)}(\mathbf{r}) = \mathbf{E}\cdot\mathbf{r} + \sum_{p=1}^{N}\mathbf{u}_{dis}^{*(p)}(\mathbf{r}_p), \tag{3.43}$$

where

$$\mathbf{u}_{dis}^{*(p)}(\mathbf{r}_p) = \sum_{\mathbf{k}}\mathbf{u}_{dis}^{(p)}\left(\mathbf{r}_p + a\mathbf{k}\right) \tag{3.44}$$

and

$$\mathbf{u}_{dis}^{(p)}(\mathbf{r}) = \sum_{i,t,s}A_{ts}^{(i)(p)}\mathbf{U}_{ts}^{(i)}(\mathbf{r}). \tag{3.45}$$

Again, by analogy with (2.38), $\mathbf{u}_{dis}^{*(p)}$ can be expressed in terms of the periodic functions $\mathbf{U}_{ts}^{*(i)}$ [30]:

$$\mathbf{u}_{dis}^{*(p)}(\mathbf{r}_p) = \sum_{i,ts}^{\infty}A_{ts}^{(p)}\mathbf{U}_{ts}^{*(i)}(\mathbf{r}_p),\;\; \mathbf{U}_{ts}^{*(i)}(\mathbf{r}_p) = \sum_{\mathbf{k}}\mathbf{U}_{ts}^{(i)}(\mathbf{r}_p + a\mathbf{k}). \tag{3.46}$$

An alternate, mathematically equivalent set of vectorial periodic functions have been written [76] in terms of fundamental periodic solutions [21]. The resulting linear system closely resembles that for FCM (3.38):

$$(t-s)!(t+s)!(\mathbf{R}\mathbf{M}_t^{(q)})^{-1}\mathbf{R}\mathbf{G}_t^{(q)}\cdot\mathbf{A}_{ts}^{(q)}+\sum_{p=1}^{N}\sum_{k,l}\left[\eta_{ktls}^*(\mathbf{R}_{pq})\right]^T\cdot\mathbf{A}_{kl}^{(p)}=-\mathbf{c}_{ts}, \quad (3.47)$$

where $c_{ts}^{(i)}$ are given by (3.19). The matrix coefficients $\eta_{ktls}^*(\mathbf{R}_{pq})$ are the lattice sums of corresponding expansion coefficients $\eta_{ktls}^*(\mathbf{R}_{pq})$ (3.27). Their evaluation is mostly based on the relevant results for scalar potential (2.40). The only new feature here is the biharmonic lattice sum

$$Z_{k+t,l-s}^*(\mathbf{R}_{pq})=\sum_{\mathbf{k}}Z_{k+t}^{l-s}\left(\mathbf{R}_{pq}+a\mathbf{k}\right); \quad (3.48)$$

for Z_t^s definition, see Sect. 3.3. This sum closely relates the fundamental solution S_2 in [21]: its evaluation with aid of Evald's technique is discussed there.

3.5.2 Effective Stiffness Tensor

The fourth rank effective elastic stiffness tensor $\mathbf{C}^*=\left\{C_{ijkl}^*\right\}$ is defined by (1.10). In order to evaluate C_{ijkl}^* for a given geometry of composite, one must conduct a series of numerical tests with different macro strains E_{ij} and evaluate the macro stress $\langle\sigma\rangle$ Eq. (1.12). Specifically, $C_{ijkl}^*=\left\langle\sigma_{ij}\right\rangle$ for $\langle\varepsilon_{mn}\rangle=\delta_{mk}\delta_{nl}$. For this purpose, we need the explicit expressions of average strain and stress corresponding to our displacement solution.

Evaluation of the macroscopic strain tensor, $\langle\varepsilon\rangle$ is elementary. First, we recall that we have taken RUC with $S_0\in V_0$, so $\mathbf{u}=\mathbf{u}^{(0)}$ in (1.12). Next, we observe that for the periodic part of solution in the boundary points $\mathbf{r}_a\in S_0$ and $\mathbf{r}_b=\mathbf{r}_a+a\mathbf{i}_j\in S_0$ belonging to the opposite cell faces we have $\mathbf{u}_{dis}(\mathbf{r}_b)=\mathbf{u}_{dis}(\mathbf{r}_a)$ whereas the normal unit vector changes the sign: $\mathbf{n}(\mathbf{r}_b)=-\mathbf{n}(\mathbf{r}_a)$. Hence, the integrals of $\mathbf{u}_{dis}$ over the opposite faces cancel each other and the total integral over S_0 equals to zero. Integration of the linear part of $\mathbf{u}^{(0)}$ is elementary: the divergence theorem gives the expected $\langle\varepsilon\rangle=\mathbf{E}$.

A suitable for our purpose expression of the macroscopic stress tensor $\langle\sigma\rangle$ is given by (1.13). Noteworthy, surface integration in (1.13) is greatly simplified by taking $\mathbf{u}_{ts}^{(i)}$ as a trial displacement vector $\mathbf{u}'$. It follows from (3.19) that $\mathbf{u}_{00}^{(3)}$ and $\mathbf{u}_{2s}^{(1)}$ are the linear function of coordinates. For example, $\mathbf{u}_{00}^{(3)}=r\gamma_0\mathbf{S}_{00}^{(3)}=\gamma_0\mathbf{r}=\gamma_0\mathbf{u}_{kk}'$; also, $\mathbf{T}_n(\mathbf{u}_{00}^{(3)})=\frac{3k}{r}\mathbf{u}_{00}^{(3)}$, k being the bulk modulus. We get from Eq. (1.13)

$$\langle\sigma_{ii}\rangle=C_{iikk}^{(0)}\langle\varepsilon_{kk}\rangle+\frac{1}{V}\sum_{q=1}^{N}\int_{S_q}\left[\mathbf{T}_n(\mathbf{u}^{(0)})-\frac{3k_0}{r}\mathbf{u}^{(0)}\right]\cdot\frac{\mathbf{u}_{00}^{(3)}}{\gamma_0}dS. \quad (3.49)$$

Now, we put here the local expansion of $\mathbf{u}^{(0)}$ given by Eq. (3.35) and the analogous expansion of $\mathbf{T}_n\left[\mathbf{u}^{(0)}\right]$:

$$\mathbf{T}_n\left[\mathbf{u}^{(0)}(\mathbf{r}_q)\right] = \sum_{i,t,s}\left\{A_{ts}^{(i)(q)}\mathbf{T}_n\left[\mathbf{U}_{ts}^{(i)}(\mathbf{r}_q)\right] + \left(a_{ts}^{(i)(q)} + c_{ts}^{(i)}\right)\mathbf{T}_n\left[\mathbf{u}_{ts}^{(i)}(\mathbf{r}_q)\right]\right\}. \quad (3.50)$$

For the explicit expression of $\mathbf{T}_n(\mathbf{U}_{ts}^{(i)})$ and $\mathbf{T}_n(\mathbf{u}_{ts}^{(i)})$ in terms of vectorial spherical harmonics $\mathbf{S}_{ts}^{(i)}$, see [54]. By taking orthogonality of these harmonics on the sphere into account we find that the only function giving non-zero contribution to the integral in (3.49) is $\mathbf{U}_{00}^{(1)} = -\frac{2}{r^2}\mathbf{S}_{00}^{(3)}$ for which $\mathbf{T}_n\left(\mathbf{U}_{00}^{(1)}\right) = -\frac{4\mu}{r}\mathbf{U}_{00}^{(1)} = \frac{8\mu}{r^3}\mathbf{S}_{00}^{(3)}$. Thus, we obtain

$$\int_{S_q}\left[\mathbf{T}_n(\mathbf{u}^{(0)}) - \frac{3k_0}{r}\mathbf{u}^{(0)}\right]\cdot\frac{\mathbf{u}_{00}^{(3)}}{\gamma_0}dS$$
$$= A_{00}^{(1)(q)}\frac{2}{R^2}(4\mu_0 + 3k_0)\int_{S_q}\left(\mathbf{S}_{00}^{(3)}\cdot\mathbf{S}_{00}^{(3)}\right)dS = 8\pi(4\mu_0 + 3k_0)A_{00}^{(1)(q)}.$$

By using in the same way the (simple shear mode) functions

$$\begin{aligned}
\mathbf{u}_{20}^{(1)} &= \mathbf{u}'_{33} - (\mathbf{u}'_{11} + \mathbf{u}'_{22})/2;\\
\mathbf{u}_{21}^{(1)} &= \left(\mathbf{u}'_{13} + \mathbf{u}'_{31}\right)/2 + \mathrm{i}(\mathbf{u}'_{23} + \mathbf{u}'_{32})/2;\\
\mathbf{u}_{22}^{(1)} &= \left(\mathbf{u}'_{11} - \mathbf{u}'_{22}\right)/4 + \mathrm{i}\left(\mathbf{u}'_{12} + \mathbf{u}'_{21}\right)/4;
\end{aligned}$$

for which $\mathbf{T}_r(\mathbf{u}_{2s}^{(1)}) = \frac{2\mu}{r}\mathbf{u}_{2s}^{(1)}$, we come to the finite exact formulas

$$S_{11} + S_{22} + S_{33} = \frac{(1+\nu_0)}{(1-2\nu_0)}(E_{11} + E_{22} + E_{33}) + \frac{12\pi}{a^3}\frac{(1-\nu_0)}{(1-2\nu_0)}\sum_{q=1}^{N}A_{00}^{(1)(q)};$$

$$2S_{33} - S_{11} - S_{22} = 2E_{33} - E_{11} - E_{22} - \frac{16\pi}{a^3}(1-\nu_0)\sum_{q=1}^{N}A_{20}^{(3)(q)};$$

$$S_{11} - S_{22} - 2\mathrm{i}S_{12} = E_{11} - E_{22} - 2\mathrm{i}E_{12} - \frac{32\pi}{a^3}(1-\nu_0)\sum_{q=1}^{N}A_{22}^{(3)(q)};$$

$$S_{13} - \mathrm{i}S_{23} = E_{13} - \mathrm{i}E_{23} - \frac{8\pi}{a^3}(1-\nu_0)\sum_{q=1}^{N}A_{21}^{(3)(q)}. \quad (3.51)$$

where $S_{ij} = \langle\sigma_{ij}\rangle/2\mu_0$. The coefficients $A_{ts}^{(i)(q)}$ are linearly proportional to $\mathbf{E}$. The $\langle\sigma_{ij}\rangle$ are uniquely determined from Eqs. (3.47, 3.51) for E_{kl} given and, thus, these equations together with (1.10) are sufficient for evaluation of the effective stiffness tensor, $\mathbf{C}^*$. (3.51) involves only the expansion coefficients $A_{i-1,s}^{(i)(q)}$ which can be regarded as the dipole moments. The effective elastic moduli of composite with simple cubic array of spherical inclusions have been found by [31, 76], for the RUC type structure—by [17, 78].

4 Composite with Spheroidal Inclusions

4.1 Scalar Spheroidal Solid Harmonics

The spheroidal coordinates (ξ, η, φ) relate the Cartesian coordinates (x_1, x_2, x_3) by [23]

$$x_1 + \mathrm{i}x_2 = d\bar{\xi}\bar{\eta}\exp(\mathrm{i}\varphi), \quad x_3 = d\xi\eta, \tag{4.1}$$

where

$$\bar{\xi}^2 = \xi^2 - 1, \quad \bar{\eta}^2 = 1 - \eta^2 \; (1 \leq \xi < \infty, \; -1 \leq \eta \leq 1, \; 0 \leq \varphi < 2\pi). \tag{4.2}$$

At $\mathrm{Re}(d) > 0$, the formulas (4.1) and (4.2) define a family of confocal prolate spheroids with inter-foci distance $2d$: to be specific, we expose all the theory for this case. In the case of oblate spheroid, one must replace ξ with $\mathrm{i}\bar{\xi}$ and d with $(-\mathrm{i}d)$ in these and all following formulas. For $d \to 0$ and $d\bar{\xi} \to r$, the spheroidal coordinates system degenerates into spherical one, with $\eta \to \cos\theta$.

Separation of variables in Laplace equation in spheroidal coordinates gives us the sets of partial solutions, or solid spheroidal harmonics: regular

$$f_t^s(\mathbf{r}, d) = P_t^{-s}(\xi)\chi_t^s(\eta, \varphi) = \frac{(t-s)!}{(t+s)!}P_t^s(\xi)\chi_t^s(\eta, \varphi) \tag{4.3}$$

and irregular

$$F_t^s(\mathbf{r}, d) = Q_t^{-s}(\xi)\chi_t^s(\eta, \varphi) = \frac{(t-s)!}{(t+s)!}Q_t^s(\xi)\chi_t^s(\eta, \varphi) \tag{4.4}$$

Here, Q_t^s are the associate Legendre's functions of second kind [23]. The functions (4.3) and (4.4) obey the properties analogous to (2.7). The functions $F_t^s \to 0$ with $\|\mathbf{r}\| \to \infty$ and, by analogy with Y_t^s, can be regarded as *spheroidal* multipole potentials. We mention the multipole type re-expansions between Y_t^s and F_t^s [13] which, in our notations, take the form

$$F_t^s(\mathbf{r}, d) = (-1)^s \sum_{k=t}^{\infty} K_{tk}^{(1)}(d) Y_k^s(\mathbf{r}) \quad (\|\mathbf{r}\| > d); \tag{4.5}$$

$$Y_t^s(\mathbf{r}) = (-1)^s \sum_{k=t}^{\infty} K_{tk}^{(2)}(d) F_k^s(\mathbf{r}, d);$$

where

$$K_{tk}^{(1)}(d) = \left(\frac{d}{2}\right)^{k+1} \frac{\sqrt{\pi}}{\Gamma\left(\frac{k-t}{2}+1\right)\Gamma\left(\frac{k+t}{2}+\frac{3}{2}\right)}, \tag{4.6}$$

$$K_{tk}^{(2)}(d) = \left(\frac{2}{d}\right)^{k+1} \frac{(-1)^{(k-t)/2}\left(k+\frac{1}{2}\right)\Gamma\left(\frac{k+t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k-t}{2}+1\right)}$$

for $(t-k)$ even and equal to zero otherwise. The analogous relations are also available for the regular solid harmonics, y_t^s and f_t^s [39].

4.2 Single Inclusion: Conductivity Problem

Let an unbounded solid contains a single prolate spheroidal inclusion with boundary defined by $\xi = \xi_0$. The matrix-inclusion thermal contact obeys the conditions (2.9) where q_r is replaced with $q_\xi = -\lambda \nabla T \cdot \mathbf{e}_\xi = d^{-1}(\overline{\xi}^2 - \eta^2)^{-1/2} \partial T / \partial \xi$. The solution flow is quite analogous to that for the spherical inclusion, so we outline it briefly.

4.2.1 Series Solution

The temperature field inside the inclusion is given by a series

$$T^{(1)}(\mathbf{r}) = \sum_{t=0}^{\infty} \sum_{s=-t}^{t} d_{ts} f_t^s(\mathbf{r}, d). \tag{4.7}$$

The temperature field outside the inclusion is a sum of far and disturbance fields:

$$T^{(0)}(\mathbf{r}) = T_{far}(\mathbf{r}) + T_{dis}(\mathbf{r}), \quad T_{dis}(\mathbf{r}) = \sum_{t=1}^{\infty} \sum_{s=-t}^{t} A_{ts} F_t^s(\mathbf{r}, d). \tag{4.8}$$

The second term in (4.8) can be thought as the spheroidal multipole expansion of the disturbance field T_{dis}. Noteworthy, at some distance from inclusion (namely, where $\|\mathbf{r}\| > d$) it can be also expanded, by applying formula (4.5), over the spherical multipoles Y_t^s. The T_{far} series expansion in a vicinity of inclusion is analogous to (4.7). In view of (2.6), the c_{ts} coefficients are equal to

$$c_{ts} = \frac{(2t+1)}{4\pi P_t^s(\xi_0)} \int_0^{2\pi} d\varphi \int_{-1}^{1} T_{far} \overline{\chi_t^s(\eta, \varphi)}\, d\eta. \tag{4.9}$$

4.2.2 Resolving Equations

The temperature continuity at $\xi = \xi_0$ gives, in view of $\chi_t^s(\eta, \varphi)$ orthogonality (2.6), a set of linear algebraic equations

$$Q_t^s(\xi_0) A_{ts} + P_t^s(\xi_0) c_{ts} = P_t^s(\xi_0) d_{ts}. \tag{4.10}$$

The second, normal flux continuity condition gives us also

$$Q_t'^s(\xi_0) A_{ts} + P_t'^s(\xi_0) c_{ts} = \omega P_t'^s(\xi_0) d_{ts}, \tag{4.11}$$

where $\omega = \lambda_1/\lambda_0$ and stroke means derivative with respect to argument. The final formula for the coefficients A_{ts} is

$$\frac{1}{(\omega - 1)} \left[\omega \frac{Q_t^s(\xi_0)}{P_t^s(\xi_0)} - \frac{Q_t'^s(\xi_0)}{P_t'^s(\xi_0)} \right] A_{ts} = -c_{ts}. \tag{4.12}$$

4.3 Re-expansion Formulas

We define the inclusion-related local coordinate system in a way that coordinate surface coincides with the surface of inclusion. Spheroidal shape of inclusion dictates the origin and, in contrast to spherical case, orientation of this system. A general transformation of coordinates can be splitted into a sum of translation and rotation, i.e. $\mathbf{r}_1 = \mathbf{R} + \mathbf{O} \cdot \mathbf{r}_2$ where $\mathbf{O}$ is a symmetrical positively definite matrix with $\det \mathbf{O} = 1$. Analogously, the general re-expansion formula is given by superposition of two more simple formulas, one written for translation and another—for rotation.

4.3.1 Translation

In the case of co-axial ($\mathbf{O} = \mathbf{I}$) coordinate systems $(d_1, \xi_1, \eta_1, \varphi_1)$ and $(d_2, \xi_2, \eta_2, \varphi_2)$ centered at the points O_1 and O_2, respectively, the re-expansion formulas (addition theorems) of three kinds for the spheroidal solid harmonics are [32, 35, 37]:

$$F_t^s(\mathbf{r}_1, d_1) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \eta_{tk,s-l}(\mathbf{R}, d_1, d_2) f_k^l(\mathbf{r}_2, d_2); \tag{4.13}$$

$$f_t^s(\mathbf{r}_1, d_1) = \sum_{k=0}^{t} \sum_{l=-k}^{k} \mu_{tk,s-l}(\mathbf{R}, d_1, d_2) f_k^l(\mathbf{r}_2, d_2); \tag{4.14}$$

$$F_t^s(\mathbf{r}_1, d_1) = \sum_{k=t}^{\infty} \sum_{l=-k}^{k} \nu_{tk,s-l}(\mathbf{R}, d_1, d_2) F_k^l(\mathbf{r}_2, d_2). \tag{4.15}$$

A simple way to derive these formulas consists in combining the formulas (4.5), (4.6) with the relevant re-expansions for spherical harmonics (2.20)–(2.22). Say, for (4.13) this procedure yields

$$\eta_{tks}^{(1)} = a_{tk} \sum_{r=0}^{\infty} \left(\frac{d_1}{2} \right)^{2r} M_{tkr} \left(d_1, d_2 \right) Y_{t+k+2r}^{s} \left(\mathbf{R} \right), \tag{4.16}$$

where

$$a_{tk} = (-1)^{k+l} \pi \left(k + \frac{1}{2} \right) \left(\frac{d_1}{2} \right)^{t+1} \left(\frac{d_2}{2} \right)^{k}, \tag{4.17}$$

$$M_{tkr} = \sum_{j=0}^{r} \frac{(d_2/d_1)^{2j}}{j!(r-j)!\Gamma\left(t+r-j+3/2\right) \Gamma\left(k+j+3/2\right)} \tag{4.18}$$

and $\Gamma(z)$ is the gamma-function [1]. For $d_1 = d_2$, this expression reduces to

$$M_{tkr} = \frac{(t+k+r+2)_r}{r!\,\Gamma\left(t+r+3/2\right)\, \Gamma\left(k+r+3/2\right)}, \tag{4.19}$$

where $(n)_m$ is the Pochhammer's symbol. Derived this way the coefficients in (4.14) and (4.15) are

$$\mu_{tks}^{(1)} = \left(k + \frac{1}{2} \right) \left(\frac{d_1}{2} \right)^{-t} \left(\frac{d_2}{2} \right)^{k} \sum_{r=0}^{\sigma} \left(\frac{d_1}{2} \right)^{2r} y_{t-k-2r}^{s} \left(\mathbf{R} \right) \tag{4.20}$$
$$\times \sum_{j=0}^{r} (-1)^{r-j} \frac{\Gamma\left(t+j-r+1/2\right) (d_2/d_1)^{2j}}{j!(r-j)!\,\Gamma\left(k+j+3/2\right)}$$

and

$$\nu_{tks}^{(1)} = \left(k + \frac{1}{2} \right) \left(\frac{d_1}{2} \right)^{t+1} \left(\frac{d_2}{2} \right)^{-(k+1)} \sum_{r=0}^{\sigma} \left(\frac{d_1}{2} \right)^{2r} y_{k-t-2r}^{s} \left(\mathbf{R} \right) \tag{4.21}$$
$$\times \sum_{j=0}^{r} (-1)^{j} \frac{\Gamma\left(k-j+1/2\right) (d_2/d_1)^{2j}}{j!(r-j)!\,\Gamma\left(t+r-j+3/2\right)},$$

respectively. Here, $\sigma = |t-k| - |s|$. They are consistent with the formulas reported by [13].

It appears, however, that formula (4.13) with coefficients (4.16) is valid only for $\|\mathbf{R}\| > d_1 + d_2$; this fact is due to the geometry restriction in (4.5) and (2.20) used for derivation. A general, geometrical restrictions-free expression [32] is

$$\eta_{tks}^{(2)} = \frac{a_{tk}}{\sqrt{\pi}} \left(\frac{2}{d_3}\right)^{t+k+1} \sum_{r=0}^{\infty} F_{t+k+2r}^{s} (\mathbf{R}, d_3) \left(t + k + 2r + \frac{1}{2}\right) \times \sum_{j=0}^{r} \frac{(-1)^{r-j}}{(r-j)!} \left(\frac{d_1}{d_3}\right)^{2j} \Gamma (t + k + r + j + 1/2)\, M_{tkj} (d_1, d_2) \tag{4.22}$$

The series (4.13) with coefficients (4.22) for $d_3 > d_1$ converges in all points inside the spheroid $\xi_2 = \xi_{20}$ with the center at a point O_2 and inter-foci distance $2d_2$ if point O_1 lies outside the spheroid with semiaxes $d_2\overline{\xi_{20}}$ and $d_2\xi_{20} + d_1$ ($\xi_{12} > \xi_{12}^0$). Here,

$$\xi_{12}^0 = \cosh\left(\operatorname{arctanh} \frac{d_2\overline{\xi_{20}}}{d_2\xi_{20} + d_1}\right) \tag{4.23}$$

where $(d_{12}, \xi_{12}, \eta_{12}, \varphi_{12})$ are spheroidal coordinates of vector $\mathbf{R}$ in a system with origin in the point O_2 and $d_{12} = (d_2\xi_{20} + d_1)/\xi_{12}^0$. This is condition of non-intersecting the spheroidal surface $\xi_2 = \xi_{20}$ and infinitely thin spheroid with inter-foci distance $2d_1$ centered at point O_1 and holds true for any two non-intersecting spheroids of finite size.

In the case $\|\mathbf{R}\| > d_1 + d_2$, both the representations, $\eta_{tk,s-l}^{(1)}$ and $\eta_{tk,s-l}^{(2)}$ are valid and using the simpler expression (4.16) is preferable. For $\|\mathbf{R}\| < d_1 + d_2$, $\eta_{tk,s-l}^{(1)}$ diverges and $\eta_{tk,s-l}^{(2)}$ should be used instead. Noteworthy, convergence rate of (4.23) is geometry-dependent and the numerical difficulties may arise, e.g., for the closely placed very elongated spheroids. In this situation, using the numerical integration formula

$$\eta_{tk,s-l} = \frac{(2t+1)}{4\pi P_k^l(\xi_{20})} \int_0^{2\pi} d\varphi_2 \int_{-1}^{1} F_t^s (\mathbf{r} + \mathbf{R}, d_1)\, \overline{\chi_k^l(\eta_2, \varphi_2)}\, d\eta_2 \tag{4.24}$$

analogous to (4.9) would be the most appropriate option.

