Chapter 2

Large Displacements and Large Strains

2.1 Introduction

Here, we focus on the kinematic and static descriptions which are independent of the constitutive equations, which connect the strains (deformations in the body) with the stresses (internal forces in the body). For the sake of preventing misunderstandings regarding the issue of this I emphasize that, in order to find the solution to any static problem in solid or structural mechanics, we need the full set of equations, namely the kinematic, the static, and the constitutive equations. Otherwise, it is not possible to set up the governing boundary-value problem. In most cases, we shall formulate boundary-value problems in terms of a variational principle, e.g. some *Principle of Virtual Work*, or sometimes as a set of differential equations with associated boundary conditions.

Lagrange Strains, Piola-Kirchhoff Stresses

Principle of Virtual Work

In the first part of this chapter we introduce three-dimensional continuum mechanics using *Lagrange Strains* and *Piola-Kirchhoff Stresses* as measures of internal deformations and internal forces, respectively. The *Principle of Virtual Work* is then derived. When dealing with specialized theories, e.g. theories for beams, plates, or shells the postulate of a Principle of Virtual Work—in particular the *Principle of Virtual Displacements*—together with a definition of the "generalized" strains, will serve as the basis, while the definition of the associated "generalized" stresses will follow from these postulates.[2.1]

As mentioned above, both the kinematics and the statics of a body entail establishing field equations and boundary conditions. For three-dimensional bodies the kinematic boundary conditions rarely present problems, while the static boundary conditions turn out to be somewhat more involved, see Section 2.3.2.

[2.1] By the term "generalized strains and stresses" we mean strains and stresses that are work conjugate in a principle of virtual work.
If this sounds cryptic, don't worry. You will realize that it is all quite straightforward, see Part II and Chapter 33.

E. Byskov, *Elementary Continuum Mechanics for Everyone*,
Solid Mechanics and Its Applications 194, DOI: 10.1007/978-94-007-5766-0_2,

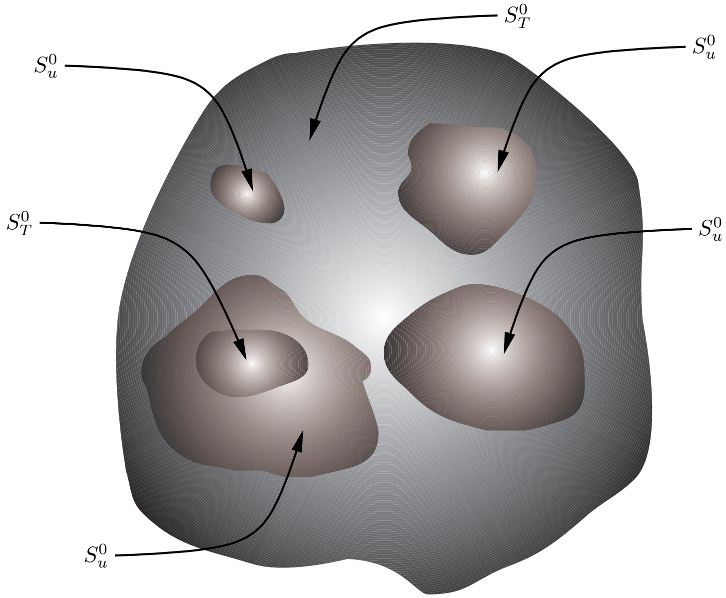

Fig. 2.1: Three-dimensional body.

Continuum "Potatoes" instead of real structures

To set the stage, consider the deformation and equilibrium of a three-dimensional body, see Fig. 2.1. In introductions to continuum mechanics such "potatoes" are often used instead of bodies of more regular configurations. The reason is that the author does not want the reader to put too much emphasis on the shape of the body because the reader might object to the practical relevance of the shape of the body which the author chooses. I shall stick with this fairly common habit.

Kinematic boundary S_u^0

Static boundary S_T^0

The body may be subjected to prescribed displacements on the boundary and, occasionally, in the interior as well as prescribed forces on the boundary and in the interior. The objective of continuum mechanics then is to set up equations that determine the deformed configuration of the body. For later purposes here we provide a classification of the different parts of the body. It consists of the *interior* V^0 and the surface S^0 where upper index ^{0} indicates that the quantity is associated with the configuration before any deformation occurs. Further, the surface is divided into two parts, namely the so-called *Kinematic Boundary* S_u^0, where the displacements are prescribed, and the so-called *Static Boundary* S_T^0, where the surface loads, the surface tractions are prescribed. It is well worth mentioning that S_u^0 and S_T^0 may occupy the same area of the surface, but in that case only some of the displacement components and some unrelated stress component(s) are prescribed over the same area.