4.3.2 Rotation

The re-expansion of spheroidal solid harmonics (4.3) and (4.4) due to the rotation of the coordinate system ($\mathbf{R} = \mathbf{0}$, $d_1 = d_2 = d$) is determined by

$$f_t^s (\mathbf{r}_1, d) = \sum_{k=0}^{t}{}' \sum_{l=-k}^{k} f_k^l (\mathbf{r}_2, d) (-1)^{s+l} \left(k + \frac{1}{2}\right) \times \sum_{r=k}^{\infty}{}' \frac{(-1)^{\frac{t-r}{2}} \Gamma\left(\frac{t+r}{2} + \frac{1}{2}\right)}{\left(\frac{t-r}{2}\right)! \left(\frac{r-k}{2}\right)! \Gamma\left(\frac{r+k}{2} + \frac{3}{2}\right)} \frac{(r-l)!(r+l)!}{(r-s)!(r+s)!} S_{2r}^{r-s,r-l}(\mathbf{w}); \tag{4.25}$$

$$F_t^s(\mathbf{r}_1, d) = \sum_{k=t}^{\infty}{}' \sum_{l=-k}^{k} F_k^l(\mathbf{r}, d)(-1)^{s+l}\left(k + \frac{1}{2}\right) \times \sum_{r=k}^{\infty}{}' \frac{(-1)^{\frac{t-r}{2}}\Gamma\left(\frac{t+r}{2} + \frac{1}{2}\right)}{\left(\frac{r-t}{2}\right)!\left(\frac{k-r}{2}\right)!\Gamma\left(\frac{r+t}{2} + \frac{3}{2}\right)} S_{2r}^{r-s, r-l}(\mathbf{w}); \quad (4.26)$$

where primes over the sums denote that they contain only terms with even $(t - k)$ and $(t - r)$. Also, $S_t^{s,l}$ are the spherical harmonics [3] in four-dimensional space, $\mathbf{w} = \{w_i\}$ is the unit four-dimensional vector determining uniquely the rotation matrix

$$\mathbf{O} = \begin{pmatrix} w_2^2 - w_1^2 - w_3^2 + w_4^2 & 2(w_2w_3 - w_1w_4) & 2(w_1w_2 + w_3w_4) \\ 2(w_2w_3 + w_1w_4) & w_3^2 - w_1^2 - w_2^2 + w_4^2 & 2(w_1w_3 - w_2w_4) \\ 2(w_1w_2 - w_3w_4) & 2(w_1w_3 + w_2w_4) & w_1^2 - w_2^2 - w_3^2 + w_4^2 \end{pmatrix}. \quad (4.27)$$

These results are similar to those derived by [13]. Coupled with (4.13)–(4.15), they provide the representation of spheroidal solid harmonics in any arbitrary positioned and oriented coordinate system and thus form a theoretical basis of solving the potential theory problems for multiply connected domains with spheroidal boundaries by the Multipole Expansion Method.

4.4 FCM

Let consider an unbounded solid containing N spheroidal inclusions of conductivity λ_q centered in the points O_q. Geometry of qth inclusion $(q = 1, 2, \ldots, N)$ is defined by two numbers: $\xi_0 = \xi_{q0}$ and $d = d_q$. Its position and orientation are given by the vector $\mathbf{R}_q$, and matrix $\mathbf{O}_q$, respectively. The local Cartesian (x_{1q}, x_{2q}, x_{3q}) and spheroidal $(d_q, \xi_q, \eta_q, \varphi_q)$ coordinates are related by (4.1); $\mathbf{r} = \mathbf{R}_q + \mathbf{O}_q \cdot \mathbf{r}_q$. The linear far field $T_{far} = \mathbf{G} \cdot \mathbf{r}$ is prescribed.

Temperature in the matrix solid is given by the superposition sum (2.27), where the disturbance field caused by pth inclusion is written now in terms of spheroidal multipoles:

$$T_{dis}^{(p)}(\mathbf{r}_p) = \sum_{t=1}^{\infty} \sum_{s=-t}^{t} A_{ts}^{(p)} F_t^s(\mathbf{r}_p, d_p). \quad (4.28)$$

In a vicinity of O_q, the following expansions are valid:

$$T_{far}(\mathbf{r}_q) = \sum_{t=0}^{\infty} \sum_{s=-t}^{t} c_{ts}^{(q)} f_t^s(\mathbf{r}_q), \quad (4.29)$$

where $c_{00}^{(q)} = \mathbf{G} \cdot \mathbf{R}_q$, $c_{10}^{(q)} = G_i O_{i3q} d_q$, $c_{11}^{(q)} = 2d_q G_i (O_{i1q} - \mathrm{i} O_{i2q})$, $c_{1,-1}^{(q)} = -\overline{c_{11}^{(q)}}$ and $c_{ts}^{(q)} = 0$ otherwise. The term $T_{dis}^{(q)}$ is already written in qth basis; for $p \neq q$, we apply the re-expansion formulas for irregular solid harmonics to get

$$T_{dis}^{(p)}(\mathbf{r}_q) = \sum_{t=0}^{\infty} \sum_{s=-t}^{t} a_{ts}^{(q)} f_t^s(\mathbf{r}_q), \; a_{ts}^{(q)} = \sum_{p \neq q}^{N} \sum_{k=1}^{\infty} \sum_{l=-k}^{k} A_{kl}^{(p)} \eta_{k+t}^{l-s}(\mathbf{R}_{pq}, \mathbf{O}_{pq}). \tag{4.30}$$

Here, $\eta_{k+t}^{l-s}(\mathbf{R}_{pq}, \mathbf{O}_{pq})$ is the expansion coefficient obtained by consecutive application of (4.13) and (4.25). In their calculation, one of three options—(4.13), (4.22) or (4.24)—is used depending on the relative position of pth and qth particles.

By putting all the parts together, we get

$$T^{(0)}(\mathbf{r}_q) = \sum_{t=1}^{\infty} \sum_{s=-t}^{t} A_{ts}^{(q)} F_t^s(\mathbf{r}_q) + \sum_{t=0}^{\infty} \sum_{s=-t}^{t} \left(a_{ts}^{(q)} + c_{ts}^{(q)} \right) f_t^s(\mathbf{r}_q). \tag{4.31}$$

Substitution of (4.31) and $T^{(q)}$ in the form (4.7) but written in qth local basis into (2.12) gives an infinite set of linear equations with unknowns $A_{ts}^{(q)}$, analogous to (4.12) ($\omega_q = \lambda_q / \lambda_q$):

$$\frac{1}{(\omega_q - 1)} \left[\omega_q \frac{Q_t^s(\xi_{q0})}{P_t^s(\xi_{q0})} - \frac{Q_t'^s(\xi_{q0})}{P_t'^s(\xi_{q0})} \right] A_{ts}^{(q)} + a_{ts}^{(q)} = -c_{ts}^{(q)}. \tag{4.32}$$

4.5 RUC

Likewise (2.33), the above solution can be used for evaluation of the effective conductivity of composite with spheroidal inclusions by combining it with Maxwell's model. An alternate, advanced tool for this purpose is RUC model: here, we discuss it briefly. For simplicity sake, we assume all the spheroids entering the unit cell oriented in x_3-direction: considering the case of arbitrarily oriented spheroids involves some extra algebra but does not meet any other difficulties [39].

4.5.1 Periodic Potentials

To apply the procedure described in Sect. 2.5, we first introduce the periodic potentials analogous to Y_{ts}^* (2.38):

$$F_{ts}^*(\mathbf{r}, d) = \sum_{\mathbf{k}} F_t^s(\mathbf{r} + a\mathbf{k}, d), \tag{4.33}$$

where summation is made over all the integer k_1, k_2 and k_3. The solution in the matrix domain is given by the (2.36), (2.37), where now the pth lattice disturbance

$$T_{dis}^{*(p)}(\mathbf{r}_p) = \sum_{t=1}^{\infty}\sum_{s=-t}^{t} A_{ts}^{(p)} F_{ts}^{*}(\mathbf{r}_p, d_p). \tag{4.34}$$

The solution in the form (2.36), (2.37) satisfies the periodicity conditions. In fact, what we need to complete the solution is to find local expansion of the functions $F_{ts}^{*}(\mathbf{r}_p, d_p)$ in a vicinity of qth inclusion in terms of spheroidal solid harmonics. In the case $p = q$, we apply the formula (4.13) to all the terms of the sum (4.33) but one with $\mathbf{k} = 0$ to obtain, after change of summation order,

$$F_{ts}^{*}(\mathbf{r}_q, d_q) = F_t^s\left(\mathbf{r}_q, d_q\right) + \sum_{k=0}^{\infty}\sum_{|l|\le k} \eta_{tk,s-l}^{*}\left(0, d_q, d_q\right)\ f_k^l\left(\mathbf{r}_q, d_q\right), \tag{4.35}$$

where the expansion coefficients are the triple infinite (lattice) sums

$$\eta_{tk}^{*s-l}\left(\mathbf{R}, d_1, d_2\right) = \sum_{\mathbf{k}\neq 0} \eta_{kt}^{s-l}\left(\mathbf{R} + a\mathbf{k}, d_1, d_2\right). \tag{4.36}$$

In the case $p \neq q$, $\mathbf{r}_p = \mathbf{R}_{pq} + \mathbf{r}_q$ and

$$F_{ts}^{*}(\mathbf{r}_p, d_p) = \sum_{k=0}^{\infty}\sum_{|l|\le k} \left[\eta_{tk}^{s-l}\left(\mathbf{R}_{pq}, d_p, d_q\right) + \eta_{tk,s-l}^{*}\left(\mathbf{R}_{pq}, d_p, d_q\right)\right] f_k^l\left(\mathbf{r}_q, d_q\right). \tag{4.37}$$

Recall, in RUC model $\mathbf{R}_{pq}$ is taken as a minimal distance between any two particles of pth and qth lattices, so $\left\|\mathbf{R}_{pq}\right\| < \left\|\mathbf{R}_{pq} + a\mathbf{k}\right\|$ for any $\mathbf{k} \neq 0$. As was discussed above, the first term, η_{tk}^{s-l} in (4.33) can be evaluated from the appropriate formula. Provided the RUC size a is sufficiently large, the formula (4.13) can be used for evaluating the lattice sum η_{tk}^{*s-l}. Specifically,

$$\begin{aligned}\eta_{tk,s-l}^{*}\left(\mathbf{R}_{pq}, d_p, d_q\right) &= \sum_{\mathbf{k}\neq 0} {}^{(1)}\eta_{tk}^{s-l}\left(\mathbf{R}_{pq} + a\mathbf{k}, d_p, d_q\right) \\ &= a_{tk}\sum_{r=0}^{\infty}\left(\frac{d_1}{2}\right)^{2r} M_{tkr}\left(d_p, d_q\right) Y_{t+k+2r,s-l}^{*}\left(\mathbf{R}_{pq}\right),\end{aligned} \tag{4.38}$$

where $Y_{t+k+2r,s-l}^{*}\left(\mathbf{R}_{pq}\right)$ are the standard lattice sums of spherical multipoles (2.40).

After we found the local expansions (4.35) and (4.37), the problem is effectively reduced to a set of single inclusion problems. An infinite linear set of equations with unknowns $A_{ts}^{(q)}$ has the form (4.32), where now

$$a_{ts}^{(q)} = \sum_{p=1}^{N} \sum_{k=1}^{\infty} \sum_{l=-k}^{k} A_{kl}^{(p)} \eta_{tk,s-l}^{*} \left(\mathbf{R}_{pq}, d_p, d_q \right). \tag{4.39}$$

4.5.2 Effective Conductivity

The effective conductivity tensor $\mathbf{\Lambda}^*$ of composite is given by (1.1). In view of the temperature field periodicity, averaging the temperature gradient in composite with spheroidal inclusions follows the same way as in the spherical case and yields $\langle \nabla T \rangle = \mathbf{G}$. The macroscopic flux $\langle \mathbf{q} \rangle$ is given by (1.9) where integration is made now over the spheroidal surfaces $S_q : \xi_q = \xi_{q0}$. To simplify derivation, we perform it for $N = 1$ omitting the q index and, then, will write the formula for a general case.

For the composite with isotropic matrix we consider, the normal component heat flux $q_n = \mathbf{q} \cdot \mathbf{n}$ at the spheroidal surface equals

$$q_n = q_\xi = -\lambda \nabla T \cdot \mathbf{e}_\xi = -\lambda \frac{\bar{\xi} h}{d} \frac{\partial T}{\partial \xi} \tag{4.40}$$

In this case, (1.9) simplifies to

$$\frac{\langle \mathbf{q} \rangle}{\lambda_0} = -\mathbf{G} - \frac{1}{V} \int_S \left(T^{(0)} \mathbf{n} - \frac{\partial T^{(0)}}{\partial n} \mathbf{r} \right) dS. \tag{4.41}$$

The surface integration uses the local expansion (4.31), from where

$$T^{(0)} = \sum\nolimits_{t=1}^{\infty} \sum\nolimits_{s=-t}^{t} \frac{(t-s)!}{(t+s)!} \left[A_{ts} Q_t^s(\xi) + (a_{ts} + c_{ts}) P_t^s(\xi) \right] \chi_t^s, \tag{4.42}$$

$$\frac{\partial T}{\partial n} = \frac{\bar{\xi} h}{d} \sum\nolimits_{t=1}^{\infty} \sum\nolimits_{s=-t}^{t} \frac{(t-s)!}{(t+s)!} \left[A_{ts} Q_t^{\prime s}(\xi) + (a_{ts} + c_{ts}) P_t^{\prime s}(\xi) \right] \chi_t^s.$$

Here, $\chi_t^s = \chi_t^s(\eta, \varphi)$; a_{ts} are given by (4.39). At the interface $S : \xi = \text{const}$,

$$\begin{aligned} \mathbf{r} &= d\left(-2\bar{\xi} \chi_1^{-1} \mathbf{e}_1 + \bar{\xi} \chi_1^1 \mathbf{e}_2 + \xi \chi_1^0 \mathbf{e}_3 \right), \\ \mathbf{n} = \mathbf{e}_\xi &= h\left(-2\xi \chi_1^{-1} \mathbf{e}_1 + \xi \chi_1^1 \mathbf{e}_2 + \bar{\xi} \chi_1^0 \mathbf{e}_3 \right). \end{aligned} \tag{4.43}$$

By substituting these formulas into (4.41) we get

$$\begin{aligned} \frac{1}{h} \left(T^{(0)} \mathbf{n} - \frac{\partial T^{(0)}}{\partial n} \mathbf{r} \right) &= 2 \left(\overline{\xi\xi} \frac{\partial T^{(0)}}{\partial \xi} - \xi T^{(0)} \right) \chi_1^{-1} \mathbf{e}_1 \\ &+ \left(\xi T^{(0)} - \overline{\xi\xi} \frac{\partial T^{(0)}}{\partial \xi} \right) \chi_1^1 \mathbf{e}_2 + \bar{\xi} \left(T^{(0)} - \xi \frac{\partial T^{(0)}}{\partial \xi} \right) \chi_1^0 \mathbf{e}_3 \end{aligned} \tag{4.44}$$

Before integrating this expression, we note that

$$f_1^0(\mathbf{r},d) - \xi \frac{\partial f_1^0(\mathbf{r},d)}{\partial \xi} \equiv 0; \quad \xi f_1^1(\mathbf{r},d) - \overline{\xi\xi} \frac{\partial f_1^1(\mathbf{r},d)}{\partial \xi} \equiv 0. \tag{4.45}$$

Also, it is directly testable that

$$\bar{\xi}\left(\xi Q_1'^0 - Q_1^0\right) = \frac{1}{\bar{\xi}}; \quad \frac{1}{2}\left(\xi Q_1^1 - \overline{\xi\xi} Q_1'^1\right) = \frac{1}{\bar{\xi}}; \tag{4.46}$$

and hence

$$F_1^0(\mathbf{r},d) - \xi \frac{\partial F_1^0(\mathbf{r},d)}{\partial \xi} = -\frac{1}{\bar{\xi}^2}\chi_1^0(\eta,\varphi); \tag{4.47}$$
$$\xi F_1^1(\mathbf{r},d) - \overline{\xi\xi} \frac{\partial F_1^1(\mathbf{r},d)}{\partial \xi} = \frac{1}{\bar{\xi}}\chi_1^1(\eta,\varphi).$$

The formula (4.41) readily transforms to

$$\frac{\langle \mathbf{q} \rangle}{\lambda_0} = -\mathbf{G} - \frac{1}{V} d^2 \int_0^{2\pi}\int_{-1}^{1} \left(T^{(0)}\mathbf{n} - \frac{\partial T^{(0)}}{\partial n}\mathbf{r} \right) \frac{\overline{\xi_0}}{h} d\eta d\varphi.$$

By taking Eqs. (4.42)–(4.47) and orthogonality of the surface harmonics χ_t^s into account we find readily that

$$\int_0^{2\pi}\int_{-1}^{1} \left(T^{(0)} - \xi \frac{\partial T^{(0)}}{\partial \xi} \right) \chi_1^0 d\eta d\varphi = -\frac{4\pi}{3\bar{\xi}^2} A_{10}; \tag{4.48}$$
$$\int_0^{2\pi}\int_{-1}^{1} \left(\overline{\xi\xi} \frac{\partial T^{(0)}}{\partial \xi} - \xi T^{(0)} \right) \chi_1^{-1} d\eta d\varphi = \frac{4\pi}{3\bar{\xi}} A_{11};$$

and, finally,

$$\frac{\langle \mathbf{q} \rangle}{\lambda_0} = -\mathbf{G} + \frac{4\pi d^2}{3a^3} \mathrm{Re}\left(A_{10}\mathbf{e}_3 - 2A_{11}\mathbf{e}_1 \right). \tag{4.49}$$

In the limiting case where a spheroid degenerates into a sphere, Eq. (4.49) is expected to reduce to (2.47). It is pertinent to mention here that (4.49) can be *derived* directly from Eq. (2.47)—and this derivation is much easier as compared with that performed above. First, we recognize that the overall structure of Eq. (4.49) is pre-determined by (1.9), so we only need to find the dipole moments entering this formula linearly. Next, (4.5) provides the necessary relationship between the moments of spherical and spheroidal multipole fields. For $t = 1$, the scaling factor equals $(-1)^s d^2/3$: as would be expected, by introducing this factor into (2.47) we immediately get (4.49).

Noteworthy, the effective conductivity of composite with spheroidal inclusions depends not only on the volume fraction and conductivity of disperse phase but also on the shape, arrangement and orientation of inclusions. Even for simplest cubic symmetry case, this composite is anisotropic at macro level [35]. The infinitely thin oblate spheroid can be used to model penny-shape crack or superconducting platelet. Overall conductivity of solid is greatly affected by presence of these inhomogeneities although their volume content is equal to zero [40].

4.6 Elasticity Problem

4.6.1 Vectorial Partial Solutions of Lame Equation

The following regular vectorial partial solutions $\mathbf{f}_{ts}^{(i)}$ have been introduced by [33, 34]:

$$
\begin{aligned}
\mathbf{f}_{ts}^{(1)} &= \mathbf{e}_1 f_{t-1}^{s-1} - \mathbf{e}_2 f_{t-1}^{s+1} + \mathbf{e}_3 f_{t-1}^{s}; \qquad (4.50)\\
\mathbf{f}_{ts}^{(2)} &= \frac{1}{(t+1)}\left[\mathbf{e}_1 (t-s+1) f_t^{s-1} + (t+s+1)\mathbf{e}_2 f_t^{s+1} - \mathbf{e}_3 s f_t^s\right];\\
\mathbf{f}_{ts}^{(3)} &= \mathbf{e}_1\left[-(x_1 - \mathrm{i}x_2)\mathrm{D}_2 f_{t+1}^{s-1} - (\xi_0^2-1)\mathrm{D}_1 f_t^s + (t-s+1)(t-s+2)\beta_t f_{t+1}^{s-1}\right]\\
&\quad + \mathbf{e}_2\left[(x_1 + \mathrm{i}x_2)\mathrm{D}_1 f_{t+1}^{s+1} - (\xi_0^2-1)\mathrm{D}_2 f_t^s + (t+s+1)(t+s+2)\beta_t f_{t+1}^{s+1}\right]\\
&\quad + \mathbf{e}_3\left[x_3 \mathrm{D}_3 f_{t+1}^s - \xi_0^2 \mathrm{D}_3 f_t^s - (t-s+1)(t+s+1)\beta_t f_{t+1}^s\right].
\end{aligned}
$$

In (4.50), $\mathbf{e}_i$ are the complex Cartesian vectors (2.11) and D_i are the differential operators (2.10). The irregular vectorial partial solutions $\mathbf{F}_{ts}^{(i)}$ are obtained from (4.50) by replacing the index$t$ with $-(t+1)$ and f_t^s with F_t^s. In (4.50), $\mathbf{f}_{ts}^{(1)}$ and $\mathbf{f}_{ts}^{(2)}$ are the vectorial harmonic functions whereas $\mathbf{f}_{ts}^{(3)}$ is the vectorial biharmonic function. It contains the harmonic term $\xi_0^2 \nabla f_t^s$ added in spirit of [70] to simplify the $\mathbf{f}_{ts}^{(3)}$ and $\mathbf{T}_\xi(\mathbf{f}_{ts}^{(3)})$ expressions at the interface $\xi = \xi_0$.

The vectorial harmonics (3.1) do not apply to spheroid; instead we will use the vectorial surface harmonics in the form $\mathbf{C}_{ts}^{(j)}(\eta, \varphi) = \mathbf{e}_j \chi_t^{s_j}(\eta, \varphi)$, where $s_1 = s-1$, $s_2 = s+1$ and $s_3 = s$ [43]. At the spheroidal surface $\xi = \xi_0$, the functions $\mathbf{f}_{ts}^{(i)}$ can be written as

$$
\begin{aligned}
\mathbf{f}_{ts}^{(1)} &= P_{t-1}^{-s+1}\mathbf{C}_{t-1,s}^{(1)} - P_{t-1}^{-s-1}\mathbf{C}_{t-1,s}^{(2)} + P_{t-1}^{-s}\mathbf{C}_{t-1,s}^{(3)} \qquad (4.51)\\
\mathbf{f}_{ts}^{(2)} &= \frac{1}{(t+1)}\left[(t-s+1)P_t^{-s+1}\mathbf{C}_{ts}^{(1)} + (t+s+1)P_t^{-s-1}\mathbf{C}_{ts}^{(2)} - sP_t^{-s}\mathbf{C}_{ts}^{(3)}\right];
\end{aligned}
$$

$$\mathbf{f}_{ts}^{(3)} = \left\{(t-s+2)\xi_0 P_t^{-s+1} + (t-s+2)\left[-1+(t-s+1)\beta_t\right]P_{t+1}^{-s+1}\right\}\mathbf{C}_{t+1,s}^{(1)}$$
$$-\left\{(t-s)\xi_0 P_t^{-s-1} + (t+s+2)\left[-1+(t+s+1)\beta_t\right]P_{t+1}^{-s-1}\right\}\mathbf{C}_{t+1,s}^{(2)}$$
$$+\left\{(t-s+1)\xi_0 P_t^{-s} - C_{ts}P_{t+1}^{-s}\right\}\mathbf{C}_{t+1,s}^{(3)}$$

where $P_t^s = P_t^s(\xi_0)$, $\mathbf{C}_{ts}^{(j)} = \mathbf{C}_{ts}^{(j)}(\eta,\varphi)$ and C_{ts} is defined in (3.31). In compact form,

$$\mathbf{f}_{ts}^{(i)}\Big|_S = \sum_{j=1}^{3} UM_{ts}^{ji}(\xi_0,\nu)\,\mathbf{C}_{t+j-2,s}^{(j)}. \tag{4.52}$$

Analogously, for the irregular solutions we have

$$\mathbf{F}_{ts}^{(i)}\Big|_S = \sum_{j=1}^{3} UG_{ts}^{ji}\,(\xi_0,\nu)\,\mathbf{C}_{t-j+2,s}^{(j)}. \tag{4.53}$$

At the spheroidal surface $\xi = const$, the traction vector $\mathbf{T}_n = \sigma\cdot\mathbf{n}$ can be written as

$$\frac{1}{2\mu}\mathbf{T}_\xi(\mathbf{u}) = \frac{\nu}{1-2\nu}\mathbf{e}_\xi\,(\nabla\cdot\mathbf{u}) + \frac{\bar{\xi}\eta}{d}\frac{\partial}{\partial\xi}\mathbf{u} + \frac{1}{2}\mathbf{e}_\xi\times(\nabla\times\mathbf{u})\,. \tag{4.54}$$

For the vectorial partial solutions (4.50), this yields

$$\frac{1}{2\mu}\mathbf{T}_\xi\left(\mathbf{f}_{ts}^{(i)}\right)\Big|_S = \sum\nolimits_{j=1}^{3} TM_{ts}^{ji}(\xi_0,\nu)\,\mathbf{C}_{t+j-2,s}^{(j)}; \tag{4.55}$$
$$\frac{1}{2\mu}\mathbf{T}_\xi\left(\mathbf{F}_{ts}^{(i)}\right)\Big|_S = \sum\nolimits_{j=1}^{3} TG_{ts}^{ji}(\xi_0,\nu)\,\mathbf{C}_{t-j+2,s}^{(j)}.$$

For the explicit expressions of TM_{ts}^{ji} and TG_{ts}^{ji}, see [34, 37].

The net force vector $\mathbf{T}$ is non-zero for the functions $\mathbf{F}_{1s}^{(3)}$ only:

$$\mathbf{T}\left(\mathbf{F}_{10}^{(3)}\right) = 16\mu\pi d(\nu-1)\mathbf{e}_3;\ \mathbf{T}\left(\mathbf{F}_{11}^{(3)}\right) = -\overline{\mathbf{T}\left(\mathbf{F}_{1,-1}^{(3)}\right)} = -32\mu\pi d(\nu-1)\mathbf{e}_1 \tag{4.56}$$

Three functions with non-zero net moment $\mathbf{M}$ are

$$\mathbf{M}\left(\mathbf{F}_{10}^{(2)}\right) = -\frac{2\mu\pi}{3}d^2\mathbf{e}_3;\ \mathbf{M}\left(\mathbf{F}_{11}^{(2)}\right) = \overline{\mathbf{M}\left(\mathbf{F}_{1,-1}^{(2)}\right)} = \frac{4\mu\pi}{3}d^2\mathbf{e}_1. \tag{4.57}$$

Hence, the singular solutions $\mathbf{S}_{1s}^{(3)}$ and $\mathbf{S}_{1s}^{(3)}$ enter the series expansion only if the total force and torque, respectively, is non-zero.

4.6.2 Single Inclusion Problem

The problem statement is analogous to that in Sect. 3.2, with $\mathbf{T}_\xi$ instead of $\mathbf{T}_r$ in the interface conditions (3.12). The far load in the form of displacement field $\mathbf{u}_{far}$ and the displacement in the inclusion $\mathbf{u}^{(1)}$ are given by the series (3.13) and (3.15), respectively, where the regular solutions $\mathbf{u}_{ts}^{(i)}$ (3.6) are replaced with their spheroidal counterparts, $\mathbf{f}_{ts}^{(i)}$. In the case of linear $\mathbf{u}_{far} = \mathbf{E}\cdot\mathbf{r}$, the non-zero expansion coefficients $c_{ts}^{(i)}$ are

$$c_{00}^{(3)} = \frac{d}{3\gamma_0(\nu_0)}(E_{11}+E_{22}+E_{33}), \quad c_{20}^{(1)} = \frac{d}{3}(2E_{33}-E_{11}-E_{22}), \tag{4.58}$$
$$c_{21}^{(1)} = -\overline{c_{2,-1}^{(1)}} = d\,(E_{13}-iE_{23}), \quad c_{22}^{(1)} = \overline{c_{2,-2}^{(1)}} = d\,(E_{11}-E_{22}-2iE_{12}).$$

In the matrix domain, $\mathbf{u}^{(0)} = \mathbf{u}_{far} + \mathbf{u}_{dis}$, where the disturbance field $\mathbf{u}_{dis}(\mathbf{r})$ is written as the multipole expansion series

$$\mathbf{u}_{dis}(\mathbf{r}) = \sum_{i,t,s} A_{ts}^{(i)}\mathbf{F}_{ts}^{(i)}(\mathbf{r},d). \tag{4.59}$$

We substitute (4.59), together with (3.13) and (3.15), into the first of conditions (3.12). By applying (4.52) and (4.53), we get

$$\sum_{i,t,s} c_{ts}^{(i)} \sum_{j=1}^{3} UM_{ts}^{ji}(\xi_0,\nu_0)\,\mathbf{C}_{t+j-2,s}^{(j)} \tag{4.60}$$
$$+\sum_{i,t,s} A_{ts}^{(i)} \sum_{j=1}^{3} UG_{ts}^{ji}(\xi_0,\nu_0)\,\mathbf{C}_{t-j+2,s}^{(j)} = \sum_{i,t,s} d_{ts}^{(i)} \sum_{j=1}^{3} UM_{ts}^{ji}(\xi_0,\nu_1)\,\mathbf{C}_{t+j-2,s}^{(j)}.$$

The orthogonality property of the complex-valued vectors $\mathbf{C}_{ts}^{(j)}$ enables reducing the vectorial functional equality (4.60) to an infinite set of linear algebraic equations. Convenient for the computer algebra form of this system is

$$\mathbf{UG}_t(\xi_0,\nu_0)\mathbf{A}_t + \mathbf{UM}_t(\xi_0,\nu_0)\mathbf{c}_t = \mathbf{UM}_t(\xi_0,\nu_1)\mathbf{d}_t \quad (t = 1,2,\ldots); \tag{4.61}$$

where the vector $\mathbf{A}_t$ contains the unknowns $A_{t+i-2,s}^{(i)}$, the vectors $\mathbf{a}_t$ and $\mathbf{d}_t$ include $a_{t-i+2,s}^{(i)}$ and $d_{t-i+2,s}^{(i)}$, respectively. Obtaining the second set of equations from the traction vector continuity condition

$$\mathbf{TG}_t(\xi_0,\nu_0)\mathbf{A}_t + \mathbf{TM}_t(\xi_0,\nu_0)\mathbf{c}_t = \mathbf{TM}_t(\xi_0,\nu_1)\mathbf{d}_t. \tag{4.62}$$

follows the same way (for more details, see [34]).

The solution we obtain is complete and valid for any non-uniform far field. For the polynomial far field of order $t_{\max}$, this solution is exact and *conservative*, i.e., is given by the finite number of terms with $t \leq t_{\max}$. This result is consistent with [28]; in particular, $t_{\max} = 1$ gives us a complete solution of the Eshelby problem. And, to complete this section, we note that the derived solution is also valid in the limiting case of infinitely thin oblate spheroid often used to model the penny-shaped cracks. The normal opening mode stress intensity factor (SIF) K_I is defined by

$$K_I = \lim_{r \to 0} \sqrt{2\pi r}\sigma_{33} = \sqrt{\pi d} \lim_{\xi \to 1} \overline{\xi}\sigma_{33}, \tag{4.63}$$

where r is the distance from the point in the plane $z = 0$ outside the crack to the crack's tip. An asymptotic analysis of the stress field near the crack's tip gives a series expansion of SIF:

$$\begin{aligned}\sqrt{\frac{d}{\pi}} \frac{K_I(\varphi)}{2\mu} = \sum_{t=0}^{\infty} \sum_{s=-t}^{t}{}' (-1)^{\frac{(t+s)}{2}} \Bigg\{ A_{ts}^{(1)} + \frac{\mathrm{i}s}{(t+1)} A_{t+1,s}^{(2)} \\ + \left[\frac{4(1-\nu)}{t(2t-1)} - (1-2\nu) \right] A_{ts}^{(3)} \Bigg\} \exp(\mathrm{i}s\varphi),\end{aligned} \tag{4.64}$$

where the prime over the internal sum means that it contains only the terms with $(t+s)$ even. For the details of derivation, see [38].

4.6.3 Re-expansion Formulas for the Lame Solutions

Translation. Let $\mathbf{r}_1 = \mathbf{R} + \mathbf{r}_2$ Then,

$$\mathbf{F}_{ts}^{(i)}(\mathbf{r}_1) = \sum_{j=1}^{3} \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \eta_{tksl}^{(i)(j)}(\mathbf{R}, d_1, d_2)\; \mathbf{f}_{kl}^{(j)}(\mathbf{r}_2,\, d_2), \tag{4.65}$$

where

$$\begin{aligned}
&\eta_{tksl}^{(1)(2)} = \eta_{tksl}^{(1)(3)} = \eta_{tksl}^{(2)(3)} = 0; \quad \eta_{tksl}^{(i)(i)} = \eta_{t+2-i,k-2+i}^{s-l}; \\
&\eta_{tksl}^{(2)(1)} = \left(\frac{s}{t} + \frac{l}{k} \right) \eta_{t,k-1}^{s-l} \quad \eta_{tksl}^{(3)(2)} = 2\left(\frac{s}{t} + \frac{l}{k} \right) \eta_{t-1,k}^{s-l}; \\
&\eta_{tksl}^{(3)(1)} = \left\{ 2\frac{l}{k} \left[\frac{s}{t} + \frac{l}{(k-1)} \right] + C_{k-2,l} - C_{-(t+1),s} \right\} \eta_{t-1,k-1}^{s-l} + (2k-1) \\
&\quad \times \sum_{m=0}^{\infty} \left[(-1)^m \frac{X_3}{d_2} \eta_{t-1,k+2m}^{s-l} - d_1(\xi_{10})^2 \eta_{t,k+2m}^{s-l} + d_1(\xi_{20})^2 \eta_{t-1,k+2m+1}^{s-l} \right].
\end{aligned} \tag{4.66}$$

In (4.66) η_{tk}^{s-l} are the expansion coefficients for the scalar spheroidal harmonics F_t^s (4.13). These, as well as the other (R2R and S2S) expansions have been derived by [33].

Rotation. The re-expansion formulas for the partial irregular solutions $\mathbf{F}_{ts}^{(i)}$ due to rotation of coordinate frame $\mathbf{r}_1 = \mathbf{O}_{12} \cdot \mathbf{r}_2$ are:

$$\mathbf{F}_{ts}^{(i)}(\mathbf{r}_1, d_1) = \sum_{j=1}^{3} \sum_{k=0}^{t+i-j} \sum_{l=-k}^{k} R_{tksl}^{(i)(j)}(\mathbf{w}_{12}, d_1, d_2)\ \mathbf{F}_{kl}^{(j)}(\mathbf{r}_2, d_2), \tag{4.67}$$

where

$$\begin{aligned} R_{tksl}^{(i)(j)} &= \sum_{\alpha=j}^{i} \sum_{p=k+j-\alpha}^{t+i-\alpha} K_{t,\,p+\alpha-i,\,s}^{(1)(i)(\alpha)}(d_1)\ K_{p,\,k+j-\alpha,\,l}^{(2)(\alpha)(j)}(d_2)\ S_{2p}^{p-s,p-l}(\mathbf{w}_{12}) \\ &+ \delta_{i3}\delta_{j1}(2k-1) \sum_{n=k}^{t}{}' \left[(\xi_{10})^2 R_{tnsl}^{(3)(3)} - (\xi_{20})^2 R_{tnsl}^{(2)(2)} \right] \end{aligned} \tag{4.68}$$

and

$$\begin{aligned} &K_{tks}^{(\beta)(i)(j)} = 0 \quad \text{for} \quad i < j; \quad K_{tks}^{(\beta)(i)(i)} = K_{t-i+2,k-i+2}^{(\beta)}; \\ &K_{tks}^{(\beta)(2)(1)} = \left(\frac{s}{k} - \frac{s}{t}\right) K_{tk}^{(\beta)}, \quad K_{tks}^{(\beta)(3)(2)} = 4(1-\nu)\left(\frac{s}{k} - \frac{s}{t}\right) K_{t-1,k-1}^{(\beta)}; \\ &K_{tks}^{(\beta)(3)(1)} = \left[C_{-(k+1),s} - C_{-(t+1),s}\right] K_{t-1,k-1}^{(\beta)} - \frac{s}{(k-1)} K_{tks}^{(\beta)(3)(2)}. \end{aligned} \tag{4.69}$$

Here, $K_{tk}^{(\beta)}$ are given by the formulas (4.6), $\beta = 1, 2$. In (4.67), we keep in mind that the vectorial functions standing in the opposite sides of equality are written in their local coordinates and components. These formulas are written for the most general case $d_1 \neq d_2$ and $\xi_{10} \neq \xi_{20}$. The analogous formulas for the regular solutions are exact and finite: for their explicit form, see [34, 39].

4.6.4 FCM and RUC

Now, we have in hands all the necessary theory to solve the elasticity problems for multiple spheroidal inclusions. We take the displacement field in the matrix domain in the form (3.32), where now multipole expansion of the disturbance caused by p-th inclusion is written as

$$\mathbf{u}_{dis}^{(p)}(\mathbf{r}_p) = \sum_{i,t,s} A_{ts}^{(i)(p)} \mathbf{F}_{ts}^{(i)}(\mathbf{r}_p, d_p). \tag{4.70}$$

Fulfilling the interface conditions gives us the linear set of equations similar to (4.61), (4.62)

$$\mathbf{UG}_t(\xi_{q0},\nu_0)\mathbf{A}_t^{(q)}+\mathbf{UM}_t(\xi_{q0},\nu_0)\left(\mathbf{a}_t^{(q)}+\mathbf{c}_t^{(q)}\right)=\mathbf{UM}_t(\xi_{q0},\nu_q)\mathbf{d}_t^{(q)};\quad (4.71)$$
$$\mathbf{TG}_t(\xi_{q0},\nu_0)\mathbf{A}_t^{(q)}+\mathbf{TM}_t(\xi_{q0},\nu_0)\left(\mathbf{a}_t^{(q)}+\mathbf{c}_t^{(q)}\right)=\mathbf{TM}_t(\xi_{q0},\nu_q)\mathbf{d}_t^{(q)}.$$

Here, $\mathbf{c}_t$ is given by (4.58) and $\mathbf{a}_t^{(q)}$ is the contribution of all other inclusions (with $p\neq q$) to the field around the qth inclusion:

$$\mathbf{a}_t^{(q)}=\sum_{p\neq q}\sum_{k=1}^{\infty}\eta_{kt}^{(p)(q)}\mathbf{A}_k^{(p)},\quad (4.72)$$

in scalar form,

$$a_{ts}^{(i)(q)}=\sum_{p\neq q}^{N}\sum_{j=1}^{3}\sum_{k=0}^{\infty}\sum_{l=-k}^{k}A_{kl}^{(j)(p)}\eta_{klts}^{(j)(i)}\left(\mathbf{R}_{pq},d_p,d_q\right).\quad (4.73)$$

Here, we assumed all the spheroids equally oriented: $\mathbf{O}_{pq}=\mathbf{I}$. The case of arbitrarily oriented spheroids is considered elsewhere [39]. By analogy with (3.39)–(3.41), one can use this model to evaluate the effective stiffness of composite in spirit of Maxwell's method.

Extension of this solution to the periodic, RUC type model is analogous to that for the spherical inclusions and so we do not repeat it here. In fact, it mostly consists in replacing the expansion coefficients (4.66) with their periodic counterparts in the matrix of linear system (4.71).

4.6.5 Effective Stiffness Tensor

The effective stiffness tensor $\mathbf{C}^*$ of composite is given by (1.10). In view of the displacement field periodicity, averaging the strain in composite with spheroidal inclusions follows the same way as in the spherical case and yields $\langle\varepsilon\rangle=\mathbf{E}$. The macroscopic stress $\langle\sigma\rangle$ is given by Eq. (1.13) where integration is made now over the spheroidal surfaces $S_q:\xi_q=\xi_{q0}$. The integration procedure closely resembles that exposed in Sect. 3 and is straightforward although somewhat laborious.

An alternate approach to obtaining the expressions for effective properties was discussed already in this Section, in the conductivity context. This approach extends to the elasticity problem: namely, an expression for the macroscopic stress in spherical particle composite (3.51) transforms to the analogous formula for the *aligned* spheroidal particle composite by taking into account that for $d\to 0$

$$\mathbf{V}_{00}^{(1)}(\mathbf{r},d)\approx N_0\mathbf{U}_{00}^{(1)}(\mathbf{r});\quad \mathbf{V}_{2,s}^{(3)}(\mathbf{r},d)\approx N_s\mathbf{U}_{2,s}^{(3)}(\mathbf{r});$$

where the scaling factor $N_s = (-1)^s d^2/3$. By inserting this factor into (3.51) we obtain immediately the desired formulas:

$$\frac{\langle\sigma_{11}\rangle + \langle\sigma_{22}\rangle + \langle\sigma_{33}\rangle}{3k_0} = (E_{11} + E_{22} + E_{33}) + \left(1 + \frac{4\mu_0}{3k_0}\right) \sum_{q=1}^{N} \widetilde{d}_p A_{00}^{(1)(q)}; \tag{4.74}$$

$$\frac{2\langle\sigma_{33}\rangle - \langle\sigma_{11}\rangle - \langle\sigma_{22}\rangle}{2\mu_0} = (2E_{33} - E_{11} - E_{22}) - 4(1-\nu_0) \sum_{q=1}^{N} \widetilde{d}_p A_{20}^{(3)(q)};$$

$$\frac{\langle\sigma_{11}\rangle - \langle\sigma_{22}\rangle - 2\mathrm{i}\langle\sigma_{12}\rangle}{2\mu_0} = (E_{11} - E_{22} - 2\mathrm{i}E_{12}) - 8(1-\nu_0) \sum_{q=1}^{N} \widetilde{d}_p A_{22}^{(3)(q)};$$

$$\frac{\langle\sigma_{13}\rangle - \mathrm{i}\langle\sigma_{23}\rangle}{2\mu_0} = (E_{13} - \mathrm{i}E_{23}) + 2(1-\nu_0) \sum_{q=1}^{N} \widetilde{d}_p A_{21}^{(3)(q)}.$$

where $\widetilde{d}_p = 4\pi \left(d_p\right)^2 / 3a^2$ and $k = \frac{2\mu(1+\nu)}{3(1-2\nu)}$.

5 Spherical Particles Reinforced Composite with Transversely-Isotropic Phases

In the Cartesian coordinate system $Ox_1x_2x_3$ with Ox_3 axis being the anisotropy axis of transversely isotropic material, an explicit form of the generalized Hooke's law is

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12}) \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}. \tag{5.1}$$

Here, two-indices notation C_{ij} is adopted. The components of stress tensor σ satisfy the equilibrium equations $\nabla \cdot \sigma = 0$ and the small elastic strain tensor ε is related to the displacement vector $\mathbf{u}$ by $\varepsilon = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$.

5.1 Background Theory

Based on representation of general solution in terms of three potential functions [70], Kushch [43] has introduced the following, full for $\nu_1 \neq \nu_2$ set of regular vectorial solutions $\mathbf{v}_{ts}^{(j)}$:

$$\mathbf{v}_{ts}^{(j)}(\mathbf{r}) = f_t^{s-1}(\mathbf{r}_j, d_j)\mathbf{e}_1 - f_t^{s+1}(\mathbf{r}_j, d_j)\mathbf{e}_2 + \frac{k_j}{\sqrt{\nu_j}} f_t^s(\mathbf{r}_j, d_j)\mathbf{e}_3 \quad (j = 1, 2);$$
$$\mathbf{v}_{ts}^{(3)}(\mathbf{r}) = f_t^{s-1}(\mathbf{r}_3, d_3)\mathbf{e}_1 + f_t^{s+1}(\mathbf{r}_3, d_3)\mathbf{e}_2; \quad t = 0, 1, 2, \ldots; \quad |s| \le t + 1. \tag{5.2}$$

In (5.2), $\nu_3 = 2C_{44}/(C_{11} - C_{12})$ whereas ν_1 and ν_2 are the roots of equation

$$C_{11}C_{44}\nu^2 - \left[(C_{44})^2 - C_{11}C_{33} - (C_{13} + C_{44})^2\right]\nu + C_{33}C_{44} = 0. \tag{5.3}$$

Expressions of k_1 and k_2 are

$$k_j = \frac{C_{11}\nu_j - C_{44}}{C_{13} + C_{44}} = \frac{\nu_j\,(C_{13} + C_{44})}{C_{33} - \nu_j C_{44}}, \ j = 1, 2\,. \tag{5.4}$$

The modified Cartesian $\mathbf{r}_j$ and prolate spheroidal coordinates $\left(\xi_j, \eta_j, \varphi_j\right)$ are given for $\nu_j < 1$ by

$$x_1 + ix_2 = x_{1j} + \mathrm{i}x_{2j} = d_j\bar{\xi}_j\bar{\eta}_j \ \exp(i\varphi_j), \quad x_3 = \sqrt{\nu_j}x_{3j} = \sqrt{\nu_j}d_j\xi_j\eta_j; \tag{5.5}$$

In the case $\nu_j > 1$, the oblate spheroidal coordinates must be used. The system (5.5) is chosen in a way that $\xi_j = \xi_{j0} = const$ at the spherical surface $r = R$; i.e., S is the ξ-coordinate surface of (5.5). We provide this by defining

$$d_j = R/\xi_{j0}, \quad \xi_{j0} = \sqrt{\nu_j/\left|\nu_j - 1\right|}. \tag{5.6}$$

In this case, moreover, we have $\eta_j = \theta$ and $\varphi_j = \varphi$ for $r = R$, where (r, θ, φ) are the ordinary spherical coordinates corresponding to the Cartesian ones (x_1, x_2, x_3).

This is the key point: no matter how complicated solution in the bulk is, at the interface we get the linear combination of regular spherical harmonics $Y_t^s(\theta, \varphi)$. Under this circumstance, fulfilling the contact conditions at interface is straightforward.

The explicit form of the irregular vectorial solutions $\mathbf{V}_{ts}^{(j)}$ is given by the Eq. (5.2), with the replace f_t^s by F_t^s. According to [23], $F_t^s = f_t^s \equiv 0$ for $|s| > t$. This condition makes it impossible to represent some irregular solutions in the form (5.2). To resolve this issue, Kushch [43] has introduced the following, additional to (4.4) functions of the form

$$F_t^{t+k}(\mathbf{r}, d) = \frac{1}{(2t + k)!} Q_t^{t+k}(\xi)\, P_t^{t+k}(\eta) \exp\left[i\,(t + k)\,\varphi\right], \tag{5.7}$$

$(k = 0, 1, 2, \ldots)$ where

$$P_t^{t+k}(p) = \frac{(2t+k)!}{(1-p^2)^{(t+k)/2}} \underbrace{\int_p^1 \int_p^1 \cdots \int_p^1}_{t+k} P_t(p)\,(dp)^{t+k}$$
$$= \frac{(2t+k)!}{(1-p^2)^{(t+k)/2}} I_{t+k} \tag{5.8}$$

for $0 \leq p \leq 1$; for $p < 0$, $P_t^{t+k}(p) = (-1)^k P_t^{t+k}(-p)$. It is straightforward to show that the functions (5.7) are the irregular solutions of Laplace equation. In contrast to (4.4), they are discontinuous at $x_3 = 0$. In the general series solution, however, these breaks cancel each other and give the continuous and differentiable expressions of the displacement and stress fields. Remarkably, the functions F_t^{t+k} are introduced in a way that for them are valid all the principal results including the re-expansion formulas (4.13) and (4.15).

At the spherical surface $r = R$, the functions $\mathbf{V}_{ts}^{(i)}$ and $\mathbf{v}_{ts}^{(i)}$ can be written in the compact form as

$$\mathbf{V}_{ts}^{(i)}\Big|_S = \sum_{j=1}^{3} UG_{ts}^{ji}\,\mathbf{C}_{ts}^{(j)}, \quad \mathbf{v}_{ts}^{(i)}\Big|_S = \sum_{j=1}^{3} UM_{ts}^{ji}\,\mathbf{C}_{ts}^{(j)}, \tag{5.9}$$

where $\mathbf{C}_{ts}^{(j)}(\theta, \varphi)$ is defined in (4.51). In (5.9), the matrix

$$\mathbf{UG}_{ts} = \left\{UG_{ts}^{ij}\right\} = \begin{pmatrix} Q_t^{s-1}(\xi_{10}) & Q_t^{s-1}(\xi_{20}) & Q_t^{s-1}(\xi_{30}) \\ -Q_t^{s+1}(\xi_{10}) & -Q_t^{s+1}(\xi_{20}) & Q_t^{s+1}(\xi_{30}) \\ \frac{k_1}{\sqrt{\nu_1}} Q_t^{s}(\xi_{10}) & \frac{k_2}{\sqrt{\nu_2}} Q_t^{s}(\xi_{20}) & 0 \end{pmatrix}; \tag{5.10}$$

$\mathbf{UM}_{ts}$ has the form (5.10), where Q_t^s are replaced with P_t^s.

To satisfy the stress boundary conditions, we need the similar representation for the traction vector $\mathbf{T_n} = \sigma \cdot \mathbf{n}$. For the explicit expressions of $\mathbf{T_n}(\mathbf{v}_{ts}^{(j)})$ at the surface S, see [43]. These and analogous expressions for $\mathbf{T_n}(\mathbf{V}_{ts}^{(j)})$ also can be written in the compact form

$$\mathbf{T_n}(\mathbf{V}_{ts}^{(j)})\Big|_S = \sum_{j=1}^{3} TG_{ts}^{ji}\,\mathbf{C}_{ts}^{(j)}, \quad \mathbf{T_n}(\mathbf{v}_{ts}^{(j)})\Big|_S = \sum_{j=1}^{3} TM_{ts}^{ji}\,\mathbf{C}_{ts}^{(j)}. \tag{5.11}$$

For the definiteness sake, we assume here and below $\nu_1 \neq \nu_2$. In the case of equal roots $\nu_1 = \nu_2$, solution (5.2) is not general because of $\mathbf{f}_{ts}^{(1)} \equiv \mathbf{f}_{ts}^{(2)}$. In this case, to get a complete set of independent solutions, $\mathbf{v}_{ts}^{(2)}$ can be taken in the form [43]

$$\mathbf{v}_{ts}^{(2)}(\mathbf{r}) = d_1\left(x_3\nabla - \frac{C_{13}+3C_{44}}{C_{13}+C_{44}}\mathbf{i}_3\right) f_t^s(\mathbf{r}_1, d_1) + \sqrt{\nu_1} d_1 \left(\xi_{10}\right)^2 \nabla f_{t-1}^s(\mathbf{r}_1,\ d_1). \tag{5.12}$$

For the expression of the corresponding traction vector, see [70].