2.2 Kinematics and Deformation

We assume that we know the configurations of a structure in two states, namely the initial, undeformed—or *virgin*—state and the deformed state.[2.2] Any strain measure serves the purpose of telling how much the material at a point of the structure suffers because of the deformation. Physically, the strains at a point in the body are measures of the intensity of the deformation at that point. There is no "best" way to define the strain measure, in particular in kinematically nonlinear problems[2.3] there exists a wealth of useful strain definitions. Here, I shall consider only the strains that are known as *Lagrange Strains* because usually they are the most convenient for our later purposes, in particular see Part IV.

Virgin state

No "best" strain definition

Lagrange strains

Mathematically speaking, a strain measure describes the deformation of the immediate neighborhood of a point. Once we know the strain at all points of the structure, we can compute the shape of the deformed structure. We may also check to see whether the magnitude of the strains have exceeded some maximum criterion, which then would tell us if the material has ruptured.

The kinematic description entails two sets of equations, namely the *Kinematic Field Equations* and *Kinematic Boundary Conditions* which are derived below.

Kinematic field equations
Kinematic boundary conditions

2.2.1 Kinematics and Strain

There are a number of ways in which we can define the Lagrange Strain. Here, we shall employ the one that seems to be the most satisfactory, namely one that considers deformation of an infinitesimal sphere, which in the initial, undeformed, state has the center P^0 and the radius ds^0—recall that upper index ^{0} indicates the *undeformed* state, see Fig. 2.2. After deformation, P^0 has moved to the position P, and the sphere has changed shape. Once we know how the infinitesimal sphere is deformed, we possess sufficient *local* information about the deformed state. This amounts to, for all directions, computing the change in length and direction of an infinitesimal vector from P^0 to a neighboring point Q^0.

Initial state
= Virgin state

[2.2] Usually we do *not* know the deformed configuration in advance, but this is a necessary assumption here. Later we shall see that we do get the tools to compute the deformed state once the undeformed geometry, the constitutive laws and the loads are known.

[2.3] Kinematic *linearity* implies that all deformations are *infinitesimal*, and thus all nonlinearity has been excluded from the description of the problem with the result that the reasonable choices are much more limited.

Cartesian coordinate system

For the description we employ a three-dimensional *Cartesian* coordinate system, i.e. one that has three orthogonal axes x_1, x_2 and x_3 with base

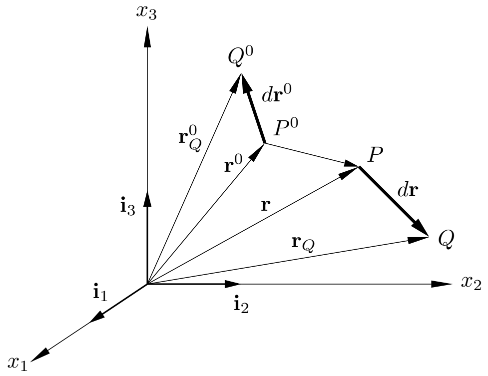

Fig. 2.2: Geometry—Kinematics.

vectors $\mathbf{i}_j$, $j = [1, 2, 3]$, of unit length, i.e.[2.4]

Base vectors $\mathbf{i}_j$

$$|\mathbf{i}_j| = 1 \tag{2.1}$$

Summation convention

Dummy index = Repeated index

In the following—as in most of this book—we utilize the *Summation Convention*, see Part VI, which states that a repeated *lowercase* index indicates summation over the range of that index. The repeated index, the *Dummy Index*, must appear exactly twice in each product in order that the operation is defined. Thus, for a three-dimensional coordinate system we may write the following expression

$$dx_i dx_i = dx_1 dx_1 + dx_2 dx_2 + dx_3 dx_3 \tag{2.2}$$

In the undeformed configuration the position vectors of point P^0 and the neighboring point Q^0 are $\mathbf{r}^0$ and $\mathbf{r}_Q^0$, respectively, see Fig. 2.2. The distance between P^0 and Q^0 is ds^0 and thus

$$\left(ds^0\right)^2 = d\mathbf{r}^0 \cdot d\mathbf{r}^0 = dx_i\, dx_i = \delta_{ij}\, dx_i\, dx_j \tag{2.3}$$