5.2 Series Solution

Let us consider an infinite solid with a single spherical inclusion of radius R embedded. The matrix and inclusion are made from transversely isotropic materials and perfectly bonded:

$$\left(\mathbf{u}^+ - \mathbf{u}^-\right)\big|_S = 0; \quad \left(\mathbf{T}_\mathbf{n}\left(\mathbf{u}^+\right) - \mathbf{T}_\mathbf{n}\left(\mathbf{u}^-\right)\right)\big|_S = 0. \tag{5.13}$$

In this Section, all the parameters associated with the matrix and inclusion are denoted by the superscript "−" and "+", respectively. We assume the anisotropy axes of both the matrix and inclusion materials to be *arbitrarily* oriented and introduce the material-related Cartesian coordinate systems $Ox_1^- x_2^- x_3^-$ and $Ox_1^+ x_2^+ x_3^+$ with common origin in the center of inclusion. The point coordinates and the vector components in these coordinate systems are related by

$$x_i^+ = O_{ij} x_j^-, \quad u_i^+ = O_{ij} u_j^- \tag{5.14}$$

where $\mathbf{O}$ is the rotation matrix. Transformation of the vectors $\mathbf{i}_i$ uses the formula

$$\mathbf{i}_i^+ = \Omega_{ij}^* \mathbf{i}_j^-, \quad \text{where} \quad \mathbf{O}^* = \mathbf{D}^{-1}\mathbf{O}\mathbf{D} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.15}$$

The displacement field within the inclusion $\mathbf{u}^+$ can be expanded into a series over the regular solutions $\mathbf{v}_{ts}^{(j)}$ (5.2):

$$\mathbf{u}^+ = \sum_{j,t,s} d_{ts}^{(j)} \mathbf{v}_{ts}^{(j)}\left(\mathbf{r}^+\right), \left(\sum_{j,t,s} = \sum_{j=1}^{3}\sum_{t=0}^{\infty}\sum_{|s|\le t+1}\right) \tag{5.16}$$

where $d_{ts}^{(j)}$ are the expansion coefficients to be determined from the contact conditions (5.13). The displacement vector in the matrix domain is a sum of far field $\mathbf{u}_{far} = \mathbf{E}\cdot\mathbf{r}^-$ and disturbance $\mathbf{u}_{dis}$ whose multipole type expansion in terms of $\mathbf{V}_{ts}^{(j)}$ is given by

$$\mathbf{u}^- = \mathbf{u}_{far} + \mathbf{u}_{dis} = \mathbf{E}\cdot\mathbf{r}^- + \sum_{j,t,s} A_{ts}^{(j)} \mathbf{V}_{ts}^{(j)}\left(\mathbf{r}^-\right). \tag{5.17}$$

Expression (5.16) is ready for use; the similar expansion of $\mathbf{u}_{far}$ follows from (5.2). Specifically,

$$\mathbf{u}_{far} = \sum_{j,t,s} c_{ts}^{(j)} \mathbf{v}_{ts}^{(j)} \left(\mathbf{r}^{-}\right), \tag{5.18}$$

where

$$\begin{aligned}
c_{10}^{(1)} &= \frac{d_1^- \nu_1^-}{k_1^- \nu_2^- - k_2^- \nu_1^-} \left[E_{33}\nu_2^- + k_2^- \left(E_{11} + E_{22}\right)\right]; \\
c_{11}^{(1)} &= -\overline{c_{1,-1}^{(1)}} = \frac{d_1^- \sqrt{\nu_1^-}}{k_1^-} \left(E_{13} - iE_{23}\right); \\
c_{12}^{(1)} &= \overline{c_{1,-2}^{(1)}} = \left(E_{11} - E_{22} - 2iE_{12}\right); \\
c_{10}^{(2)} &= -\frac{\nu_2^- d_2^-}{k_1^- \nu_2^- - k_2^- \nu_1^-} \left[E_{33}\nu_1^- + k_1^- \left(E_{11} + E_{22}\right)\right]; \\
c_{11}^{(3)} &= \overline{c_{1,-1}^{(3)}} = \frac{\sqrt{\nu_3^-} d_3^-}{k_1^-} \left(1 - k_1^-\right) \left(E_{13} - iE_{23}\right);
\end{aligned} \tag{5.19}$$

all other coefficients $c_{ts}^{(i)}$ are equal to zero.

We note first that the functions $\mathbf{V}_{0s}^{(i)}$ are the fundamental solutions representing action of the point body forces in an infinite solid. Because no body forces is suggested in the problem statement, we get immediately $A_{0s}^{(i)} \equiv 0$. The remaining coefficients $A_{ts}^{(i)}$ and $D_{ts}^{(i)}$ will be determined from the interface boundary conditions (5.13). We use (5.9) to write

$$\mathbf{u}^{-} = \sum_{j,t,l} \left(\sum_{\alpha=1}^{3} UG_{tl}^{j\alpha-} A_{tl}^{(\alpha)} + UM_{tl}^{j\alpha-} e_{tl}^{(\alpha)} \right) \mathbf{C}_{tl}^{(j)} \left(\theta^{-}, \varphi^{-}\right) \tag{5.20}$$

and

$$\mathbf{u}^{+} = \sum_{i,t,s} \left(\sum_{\alpha=1}^{3} UM_{ts}^{i\alpha+} d_{ts}^{(\alpha)} \right) \mathbf{C}_{ts}^{(i)} \left(\theta^{+}, \varphi^{+}\right). \tag{5.21}$$

Note that $\mathbf{u}^{-}$ (5.20) and $\mathbf{u}^{+}$ (5.21) are still written in the different coordinate systems. Therefore, before substituting them into (5.13), $\mathbf{u}^{+}$ has to be expressed in terms of the variables θ^{-}, φ^{-} and vectors $\mathbf{e}_j^{-}$. For this purpose, we apply the Bateman's transformation formula of the surface spherical harmonics due to rotation of coordinate basis [3] and (5.15) to derive:

$$\mathbf{C}_{ts}^{(i)}(\theta^+,\varphi^+) = \sum_{j=1}^{3} \Omega_{ij}^* \sum_{|l_j|\le t+1} \frac{(t+l_j)!}{(t+s_i)!} S_{2t}^{t-s_i,t-l_j}(\mathbf{w})\,\mathbf{C}_{tl}^{(j)}\left(\theta^-,\varphi^-\right), \tag{5.22}$$

where S_{2t}^{sl} are the spherical harmonics in four-dimensional space and $\mathbf{w}$ is the vector of Euler's parameters related to the rotation matrix $\mathbf{O}$ by (4.27). By substitution (5.22) into (5.21), we get

$$\mathbf{u}^+ = \sum_{j,t,l} \left[\sum_{i=1}^{3} O_{ij}^* \sum_{|s|\le t+1} \frac{(t+l_j)!}{(t+s_i)!} S_{2t}^{t-s_i,t-l_j}(\mathbf{w}) \sum_{\alpha=1}^{3} UM_{ts}^{i\alpha+} d_{ts}^{(\alpha)} \right] \mathbf{C}_{tl}^{(j)}\left(\theta^-,\varphi^-\right). \tag{5.23}$$

Now, we put $\mathbf{u}^-$ (5.20) and transformed expression of $\mathbf{u}^+$ (5.23) into the first of conditions (5.13) and make use of the orthogonality property of spherical harmonics χ_t^s on the surface S to decompose vectorial functional equality $\mathbf{u}^+ = \mathbf{u}^-$ into a set of linear algebraic equations. It is written in the matrix-vector form as

$$\mathbf{UG}_{tl}^- \cdot \mathbf{A}_{tl} + \mathbf{UM}_{tl}^- \cdot \mathbf{e}_{tl} = \sum_{|s|\le t+1} \mathbf{UM}_{tsl}^* \cdot \mathbf{d}_{ts}, \tag{5.24}$$
$$t = 0, 1, 2, \dots;\ |l| \le t+1;$$

where

$$\mathbf{UM}_{tsl}^* = \mathbf{W}_{tsl}\mathbf{UM}_{ts}^+, \quad W_{tsl}^{ji} = O_{ij}^* \frac{(t+l_j)!}{(t+s_i)!} S_{2t}^{t-s_i,t-l_j}(\mathbf{w}), \tag{5.25}$$
$$\mathbf{A}_{tl} = \left\{A_{tl}^{(i)}\right\}^T,\ \mathbf{d}_{tl} = \left\{d_{tl}^{(i)}\right\}^T,\ \mathbf{e}_{tl} = \left\{e_{tl}^{(i)}\right\}^T.$$

Obtaining the second set of equations follows the same procedure where, instead of (5.9), the representation (5.11) of the normal traction vectors $\mathbf{T_n}(\mathbf{v}_{ts}^{(j)})$ and $\mathbf{T_n}(\mathbf{V}_{ts}^{(j)})$ on the surface $r = R$ should be used. After transformations, we obtain

$$\mathbf{TG}_{tl}^- \cdot \mathbf{A}_{tl} + \mathbf{TM}_{tl}^- \cdot \mathbf{e}_{tl} = \sum_{|s|\le t+1} \mathbf{TM}_{tsl}^* \cdot \mathbf{d}_{ts}, \tag{5.26}$$

where $\mathbf{TM}_{tsl}^* = \mathbf{W}_{tsl}\mathbf{TM}_{ts}^+$. Form of the matrices $\mathbf{TG}_{tl}$ and $\mathbf{TM}_{tl}$ is clear form (5.11). The Eqs. (5.24) and (5.26) together form a complete set of linear equations from where $A_{ts}^{(i)}$ and $d_{ts}^{(i)}$ can be determined. For more details, see [43].

5.3 FCM and RUC

Let us consider now an unbounded domain containing N non-touching spherical particles of radius R_q with the centres located in the points $O_q,\ q = 1, 2, \dots, N$ and the elastic stiffness tensors $\mathbf{C}_q^+$. We introduce the local material-related coordinate systems $O_q x_{1q}^+ x_{2q}^+ x_{3q}^+$ which origin and orientation with respect to the global Cartesian coordinate system $Ox_1^- y^- z^-$ is defined by the vector $\mathbf{R}_q$ and the rotation matrix $\mathbf{O}_q$. The matrix-inclusion interface boundary conditions ($q = 1, 2, \ldots, N$) are

$$\left(\mathbf{u}_q^+ - \mathbf{u}^-\right)\Big|_{S_q} = 0; \quad \left(\mathbf{T_n}\left(\mathbf{u}_q^+\right) - \mathbf{T_n}\left(\mathbf{u}^-\right)\right)\Big|_{S_q} = 0; \tag{5.27}$$

the stress state of the heterogeneous solid is governed, as before, by the linear far displacement field.

In (5.27), $\mathbf{u}_q^+$ is the displacement vector in the volume of qth inclusion: by analogy with (5.16),

$$\mathbf{u}_q^+ = \sum_{j,t,s} d_{ts}^{(q)(j)} \mathbf{v}_{ts}^{(j)}\left(\mathbf{r}_q^+\right), \tag{5.28}$$

The displacement vector $\mathbf{u}^-$ in the matrix domain is written as a superposition of linear far field and the disturbance fields induced by each separate inclusion (3.32): In turn, the disturbance term $\mathbf{u}_{dis}^{(p)}(\mathbf{r})$ allows the multipole expansion in the form

$$\mathbf{u}_{dis}^{(p)}\left(\mathbf{r}_p^-\right) = \sum_{j,t,s} A_{ts}^{(p)(j)} \mathbf{V}_{ts}^{(j)}\left(\mathbf{r}_p^-\right), \tag{5.29}$$

where $A_{ts}^{(p)(j)}$ as well as $d_{ts}^{(q)(j)}$ in (5.28) are the unknown coefficients. The limiting behavior of $\mathbf{u}^-$ is $\mathbf{u}^- \to \mathbf{E} \cdot \mathbf{r}$ with $\|\mathbf{r}\| \to \infty$.

The separate terms of the superposition sum are written in the different coordinate systems. To enable application the procedure described in the previous Section, we need to express $\mathbf{u}^-$ in variables of the local coordinate system. This transform uses the re-expansion formulae for the irregular vectorial solutions $\mathbf{V}_{ts}^{(j)}$ due to translation of coordinate system:

$$\mathbf{V}_{ts}^{(j)}\left(\mathbf{r}_p^-\right) = \sum_{k=0}^{\infty} \sum_{|l| \le k+1} \eta_{tk,s-l}\left(\mathbf{R}_{pq}, d_{pj}^-, d_{qj}^-\right) \mathbf{v}_{kl}^{(j)}\left(\mathbf{r}_q^-\right), \tag{5.30}$$
$$t = 0, 1, 2, \ldots; \quad |s| \le t + 1;$$

The formulas (5.30) follow directly from the corresponding result for the scalar harmonic functions F_t^s (4.13). The explicit form of the coefficients η_{tks} is given by the formulas (4.16), (4.22) or (4.24).

We apply (5.30) to all the sum terms (5.29) but that one with $p = q$ written initially in the variables of this local coordinate system. After some algebra, we find

$$\mathbf{u}^-(\mathbf{r}_q^-) = \sum_{j,t,s}\left[A_{ts}^{(q)(j)} \mathbf{V}_{ts}^{(j)}\left(\mathbf{r}_q^-\right) + \left(a_{ts}^{(q)(j)} + c_{ts}^{(q)(j)}\right) \mathbf{v}_{ts}^{(j)}\left(\mathbf{r}^-\right)\right], \tag{5.31}$$

where

$$a_{ts}^{(q)(j)} = \sum_{k=0}^{\infty} \sum_{|l| \le k+1} \sum_{p=1 p\ne q}^{N} \eta_{kt,l-s} \left(\mathbf{R}_{pq}, d_{pj}^{-}, d_{qj}^{-} \right) A_{kl}^{(p)(j)} \tag{5.32}$$

and $c_{ts}^{(q)(j)}$ are the expansion coefficients of the linear part of $\mathbf{u}^{-}$ given by the formula (5.19), with replace d_j^- to $d_{qj}^- = R_q/\xi_{j0}^-$.

After the local expansion of $\mathbf{u}^-$ in the vicinity of the point O_q was found, the remaining part of solving procedure follows the described above way. In fact, by applying the re-expansion formula (5.30) we reduced the initial, multiple inclusion problem to a coupled set of N problems for a solid with a single inclusion in the non-uniform far field. The resulting *infinite* set of linear algebraic equations is

$$\begin{gathered}
\mathbf{UG}_{tl}^{(q)-} \cdot \mathbf{A}_{tl}^{(q)} + \mathbf{UM}_{tl}^{(q)-} \cdot \left(\mathbf{a}_{tl}^{(q)} + \mathbf{c}_{tl}^{(q)} \right) = \sum_{|s| \le t+1} \mathbf{UM}_{tsl}^{(q)*} \cdot \mathbf{d}_{ts}^{(q)}, \\
\mathbf{TG}_{tl}^{(q)-} \cdot \mathbf{A}_{tl}^{(q)} + \mathbf{TM}_{tl}^{(q)-} \cdot \left(\mathbf{a}_{tl}^{(q)} + \mathbf{c}_{tl}^{(q)} \right) = \sum_{|s| \le t+1} \mathbf{TM}_{tsl}^{(q)*} \cdot \mathbf{d}_{ts}^{(q)}, \\
q = 1, 2, \ldots, N; \quad t = 0, 1, 2, \ \ldots; \quad |l| \le t+1;
\end{gathered} \tag{5.33}$$

where $\mathbf{a}_{tl}^{(q)} = \left\{ a_{ts}^{(q)(j)}, a_{ts}^{(q)(j)}, a_{ts}^{(q)(j)} \right\}^T$ and $a_{ts}^{(q)(j)}$ are given by (5.32). Its approximate solution can be obtained by the truncation method, when the unknowns and equations with $t \le t_{\max}$ only are retained in the (5.33). The solution is convergent for $t_{\max} \to \infty$ provided that the non-touching conditions $\left\| \mathbf{R}_{pq} \right\| > R_p + R_q$ hold true for each pair of inclusions. Thus, (5.33) can be solved for $\mathbf{A}_{tl}^{(q)}$ and $\mathbf{d}_{ts}^{(q)}$ with any desirable accuracy by taking $t_{\max}$ sufficiently large.

The above formalism applies also to solution of the RUC model problem. To this end, it is sufficient to replace $\mathbf{V}_{ts}^{(j)}$ with its periodic counterpart

$$\mathbf{V}_{ts}^{*(j)}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{V}_{ts}^{(j)}(\mathbf{r} - a\mathbf{k}) \tag{5.34}$$

in (5.29) and $\mathbf{a}_{ts}^{(q)}$ (5.32)—with

$$\mathbf{a}_{ts}^{(q)} = \sum_{k=0}^{\infty} \sum_{|l| \le k+1} \sum_{p=1}^{N} \eta_{kt,l-s}^{*} \left(\mathbf{R}_{pq}, d_{pj}^{-}, d_{qj}^{-} \right) \mathbf{A}_{kl}^{(p)}, \tag{5.35}$$

where η_{kts}^* is the lattice sum (4.36). These sums appear in solution of the conductivity problem for a composite with transversely isotropic phases [35]. There, the convergence rate of the series (4.36) is discussed and the fast summation technique has been proposed. The only difference here is the extended variation range of indices s

and l. These series converge for all indices in the range $|s| \le t+1$ and $|l| \le k+1$ provided the inclusions do not overlap.

5.4 Effective Stiffness Tensor

The macroscopic elastic stiffness tensor is defined by (1.10) where we consider RUC a representative volume element of composite. In this case, integration of the local strain and stress fields can be done analytically. The components of the effective stiffness tensor are found as $\mathbf{C}^*_{ijkl} = \langle \sigma_{ij} \rangle$, where the stress σ is calculated for $\langle \varepsilon_{kl} \rangle = 1$, $\langle \varepsilon_{k'l'} \rangle = 0$ $(k \ne k', l \ne l')$. It has been shown in Chap. 5 that $\mathbf{E}$ has a meaning of macroscopic strain tensor. Hence, $\mathbf{C}^*_{ijkl} = \langle \sigma_{ij} \rangle \big|_{E_{mn}=\delta_{mk}\delta_{nl}}$.

To compute the macroscopic stress (1.12), we employ Eq. (1.13) valid for the arbitrary orientation of inclusions and general type anisotropy of constituents. Integration in (1.13) has to be done over the matrix volume only: with the local series expansions (5.31) taken into account, this task is straightforward. Moreover, it follows from the Betti's theorem that these integrals are equal to zero for all regular solutions $\mathbf{v}^{(j)}_{ts}$. Among the singular solutions $\mathbf{V}^{(j)}_{ts}$, only those with $t = 1$ contribute to $\langle \sigma_{ij} \rangle$. After some algebra analogous to that in Sect. 8.1, we get the *exact* explicit formulas [44]:

$$\langle \sigma_{11} \rangle = C^-_{1k} \langle \varepsilon_{kk} \rangle - C^-_{11} \left(\sqrt{\nu^-_1}\, \widetilde{A}^{(1)}_{10} + \sqrt{\nu^-_2}\, \widetilde{A}^{(2)}_{10} \right) + \frac{C^-_{44}}{2} \frac{(1+k^-_j)}{\sqrt{\nu^-_j}} \mathrm{Re} \widetilde{A}^{(j)}_{12}; \tag{5.36}$$

$$\langle \sigma_{22} \rangle = C^-_{2k} \langle \varepsilon_{kk} \rangle - C^-_{22} \left(\sqrt{\nu^-_1}\, \widetilde{A}^{(1)}_{10} + \sqrt{\nu^-_2}\, \widetilde{A}^{(2)}_{10} \right) - \frac{C^-_{44}}{2} \frac{(1+k^-_j)}{\sqrt{\nu^-_j}} \mathrm{Re} \widetilde{A}^{(j)}_{12};$$

$$\langle \sigma_{33} \rangle = C^-_{3k} \langle \varepsilon_{kk} \rangle - C^-_{33} \left(\frac{k^-_1}{\sqrt{\nu^-_1}} \widetilde{A}^{(1)}_{10} + \frac{k^-_2}{\sqrt{\nu^-_2}} \widetilde{A}^{(2)}_{10} \right);$$

$$\langle \sigma_{12} \rangle = C^-_{66} \langle \varepsilon_{12} \rangle + \frac{C^-_{44}}{2} \frac{(1+k^-_j)}{\sqrt{\nu^-_j}} \mathrm{Im} \widetilde{A}^{(j)}_{12};$$

$$\begin{aligned} \langle \sigma_{13} \rangle - \mathrm{i} \langle \sigma_{23} \rangle &= C^-_{44} \left(\langle \varepsilon_{13} \rangle - \mathrm{i} \langle \varepsilon_{23} \rangle \right) \\ &\quad + \frac{C^-_{44}}{2} \left[(1+2k^-_1) \widetilde{A}^{(1)}_{11} + (1+2k^-_2) \widetilde{A}^{(2)}_{11} - \widetilde{A}^{(3)}_{11} \right]; \end{aligned}$$

where

$$\widetilde{A}^{(j)}_{1s} = \frac{4\pi}{3a^3} \sum_{q=1}^{N} \left(d^-_q \right)^2 R^3_q A^{(q)(j)}_{1s}$$

Again, as expected, these formulas involve only the dipole moment of the disturbance field caused by each inclusion contained in RUC.

6 Fibrous Composite with Interface Cracks

In this and subsequent Sections, we consider a series of two-dimensional (2D) multiple inclusion problems. This geometry can be viewed as the cross-section of unidirectional fibrous composite (FRC) for which, in the case of transverse loading, the 2D model is adequate. In 2D, the powerful method of complex potentials (e.g., [62, 67]) provides an efficient analysis of a wide range of the boundary value problems including those stated on the multiply-connected domains. This fact makes the method complex potentials very useful in micromechanics of fibrous composites and, in particular, in 2D version of Multipole Expansion Method ([6, 17, 24], among others).

6.1 Background Theory

In 2D plane, the point (x_1, x_2) it conveniently associated with the complex variable $z = x_1 + \mathrm{i}x_2$. In the exponential form, $z = \rho \exp(\mathrm{i}\theta)$, where the modulus $\rho = |z|$ and argument $\theta = \mathrm{Arg}(z)$ are the polar coordinates corresponding to the Cartesian x_1 and x_2. The complex-valued, analytical in a vicinity of the point z_0 function $f(z)$ can be expanded into Laurent series

$$f(z) = \sum_{\infty}^{\infty} a_n\, (z - z_0)^n, \tag{6.1}$$

being inherently the 2D multipole expansion of $f(z)$. The series (6.1) contains the negative powers only if z_0 is the singularity point of $f(z)$. By direct analogy with (2.3), the powers of z can be regarded as the *solid* harmonics, regular for $n \geq 0$ and singular otherwise. Moreover, at the circular line $\rho = R$, we get $z^n = R^n \exp(\mathrm{i}n\theta)$ where Fourier harmonics $\exp(\mathrm{i}n\theta)$ are viewed as the *surface* harmonics.

We also mention the re-expansion formulas analogous to (2.20)–(2.22). The most famous one is the Newtonian binomial:

$$(z + Z)^n = \sum_{k=0}^{n} \binom{n}{k} Z^{n-k} z^k, \tag{6.2}$$

where $\binom{n}{k} = n!/k!(n-k)!$ are the binomial coefficients. In our context, (6.2) can be viewed as translation of regular solid harmonics (R2R). The local expansion of singular harmonics (S2R) takes the form

$$(z + Z)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} Z^{-(n+k)} z^k, \quad |z| < |Z|; \tag{6.3}$$

translation of singular harmonics (S2S) is given by

$$(z + Z)^{-n} = \sum_{k=n}^{\infty} \binom{k-1}{n-1} Z^{k-n} z^{-k}, \quad |z| > |Z|. \tag{6.4}$$

These results are sufficient to solve the conductivity problem for a multiply-connected domain with circular boundaries in a way, quite analogous to that described in Sect. 1. This multipole solution can be further extended, in obvious manner, to the periodic and RUC type infinite arrays of inclusions, etc. For the details, we refer to [17, 69], among others, where this work was done. In what follows, we focus on the elasticity problem.

6.2 2D RUC Geometry

Two-dimensional RUC model of unidirectional FRC (Fig. 2) contains the centers of N aligned in x_3-direction and circular in cross-section fibers. Within a cell, the fibers are placed randomly but without overlapping. The fibers shown by the dashed line do not belong to the cell while occupy a certain area within it. The whole composite bulk is obtained by replicating the cell in two orthogonal directions. Geometry of the cell is defined by its side length a and the coordinates (X_{1q}, X_{2q}) of qth fiber center $(q = 1, 2, \ldots, N)$ in the global Cartesian coordinate system Ox_1x_2. For simplicity

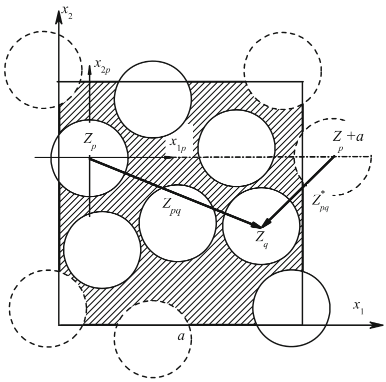


Fig. 2 RUC model of fibrous composite

sake, we assume the fibers equally sized, of radius $R = 1$, and made from the same material. The fiber volume content is $c = N\pi/a^2$.

We introduce the local, fiber-related coordinate systems $O_q x_{1q} x_{2q}$ with origins in Z_q and the associated complex variables $z_q = x_{1q} + \mathrm{i}x_{2q}$. The global complex variable $z = z_q + Z_q$, where $Z_q = X_{1q} + \mathrm{i}X_{2q}$, $q = 1, 2, \ldots, N$. The local variables $z_p = z - Z_p$ relate each other by $z_q = z_p - Z_{pq}$. Here, $Z_{pq} = X_{1pq} + \mathrm{i}X_{2pq}$ is the complex number determining relative position of the fibers with indices p and q *inside* the cell. Also, we introduce the complex number Z^*_{pq} defining the minimal distance between the *arrays* of fibers with indices p and q. Here, we do not assume that the both fibers are belonging to the same RUC, see Fig. 2. In these notations, the non-touching condition of any two fibers in entire composite space is written as $\left|Z^*_{pq}\right| > 2R$.