Kronecker delta δ_{ij}

where, δ_{ij} denotes the *Kronecker delta* δ_{ij} which is defined by (2.4) below and by (31.9) in Chapter 31. Here, we have used the Kronecker delta to rewrite the formula for the distance between points P^0 and Q^0 in a form,

[2.4] Vectors are indicated by boldface letters.

which is often useful, see e.g. (2.10).

$$\delta_{ij} \equiv \begin{cases} 1 & \text{for} \quad j = i \\ 0 & \text{for} \quad j \neq i \end{cases} \tag{2.4}$$

The Kronecker delta δ_{ij}

Because of (2.1) and the fact that the base vectors are orthogonal their *scalar product*, also denoted the *inner product*, is given by the *Kronecker delta*

$$\delta_{ij} = \mathbf{i}_i \cdot \mathbf{i}_j \tag{2.5}$$

where $\cdot$ indicates a scalar product.

After deformation the material points P^0 and Q^0 are moved to the positions P and Q with position vectors $\mathbf{r}$ and $\mathbf{r}_Q$, respectively. The length of the infinitesimal line element has changed from ds^0 to ds. Then

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (\mathbf{r}_{,i} dx_i) \cdot (\mathbf{r}_{,j} dx_j) = (\mathbf{r}_{,i} \cdot \mathbf{r}_{,j})\, dx_i dx_j \tag{2.6}$$

where the *Comma Notation* indicates partial derivatives

$$(\;)_{,j} \equiv \frac{\partial}{\partial x_j}(\;) \tag{2.7}$$

Comma notation $(\;)_{,j}$

For later purposes introduce the quantities $\mathbf{g}_i$ and g_{ij} by

$$\mathbf{g}_i \equiv \mathbf{r}_{,i} \quad \text{and} \quad g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j \tag{2.8}$$

where the geometric interpretation of $\mathbf{g}_i$ and g_{ij} will be clear subsequently. For now it suffices to think of them as convenient shorthand notations.

By (2.8a) and (2.8b) Eq. (2.6) may be written

$$(ds)^2 = \mathbf{g}_i \cdot \mathbf{g}_j \, dx_i dx_j = g_{ij} dx_i dx_j \tag{2.9}$$

The change in the square of the length of ds^0 provides as much information as the change in length itself and is more easily applied in the following. Therefore, we compute

$$(ds)^2 - (ds^0)^2 = (g_{ij} - \delta_{ij})\, dx_i dx_j \tag{2.10}$$

Change in square of length of ds^0

2.2.2 Kinematic Field Equations—Lagrange Strain

Now, we define the *Lagrange Strain Measure* γ_{ij} by

$$\gamma_{ij} \equiv \tfrac{1}{2}(g_{ij} - \delta_{ij}) \tag{2.11}$$

Lagrange strain γ_{ij}

which may be introduced into (2.10) to give

$$(ds)^2 - (ds^0)^2 = 2\gamma_{ij} dx_i dx_j \tag{2.12}$$

Since both g_{ij} and δ_{ij} are symmetric it is obvious from its definition that

The Lagrange strain γ_{ij} is symmetric

γ_{ij} is symmetric in its indices

$$\gamma_{ij} = \gamma_{ji} \tag{2.13}$$

Lagrange strain γ_{ij} Infinitesimal strains ε_{ij} or e_{ij}

It is a fact that there exist more different notations for strains than anyone could wish for, which occasionally causes some confusion.[2.5] For instance, sometimes we employ the notation ε_{ij} for nonlinear strains instead of γ_{ij}, see Part II. Here, however, we retain γ_{ij} because we wish to emphasize that the Lagrange Strains are nonlinear. It is unfortunate that the infinitesimal strains, which often are denoted e_{ij}, see (2.36), also are designated ε_{ij}. Since these notations, confusing as they are, all are very common, I have decided to use them and in each case try to be careful to note when ε_{ij} indicates a nonlinear or a linear strain, respectively.

It may be worthwhile mentioning that to lowest order there is no difference between the infinitesimal strain measure and the Lagrange Strain, see Chapter 4, Infinitesimal Theory.

In the following we assume that the displacement field, given by the vector field $\mathbf{u}(\mathbf{r}^0)$ or equivalently by its components $u_i(x_k)$, is known[2.6] and establish expressions for the strains $\gamma_{ij}(x_k)$ in terms of the displacement gradients $u_{i,j}(x_k)$.