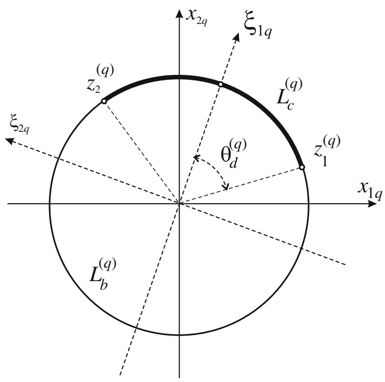
This model is thoroughly studied ([9, 17, 22, 47, 69], among many others) in the case of perfect matrix-fiber interfaces L_q, $q = 1, 2, \ldots, N$. Here, we consider more general problem where one part $L_c^{(q)}$ of interface L_q defined by the endpoints $z_j^{(q)} = \exp(\mathrm{i}\theta_j^{(q)})$ $(j = 1, 2)$ (Fig. 3) is separated and another part $L_b^{(q)} = L_q \backslash L_c^{(q)}$ is perfectly bonded.

To simplify the subsequent algebra, we introduce the crack-related complex variables $\zeta_q = z_q / z_c^{(q)}$, where $z_c^{(q)} = \exp(i\theta_c^{(q)})$ is the crack midpoint: $\theta_c^{(q)} = (\theta_1^{(q)} + \theta_2^{(q)})/2$. The interface crack size is measured by the angle $\theta_2^{(q)} - \theta_1^{(q)} = 2\theta_d^{(q)}$; in the prefect bonding case, $\theta_d^{(q)} = 0$.

6.3 Model Problem

We consider the plane strain problem ($u_3 = 0$). Both the matrix and fiber materials are taken isotropic and linearly elastic. The complex displacement $u = u_1 + iu_2$ corresponds to the Cartesian displacement vector $\mathbf{u} = (u_1, u_2)^T$. We denote $u = u_0$

Fig. 3 A fiber with interface crack



in the matrix material with a shear modulus μ_0 and Poisson ratio ν_0; u_q, μ_1 and ν_1 refer to displacement and elastic moduli, respectively, of qth fiber. The components of the stress tensor $\sigma = \{\sigma_{ij}\}$ are given by

$$\frac{\sigma_{ij}}{2\mu} = \varepsilon_{ij} + \delta_{ij}\frac{(3-\kappa)}{2(\kappa-1)}(\varepsilon_{11}+\varepsilon_{22}), \tag{6.5}$$

where ε_{ij} are the components of the strain tensor and $\kappa = 3-4\nu$ for the plane strain problem. In the adopted by us open-crack model, the traction-free crack surface is assumed. The relevant boundary conditions are:

$$[[u]]_{L_b^{(q)}} = [[T_r]]_{L_b^{(q)}} = 0,\ T_r|_{L_c^{(q)}} = 0, \quad q = 1, 2, \ldots, N. \tag{6.6}$$

where $T_r = \sigma_{rr} + i\sigma_{r\theta}$ is the complex traction at the circular interface $L^{(q)}$.

We consider the macroscopically uniform stress state of the composite bulk, which implies constancy of the volume-averaged strain $\mathbf{E} = \{E_{ij}\} = \{\langle\varepsilon_{ij}\rangle\}$ and stress $\mathbf{S} = \{S_{ij}\} = \{\langle\sigma_{ij}\rangle\}$ tensors, where $\langle f\rangle = A^{-1}\int_A f\, dA$ and $A = a^2$ is the cell area. We take $\mathbf{E}$ a load governing parameter. Under the assumptions made, periodicity of geometry gives rise to periodicity of local strain and stress fields ([10, 17, 68]; among others)

$$\varepsilon_{ij}(z+a) = \varepsilon_{ij}(z+ia) = \varepsilon_{ij}(z); \quad \sigma_{ij}(z+a) = \sigma_{ij}(z+ia) = \sigma_{ij}(z). \tag{6.7}$$

The corresponding to (6.7) displacement field is a quasi-periodic function of coordinates:

$$u(z+a) - u(z) = (E_{11} + \mathrm{i}E_{12})a; \quad u(z+\mathrm{i}a) - u(z) = (E_{12} + \mathrm{i}E_{22})a. \tag{6.8}$$

Hence, (6.8) can be decomposed into a sum of the linear part u^∞ being the far field determined entirely by the $\mathbf{E}$ tensor and the periodic disturbance field caused by the inhomogeneities.

6.4 Displacement Solution

The general displacement solution in a vicinity of the partially debonded fiber (6.6) can be written in the form (e.g., [51])

$$2\mu_j\, u_j(\zeta) = \kappa_j\varphi_j(\zeta) - \left(\zeta - \frac{1}{\bar{\zeta}}\right)\overline{\varphi_j'(\zeta)} - \omega_j\left(\frac{1}{\bar{\zeta}}\right), \tag{6.9}$$

$j = 0$ for matrix and $j = 1$ for fiber. Representation (6.9) is valid for any inhomogeneous far field and so can be applied equally to the single- and multiple-fiber problem.

In (6.9), ζ is the complex plane variable, φ_j and ω_j are the complex potentials and $\kappa_j = 3 - 4\nu_j$. Also, the prime means a derivative with respect to the argument and overbar means a complex conjugate. Expression (6.9) is advantageous in that it simplifies greatly at the circle L defined by the condition $|z| = 1$. The complex interface traction $T_r = \sigma_{rr} + \mathrm{i}\sigma_{r\theta}$ also takes a simple form for $t \in L$, in terms of the derivative complex potentials $\varphi'(z)$ and $\omega'(z)$.

The potentials φ_r and ω_r of the regular in a vicinity of the point $z = 0$ displacement field $u = u_r(z)$ are given by power series

$$\varphi_r(z) = \sum_{n=0}^{\infty} a_n\, z^n, \quad \omega_r(z) = \sum_{n=-\infty}^{1} c_n\, z^n. \tag{6.10}$$

Provided $u_r(t)$ is known, the series expansion coefficients a_n and c_n are given by

$$\kappa a_k - \delta_{k1}\overline{a_1} = I_k\ , k > 0; \quad c_k = -I_k, \quad k < 0; \quad c_1 = \overline{a_1}; \tag{6.11}$$

where δ_{kl} is the Kronecker delta and I_k are the Fourier coefficients

$$I_k = \frac{2\mu}{2\pi} \int_0^{2\pi} u_r(t) t^{-k} d\theta. \tag{6.12}$$

6.5 Single Partially Debonded Fiber

Kushch et al. [51] have suggested the following form of the potentials φ_j and ω_j in (6.9) for the matrix and fiber area, respectively:

$$\begin{aligned}
\varphi_0(\zeta) &= \frac{(1-\alpha)}{2} f(\zeta) + \frac{(1-\beta)}{2} h(\zeta) R_\lambda(\zeta); \\
\omega_0(\zeta) &= -\frac{(1-\alpha)}{2} f(\zeta) + \frac{(1+\beta)}{2} h(\zeta) R_\lambda(\zeta); \\
\varphi_1(\zeta) &= \frac{(1+\alpha)}{2} f(\zeta) + \frac{(1+\beta)}{2} h(\zeta) R_\lambda(\zeta); \\
\omega_1(\zeta) &= -\frac{(1+\alpha)}{2} f(\zeta) + \frac{(1-\beta)}{2} h(\zeta) R_\lambda(\zeta).
\end{aligned} \tag{6.13}$$

Here,

$$\alpha = \frac{\mu_1(\kappa_0 + 1) - \mu_0(\kappa_1 + 1)}{\mu_1(\kappa_0 + 1) + \mu_0(\kappa_1 + 1)}, \quad \beta = \frac{\mu_1(\kappa_0 - 1) - \mu_0(\kappa_1 - 1)}{\mu_1(\kappa_0 + 1) + \mu_0(\kappa_1 + 1)} \tag{6.14}$$

are known as the bi-material constants [12],

$$R_\lambda(\zeta) = (\zeta - \zeta_d)^{\frac{1}{2}+\mathrm{i}\lambda}(\zeta - \overline{\zeta_d})^{\frac{1}{2}-\mathrm{i}\lambda}, \tag{6.15}$$

$f(\zeta)$ and $h(\zeta)$ are the analytical functions to be found. In (6.15), $\zeta_d = \exp(i\theta_d)$ and $\lambda = -\log\left(\frac{1-\beta}{1+\beta}\right)/2\pi$. The boundary conditions (6.6) are fulfilled exactly provided the potentials φ_j and ω_j taken in the form (6.13). The solution (6.9), (6.13) is valid for the arbitrary non-uniform far field $u_r = u(\varphi_r, \omega_r)$ considered as input parameter.

The potentials $f(\zeta)$ and $h(\zeta)$ in (6.13) are written as Laurent series:

$$f(\zeta) = \sum_k f_k\,\zeta^k, \quad h(\zeta) = \sum_k h_k\,\zeta^k. \tag{6.16}$$

The f_k coefficients are given by the simple formulas

$$f_k = -c_k, \quad k < 0; \quad f_k = a_k, \quad k > 1; \quad f_1 = \frac{\mathrm{Re}M_1}{M_2 + M_3} + \mathrm{i}\frac{\mathrm{Im}M_1}{M_2 - M_3}; \tag{6.17}$$

for the explicit expression of M_j, see [51]. The h_k coefficients are given by the series

$$h_k = \frac{(1+\alpha)}{(1-\beta)} \sum_{l=k+1}^{\infty} X_{l-k-1}(\zeta_d, -\lambda)a_l,\ k > 0; \tag{6.18}$$
$$h_k = \frac{(1+\alpha)}{(1+\beta)} {\zeta_d}^{-2\mathrm{i}\lambda} \sum_{l=-\infty}^{k} X_{k-l}(\zeta_d, \lambda)c_l,\ k < 0;$$

where

$$X_n(\zeta_d, \lambda) = (-1)^n \sum_{k=0}^{n} G_k(\lambda)\overline{G_{n-k}(\lambda)}\zeta_d^{2k-n}, \quad G_k(\lambda) = \prod_{j=1}^{k}\left(\frac{1+2\mathrm{i}\lambda}{2j} - 1\right). \tag{6.19}$$

6.6 *Finite Array of Partially Debonded Fibers*

In accordance with the superposition principle, we write the total displacement field u_0 in the matrix domain as a sum of the far field u^∞ and the disturbance fields $u_s^{(p)}$ caused by each individual fiber:

$$u_0(z) = u^\infty(z) + u_s(z), \quad u_s(z) = \sum_{p=1}^{N} z_c^{(p)} u_s^{(p)}(\zeta_p). \tag{6.20}$$

In (6.20), $\zeta_p = z_p / z_c^{(p)}$; the disturbance displacements are summed up in the global coordinate system. We divide the local expansion of $u_0(z)$ in a vicinity of qth fiber into the regular $u_r^{(q)}$ and singular $u_s^{(q)}$ parts:

$$u_0(\zeta_q) = u_r^{(q)}(\zeta_q) + u_s^{(q)}(\zeta_q), \tag{6.21}$$

Here, $u_0(\zeta_q)$ and $u_r^{(q)}(\zeta_q)$ are given by (6.9), with the potentials (6.13) and (6.10), respectively. The disturbance field vanishes at infinity: $u_s^{(q)}(\zeta_q) \to 0$ for $\zeta_q \to \infty$. The regular part $u_r^{(q)}(\zeta_q)$ is a sum of the far field and the disturbance fields from all fibers excluding that one with $p = q$:

$$z_c^{(q)} u_r^{(q)}(\zeta_q) = u^{\infty}(z) + \sum_{p \neq q}^{N} z_c^{(p)} u_s^{(p)}(\zeta_p). \tag{6.22}$$

Provided the right hand side of (6.22) is known, the $u_r^{(q)}$ series expansion coefficients $a_k^{(q)}$ and $c_k^{(q)}$ can be found from (6.11) where, in view of (6.22), I_k (6.12) are calculated by integrating along the interface L_q:

$$I_k^{(q)} = \frac{2\mu_0}{2\pi z_c^{(q)}} \int_0^{2\pi} \left\{ u^{\infty}(z) + \sum_{p \neq q}^{N} z_c^{(p)} \left[u_0\left(\zeta_p\right) - u_r^{(p)}\left(\zeta_p\right) \right] \right\} t_q^{-k} d\theta_q, \tag{6.23}$$
$$-\infty < k < \infty, \quad q = 1, 2, \ldots, N.$$

The Eqs. (6.23), (6.11) and (6.17), (6.18) taken together form an infinite system of linear algebraic equations, from where all the expansion coefficients are determined uniquely provided u^{∞} is known.

6.6.1 Evaluation of the Integrals in (6.23)

The integrals entering (6.23) can be evaluated either analytically or numerically. In what follows, $u^{\infty}(z)$ is assumed to be linear function of coordinates, i.e.,

$$u^{\infty}(z) = \Gamma_1 z - \overline{\Gamma_2} \overline{z}. \tag{6.24}$$

where Γ_1 and Γ_2 are the arbitrary complex constants. Its analytical integration is elementary and yields

$$I_k^{(q)} = \frac{2\mu_0}{2\pi z_c^{(q)}} \int_0^{2\pi} u^{\infty}(Z_q + z_c^{(q)} t_q) t_q^{-k} d\theta_q = \begin{cases} -2\mu_0 \overline{\Gamma_2} \left(z_c^{(q)}\right)^{-2}, k = -1; \\ 2\mu_0 u^{\infty}(Z_q)/z_c^{(q)}, k = 0; \\ 2\mu_0 \Gamma_1, k = 1; \end{cases} \tag{6.25}$$

and $I_k^{(q)} \equiv 0$ otherwise.

Numerical integration is straightforward but not necessarily the perfect way of evaluating the sum terms in (6.23). First, computational cost of this procedure is relatively high. Second, the fact is that with Z_{pq} (and, hence, ζ_p) increase, $u_0\left(\zeta_p\right)$ tends to $u_r^{(p)}\left(\zeta_p\right)$. This case requires a special attention: evaluation of $u_s^{(p)}$ by subtraction of two close numbers is not advisable because it may lead to substantial numerical error. To avoid this problem, we derive an asymptotic expansion of $u_s^{(p)}$ using the following Laurent series [51]:

$$h(\zeta) R_{\lambda}(\zeta) = \begin{cases} \sum_n g_n^{(1)} \zeta^n, \ |\zeta| > 1; \\ \sum_n g_n^{(2)} \zeta^n, \ |\zeta| < 1; \end{cases} \tag{6.26}$$

In (6.26),

$$g_n^{(1)} = \sum_{m=0}^{\infty} R_m(\zeta_d, \lambda) h_{n+m-1}, \quad g_n^{(2)} = -(\zeta_d)^{2i\lambda} \sum_{m=0}^{\infty} R_m(\zeta_d, -\lambda) h_{n-m}, \tag{6.27}$$

where

$$R_n(\zeta_d, \lambda) = (-1)^n \sum_{k=0}^{n} H_k(\lambda) \overline{H_{n-k}(\lambda)} \zeta_d^{2k-n}, \quad H_k(\lambda) = \prod_{j=1}^{k} \left(\frac{3 + 2i\lambda}{2j} - 1 \right). \tag{6.28}$$

In fact, it suffices to derive the asymptotic expansion of the singular potentials $\varphi_s^{(p)} = \varphi_0 - \varphi_r^{(p)}$ and $\omega_s^{(p)} = \omega_0 - \omega_r^{(p)}$. By combining the formulas (6.13), (6.26) (6.10) and (6.17), one finds

$$\begin{aligned} \varphi_s^{(p)}\left(\zeta_p\right) &= \frac{(1-\alpha)}{2} f(\zeta_p) + \frac{(1-\beta)}{2} h(\zeta_p) R_{\lambda}(\zeta_p) - \sum_{n=0}^{\infty} a_n^{(p)} \zeta_p^n \\ &= \sum_{n=1}^{\infty} a_{-n}^{(p)} \zeta_p^{-n}, \end{aligned} \tag{6.29}$$

where

$$a_{-n}^{(p)} = \frac{(1-\alpha)}{2} f_n^{(p)} + \frac{(1-\beta)}{2} g_n^{(1)(p)} \quad (n = 1, 2, \ldots). \tag{6.30}$$

Analogously,

$$\omega_s^{(p)}\left(\zeta_p\right) = \sum_{n=1}^{\infty} c_n^{(p)}\, \zeta_p^{\,n}, \tag{6.31}$$

where

$$c_n^{(p)} = -\frac{(1-\alpha)}{2} f_n^{(p)} + \frac{(1+\beta)}{2} g_n^{(2)(p)} - \delta_{n1}\overline{a_1^{(p)}} \quad (n = 1, 2, \ldots). \tag{6.32}$$

In these notations, $u_s^{(p)}$ takes the same form as in the case of perfectly bonded fibers, i.e.

$$2\mu_0 u_s^{(p)}(\zeta_p) = \sum_{n=1}^{\infty}\left[\kappa_0 a_{-n}^{(p)}\, \zeta_p^{\,-n} + \left(\zeta_p - \overline{\zeta_p}^{-1}\right)\overline{a_{-n}^{(p)}}\; \overline{\zeta_p}^{\,-(n+1)} - c_n^{(p)}\overline{\zeta_p}^{\,-n}\right]. \tag{6.33}$$

The expression (6.33) is appropriate for $I_k^{(q)}$ evaluation provided $\left|Z_{pq}\right|$ is sufficiently large. Moreover, in this case integration can be done analytically, with aid of the re-expansion formula (6.3) written for ζ_p as

$$\zeta_p^{-n} = \sum_k \eta_{nk}^{pq}\, \zeta_q^{\,k}, \quad \left|\zeta_q\right| < \left|Z_{pq}\right|; \tag{6.34}$$

where

$$\eta_{nk}^{pq} = (-1)^k \frac{(n+k-1)!}{(n-1)!k!}\left(z_c^{(p)}\right)^n \left(z_c^{(q)}\right)^k Z_{pq}^{-(n+k)} \tag{6.35}$$

for $k \geq 0$ and equal to zero otherwise. After some algebra, we get the explicit expression of $I_k^{(q)}$ (6.23):

$$I_k^{(q)} = 2\mu_0\Gamma_1\delta_{k1} - 2\mu_0\overline{\Gamma_2}\left(z_c^{(q)}\right)^{-2}\delta_{k,-1} \tag{6.36}$$
$$+ \sum_{p\neq q}^{N}\frac{z_c^{(p)}}{z_c^{(q)}}\sum_{n=1}^{\infty}\left\{\kappa_0 A_{nk}^{pq} a_{-n}^{(p)} + n(\overline{B_{nk}^{pq}} + \overline{C_{nk}^{pq}})\overline{a_{-n}^{(p)}} + \overline{D_{nk}^{pq}} c_n^{(p)}\right\},$$

where

$$A_{nk}^{pq} = \begin{cases} \eta_{nk}^{pq}, & k > 0; \\ 0, & \text{otherwise}; \end{cases} \tag{6.37}$$

$$B_{nk}^{pq} = \begin{cases} (-1)^{k+1}\frac{(n-k+2)!}{(n+1)!(1-k)!}\left(z_c^{(p)}\right)^{n+2}\left(z_c^{(q)}\right)^{-k} Z_{pq}^{-(n-k+2)}, k \leq 1; \\ 0, \quad \text{otherwise}; \end{cases} \tag{6.38}$$

$$C_{nk}^{pq} = \begin{cases} (-1)^k \frac{(n-k)!}{(n-1)!(-k)!} \left(z_c^{(p)}\right)^{n+2} \left(z_c^{(q)}\right)^{-k} \left|Z_{pq}\right|^2 Z_{pq}^{-(n-k+2)}, k < 0; \\ 0, \quad \text{otherwise}; \end{cases} \tag{6.39}$$

$$D_{nk}^{pq} = \begin{cases} \eta_{n,-k}^{pq}, \quad k < 0; \\ 0, \quad \text{otherwise}. \end{cases} \tag{6.40}$$

Theoretically, the obtained formulas are valid for any two non-touching fibers with indices p and q. However, for the closely placed fibers ($\left|Z_{pq}\right| \approx 2R$) the series (6.33) and (6.34) converge rather slow: in this case, direct numerical integration is preferable. The following empirical rule provides the reasonable compromise between computational effort and accuracy: for $\left|Z_{pq}\right| \leq 2.5R$, the integrals (6.23) are evaluated numerically whereas for $\left|Z_{pq}\right| > 2.5R$ the explicit analytical formula (6.36) is applied.

6.7 RUC Model of Fibrous Composite with Interface Cracks

Now, we consider the model problem for RUC geometry, with the macroscopic strain **E** as a governing parameter. The periodicity conditions (6.7), (6.8) are fulfilled by taking the displacement in matrix domain as a sum of linear mean field u^∞ and periodic disturbance field u_s^*. Specifically,

$$u_0(z) = u^\infty(z) + u_s^*(z), \quad u_s^*(z) = \sum_{p=1}^{N} z_c^{(p)} u_s^{*(p)}(\zeta_p). \tag{6.41}$$

In comparison with (6.20), the single fiber disturbance terms u_p^s are replaced with their periodic counterparts given by the sums over all the square lattice nodes $\mathbf{k} = \{k_1, k_2\}$ $(-\infty < k_1, k_2 < \infty)$:

$$u_s^{*(p)}(\zeta_p) = \sum_{\mathbf{k}} u_s^{(p)} \left(\frac{z_p + W_{\mathbf{k}}}{z_c^{(p)}} \right), \quad W_{\mathbf{k}} = (k_1 + \mathrm{i}k_2)a. \tag{6.42}$$

In view of (6.33), $u_s^{*(p)}$ can be expressed in terms of the periodic complex (harmonic) potentials $S_n^*(z)$ and their biharmonic $S_n^{**}(z)$ counterparts:

$$S_n^*(z) = \sum_{\mathbf{k}} (z + W_{\mathbf{k}})^{-n} \, ; S_n^{**}(z) = \sum_{\mathbf{k}} |z + W_{\mathbf{k}}|^2 \, (z + W_{\mathbf{k}})^{-(n+2)} \tag{6.43}$$

([17, 65], among others). In particular,

$$\varphi_s^{*(p)}(\zeta_p) = \sum_{n=1}^{\infty} a_{-n}^{(p)} \left(z_c^{(p)}\right)^n S_n^*(z_p). \tag{6.44}$$

The above procedure, with two amendments, applies to this problem. First, we require $u_0(z)$ (6.41) to obey the periodicity conditions (6.8). This condition gives us the far field $u^\infty(z)$ (6.24) coefficients Γ_1 and Γ_2. It has been established elsewhere [17, 65] that

$$\begin{aligned} S_n^*(z) &= S_n^*(z+a) = S_n^*(z+ia) - \delta_{n1}2\pi\mathrm{i}/a; \\ S_n^{**}(z) &= S_n^{**}(z+a) = S_n^{**}(z+ia); \end{aligned} \tag{6.45}$$

where δ_{nm} is the Kronecker delta. By substituting (6.41) into (6.8) and taking (6.45) into consideration we find

$$\Gamma_1 = \frac{E_{11}+E_{22}}{2} + \mathrm{Re}\Gamma_\Sigma; \quad \Gamma_2 = \frac{E_{22}-E_{11}}{2} + iE_{12} + \overline{\Gamma_\Sigma}; \tag{6.46}$$

where

$$\Gamma_\Sigma = \frac{\pi}{V} \sum_{p=1}^{N} \left[\kappa_0 a_{-1}^{(p)} \left(z_c^{(p)}\right)^2 - \overline{a_{-1}^{(p)} \left(z_c^{(p)}\right)^2} + c_1^{(p)} \right]. \tag{6.47}$$

Second, the formula (6.23) should be modified to take into account interactions of an infinite array of fibers. To avoid the discussed above integration issues, we write it as

$$\begin{aligned} I_k^{(q)} = \frac{2\mu_0}{2\pi z_c^{(q)}} \int_0^{2\pi} &\left\{ u^\infty(z) + \sum_{p\neq q}^{N} z_c^{(p)} u_s^{(p)}\left(\zeta_p\right) \right. \\ &\left. + \sum_{p=1}^{N} z_c^{(p)} \sum_{\mathbf{k}\neq 0} u_s^{(p)} \left(\frac{z_p + W_{\mathbf{k}}}{z_c^{(p)}} \right) \right\} t_q^{-k} d\theta_q. \end{aligned} \tag{6.48}$$

Here, the first sum contains contributions from the nearest neighboring fibers of pth and qth arrays defined by Z_{pq}^*, see Fig. 2. It can be integrated either analytically or numerically, in accordance with the rule formulated in Sect. 6.6. The second sum contains contributions from the "far" fibers. Provided a number of fibers in the cell is sufficiently large, the cell size $a \gg 2R$ and $\left|Z_q - Z_p\right| \gg 2R$ for all fibers contributing to this sum. In this case analytical integration is justified and yields

$$I_k^{(q)} = \sum_{p=1}^{N} \frac{z_c^{(p)}}{z_c^{(q)}} \sum_{n=1}^{\infty} \left\{ \kappa_0 A_{nk}^{*pq} a_{-n}^{(p)} + n(\overline{B_{nk}^{*pq}} + \overline{C_{nk}^{*pq}})\overline{a_{-n}^{(p)}} + \overline{D_{nk}^{*pq}} c_n^{(p)} \right\} \tag{6.49}$$

In (6.49), the coefficients A_{nk}^{*pq}, B_{nk}^{*pq}, C_{nk}^{*pq} and D_{nk}^{*pq} are given by the formulas from (6.37) to (6.40), respectively, with replacing the $Z_{pq}^{-(n+k)}$ to Σ_{n+k}^{pq*} and $\left|Z_{pq}\right|^2 Z_{pq}^{-(n-k+2)}$ to Σ_{n-k}^{pq**}, where

$$\Sigma_n^{pq*} = \sum_{\mathbf{k}\neq 0} (Z_{pq}^* + W_{\mathbf{k}})^{-n}, \quad \Sigma_n^{pq**} = \sum_{\mathbf{k}\neq \mathbf{0}} \frac{\left|Z_{pq}^* + W_{\mathbf{k}}\right|^2}{(Z_{pq}^* + W_{\mathbf{k}})^{n+2}} \tag{6.50}$$

are the standard harmonic and biharmonic lattice sums, respectively ([65]). Now, with $u^{\infty}(z)$ and $I_k^{(q)}$ expressed in terms of $a_{-n}^{(p)}$ and $c_n^{(p)}$ (formulas (6.24), (6.46), (6.47) and (6.49), respectively), the problem is reduced to considered above [51].