From the geometry of the undeformed and the deformed configurations we may get

$$\mathbf{r} = \mathbf{r}^0 + \mathbf{u} = (x_j + u_j)\,\mathbf{i}_j \tag{2.14}$$

which by differentiation with respect to x_m furnishes

$$\mathbf{r}_{,m} = (\delta_{jm} + u_{j,m})\,\mathbf{i}_j \tag{2.15}$$

Because of (2.8a) we may get

$$\mathbf{g}_m = (\delta_{jm} + u_{j,m})\,\mathbf{i}_j \tag{2.16}$$

From Fig. 2.3 it is observed that by the deformation the base vector $\mathbf{i}_m$, where $\mathbf{i}_m$ indicates any of the base vectors, is displaced and deformed into the vector $\mathbf{j}_m$, where

$$\mathbf{j}_m = (\mathbf{u} + \mathbf{u}_{,m}\,|\mathbf{i}_j|) - \mathbf{u} + \mathbf{i}_m = (\delta_{jm} + u_{j,m})\,\mathbf{i}_j \tag{2.17}$$

where it has been exploited that the length $|\mathbf{i}_j|$ of the base vectors is unity, see (2.1).

Comparison between (2.16) and (2.17) shows that $\mathbf{j}_m = \mathbf{g}_m$, which means that $\mathbf{g}_m$ is the base vector $\mathbf{i}_m$ after deformation and is therefore called the

[2.5] I only expose you to three of them, namely γ_{ij}, ε_{ij} and e_{ij}, see below.
[2.6] See footnote on page 35.

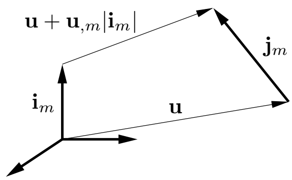

Fig. 2.3: Connection between $\mathbf{i}_m$ and $\mathbf{g}_m$.

deformed base vector. Note that, in general, $\mathbf{g}_m$ is not a unit vector.

Deformed base vector $\mathbf{g}_m$

From (2.8b) and (2.17) we may get

$$\begin{aligned} g_{mn} &= \mathbf{g}_m \cdot \mathbf{g}_n = (\delta_{jm} + u_{j,m})(\delta_{kn} + u_{k,n})\,\mathbf{i}_j \cdot \mathbf{i}_k \\ &= (\delta_{jm}\delta_{kn} + \delta_{jm}u_{k,n} + \delta_{kn}u_{j,m} + u_{j,m}u_{k,n})\,\delta_{jk} \\ \Rightarrow \quad g_{mn} &= \delta_{mn} + u_{m,n} + u_{n,m} + u_{k,m}u_{k,n} \end{aligned} \tag{2.18}$$

and thus the components of the Lagrange Strain are given by

$$\gamma_{mn} = \tfrac{1}{2}(u_{m,n} + u_{n,m}) + \tfrac{1}{2}u_{k,m}u_{k,n} \tag{2.19}$$

Lagrange strain γ_{mn}

2.2.2.1 "Fiber" Elongation

The relative elongation of a "fiber" of the material may be given by the quantity γ, where

$$\gamma \equiv \frac{(ds)^2 - (ds^0)^2}{2(ds^0)^2} \tag{2.20}$$

Fiber elongation γ = Change of length

where the reason for the factor 2 in the denominator may be seen from the expansion of (2.20) and from the definition of the Lagrange strain

$$\gamma = \frac{(ds + ds^0)(ds - ds^0)}{2ds^0 ds^0} \approx \frac{ds - ds^0}{ds^0} \quad \text{for} \quad ds \approx ds^0 \tag{2.21}$$

Fiber elongation γ for infinitesimal deformation

which shows that γ indeed is equal to the relative change in length for small deformations.

You may, of course, ask: why not use the relative change in length, i.e. $(ds - ds^0)/ds^0$, instead of the quantity γ defined by (2.20)? The reason is that our strain measure is the Lagrange strain, which is associated with the difference between the square of the length of the line element after and before deformation, and we wish to express the elongation in terms of the Lagrange strain. Utilize (2.12) to get

$$\gamma = \frac{2\gamma_{ij}dx_i dx_j}{2ds^0 ds^0} \tag{2.22}$$

and realize that the unit vector $\overset{0}{\mathbf{n}}$ in the direction of the "fiber" before deformation has the components $\overset{0}{n}_j$

$$\overset{0}{n}_j = \frac{dx_j}{ds^0} \tag{2.23}$$

to get

$$\gamma = \gamma_{ij}\overset{0}{n}_i\overset{0}{n}_j \tag{2.24}$$

Thus, when we have determined the Lagrange strain γ_{ij} we may determine the change in length of an arbitrary unit vector.

If the vector n_j is directed along one of the coordinate axes, say number I, then (2.24) provides

$$\gamma = \gamma_{II} \quad \text{(no sum over capital indices)} \tag{2.25}$$

An obvious question to ask is: in which direction do we find the maximum (or minimum) elongation? We shall not pursue this question here, but refer to Section 4.2.6, where this subject is covered for the kinematically linear case.

2.2.2.2 Change of Angle

Change of Angle

The change of angle between two initially orthogonal directions is another important measure of deformation. In the undeformed configuration introduce two orthogonal unit vectors $\overset{0}{\mathbf{n}}^{(1)}$ and $\overset{0}{\mathbf{n}}^{(2)}$ with components $\overset{0}{n}{}_j^{(1)}$ and $\overset{0}{n}{}_j^{(2)}$. Then,

$$\overset{0}{n}{}_j^{(1)}\overset{0}{n}{}_j^{(2)} = 0\,, \quad \overset{0}{n}{}_j^{(1)}\overset{0}{n}{}_j^{(1)} = 1 \quad \text{and} \quad \overset{0}{n}{}_j^{(2)}\overset{0}{n}{}_j^{(2)} = 1 \tag{2.26}$$

Let us denote these vectors after deformation by $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ with components $m_j^{(1)}$ and $m_j^{(2)}$, respectively. Then

$$m_j^{(1)}m_j^{(2)} = |\mathbf{m}^{(1)}||\mathbf{m}^{(2)}|\cos(\psi^{(12)}) \tag{2.27}$$

where $\psi^{(12)}$ denotes the angle between $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$.

The components $m_j^{(1)}$ and $m_j^{(2)}$ may be found to be

$$m_j^{(1)} = \overset{0}{n}{}_j^{(1)} + \overset{0}{n}{}_i^{(1)}u_{j,i} \quad \text{and} \quad m_j^{(2)} = \overset{0}{n}{}_j^{(2)} + \overset{0}{n}{}_i^{(2)}u_{j,i} \tag{2.28}$$

Another way of expressing (2.27) therefore is

$$\begin{aligned} m_j^{(1)}m_j^{(2)} &= (\overset{0}{n}{}_j^{(1)} + \overset{0}{n}{}_i^{(1)}u_{j,i})(\overset{0}{n}{}_j^{(2)} + \overset{0}{n}{}_k^{(2)}u_{j,k}) \\ &= \overset{0}{n}{}_j^{(1)}\overset{0}{n}{}_j^{(2)} + \overset{0}{n}{}_j^{(1)}\overset{0}{n}{}_k^{(2)}u_{j,k} + \overset{0}{n}{}_i^{(1)}\overset{0}{n}{}_j^{(2)}u_{j,i} + \overset{0}{n}{}_i^{(1)}\overset{0}{n}{}_k^{(2)}u_{j,i}u_{j,k} \\ &= 0 + \overset{0}{n}{}_j^{(1)}\overset{0}{n}{}_i^{(2)}(u_{j,i} + u_{i,j} + u_{k,j}u_{k,i}) \end{aligned} \tag{2.29}$$

where we have changed dummy indices in several places. By use of (2.19) we may introduce the Lagrange strain γ_{ij} and get

$$m_j^{(1)} m_j^{(2)} = 2 \overset{0}{n}{}_i^{(1)} \overset{0}{n}{}_j^{(2)} \gamma_{ij} \tag{2.30}$$

In order to find an expression for $\cos(\psi^{(12)})$ we need expressions for the length of $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$, see (2.27). First, let us determine $|\mathbf{m}^{(1)}|^2$

$$\begin{aligned} |\mathbf{m}^{(1)}|^2 &= m_j^{(1)} m_j^{(1)} \\ &= (\overset{0}{n}{}_j^{(1)} + \overset{0}{n}{}_i^{(1)} u_{j,i})(\overset{0}{n}{}_j^{(1)} + \overset{0}{n}{}_k^{(1)} u_{j,k}) \\ &= \overset{0}{n}{}_j^{(1)} \overset{0}{n}{}_j^{(1)} + \overset{0}{n}{}_j^{(1)} \overset{0}{n}{}_k^{(1)} u_{j,k} + \overset{0}{n}{}_i^{(1)} \overset{0}{n}{}_j^{(1)} u_{j,i} + \overset{0}{n}{}_i^{(1)} \overset{0}{n}{}_k^{(1)} u_{j,i} u_{j,k} \\ &= 1 + \overset{0}{n}{}_j^{(1)} \overset{0}{n}{}_i^{(1)} (u_{j,i} + u_{i,j} + u_{k,j} u_{k,i}) \\ &= 1 + 2 \overset{0}{n}{}_i^{(1)} \overset{0}{n}{}_j^{(1)} \gamma_{ij} \end{aligned} \tag{2.31}$$