6.8 Effective Stiffness Tensor

The obtained analytical solution provides an accurate evaluation of the local strain and stress fields in any point of RUC and thus enables a comprehensive study of stress concentration, stress intensity factors and energy release rate at the interface crack tips in FRC. On the other hand, the strain and stress fields can be integrated analytically to get the finite expression of the macroscopic stiffness tensor $\mathbf{C}^*$ (1.10). From the plane strain problem, the effective transverse elastic moduli C_{1111}^*, C_{1122}^*, C_{2222}^* and C_{1212}^* can be determined.

To simplify the integration procedure, we write the bulk and shear components of the strain tensor as

$$\varepsilon_{11} + \varepsilon_{22} = 2\mathrm{Re}\frac{\partial u}{\partial z}; \quad \varepsilon_{22} - \varepsilon_{11} + 2\mathrm{i}\varepsilon_{12} = -2\frac{\partial \overline{u}}{\partial z}. \tag{6.51}$$

In what follows, we will use also the Gauss formula written in complex variables as

$$\int_A \frac{\partial u}{\partial z} dV = \frac{1}{2}\int_L u(n_1 - \mathrm{i}n_2)dL, \tag{6.52}$$

where L is the boundary of A and $(n_1, n_2)^T$ is the outer normal to L unit vector.

First, by integrating the bulk stress

$$\frac{\sigma_{11} + \sigma_{22}}{2} = \frac{4\mu}{(\kappa - 1)}\mathrm{Re}\frac{\partial u}{\partial z} \tag{6.53}$$

we get

$$A\frac{\langle\sigma_{11}\rangle + \langle\sigma_{22}\rangle}{2} = \frac{4\mu_0}{(\kappa_0 - 1)}\int_A \mathrm{Re}\frac{\partial u_0}{\partial z} dV + \frac{4\mu_1}{(\kappa_1 - 1)}\sum_{q=1}^{N}\int_{A_q}\mathrm{Re}\frac{\partial u_q}{\partial z} dV \tag{6.54}$$

where $A_q = \pi R^2$ is the area of qth fiber cross-section and $A_0 = a^2 - N\pi R^2$ is the matrix area inside the cell: $A = \sum_{q=0}^{N} A_q$. Next, by applying the formula (6.52) we obtain

$$A\frac{\langle\sigma_{11}\rangle + \langle\sigma_{22}\rangle}{2} = \frac{4\mu_0}{(\kappa_0 - 1)}\left(\int_{L_0} - \sum_{q=1}^{N}\int_{L_q}\right)\mathrm{Re}\left[u_0\left(n_1 - \mathrm{i}n_2\right)\right]dL \qquad (6.55)$$
$$+ \frac{4\mu_1}{(\kappa_1 - 1)}\sum_{q=1}^{N}\int_{L_q}\mathrm{Re}\left[u_q\left(n_1 - \mathrm{i}n_2\right)\right]dL,$$

where L_0 is the outer boundary of the cell. In the case L_q involves the interface crack $L_c^{(q)}$, u_0 and u_q in (6.55) are integrated along the matrix and fiber, respectively, side of the crack.

In view of (6.8), integration along L_0 is elementary and yields

$$\int_{L_0}\mathrm{Re}\left[u_0\left(n_1 - \mathrm{i}n_2\right)\right]dL = a^2\left(E_{11} + E_{22}\right). \qquad (6.56)$$

We recognize also that at the interface L_q, the integrands simplify to

$$\frac{4\mu_j}{(\kappa_j - 1)}\mathrm{Re}\left[u_j\left(n_1 - \mathrm{i}n_2\right)\right] = 2\mathrm{Re}\left[\frac{\kappa_j\varphi_j(t_q) - \omega_j(1/\overline{t_q})}{(\kappa_j - 1)}t_q^{-1}\right]. \qquad (6.57)$$

In view of (6.57), integration along L_q is straightforward. With (6.13) and (6.32) taken into consideration we come, after some algebra, to

$$\int_{Lq}\mathrm{Re}\left\{\left[\frac{4\mu_1 u_q}{(\kappa_1 - 1)} - \frac{4\mu_0 u_0}{(\kappa_0 - 1)}\right]\left(n_1 - \mathrm{i}n_2\right)\right\}dL = 2\pi\frac{(\kappa_0 + 1)}{(\kappa_0 - 1)}c_1^{(q)}, \qquad (6.58)$$

where $c_1^{(q)}$ is given by (6.32). Now, collecting all the terms gives us the exact formula:

$$\frac{\langle\sigma_{11}\rangle + \langle\sigma_{22}\rangle}{2} = \frac{2\mu_0\left(E_{11} + E_{22}\right)}{(\kappa_0 - 1)} + \frac{2\pi}{a^2}\frac{(\kappa_0 + 1)}{(\kappa_0 - 1)}\sum_{q=1}^{N}c_1^{(q)}. \qquad (6.59)$$

Averaging the shear part of stress tensor is quite analogous. It follows from (6.51) that

$$\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2\mu(\varepsilon_{22} - \varepsilon_{11} + 2\mathrm{i}\varepsilon_{12}) = -2\mu\frac{\partial\overline{u}}{\partial z}. \qquad (6.60)$$

By applying the formula (6.52) and performing the transformations similar to those exposed above, we get also

$$\langle\sigma_{22}\rangle - \langle\sigma_{11}\rangle - 2\mathrm{i}\langle\sigma_{12}\rangle = 2\mu_0\left(E_{22} - E_{11} - 2\mathrm{i}E_{12}\right) + \frac{2\pi}{a^2}(\kappa_0+1)\sum_{q=1}^{N} a_{-1}^{(q)}\left(z_c^{(q)}\right)^2, \tag{6.61}$$

where $a_{-1}^{(q)}$ is given by (6.30). Together with (1.10), relations (6.59) and (6.61) enable evaluation of the effective transverse elastic moduli C^*_{1111}, C^*_{1122}, C^*_{2222} and C^*_{1212} of unidirectional FRC with interface cracks [53]. To find C^*_{2323} and C^*_{1313}, one has to consider longitudinal shear in the fiber axis (x_3) direction. In mathematical sense, this problem is equivalent to the transverse conductivity (2D Laplace) problem, solved recently in [52].

7 Composite with Elliptic in Cross-Section Fibers

In this Section, we consider a plane containing a finite array of elliptic inclusions. No restrictions is imposed on their number, size, aspect ratio, elastic properties and arrangement. For the sake of simplicity, the inclusions are assumed equally oriented; extension of the below analysis to the case of arbitrarily oriented inclusions is straightforward. We apply MEM to solve the elasticity problem: for the multipole-type solution of the counterpart conductivity problem, see [86, 87].

7.1 Single Elliptic Inclusion in Non-uniform Far Field

Let consider an isotropic elastic plane with a single elliptic inclusion. The Cartesian coordinate system Ox_1x_2 is defined so that its origin coincides with the centroid of ellipse whereas the Ox_1 and Ox_2 axes are directed along the major and minor axes of the ellipse. An aspect ratio of the ellipse is $e = l_2/l_1$, where l_1 and l_2 are the major and minor, respectively, semi-axes of the ellipse. Another derivative geometric parameter is the inter-foci distance $2d$, where $d = \sqrt{l_1^2 - l_2^2}$.

7.1.1 Background Theory

We introduce the complex variables of two kinds. The first one is $z = x_1 + ix_2$, the second variable $\xi = \zeta + i\eta$ is defined by

$$z = \omega(\xi) = d\cosh\xi. \tag{7.1}$$

Equation (7.1) specifies an elliptic coordinate system with ζ and η as "radial" and "angular" coordinates, respectively. So, the boundary of the ellipse is the coordinate line given by the equation

$$\zeta = \zeta_0 = \frac{1}{2}\ln\left(\frac{1+e}{1-e}\right); \tag{7.2}$$

i.e., the points at matrix-elliptic fiber interface are the functions of angular coordinate η only. This fact makes the elliptic complex variable ξ particularly useful in the problems formulated on domains with elliptic boundaries/interfaces.

The complex displacement $u = u_1 + \mathrm{i}u_2$ is expressed in terms of complex potentials φ and ψ as

$$u = \kappa\varphi(z) - (z - \overline{z})\,\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{7.3}$$

The expression (7.3) is slightly different in form but equivalent to that originally suggested in [67]. Cartesian components of the corresponding to u (7.3) stress tensor σ are given by

$$\sigma_{11} + \sigma_{22} = 4\mu\left(\varphi'(z) + \overline{\varphi'(z)}\right); \tag{7.4}$$
$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 4\mu\left[(\overline{z} - z)\varphi''(z) - \varphi'(z) + \psi'(z)\right].$$

The displacement u and traction T_n are assumed to be continuous through the elliptic matrix-inclusion interface $S : \zeta = \zeta_0$. Satisfying these conditions can be greatly simplified by writing the complex displacement and traction in terms of their curvilinear (actually, normal and tangential to interface $\zeta = \zeta_0$) components:

$$u = u_\zeta + iu_\eta \text{ and } T_n = \sigma_{\zeta\zeta} + i\sigma_{\zeta\eta}. \tag{7.5}$$

In terms of complex potentials [67],

$$u_\zeta + iu_\eta = \frac{\overline{\varpi'(\xi)}}{|\varpi'(\xi)|}\left[\kappa\varphi(z) - (z - \overline{z})\,\overline{\varphi'(z)} - \overline{\psi(z)}\right]; \tag{7.6}$$
$$\sigma_{\zeta\zeta} - i\sigma_{\zeta\eta} = 2G\left\{\varphi'(z) + \overline{\varphi'(z)} - \frac{\varpi'(\xi)}{\overline{\varpi'(\xi)}}\left[(\overline{z} - z)\varphi''(z) - \varphi'(z) + \psi'(z)\right]\right\},$$

where, from (7.1), $\varpi'(\xi) = dz/d\xi = d\sinh\xi$.

7.1.2 Formal Solution

The key point is the proper choice of the form of potential functions φ and ψ. We take φ, by analogy with [61], as

$$\varphi = \sum_n A_n \upsilon^{-n}. \tag{7.7}$$

The ψ potential, by analogy with the 3D case (4.50), is taken in the form [45]

$$\psi = \psi_0 - \psi_1, \ \psi_0 = \sum_n B_n \upsilon^{-n}, \quad \psi_1 = \frac{\sinh \zeta_0}{\sinh \xi}\left(\frac{\upsilon}{\upsilon_0} - \frac{\upsilon_0}{\upsilon}\right)\sum_n n A_n \upsilon^{-n}, \tag{7.8}$$

where A_n and B_n are the complex coefficients, $\upsilon = \exp \xi$ and $\upsilon_0 = \exp(\zeta_0)$. The potentials φ_i and ψ_i for inclusion are also given by (7.7) and (7.8), with replacing A_n and B_n to C_n and D_n, respectively.

With φ in the form (7.7) and ψ in the form (7.8), the expressions of u and T_n (7.6) at the interface $\zeta = \zeta_0$ are simplified dramatically: specifically,

$$u\,\big|_{\zeta=\zeta_0} = \kappa\varphi - \overline{\psi_0} = \sum_n \left(\kappa A_n \upsilon^{-n} - \overline{B_n \upsilon^{-n}}\right) \tag{7.9}$$

and

$$\varpi' \frac{T_n}{2G}\,\big|_{\zeta=\zeta_0} = \varphi' - \overline{\psi_0'} = \sum_n (-n)\left(A_n \upsilon^{-n} - \overline{B_n \upsilon^{-n}}\right). \tag{7.10}$$

For the details of derivation, see [45]. By virtue of (7.9), from the displacement continuity condition one finds

$$\sum_n (\kappa_0 A_n e^{-n\zeta_0 - in\eta} - \overline{B_n} e^{-n\zeta_0 + in\eta}) = \sum_n (\kappa_1 C_n e^{-n\zeta_0 - in\eta} - \overline{D_n} e^{-n\zeta_0 + in\eta}), \tag{7.11}$$

where $\kappa_i = \kappa(\nu_i)$. Orthogonality of Fourier harmonics $\exp(in\eta)$ enables splitting the functional equality (7.11) into an infinite set of linear algebraic equations

$$\kappa_0 A_n \upsilon_0^{-n} - \overline{B}_{-n} \upsilon_0^n = \kappa_1 C_n \upsilon_0^{-n} - \overline{D}_{-n} \upsilon_0^n. \tag{7.12}$$

By applying the same procedure to the interface traction continuity condition, we get another set of linear equations:

$$A_n \upsilon_0^{-n} + \overline{B}_{-n} \upsilon_0^n = \omega(C_n \upsilon_0^{-n} + \overline{D}_{-n} \upsilon_0^n), \tag{7.13}$$

where $\omega = \mu_{1/}\mu_0$.

It is advisable, for computational purpose, to introduce new scaled variables $\widetilde{A}_n = A_n \upsilon_0^{-n}$, etc.; we obtain

$$
\begin{aligned}
\kappa_0 \widetilde{A}_n - \overline{\widetilde{B}}_{-n} &= \kappa_1 \widetilde{C}_n - \overline{\widetilde{D}}_{-n}, \\
\widetilde{A}_n + \overline{\widetilde{B}}_{-n} &= \omega(\widetilde{C}_n + \overline{\widetilde{D}}_{-n}), \quad -\infty < n < \infty.
\end{aligned} \tag{7.14}
$$

The form of Eq. (7.14) is remarkably simple, if not simplest (see, for comparison, [61]) which clearly indicates rational choice of the potential functions (7.8).

The obtained solution is general and contains the extra degrees of freedom which must be excluded by imposing the constraints drawn from physical considerations. One obvious condition consists in that the displacement field must be regular, i.e., continuous and finite inside the inclusion. It means the Laurent series expansions of corresponding complex potentials contain the terms with non-negative powers of z only. It has been shown elsewhere [45] that the following relations between C_n and D_n with positive and negative index n,

$$
C_n = C_{-n}; \quad D_n = D_{-n} + 2n \sinh(2\zeta_0) C_{-n}, \quad n > 0; \tag{7.15}
$$

provide regularity of the displacement and stress fields inside the inclusion.

The displacement solution in the matrix domain comprises the regular and irregular parts: $u^{(0)} = u^r + u^s$. Here, u^r is the regular far field whereas the disturbance $u^s \to 0$ as $|z| \to \infty$. The potentials φ and ψ also can be splitted onto irregular and regular parts:

$$
\varphi = \varphi^s + \varphi^r, \quad \psi = \psi^s + \psi^r. \tag{7.16}
$$

The expression of φ^s and ψ^s is given by Eqs. (7.7) and (7.8), respectively, where we keep the terms with negative powers of υ only to provide the required asymptotic behavior, so

$$
A_n = B_n \equiv 0 \quad \text{for} \quad n \le 0. \tag{7.17}
$$

On the contrary, u^r is assumed to be regular, with the potentials

$$
\varphi^r = \sum_n a_n \upsilon^{-n}, \quad \psi^r = \sum_n \left[b_n - 2n a_n \frac{\sinh \zeta_0}{\sinh \xi} \sinh(\xi - \zeta_0) \right] \upsilon^{-n}, \tag{7.18}
$$

where a_n and b_n comply (7.15) as well.

7.1.3 Resolving Linear System

Representation of the linear displacement field corresponding to the constant far strain tensor $\mathbf{E}$

$$
u^r = (E_{11} x_1 + E_{12} x_2) + i(E_{12} x_1 + E_{22} x_2) \tag{7.19}
$$

takes the form (7.3) with the potentials (7.18), where

$$a_{-1} = \frac{d}{4}\frac{E_{11}+E_{22}}{(\kappa_0-1)}; \quad b_{-1} = a_{-1}\upsilon_0^{-2} + \frac{d}{4}(E_{22}-E_{11}+2iE_{12}); \tag{7.20}$$

a_1 and b_1 are given by (7.15) and all other a_n and b_n for $n \neq \pm 1$ are equal to zero. The displacement u^r and corresponding traction T_n^r at the interface $\zeta = \zeta_0$ take the form (7.9) and (7.10), respectively. Applying the procedure analogous to that described above gives us an infinite linear system

$$\begin{aligned} &\kappa_0 A_n \upsilon_0^{-n} - \overline{B}_{-n}\upsilon_0^n + \kappa_0 a_n \upsilon_0^{-n} - \overline{b}_{-n}\upsilon_0^n = \kappa_1 C_n \upsilon_0^{-n} - \overline{D}_{-n}\upsilon_0^n; \\ &A_n \upsilon_0^{-n} + \overline{B}_{-n}\upsilon_0^n + a_n \upsilon_0^{-n} + \overline{b}_{-n}\upsilon_0^n = \omega(C_n \upsilon_0^{-n} + \overline{D}_{-n}\upsilon_0^n). \end{aligned} \tag{7.21}$$

In the single-inclusion problem, we assume u^r (or, the same, a_n and b_n) to be known. In this case, the equations (7.21) together with (7.15) and (7.17) form a closed set of linear algebraic equations possessing a unique solution. By substituting (7.15) and (7.17) into (7.21) we come to the linear system

$$\begin{aligned} &\kappa_0 A_n - \kappa_1 C_n + (\overline{D_n} - 2n\sinh 2\zeta_0 \overline{C_n})\upsilon_0^{2n} = -\kappa_0 a_n + \overline{b_{-n}}\upsilon_0^{2n}; \\ &\overline{B_n} + \kappa_1 C_n \upsilon_0^{2n} - \overline{D_n} = \kappa_0 a_{-n}\upsilon_0^{2n} - \overline{b_n}; \\ &A_n - \omega C_n - \omega(\overline{D_n} - 2n\sinh 2\zeta_0 \overline{C_n})\upsilon_0^{2n} = -a_n - \overline{b_{-n}}\upsilon_0^{2n}; \\ &\overline{B_n} - \omega C_n \upsilon_0^{2n} - \omega\overline{D_n} = -a_{-n}\upsilon_0^{2n} - \overline{b_n}; \\ &n = 1, 2, \ldots . \end{aligned} \tag{7.22}$$

with the unknowns A_n, B_n, C_n and D_n ($n > 0$) and with the coefficients a_n and b_n entering the right-hand side vector. For the Eshelby-type problem, these coefficients are given by (7.20). The corresponding resolving system (7.22) consists of four equations for $n = 1$.

7.1.4 Stress Intensity Factors

The solution we derived is valid for any $0 < e < 1$. To complete this section, we consider the limiting case $e \to 0$ where an ellipse degenerates into the cut $|x_1| \leq d$ in the complex plane (another limit, $e \to 1$ where an ellipse becomes a circle, is trivial). By putting $\omega = 0$ we get a straight crack, the stress field around which is known to have singularity at the crack tips. In the linear fracture mechanics, the stress intensity factor (SIF) defined as

$$K_{\mathrm{I}} + iK_{\mathrm{II}} = \lim_{z \to d} \sqrt{2\pi(z-d)}\,(\sigma_{22} + i\sigma_{12}) \tag{7.23}$$

is used to quantify the stress field around the crack tip. Taking this limit in the above solution is straighforward: for $e \to 0$ we have, from (7.2), $\zeta_0 \to 0$ and $\upsilon_0 \to 1$. After simple algebra, we get the formula

$$K_{\mathrm{I}} + iK_{\mathrm{II}} = -2G_0\sqrt{\pi/d}\sum_{n=1}^{\infty} n\left(A_n + \overline{B_n}\right), \tag{7.24}$$

valid for the arbitrary, not necessarily uniform, far load.

7.2 Finite Array of Inclusions

Now, we consider a plane containing N non-overlapping equally oriented elliptic inclusions with the semi-axes l_{1p}, l_{2p} and elastic moduli μ_p and ν_p centered in the points $Z_p = X_{1p} + iX_{2p}$. The variables of local, inclusion-associated coordinate systems are related by

$$z_p = Z_{pq} + z_q, \quad \text{where} \quad z_p = x_{1p} + ix_{2p} \quad \text{and} \quad Z_{pq} = Z_q - Z_p. \tag{7.25}$$

We define the local curvilinear coordinates $\xi_p = \zeta_p + i\eta_p$ by analogy with (7.1), i.e., $z_p = d_p \cosh \xi_p$. In these variables, the geometry of the pth inclusion is specified by a pair of parameters (ξ_{0p}, d_p), where $d_p = \sqrt{\left(l_{1p}\right)^2 - \left(l_{2p}\right)^2}$. The matrix-inclusion interfaces $\xi_p = \xi_{0p}$ are perfectly bonded. The stress in and around the inclusions is induced by the far field u^r taken in the form (7.19).

7.2.1 Direct (Superposition) Sum

We use the superposition principle to write the solution for a multiply-connected matrix domain:

$$u^{(0)} = u^r(z) + \sum_{p=1}^{N} u_p^s(z - Z_p), \tag{7.26}$$

where u_p^s is the disturbance from pth inclusion, decaying at $|z| \to \infty$. The corresponding complex potentials φ_p^s and ψ_p^s ahave the form (7.7) and (7.8), where, by analogy with (7.17), $A_{np} = B_{np} = 0$ for $n \leq 0$.

The sum in (7.26) contains the terms written in variables of different coordinate systems. In order to fulfil the interface bonding conditions, we need to find local expansion of (7.26). Our aim is to transform

$$u_p^s = \kappa_0 \varphi_p^s\left(z_p\right) - \left(z_p - \overline{z}_p\right)\overline{\varphi_p^{s\,\prime}\left(z_p\right)} - \overline{\psi_p^s\left(z_p\right)}, \tag{7.27}$$

where

$$\varphi_p^s = \sum_{n=0}^{\infty} A_{np} \upsilon_p^{-n}, \tag{7.28}$$

$$\psi_p^s = \psi_{0p}^s - \psi_{1p}^s = \sum_{n=0}^{\infty} \left[B_{np} - 2nA_{np} \frac{\sinh \zeta_{0p}}{\sinh \xi_p} \sinh(\xi_p - \zeta_{0p}) \right] \upsilon_p^{-n},$$

into u_{pq}^r, written in the same form as u_p^s, but in the qth coordinate basis, namely

$$u_{pq}^r = \kappa_0 \varphi_{pq}^r \left(z_q\right) - \left(z_q - \overline{z}_q\right) \overline{\varphi_{pq}^{r\,\prime} \left(z_q\right)} - \overline{\psi_{pq}^r \left(z_q\right)}, \tag{7.29}$$

with

$$\varphi_{pq}^r = \sum_n a_{npq} \upsilon_q^{-n}, \tag{7.30}$$

$$\psi_{pq}^r = \psi_{0pq}^r - \psi_{1pq}^r = \sum_n \left[b_{npq} - 2na_{npq} \frac{\sinh \zeta_{0q}}{\sinh \xi_q} \sinh(\xi_q - \zeta_{0q}) \right] \upsilon_q^{-n}.$$

7.2.2 Re-expansion Formulas

For this purpose, we use the re-expansion formulas for irregular complex potentials [45]

$$\upsilon_p^{-n} = \sum_m \eta_{nm}^{pq} \upsilon_q^{-m}, \quad n = 1, 2, \ldots; \tag{7.31}$$

Here, $\upsilon_p = \exp \xi_p = z_p/d_p \pm \sqrt{\left(z_p/d_p\right)^2 - 1}$, $z_p = z_q + Z_{pq}$; the expansion coefficients $\eta_{nm}^{pq} = \eta_{nm}\left(Z_{pq}, d_p, d_q\right)$

$$\eta_{nm}^{(1)pq} = (-1)^m\, n \left(\frac{d_p}{d_{pq}}\right)^n \sum_{j=0}^{\infty} \upsilon_{pq}^{-(n+m+2j)} \sum_{l=0}^{j} \frac{(-1)^{j-l}}{(j-l)!} \left(\frac{d_p}{d_{pq}}\right)^{m+2l} \times M_{nml}\left(d_p, d_q\right) \frac{(n+m+l+j-1)!}{(j-l)!}. \tag{7.32}$$

where $d_{pq} = d_p + d_q$ and $\upsilon_{pq} = Z_{pq}/d_{pq} + \sqrt{\left(Z_{pq}/d_{pq}\right)^2 - 1}$,

$$M_{nml}\left(d_p, d_q\right) = \sum_{k=0}^{l} \frac{\left(d_p/d_q\right)^{2k}}{k!\,(l-k)!\,(k+n)!\,(m+l-k)!}. \tag{7.33}$$

In the case $|z_p| > d_p$ and $|z_q| < |Z_{pq}|$ and $|Z_{pq}| > (d_p + d_q)$, the expansion coefficients simplify to

$$\eta_{nm}^{(2)pq} = nd_p^n (-1)^m \sum_{l=0}^{\infty} d_q^{2l+m} M_{nml} \left(d_p, d_q\right) \frac{(n+m+2l-1)!}{\left(2Z_{pq}\right)^{n+m+2l}}. \tag{7.34}$$

The series (7.31) converges within an ellipse centered in Z_q with inter-foci distance d_{pq} and passing the pole of pth elliptic coordinate system closest to Z_q. This convergence area is sufficient to solve for any two non-overlapping ellipses. For well-separated inclusions, both (7.34) and (7.32) give the same numerical value of η_{nm}. Therefore, when we solve numerically for many inclusions, the computational effort-saving strategy is to apply (7.32) to closest neighbors whereas interaction of the rest, more distant inclusions is evaluated using more simple formula (7.34). In the analogous to (7.31) addition theorem derived in [86], the expansion coefficients are expressed in terms of hypergeometric function.