By substituting $^{(2)}$ for $^{(1)}$ in (2.31) the expression for $|\mathbf{m}^{(2)}|^2$ is found to be

$$|\mathbf{m}^{(2)}|^2 = 1 + 2 \overset{0}{n}{}_i^{(2)} \overset{0}{n}{}_j^{(2)} \gamma_{ij} \tag{2.32}$$

We are now able to establish an expression for the angle $\psi^{(12)}$, but we are probably more interested in the *change* of angle between $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$. Let $\varphi^{(12)}$ denote this change. Note that

$$\varphi^{(12)} = \tfrac{1}{2}\pi - \psi^{(12)} \tag{2.33}$$

Change of angle $\varphi^{(12)}$

Then,

$$\sin(\varphi^{(12)}) = \cos(\psi^{(12)}) \tag{2.34}$$

i.e.

$$\sin(\varphi^{(12)}) = \frac{2\gamma_{ij} \overset{0}{n}{}_i^{(1)} \overset{0}{n}{}_j^{(2)}}{\sqrt{(1 + 2 \overset{0}{n}{}_i^{(1)} \overset{0}{n}{}_j^{(1)} \gamma_{ij})(1 + 2 \overset{0}{n}{}_m^{(2)} \overset{0}{n}{}_n^{(2)} \gamma_{mn})}} \tag{2.35}$$

Thus, when we have determined the Lagrange strain γ_{ij} we may determine the change of angle between two arbitrary orthogonal unit vectors.

2.2.3 Infinitesimal Strains and Infinitesimal Rotations

The *infinitesimal strain* tensor[2.7] e_{mn} is defined by

$$e_{mn} \equiv \tfrac{1}{2}(u_{m,n} + u_{n,m}) = e_{nm} \tag{2.36}$$

Infinitesimal strain e_{mn}

which shows that e_{mn} is symmetric.

[2.7] Do not put too much emphasis on the term *tensor*—in our case it is merely a fancy word used to impress some.

Infinitesimal rotation ω_{mn}

The *infinitesimal rotation* tensor ${\omega_{mn}}$[2.8] is defined by

$$\omega_{mn} \equiv \tfrac{1}{2}\left(u_{m,n} - u_{n,m}\right) = -\omega_{nm} \tag{2.37}$$

which shows that ω_{mn} is antisymmetric in its indices.

From (2.36) and (2.37)

$$u_{m,n} = e_{mn} + \omega_{mn} \tag{2.38}$$

The *Lagrange Strain* γ_{mn} may then be expressed in terms of e_{mn} and ω_{mn}

$$\gamma_{mn} = e_{mn} + \tfrac{1}{2}\left(e_{km} + \omega_{km}\right)\left(e_{kn} + \omega_{kn}\right) \tag{2.39}$$

or

$$\gamma_{mn} = e_{mn} + \tfrac{1}{2} e_{km} e_{kn} + \tfrac{1}{2}\left(e_{km}\omega_{kn} + e_{kn}\omega_{km}\right) + \tfrac{1}{2}\omega_{km}\omega_{kn} \tag{2.40}$$

For small strains, i.e. $|e_{mn}| \ll 1$

$$\gamma_{mn} \approx e_{mn} + \tfrac{1}{2}\left(e_{km}\omega_{kn} + e_{kn}\omega_{km}\right) + \tfrac{1}{2}\omega_{km}\omega_{kn} \tag{2.41}$$

and for small strains and moderately small rotations, i.e. $|\omega_{mn}| \ll 1$, but $|\omega_{mn}| > |e_{mn}|$

$$\gamma_{mn} \approx e_{mn} + \tfrac{1}{2}\omega_{km}\omega_{kn} \tag{2.42}$$

The approximate strain measure given by (2.42) forms the basis of *Kinematically Moderately Nonlinear Theories*, which assume that the strains are infinitesimal and that the rotations are small, but finite. Most kinematically nonlinear analyses of beam, plate and shell structures utilize a theory of this type.

2.2.4 Compatibility Equations

The compatibility equations ensure that a strain field can be integrated to provide a displacement field

It is a mathematical fact that not all strain fields are compatible in the sense that there is no guarantee that a given strain field γ_{ij} with 6 independent terms can be integrated to a displacement field u_m of 3 independent terms. In order that this is the case the so-called *Compatibility Equations* must be satisfied. Even in the linear (infinitesimal) case these equations have a rather complicated structure, and in the kinematically nonlinear cases it is, of course, even worse. But, fortunately we do not always need them for our purposes because in our applications we often determine the displacement field first and derive the strains from the displacements, and thus the strains satisfy the compatibility equations automatically. We shall therefore neither derive, nor cite the compatibility equations here, but refer the interested

[2.8] For the sake of completeness I emphasize that, as its name indicates, only for small rotations is ω_{mn} a valid measure of rotation.

reader to the book by Malvern (1969).

We shall, however, derive the compatibility equations in the case of kinematic linearity, see Chapter 4.2, Infinitesimal Theory, in particular Section 4.2.2. One reason for this is that in formulation of theories based on stress functions, such as the Airy Stress Function, see Section 9.2.5, the compatibility conditions are not satisfied a priori.

2.2.5 Kinematic Boundary Conditions

Kinematic boundary conditions

For a three-dimensional body the *Kinematic Boundary Conditions* usually are quite obvious, and we shall defer discussion of this matter to the theories for specialized continua, see Part II.

2.3 Equilibrium Equations

Static field equations and static boundary conditions

Like the kinematic equations, the equilibrium equations fall into two parts, namely the *Static Field Equations* and the *Static Boundary Conditions*. The static boundary equations usually do not present difficulties for the three-dimensional continuum, but for the specialized continua, see Part II, the situation often is quite different.

2.3.1 Static Field Equations

Equilibrium of an infinitesimal parallelepiped

We may derive the continuum equilibrium equations from the equilibrium of a deformed, infinitesimal sphere and thus apply a procedure which is analogous to the one from Section 2.2.1. Here, however, I prefer to go about the task in another way and establish the equilibrium equations of a (deformed) infinitesimal parallelepiped, see Fig. 2.4. In the undeformed state the parallelepiped, whose volume is dV^0, is spanned by the vectors $\mathbf{i}_1 dx_1$, $\mathbf{i}_2 dx_2$, and $\mathbf{i}_3 dx_3$. From Section 2.2.1 we know that these vectors deform into the vectors $\mathbf{g}_1 dx_1$, $\mathbf{g}_2 dx_2$, and $\mathbf{g}_3 dx_3$, which span a deformed parallelepiped with the volume dV, where

$$dV = (\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3)\, dV^0 \tag{2.43}$$

where we assume that $dV > 0$ because otherwise the cube would have collapsed.

Since the loads on the structure act on the deformed structure, it is the equilibrium of the deformed parallelepiped which must be analyzed. In spite of this fact, we shall measure all forces in terms of *undeformed* areas, simply because it proves to be more convenient.

The force acting on the side, which before deformation had the normal $-\mathbf{i}_1$, is $-\mathbf{F}_1$, and the force on the other side of the parallelepiped is then $\mathbf{F}_1 + d\mathbf{F}_1$.

In terms of the undeformed area.

Measure all forces in terms of *undeformed* areas

$$-\mathbf{F}_1 = -\mathbf{t}_1 dx_2 dx_3 \tag{2.44}$$

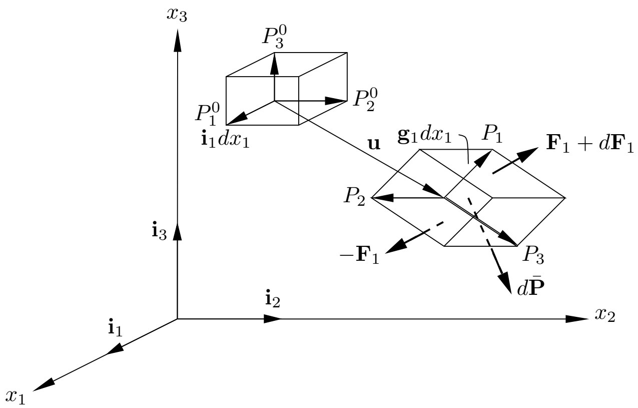

Fig. 2.4: Statics of an infinitesimal element. Not all forces acting on the element are shown.

and on the other side of the parallelepiped

$$\mathbf{F}_1 + d\mathbf{F}_1 = \mathbf{t}_1 dx_2 dx_3 + (\mathbf{t}_1 dx_2 dx_3)_{,1}\, dx_1 \tag{2.45}$$

with analogous relations for the other two directions. The total load, or total body force, acting within the volume dV is $d\bar{\mathbf{P}}$, which is measured in terms of the *undeformed* volume dV^0

Total body force in dV^0

$$d\bar{\mathbf{P}} = \bar{\mathbf{q}} dx_1 dx_2 dx_3 \tag{2.46}$$

where $\bar{\mathbf{q}}$ is the body force acting within the element per unit volume.