Also, we mention two useful consequences of the formula (7.31). The first of them can be obtained by differentiating both parts of (7.31) with respect to z_q. It gives us

$$\frac{\upsilon_p^{-n}}{\sinh \xi_p} = \frac{d_p}{d_q} \sum_m \frac{m}{n} \eta_{nm}^{pq} \frac{\upsilon_q^{-m}}{\sinh \xi_q}, \tag{7.35}$$

being, in fact, an addition theorem for the alternate set of basic functions [45]. Another differentiation of (7.31), this time with respect to Z_{pq}, results in

$$\frac{\upsilon_p^{-n}}{\sinh \xi_p} = \sum_n \mu_{nm}^{pq} \upsilon_q^{-m}, \tag{7.36}$$

where $\mu_{nm}^{pq} = \frac{d_p}{n} \frac{d}{dZ_{pq}} \eta_{nm}^{pq}$. For μ_{nm} we also have two (general and simplified) expressions obtained by differentiating (7.32) and (7.34), respectively.

7.2.3 Local Expansion

By applying (7.31) to φ_p^s, one obtains

$$\varphi_p^s = \sum_{n=0}^{\infty} A_{np} \upsilon_p^{-n} = \sum_n a_{npq} \upsilon_q^{-n} = \varphi_{pq}^r, \tag{7.37}$$

where

$$a_{npq} = \sum_{m=1}^{\infty} A_{mp} \eta_{mn}^{pq} \tag{7.38}$$

and η_{mn}^{pq} are the expansion coefficients given by (7.32). With a_{npq} in the form (7.38), the first terms in (7.27) and (7.29) coincide, $\kappa_0\varphi_p^s = \kappa_0\varphi_{pq}^r$.

Determination of b_{npq} is more involved. From (7.37) we find also $\varphi_p^{s\,\prime}(z_p) = \varphi_{pq}^{r\,\prime}(z_q)$ and thus the second term in (7.27) can be written as

$$\left(z_p - \overline{z}_p\right)\varphi_p^{s\,\prime} = \left(Z_{pq} - \overline{Z_{pq}}\right)\varphi_{pq}^{r\,\prime} + \left(z_q - \overline{z}_q\right)\varphi_{pq}^{r\,\prime}. \tag{7.39}$$

To provide $u_p^s = u_{pq}^r$, we determine b_{npq} in (7.30) from

$$\psi_p^s = \psi_{pq}^r + \left(Z_{pq} - \overline{Z_{pq}}\right)\varphi_{pq}^{r\,\prime}. \tag{7.40}$$

To this end, all the terms in (7.40) should be expanded into a series of υ_q. For the details of derivation procedure, see [45]. Here, we give the final expression of b_{npq} for $n < 0$:

$$\begin{aligned} b_{npq} = {} & \sum_{m=1}^{\infty} B_{mp}\eta_{mn}^{pq} + \sum_{m=1}^{\infty} A_{mp}\left\{\frac{m}{2}\left(\upsilon_{0p} - \upsilon_{0p}^{-1}\right)^2 \mu_{m+1,n}^{pq}\right. \\ & + \left[n\left(\upsilon_{0q}^{-2} - 1\right) - n\left(1 - \upsilon_{0p}^{-2}\right)\right]\eta_{mn}^{pq} + \left(\upsilon_{0q} - \upsilon_{0q}^{-1}\right)^2 \\ & \times \left.\sum_{k=1}^{\infty}(2k-n)\eta_{m,2k-n}^{pq} + \frac{2}{d_q}\left(\overline{Z}_{pq} - Z_{pq}\right)\sum_{k=0}^{\infty}(2k+1-n)\eta_{m,2k+1-n}^{pq}\right\}; \end{aligned} \tag{7.41}$$

for $n > 0$, in accordance with (7.15),

$$b_{npq} = b_{-n,pq} + n\left(\upsilon_{0q}^{2} - \upsilon_{0q}^{-2}\right)a_{npq}. \tag{7.42}$$

Now, we come back to (7.26) and write

$$\sum_{p=1}^{N} u_p^s(z_p) = u_q^s(z_q) + u_q^r(z_q), \tag{7.43}$$

where $u_q^r(z_q) = \sum_{p\neq q} u_{pq}^r(z_q)$ has the form (7.29), (7.30) with replace φ_{pq}^r to $\varphi_q^r = \sum_{p\neq q}\varphi_{pq}^r$, ψ_{pq}^r to $\psi_q^r = \sum_{p\neq q}\psi_{pq}^r$; also,

$$a_{nq} = \sum_{p\neq q} a_{npq}, \quad b_{nq} = \sum_{p\neq q} b_{npq} \tag{7.44}$$

No problems arise with local expansion of the linear term in (7.26):

$$u^r(z) = U_q + u^r(z_q), \tag{7.45}$$

where $U_q = (X_{1q}E_{11} + X_{2q}E_{12}) + i(X_{1q}E_{12} + X_{2q}E_{22})$ is the constant and the formulas (7.19) and (7.19) apply to expand $u^r(z_q)$.

7.2.4 Resolving System

The resolving set of equations is:

$$\kappa_0 A_{nq} - \kappa_q C_{nq} + (\overline{D_{nq}} - 2n \sinh 2\zeta_{0q}\overline{C_{nq}})\upsilon_{0q}^{2n} = -\kappa_0 a_{nq} + \overline{b_{-n,q}}\upsilon_{0q}^{2n}; \quad (7.46)$$
$$\overline{B_{nq}} + \kappa_q C_{nq}\upsilon_{0q}^{2n} - \overline{D_{nq}} = \kappa_0 a_{-n,q}\upsilon_{0q}^{2n} - \overline{b_{nq}};$$
$$A_{nq} - \omega_q C_{nq} - \omega_q(\overline{D_{nq}} - 2n \sinh 2\zeta_{0q}\overline{C_{nq}})\upsilon_{0q}^{2n} = -a_{nq} - \overline{b_{-n,q}}\upsilon_{0q}^{2n};$$
$$\overline{B}_{nq} - \omega_q C_{nq}\upsilon_{0q}^{2n} - \omega_q\overline{D_{nq}} = -a_{-n,q}\upsilon_{0q}^{2n} - \overline{b_{nq}};$$
$$n = 1, 2, \ldots; \quad q = 1, 2, \ldots, N;$$

where $\omega_q = \mu_q/\mu_0$; C_{nq} and D_{nq} are the expansion coefficients of solution in the qth inclusion. To get it in an explicit form for direct solver one needs to substitute (7.30) and (7.41) into (7.44) and then into (7.46). Alternatively, the simple iterative solving procedure can be applied here: given some initial guess of A_{nq}, B_{nq}, C_{nq} and D_{nq} for $1 \leq q \leq N$, we compute a_{nq} and b_{nq} from (7.38), (7.41) and (7.44), then substitute into the right-hand side of (7.46) and solve it for the next approximation of unknown coefficients, etc. This procedure converges for a whole range of input parameters excluding the case of nearly touching inclusions where an appropriate initial approximation must be taken to provide convergence of numerical algorithm.

An extension of the above analysis to the infinite arrays of elliptic inclusions is obvious. The appropriate periodic potentials for the conductivity problem have been obtained in [86, 87]. The RUC model of elastic solid with various statistical (both uniform and clustered) distributions of cracks is considered in [49, 50]. The developed there theory, with minor modifications, applies to the RUC model of composite reinforced by the equally oriented, elliptic in cross-section fibers.

8 Fibrous Composite with Anisotropic Constituents

In [48], the multipole expansion based approach has been developed to study the stress field and effective elastic properties of unidirectional FRC composite with anisotropic phases. The most general case of elastic anisotropy which can be considered in 2D statement is the monoclinic symmetry with the Ox_1x_2 being the symmetry plane. In two-index notation, the generalized Hooke's law for this symmetry type has the form

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix}. \tag{8.1}$$

In the case of the fibers aligned in x_3-direction, a composite possesses the same anisotropy type of macroscopic elastic moduli.

8.1 Anti-Plane Shear Problem

8.1.1 Background Theory

In the case of shear deformation along the fiber axis, u_3 is the only non-zero component of the displacement vector:

$$u_1 = u_2 = 0;\ \ u_3 = w(x_1, x_2). \tag{8.2}$$

Also, we have two non-zero components of stress tensor, σ_{13} and σ_{23}; the equilibrium Equation $\nabla \cdot \sigma = 0$ takes the form

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0. \tag{8.3}$$

This problem is mathematically equivalent to the transverse conductivity problem in FRC.

The stress function $W(x_1, x_2)$ is introduced [55] as follows:

$$\sigma_{13} = C\frac{\partial W}{\partial x_2}, \quad \sigma_{23} = -C\frac{\partial W}{\partial x_1}. \tag{8.4}$$

The stress field (8.4) satisfies the equilibrium equation (8.3) identically. Also, we need to provide the strain compatibility condition

$$\frac{\partial \varepsilon_{13}}{\partial x_2} = \frac{\partial \varepsilon_{23}}{\partial x_1} = \frac{1}{2}\frac{\partial^2 w}{\partial x_1 \partial x_2}. \tag{8.5}$$

Inversion of (8.1) gives

$$\begin{aligned} 2\varepsilon_{13} &= S_{55}\sigma_{13} + S_{45}\sigma_{23}; \\ 2\varepsilon_{23} &= S_{45}\sigma_{13} + S_{44}\sigma_{23}; \end{aligned} \tag{8.6}$$

where $S_{55} = C_{44}/C^2$, $S_{45} = -C_{45}/C^2$, $S_{44} = C_{55}/C^2$ and $C = \sqrt{C_{44}C_{55} - C_{45}^2} > 0$. Substitution of (8.4) and (8.6) into (8.5) yields

$$S_{44}\frac{\partial^2 W}{\partial x_1^2} - 2S_{45}\frac{\partial^2 W}{\partial x_1 \partial x_2} + S_{55}\frac{\partial^2 w}{\partial x_2^2} = 0. \tag{8.7}$$

By introducing new complex variable $\xi = x_1 + \mu x_2$ [55], where

$$\mu = (-C_{45} + \mathrm{i}C)/C_{44} \tag{8.8}$$

is the root of the characteristic equation

$$C_{44}\mu^2 + 2C_{45}\mu + C_{55} = 0, \tag{8.9}$$

the Eq. (8.7) reduces to 2D Laplace equation $\frac{\partial^2 W}{\partial \xi \partial \overline{\xi}} = 0$. Hence, the theory of analytical functions [67] applies to this problem. Specifically, W can be taken as $W(x_1, x_2) = \mathrm{Re}\Phi$, where Φ is an analytical function of complex variable ξ. In this case also $w = -\mathrm{Im}\Phi$ and, thus, the boundary value problem for w can be formulated as the potential theory problem, in terms of Φ. The traction at interface is written as

$$t_n = \sigma_{13}n_1 + \sigma_{23}n_2 = C\left(\frac{\partial W}{\partial x_2}\cos\varphi - \frac{\partial W}{\partial x_1}\sin\varphi\right) = \frac{C}{2\rho}\frac{\partial}{\partial \varphi}\mathrm{Re}\Phi, \tag{8.10}$$

where n_1 and n_2 are the outward normal direction cosines and (ρ, φ) are the circular coordinates defined by $x_1 + ix_2 = \rho \exp(i\varphi)$.

Introducing the new variable $\xi = x_1 + \mu x_2$ is equivalent to affine transformation of the complex plane. It reduces the elasticity theory problem to that Laplace equation in the transformed coordinates. However, any affine transformation deforms the circular matrix-fiber interface, where the bonding conditions must be fulfilled, into elliptic one.

The exposed in the precious Section theory, with some modifications, is appropriate for this purpose. The series expansion solution of the potential theory problem is

$$\Phi = \sum_n D_n \upsilon^{-n}, \quad \upsilon = \xi/d \pm \sqrt{(\xi/d)^2 - 1}. \tag{8.11}$$

In the υ expression, sign before the square root is chosen to keep $\mathrm{Im}\upsilon \geq 0$, d is the matching parameter and D_n are the complex constants. The regularity of $\Phi(\xi)$ implies $D_{-n} = D_n$ [45]. In this case, Φ is expressed in terms of Chebyshev polynomial of complex variable ξ/d:

$$\upsilon^n + \upsilon^{-n} = 2\cosh\left[n\mathrm{Arccosh}\left(\xi/d\right)\right]; \tag{8.12}$$

for $n = 1$, $2\xi/d = \upsilon + 1/\upsilon$. In contrast, the multipole expansion of the disturbance field contains the negative powers of υ only: $\Phi = \sum_n A_n \upsilon^{-n}$, where $A_n = 0$ for $n \leq 0$.

An appropriate choice of parameter d provides a remarkably simple form of (8.11) at the circular interface $\rho = R$. In particular, for $d = R\sqrt{1+\mu^2}$

$$\upsilon\left|_{\rho=R}\right. = \left(\frac{1-i\mu}{1+i\mu}\right)^{1/2} R\,\exp(i\varphi) = R_\mu \exp(i\varphi) \tag{8.13}$$

so expression (8.11) at the interface appears to be the Fourier series of φ. Noteworthy, d can be viewed as the anisotropy parameter: for an isotropic solid, $d \equiv 0$. In the limiting case $d \to 0$, $\frac{d}{2}\upsilon \to z = x_1 + ix_2$ and $\frac{d}{2}R_\mu \to R$. Hence, at least for a weakly anisotropic material, the normalized expansion coefficients $\widetilde{A_n} = \left(\frac{d}{2}\right)^{-n} A_n$ and $\widetilde{D_n} = \left(\frac{d}{2}\right)^n D_n$ must be used in order to prevent possible numerical error accumulation.

8.1.2 Single Inclusion Problem

Let consider a single inclusion of radius R embedded in an infinite matrix. Both the matrix and inclusion materials are anisotropic, with the elastic stiffness tensors $\mathbf{C}^- = \left\{C_{ij}^-\right\}$ and $\mathbf{C}^+ = \left\{C_{ij}^+\right\}$, respectively. Hereafter, we will mark all the matrix-related parameters by the sign "$-$" and the inclusion-related ones by the sign "$+$". The uniform far field loading is prescribed by the constant strain tensor $\mathbf{E} = \left\{E_{ij}\right\}$; in the anti-plane shear problem, only E_{13} and E_{23} are non-zero. The perfect bonding between the matrix and inclusion $[[w]] = [[t_n]] = 0$ is assumed.

The regular displacement field w^+ in the inclusion is expanded into series (8.11):

$$w^+ = -\mathrm{Im}\Phi^+, \quad \Phi^+ = \sum_n D_n \left(\upsilon^+\right)^{-n}, \tag{8.14}$$

where $D_{-n} = D_n$ and $\upsilon^\pm$ is defined by (8.11), with $d^\pm = R\sqrt{1+\left(\mu^\pm\right)^2}$.

The displacement field w^- in the matrix is written as the superposition sum of far linear field w_0^- and the disturbance field w_1^- caused by the inclusion. We seek w^- in the form

$$w^- = -\mathrm{Im}\Phi^-, \; \Phi^- = \Gamma\xi^- + \sum_{n=1}^{\infty} A_n \left(\upsilon^-\right)^{-n} = \sum_n \left(A_n + a_n\right)\left(\upsilon^-\right)^{-n}, \tag{8.15}$$

where $\Gamma = \Gamma_1 + i\Gamma_2$ and $a_n = \frac{1}{2}\delta_{n1}\Gamma d^-$, δ_{nm} being the Kronecker's delta. The corresponding to $\mathbf{E}$ linear displacement field is $w_0^- = E_{13}x_1 + E_{23}x_2$. On the other hand,

$$w_0^- = -\mathrm{Im}\left[(\Gamma_1 + i\Gamma_2)\left(x_1 + \mu^- x_2\right)\right] = -\Gamma_2 x_1 - \left(\Gamma_1 \frac{C^-}{C_{44}^-} - \Gamma_2 \frac{C_{45}^-}{C_{44}^-}\right) x_2. \tag{8.16}$$

This yields

$$\Gamma = -\left(\frac{C_{44}^- + iC^-}{C^-} E_{13} + \frac{C_{44}^-}{C^-} E_{23}\right). \tag{8.17}$$

The interface bonding conditions in terms of complex potentials Φ^- and Φ^+ take the form

$$\left(\Phi^+ - \overline{\Phi^+}\right)|_{\rho=R} = \left(\Phi^- - \overline{\Phi^-}\right)|_{\rho=R}, \tag{8.18}$$
$$\frac{C^+}{C^-}\frac{\partial}{\partial\varphi}\left(\Phi^+ + \overline{\Phi^+}\right)|_{\rho=R} = \frac{\partial}{\partial\varphi}\left(\Phi^- + \overline{\Phi^-}\right)|_{\rho=R}.$$

By choosing the center of inclusion as a reference point, we get $D_0 = 0$. All other coefficients are found from (8.18). Substitution of (8.14) and (8.15) into the first condition in (8.18) yields

$$\begin{aligned}\sum_n &\left[D_n \left(R_{\mu^+}\right)^{-n} e^{-in\varphi} - \overline{D_n}\left(\overline{R_{\mu^+}}\right)^{-n} e^{in\varphi}\right] \\ &= \sum_n \left[A_n \left(R_{\mu^-}\right)^{-n} e^{-in\varphi} - \overline{A_n}\left(\overline{R_{\mu^-}}\right)^{-n} e^{in\varphi}\right. \\ &\quad \left. + a_n \left(R_{\mu^-}\right)^{-n} e^{-in\varphi} - \overline{a_n}\left(\overline{R_{\mu^-}}\right)^{-n} e^{in\varphi}\right].\end{aligned} \tag{8.19}$$

By equating the Fourier coefficients in the left and right hand parts of (8.19), we obtain a set of linear equations

$$(A_n + a_n)\left(R_{\mu^-}\right)^{-n} - \overline{a_n}\left(\overline{R_{\mu^-}}\right)^n = D_n\left(R_{\mu^+}\right)^{-n} - \overline{D_n}\left(\overline{R_{\mu^+}}\right)^n, \quad n = 1, 2, \ldots. \tag{8.20}$$

The second condition in (8.18) gives rise to another set of equations:

$$(A_n + a_n)\left(R_{\mu^-}\right)^{-n} + \overline{a_n}\left(\overline{R_{\mu^-}}\right)^n = \frac{C^+}{C^-}\left[D_n\left(R_{\mu^+}\right)^{-n} + \overline{D_n}\left(\overline{R_{\mu^+}}\right)^n\right], \quad n = 1, 2, \ldots. \tag{8.21}$$

The Eqs. (8.17), (8.20) and (8.21) form a closed infinite system of linear equations with the unknowns A_n and D_n which can be solved with any desirable accuracy by the truncation method [27]. In this specific problem, $A_n = D_n = 0$ for $n > 1$; an explicit analytical solution of the problem is readily found from two equations in (8.20) and (8.21) for $n = 1$. As seen from (8.20) and (8.21), the polynomial far displacement field induces in the inclusion the polynomial displacement field of the same order, in accordance with [28].

8.1.3 Finite Array of Inclusions

Now, we consider an infinite matrix domain containing a finite number N of circular inclusions centered in the points $Z_q = X_{1q} + iX_{2q}$. For simplicity, we put $Z_1 = 0$ and take all the inclusions identical: $R_q = R$ and $\mathbf{C}_q^+ = \mathbf{C}^+$. The inclusions do not overlap: $\left|Z_{pq}\right| > 2R_1$, where $Z_{pq} = Z_p - Z_q = X_{1pq} + iX_{2pq}$. The conditions in the remote points and at the interfaces $\rho_q = R$ are the same as in the previous problem and ρ_q is defined by $x_{1q} + ix_{2q} = \rho_q \exp\left(i\varphi_q\right)$. Here, $O_q x_{1q} x_{2q}$ is the local coordinate system associated with qth inclusion. All other inclusion-related parameters are marked by subscript "q": so, $\xi_q^\pm = x_{1q} + \mu^\pm x_{2q}$ and $\upsilon_q^\pm = \xi_q^\pm / d^\pm \pm \sqrt{\left(\xi_q^\pm / d^\pm\right)^2 - 1}$.

By analogy with (8.14), the displacement field in qth inclusion is written as

$$w_q^+ = -\mathrm{Im}\Phi_q^+, \quad \Phi_q^+ = \sum_n D_{nq} \left(\upsilon_q^+\right)^{-n}. \tag{8.22}$$

The solution in the matrix domain is given by the superposition sum of far linear field w_0^- and the disturbance fields w_q^- from each individual inclusion:

$$w^- = -\mathrm{Im}\Phi^-, \quad \Phi^- = \Gamma\xi_1^- + \sum_{p=1}^{N} \Phi_p^- \text{ and } \Phi_p^- = \sum_{n=1}^{\infty} A_{np} \left(\upsilon_p^-\right)^{-n}. \tag{8.23}$$

The expansion coefficients D_{nq} (8.22) and A_{np} (8.23) are determined from the interface boundary conditions. For this purpose, we need the local expansion of w^- in a vicinity of qth inclusion.

An expansion of w_0^- is simple: $\xi^- = \xi_1^- = \xi_q^- + \Xi_{1q}$, where $\Xi_{pq} = X_{1pq} + \mu^- X_{2pq}$. Hence, $w_0^- = -\mathrm{Im}\Gamma\xi^- = -\mathrm{Im}\Gamma\xi_q^- + W_0$, where $W_0 = -\mathrm{Im}\left(\Gamma\ \Xi_{1q}\right)$ is the rigid body motion of qth inclusion. The singular in the point Z_q term Φ_q^- is initially written in variables of qth local coordinate system. The terms Φ_p^- in (8.23) are regular in a vicinity and in the point Z_q for $p \neq q$. Their local expansion in the form analogous to (8.22) is obtained by applying the re-expansion formula (7.31) [45], with the coefficients $\eta_{nm}^{pq} = \eta_{nm}\left(\Xi_{pq}, d_p, d_p\right)$ given by (7.32), (7.34). In fact, we only replace in (7.31) z_p with $\xi_p = x_p + \mu y_p$ and Z_{pq} with Ξ_{pq}. In the problem we consider, $d_p = d_q = \sqrt{1 + \mu^2}$ which simplifies η_{nm} considerably.

Applying (7.31) to the last sum in (8.23) yields

$$\begin{aligned}\Phi^- &= \Gamma \Xi_{pq} + \Gamma \frac{d^-}{2}\left(\upsilon_q^- + 1/\upsilon_q^-\right) + \Phi_q^- + \sum_{\substack{p=1\\p\neq q}}^{N}\sum_{n=1}^{\infty} A_{np}\left(\upsilon_p^-\right)^{-n} \\ &= \Gamma \Xi_{pq} + \sum_n \left(A_{nq} + a_{nq}\right)\left(\upsilon_q^-\right)^{-n},\end{aligned} \tag{8.24}$$

where

$$a_{nq} = \sum_{p=1}^{N}\sum_{m=1}^{\infty} A_{mp}\eta_{nm}^{pq} + \Gamma\frac{d^-}{2}\delta_{n,\pm 1}. \tag{8.25}$$

In (8.25), $\eta_{nm}^{pq} \equiv 0$ for $p = q$, $a_{nq} = a_{-n,q}$ and $A_{nq} = 0$ for $n \leq 0$.

The interface conditions in terms of complex potentials take the form (8.18), with replace Φ^+ to Φ_q^+ and (ρ, φ) to $\left(\rho_q, \varphi_q\right)$. Their fulfilling gives the analogous to (8.20), (8.21) infinite system of linear algebraic equations of remarkably simple form:

$$\begin{aligned}&\left(A_{nq} + a_{nq}\right)\left(R_{\mu^-}\right)^{-n} - \overline{a_{nq}}\left(\overline{R_{\mu^-}}\right)^{n} = D_{nq}\left(R_{\mu^+}\right)^{-n} - \overline{D_{nq}}\left(\overline{R_{\mu^+}}\right)^{n}, \\ &\left(A_{nq} + a_{nq}\right)\left(R_{\mu^-}\right)^{-n} + \overline{a_{nq}}\left(\overline{R_{\mu^-}}\right)^{n} = \frac{C^+}{C^-}\left[D_{nq}\left(R_{\mu^+}\right)^{-n} + \overline{D_{nq}}\left(\overline{R_{\mu^+}}\right)^{n}\right], \\ &\qquad n = 1, 2, \ldots; \quad q = 1, 2, \ldots, N.\end{aligned} \tag{8.26}$$

8.1.4 Periodic Complex Potentials

The periodic complex potentials υ_n^* are given by the 2D lattice sums:

$$\upsilon_n^*(\xi) = \sum_{\alpha,\beta}\left[\upsilon(\xi - L_{\alpha\beta})\right]^{-n}, \quad n = 1, 2, \ldots; \tag{8.27}$$

where $L_{\alpha\beta} = \alpha a + \mu\beta a$ and α and β are the integer numbers: $-\infty < \alpha, \beta < \infty$. These functions possess a countable set of cuts centered in the points $Z_{\alpha\beta} = \alpha a + i\beta b$ and yield the periodicity conditions [48]

$$\upsilon_n^*(x + a, y) - \upsilon_n^*(x, y) = 0; \quad \upsilon_n^*(x, y + b) - \upsilon_n^*(x, y) = \delta_{n1}\frac{d\pi i}{b\mu}. \tag{8.28}$$

The local expansion of $\upsilon_n^*(\xi_p)$ in terms of $\upsilon_p = \upsilon(\xi_p)$ $\left(\xi_p = x_p + \mu y_p\right)$ is obtained with aid of the formula (7.31):

$$\upsilon_n^*(\xi_p) = \left(\upsilon_p\right)^{-n} + \sum_m \eta_{nm}^{*pp}\left(\upsilon_p\right)^{-m}, \tag{8.29}$$

where

$$\eta_{nm}^{*pp} = \sum_{\alpha,\beta} {}' \eta_{nm} \left(L_{\alpha\beta}, d, d\right). \tag{8.30}$$

The upper strike means absence in this sum of the term with $\alpha = \beta = 0$. An expansion of $\upsilon_n^*(\xi_p)$ in terms of υ_q for $p \neq q$ also uses the re-expansion (7.31) and yields

$$\upsilon_n^*(\xi_p) = \sum_m \eta_{nm}^{*pq} \left(\upsilon_q\right)^{-m}, \tag{8.31}$$

where

$$\eta_{nm}^{*pq} = \sum_{\alpha,\beta} \eta_{nm} \left(\Xi_{pq} + L_{\alpha\beta}, d, d\right). \tag{8.32}$$

The following efficient way of the lattice sums (8.27) evaluation takes an advantage of two η_{nm} expressions, $\eta_{nm}^{(1)pq}$ (7.32) and $\eta_{nm}^{(2)pq}$ (7.34). We write (8.32) as

$$\begin{aligned} \eta_{nm}^{*pq} = & \sum_{\left|\Xi_{pq}+L_{\alpha\beta}\right| \leq L^*} \eta_{nm} \left(\Xi_{pq} + L_{\alpha\beta}, d, d\right) \\ & + \sum_{\left|\Xi_{pq}+L_{\alpha\beta}\right| > L^*} \eta_{nm} \left(\Xi_{pq} + L_{\alpha\beta}, d, d\right), \end{aligned} \tag{8.33}$$

where L^* is taken sufficiently large to provide applicability of the formula (7.34) to all the terms of the second sum in (8.33). Next, we re-arrange (8.33) as

$$\begin{aligned} \eta_{nm}^{*pq} &= \sum_{\left|\Xi_{pq}+L_{\alpha\beta}\right| \leq L^*} \eta_{nm}^{(1)pq} + \sum_{\left|\Xi_{pq}+L_{\alpha\beta}\right| > L^*} \eta_{nm}^{(2)pq} \\ &= \sum_{\left|\Xi_{pq}+L_{\alpha\beta}\right| \leq L^*} \left(\eta_{nm}^{(1)pq} - \eta_{nm}^{(2)pq}\right) + \sum_{\alpha,\beta} \eta_{nm}^{(2)pq}. \end{aligned} \tag{8.34}$$

The first sum in the right hand side of (8.34) is finite and no problem arise with its evaluation. In the second sum, we change the summation order to get

$$\sum_{\alpha,\beta} \eta_{nm}^{(2)pq} = n\,(-1)^m \sum_{l=0}^{\infty} S_{n+m+2l}^{pq*} \left(\frac{d}{2}\right)^{n+m+2l} M_{nml}\,(n+m+2l-1)!, \tag{8.35}$$

where

$$S_n^{pq*} = \sum_{\alpha,\beta} \left(\varXi_{pq} + L_{\alpha\beta}\right)^{-n} = \sum_{\alpha,\beta} \left[\left(X_{pq} + \alpha a\right) + \mu\left(Y_{pq} + \beta a\right)\right]^{-n}, n \geq 2. \tag{8.36}$$

The standard lattice sums S_n^{pq*} (see also (6.50)) are computed using the Evald's method or another fast summation technique ([17, 18, 65]; among others).

8.1.5 RUC Model

In the framework of RUC model, we consider macroscopically uniform stress field in FRC with anisotropic constituents. Periodicity of geometry induces the periodicity of the local stress filed

$$\sigma_{ij}\left(x_1 + a, x_2\right) = \sigma_{ij}\left(x_1, x_2 + a\right) = \sigma_{ij}\left(x_1, x_2\right) \tag{8.37}$$

and quasi-periodicity of the displacement field w:

$$w\left(x_1 + a, x_2\right) - w\left(x_1, x_2\right) = E_{23}a \text{ and } w\left(x_1, x_2 + a\right) - w\left(x_1, x_2\right) = E_{13}a. \tag{8.38}$$

To solve this model boundary value problem, it suffices to replace the singular potentials $\varPhi_p^-$ in (8.23) with their periodic counterparts

$$w^- = -\mathrm{Im}\varPhi^-, \quad \varPhi^- = \varGamma^* \xi_1^- + \sum_{p=1}^{N} \varPhi_p^* \text{ and } \varPhi_p^* = \sum_{n=1}^{\infty} A_{np} \upsilon_n^*(\xi_p), \tag{8.39}$$

where $\upsilon_n^*(\xi_p)$ are the periodic analytical functions (8.27). First, we substitute (8.39) into (8.38) to obtain, with the periodicity of υ_n^* (8.28) taken into account,

$$\varGamma^* = \frac{d^- \pi i}{ab\mu^-} \sum_{p=1}^{N} \overline{A_{1p}} + \varGamma, \tag{8.40}$$

where $\varGamma$ is given by (8.17). The subsequent flow of solution resembles the above procedure, with minor modifications. To get the local expansion, the formulas (8.29) and (8.31) are used. An infinite algebraic system has the form (8.26), where in the a_{nq} expression (8.25) the coefficients η_{nm}^{pq} must be replaced with the corresponding lattice sums η_{nm}^{*pq} given by the formula (8.30) for $p = q$ and by the formula (8.32) otherwise.

8.2 Plane Strain Problem

In the plane strain problem, $u_1 = u(x_1, x_2)$, $u_2 = v(x_1, x_2)$ and $u_3 = 0$. Non-zero components of the strain tensor

$$\varepsilon_{11} = \frac{\partial u}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial v}{\partial x_2}, \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right) \tag{8.41}$$

relate the stress tensor components by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}. \tag{8.42}$$

The stress equilibrium equations take the form

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0. \tag{8.43}$$

8.2.1 General Solution

A general solution of the plane strain problem (8.43) is written in terms of stress functions [55] as

$$\begin{aligned} u_1 &= 2\mathrm{Re}\, p_j \Phi_j; \quad u_2 = 2\mathrm{Re}\, q_j \Phi_j; \\ \sigma_{11} &= 2\mathrm{Re}\, \mu_j^2 \Phi_j'; \quad \sigma_{22} = 2\mathrm{Re}\, \Phi_j'; \quad \sigma_{12} = 2\mathrm{Re}\, \mu_j \Phi_j'. \end{aligned} \tag{8.44}$$

In (8.44), $\Phi_j = \Phi_j\left(\xi_j\right)$ $(j = 1, 2)$ are the analytical functions of the complex variable $\xi_j = x_1 + \mu_j x_2$ and $\Phi_j' = \partial \Phi_j / \partial \xi_j$. Hereafter, the summation convention is assumed. The complex numbers μ_j $\left(\mathrm{Im}\mu_j > 0\right)$ are the roots of the characteristic equation

$$C_{11}\mu^4 - 2C_{16}\mu^3 + (2C_{12} + C_{66})\,\mu^2 - 2C_{26}\mu + C_{22} = 0 \tag{8.45}$$

and

$$p_j = C_{11}\mu_j^2 - C_{16}\mu_j + C_{12}, \quad q_j = C_{12}\mu_j - C_{26} + C_{22}/\mu_j, \quad j = 1, 2. \tag{8.46}$$

Thus, the problem consists in finding two analytical functions, Φ_j. We assume the perfect bonding between the matrix and inclusions: i.e., the displacement $\mathbf{u}$ and normal traction $\mathbf{t}_n = \sigma \cdot \mathbf{n}$ vectors are continuous through the interface:

$$[[\mathbf{u}]] = [[\mathbf{t}_n]] = 0. \tag{8.47}$$

In contrast to anti-plane problem, $\mathbf{u}$ and $\mathbf{t}_n$ contain two non-zero components.

The first condition in (8.47) equates the Cartesian components of $\mathbf{u}$, from where we get

$$\left[\left[\mathrm{Re} p_j \Phi_j\right]\right] = 0; \quad \left[\left[\mathrm{Re} q_j \Phi_j\right]\right] = 0. \tag{8.48}$$

The normal stress continuity condition (8.47) is equivalent to [55]

$$\left[\left[\mathrm{Re} \Phi_j\right]\right] = 0; \quad \left[\left[\mathrm{Re} \mu_j \Phi_j\right]\right] = 0. \tag{8.49}$$

Thus, a general solution of the plane strain problem for anisotropic solid is expressed in terms of harmonic potentials and hence the technique developed for the anti-plane shear problem works equally to the plane strain problem.

8.2.2 Single Inclusion Problem

Let consider a single circular inclusion embedded into an infinite matrix. The uniform far field is prescribed by the constant strain tensor $\mathbf{E}$, with the non-zero components E_{11}, E_{22} and $E_{12} = E_{21}$. By analogy with (8.15), the matrix displacement vector $\mathbf{u}^-$ is the superposition sum of far field $\mathbf{u}_0^- = \mathbf{E} \cdot \mathbf{r}$ and disturbance field $\mathbf{u}_f^-$. The potentials in the matrix Φ_j^- and inclusion Φ_j^+ are taken in the form

$$\Phi_j^- = \Gamma_j \xi_j^- + \sum_{n=1}^{\infty} A_{nj} \left(v_j^-\right)^{-n}, \quad \Phi_j^+ = \sum_n D_{nj} \left(v_j^+\right)^{-n}, \quad j = 1, 2; \tag{8.50}$$

where Γ_j, A_{nj} and D_{nj} are the complex constants. The Γ_j are entirely determined by the far field:

$$E_{11} x_1 + E_{12} x_2 = 2\mathrm{Re} p_j^- \Gamma_j \xi_j^-; \quad E_{12} x_1 + E_{22} x_2 = 2\mathrm{Re} q_j^- \Gamma_j \xi_j^-. \tag{8.51}$$

Taking account of $\xi_j^- = x_1 + \mu_j^- x_2$ gives the resolving set of equations for Γ_j:

$$2\mathrm{Re} p_j^- \Gamma_j = E_{11}; \quad 2\mathrm{Re} q_j^- \Gamma_j = 2\mathrm{Re} \mu_j^- p_j^- \Gamma_j = E_{12}; \quad 2\mathrm{Re} \mu_j^- q_j^- \Gamma_j = E_{22}. \tag{8.52}$$

Next, we substitute (8.50) and (8.51) into (8.48), (8.49) and use (8.13) to get the linear system analogous to (8.20), (8.21). The interface bonding conditions (8.48), (8.49) are written in the unified form (no summation over k):

$$\left[\left[\mathrm{Re} \kappa_{jk} \Phi_j\right]\right] = 0 \quad (k = 1, 2, 3, 4); \tag{8.53}$$

where $\kappa_{j1} = p_j, \kappa_{j2} = q_j, \kappa_{j3} = 1$ and $\kappa_{j4} = \mu_j$. We transform each of these equations to get consequently:

$$\kappa_{jk}^{+}\Phi_j^{+} + \overline{\kappa_{jk}^{+}\Phi_j^{+}} = \kappa_{jk}^{-}\Phi_j^{-} + \overline{\kappa_{jk}^{-}\Phi_j^{-}};$$

$$\sum_{j=1}^{2}\sum_{n}\left[\kappa_{jk}^{+}D_{nj}\left(R_{\mu_j^+}\right)^{-n}e^{-in\varphi} + \overline{\kappa_{jk}^{+}}\,\overline{D_{nj}}\left(\overline{R_{\mu_j^+}}\right)^{-n}e^{in\varphi}\right] \tag{8.54}$$

$$= \sum_{j=1}^{2}\sum_{n}\left[\kappa_{jk}^{-}\left(A_{nj}+a_{nj}\right)\left(R_{\mu_j^-}\right)^{-n}e^{-in\varphi} + \overline{\kappa_{jk}^{-}}\left(\overline{A_{nj}}+\overline{a_{nj}}\right)\left(\overline{R_{\mu_j^-}}\right)^{-n}e^{in\varphi}\right];$$

where $a_{nj} = \frac{1}{2}\delta_{n1}\Gamma_j d_j^-$ and, finally,

$$\sum_{j=1}^{2}\left[\overline{\kappa_{jk}^{-}}\left(\overline{A_{nj}}+\overline{a_{nj}}\right)\left(\overline{R_{\mu_j^-}}\right)^{-n} + \kappa_{jk}^{-}a_{nj}\left(R_{\mu_j^-}\right)^{n}\right] \tag{8.55}$$

$$= \sum_{j=1}^{2}\left[\kappa_{jk}^{+}D_{nj}\left(R_{\mu_j^+}\right)^{n} + \overline{\kappa_{jk}^{+}}\,\overline{D_{nj}}\left(\overline{R_{\mu_j^+}}\right)^{-n}\right]; \quad n = 1, 2, \ldots, \quad k = 1, 2, 3, 4.$$

Together with (8.52), these equations form a complete system of equations for A_{nj} and D_{nj}. In the Eshelby's problem, $A_{nj} = D_{nj} = 0$ for $n > 1$ and the solution if given by four Eq. (8.52).

8.2.3 Array of Inclusions

An analysis of the multiple-inclusion plane strain problem is a mere compilation of the results exposed above, so here we give only a brief summary of the relevant formulas. The interface conditions are written in terms of harmonic potentials as

$$\mathrm{Re}\left(\kappa_{jk}^{+}\Phi_{qj}^{+} - \kappa_{jk}^{-}\Phi_j^{-}\right)\big|_{\rho_q=R} = 0; \quad k = 1, 2, 3, 4; \quad q = 1, 2, \ldots, N. \tag{8.56}$$

Here,

$$\Phi_{qj}^{+} = \sum_{n} D_{nqj}\left(\upsilon_{qj}^{+}\right)^{-n}, \quad j = 1, 2; \tag{8.57}$$

where $\xi_{qj}^{\pm} = x_{1q} + \mu_j^{\pm}x_{2q}$ and $\upsilon_{qj}^{\pm} = \xi_{qj}^{\pm}/d_j^{\pm} \pm \sqrt{\left(\xi_{qj}^{\pm}/d_j^{\pm}\right)^2 - 1}$. The local variables relate each other by $\xi_{pj}^{-} = \xi_{qj}^{-} + \Xi_{pqj}$, where $\Xi_{pqj} = X_{1pq} + \mu_j^{-}X_{2pq}$.

By analogy with (8.23),

$$\Phi_j^- = \Gamma_j \xi_{1j}^- + \sum_{p=1}^{N} \Phi_{pj}^-, \quad \text{where } \Phi_{pj}^- = \sum_{n=1}^{\infty} A_{npj} \left(\upsilon_{pj}^-\right)^{-n}. \tag{8.58}$$

The local expansion for Φ_j^- is analogous to that described above (see (8.24)). After some algebra, we get

$$\Phi_j^- = \Gamma_j \Xi_{pqj} + \sum_{n=0}^{\infty} \left(A_{nqj} + a_{nqj}\right) \left(\upsilon_{qj}^-\right)^{-n}, \tag{8.59}$$

where

$$a_{nqj} = \sum_{p=1}^{N} \sum_{m=1}^{\infty} A_{mpj} \eta_{nmj}^{pq} + \frac{\Gamma_j d_j^-}{2} \delta_{n,\pm 1} \text{ and } \eta_{nmj}^{pq} = \eta_{nm} \left(\Xi_{pqj}, d_j^-, d_j^-\right). \tag{8.60}$$

The resulting infinite system of linear equations is

$$\begin{gathered} \sum_{j=1}^{2} \left[\overline{\kappa_{jk}^-} \left(\overline{A_{nqj}} + \overline{a_{nqj}}\right) \left(\overline{R_{\mu_j^-}}\right)^{-n} + \kappa_{jk}^- a_{nqj} \left(R_{\mu_j^-}\right)^n \right] \\ = \sum_{j=1}^{2} \left[\kappa_{jk}^+ D_{nqj} \left(R_{\mu_j^+}\right)^n + \overline{\kappa_{jk}^+} \overline{D_{nqj}} \left(\overline{R_{\mu_j^+}}\right)^{-n} \right]; \\ n = 1, 2, \ldots; \quad q = 1, 2, \ldots, N; \quad k = 1, 2, 3, 4. \end{gathered} \tag{8.61}$$

The solution of the plane strain problem for the cell type model of FRC requires only minor modification of the above formulas; for the details, we refer to [48].

8.3 Effective Stiffness Sensor

In order to get the analytical expression for the effective stiffness tensor $\mathbf{C}^*$ (1.10) of fibrous composite with anisotropic constituents, we average the local strain and stress fields over the RUC, where $V = a^2$ is the cell volume (unit length is assumed in x_3-direction). So,

$$2V \left\langle \varepsilon_{ij} \right\rangle = \left(\int_{V_0} + \sum_{q=1}^{N} \int_{V_q} \right) \left(u_{i,j} + u_{j,i}\right) dV, \tag{8.62}$$

With aid of Gauss' theorem, we transform the volume integral into surface one:

$$2V\left\langle\varepsilon_{ij}\right\rangle=\int_{\Sigma}\left(u_i^-n_j+u_j^-n_i\right)dS \qquad (8.63)$$
$$+\sum_{q=1}^{N}\left[\int_{S_q}\left(u_i^+n_j+u_j^+n_i\right)\,dS-\int_{S_q}\left(u_i^-n_j+u_j^-n_i\right)\,dS\right],$$

where Σ is the cell outer surface and n_i are the components of the unit normal vector. From the displacement continuity condition and decomposition $\mathbf{u}^-=\mathbf{E}\cdot\mathbf{r}+\mathbf{u}_f^-$ we get $\left\langle\varepsilon_{ij}\right\rangle=E_{ij}$; as expected, $\mathbf{E}$ is the macroscopic strain tensor.

The macroscopic stress tensor is written as

$$V\left\langle\sigma_{ij}\right\rangle=\int_{V_0}\sigma_{ij}^-dV+\sum_{q=1}^{N}\int_{V_q}\sigma_{ij}^+dV \qquad (8.64)$$
$$=VC_{ijkl}^-\left\langle\varepsilon_{kl}\right\rangle+\left(C_{ijkl}^+-C_{ijkl}^-\right)\sum_{q=1}^{N}\int_{V_q}\varepsilon_{kl}^+dV.$$

where $V_q=\pi R^2$ is the volume of qth fiber and $V_0=ab-N\pi R^2$ is the matrix volume inside the cell: $V=V_0+\sum_{q=1}^{N}V_q$. By applying Gauss' theorem, the integrals in (8.64) are reduced to

$$I_{kl}=\int_{V_q}\varepsilon_{kl}^+dV=\frac{1}{2}\int_{S_q}\left(u_k^+n_l+u_l^+n_k\right)dS. \qquad (8.65)$$

and can be taken analytically. In the anti-plane shear problem, only the I_{13} and I_{23} are non-zero. It is convenient to evaluate the following combination of these two:

$$I_{13}+iI_{23}=\frac{1}{2}\int_{S_q}w_q^+\left(n_1+in_2\right)dS=\frac{R}{2}\int_0^{2\pi}w_q^+\left|_{\rho_q=R}\right.e^{i\varphi}d\varphi. \qquad (8.66)$$

Taking account the explicit form of w_q^+ series expansion at the interface $\rho_q=R$ (8.19)

$$w_q^+=2\mathrm{i}\sum_{n=1}^{\infty}\mathrm{Im}\left\{D_{nq}\left[\left(R_{\mu^+}\right)^n e^{in\varphi}+\left(R_{\mu^+}\right)^{-n}e^{-in\varphi}\right]\right\}, \qquad (8.67)$$

we find

$$I_{13}+iI_{23}=-2\pi R\,\mathrm{Im}\left(D_{1q}/R_{\mu^+}\right) \qquad (8.68)$$

This formula is sufficient for evaluation of the effective moduli C_{2323}^*, C_{1313}^* and C_{1323}^* or, in two-index notation, C_{44}^*, C_{55}^* and C_{45}^*.

The in-plane effective moduli, namely, C_{11}^*, C_{22}^*, C_{12}^*, C_{16}^*, C_{26}^* and C_{66}^* are found by integrating the strain and stress fields found from the plane strain problem. We have

$$I_{11} = \int_{V_q} \varepsilon_{11}^+ dV = \frac{1}{2}\int_{S_q} u_1^+ n_1 dS = \frac{R}{2}\int_0^{2\pi} u_q^+ \left|_{\rho_q=R} \left(e^{i\varphi} + e^{-i\varphi}\right) d\varphi, \right. \quad (8.69)$$

where, according to (8.57),

$$u_q^+ \left|_{\rho_q=R} = 2\mathrm{Re}\sum_{j=1}^{2} p_j^+ \sum_{n=0}^{\infty} D_{nqj} \left[\left(R_{\mu_j^+}\right)^n e^{in\varphi} + \left(R_{\mu_j^+}\right)^{-n} e^{-in\varphi}\right]. \right. \quad (8.70)$$

Its substitution into (8.69) gives

$$I_{11} = 2\pi R \mathrm{Re}\sum_{j=1}^{2} p_j^+ D_{1qj}\left(R_{\mu_j^+} + 1/R_{\mu^+}\right). \quad (8.71)$$

In the same way, we find

$$I_{22} = 2\pi \mathrm{Im}\sum_{j=1}^{2} q_j^+ D_{1qj}\left(R_{\mu_j^+} - 1/R_{\mu^+}\right). \quad (8.72)$$

and

$$\begin{aligned} I_{12} = \pi R \Bigg[& \mathrm{Re}\sum_{j=1}^{2} q_j^+ D_{1qj}\left(R_{\mu_j^+} + 1/R_{\mu^+}\right) \\ & + \mathrm{Im}\sum_{j=1}^{2} p_j^+ D_{1qj}\left(R_{\mu_j^+} - 1/R_{\mu^+}\right)\Bigg]. \end{aligned} \quad (8.73)$$

The effective moduli C_{13}^*, C_{23}^*, C_{33}^* and C_{36}^* are found from the generalized plane strain problem, see [16].

Summary of the Method

Here, we summarize briefly the specific features of the Multipole Expansion Method.

- The scheme of the method is simple and involves a few mandatory steps. In application to the multiple-inclusion problem, they are:
  - representation of solution in the multiply-connected domain as the superposition sum;

- local regular expansion of this sum, with aid of the relevant re-expansion formulas, in a vicinity of each inclusion;
- fulfillment of the interface bonding conditions and thereby reduction of the boundary value problem to a linear set of algebraic equations;
- numerical solution of the truncated linear system;
- evaluation of the field variables and effective properties of composite from the explicit algebraic formulas.

- Application of the method is case-dependent in the sense that the geometry of specific problem dictates the form of partial solutions. In many practically important cases, these solutions and their properties are well established. Noteworthy, with use the appropriate math—vectorial solutions in 3D and complex potentials in 2D theory—the method appears equally simple for the scalar and vectorial problems.
- The method gives a *complete* solution of the boundary value problem including the local fields and effective properties. The obtained by this method exact analytical and numerical results can be regarded as the benchmarks for other, existing or newly developed, methods of the micromechanics of composites.
- Numerical efficiency of the method is high due to the fact that the most work on solution is done analytically. The computational algorithm includes three simple steps, namely

  - evaluation of the matrix coefficients (minor computational effort),
  - iterative solution of the linear system (major computational effort),
  - evaluation of the local fields and effective moduli (negligible effort).

  Remarkably, the bigger number of inclusions under study is, the more efficient becomes the algorithm: on application of the fast multipole scheme to the large-scale models, the total computational effort scales as $O(N)$.
- Obtained by analytical integration, the exact finite formulas for the effective properties involve only the first, dipole moments of multipole expansion which can be found with high accuracy from the small-size truncated linear system.
- The wide opportunities exist for further development of the method, both in terms of theory and application area. The promising directions include the nano composites, materials with hierarchical or clustered micro structure, composites with imperfect/debonded interfaces, the multi-scale analysis of steady-state and transient phenomena, to mention only a few.

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