Force equilibrium of dV requires that

Force equilibrium of dV

$$d\mathbf{F}_1 + d\mathbf{F}_2 + d\mathbf{F}_3 + d\bar{\mathbf{P}} = \mathbf{0} \tag{2.47}$$

which, in light of (2.44), (2.45), and (2.46), provides

Force equilibrium of dV

$$\mathbf{t}_{i,i} + \bar{\mathbf{q}} = \mathbf{0} \tag{2.48}$$

We wish to express (2.48) in component form, and to this end we resolve the vectors $\mathbf{t}_i$ with respect to the *deformed* base vectors $\mathbf{g}_j$ although the loads are measured in terms of the *undeformed* area. This is, of course, not the only possible choice—and not an obvious one either—but it will prove to be convenient. Thus,

$$\mathbf{t}_i = t_{ij} \mathbf{g}_j \tag{2.49}$$

where t_{ij} is the (second)[2.9] *Piola-Kirchhoff Stress Tensor.* For a good reason it is often referred to as a *pseudo* stress because it is measured as the force per unit *undeformed* area resolved in terms of the *deformed* base vectors $\mathbf{g}_j$, which, as mentioned above, in general are not unit vectors.

The Piola-Kirchhoff *pseudo* stress t_{ij} is measured on the *undeformed* area, but resolved in terms of the *deformed* base vectors

When higher order terms in dx_i are neglected, moment equilibrium of dV requires that[2.10]

$$\mathbf{g}_1 \times \mathbf{F}_1 dx_1 + \mathbf{g}_2 \times \mathbf{F}_2 dx_2 + \mathbf{g}_3 \times \mathbf{F}_3 dx_3 = \mathbf{0} \tag{2.50}$$

or because of (2.44)

$$(\mathbf{g}_i \times \mathbf{t}_i)\, dx_1 dx_2 dx_3 = \mathbf{0} \tag{2.51}$$

and thus

$$\mathbf{g}_i \times \mathbf{t}_i = \mathbf{0} \quad \Rightarrow \quad t_{ij}\mathbf{g}_i \times \mathbf{g}_j = \mathbf{0} \tag{2.52}$$

which, written out, gives the following three equations

$$(t_{12} - t_{21})\, \mathbf{g}_1 \times \mathbf{g}_2 + (t_{23} - t_{32})\, \mathbf{g}_2 \times \mathbf{g}_3 + (t_{31} - t_{13})\, \mathbf{g}_3 \times \mathbf{g}_1 = \mathbf{0} \tag{2.53}$$

which do not seem to provide much information about the properties of the stress tensor t_{ij}. However, barring deformations that annihilate the initial element[2.11] the vectors $\mathbf{g}_1 \times \mathbf{g}_2$, $\mathbf{g}_2 \times \mathbf{g}_3$, and $\mathbf{g}_3 \times \mathbf{g}_1$ are linearly independent and therefore

The Piola-Kirchhoff stress tensor t_{ij} is symmetric

$$t_{ij} = t_{ji} \tag{2.54}$$

meaning that the Piola-Kirchhoff stress tensor is symmetric.[2.12]

We wish to express the equilibrium equation (2.48) in component form and start with resolving the body force $\bar{\mathbf{q}}$ in terms of the *undeformed* base vectors $\mathbf{i}_j$

$$\bar{\mathbf{q}} = \bar{q}_j \mathbf{i}_j \tag{2.55}$$

[2.9] There is, as you may have guessed, a first Piola-Kirchhoff stress tensor, see e.g. (Malvern 1969). For our purposes it is not the one that we want.

[2.10] We exclude possible distributed moment loads $\bar{c}_j$ because they do not appear directly in most structural problems, except in connection with e.g. magneto-elasticity, which rightfully may be considered a very specialized field that we do not intend to discuss.

On the other hand, since about 1980 there has been great interest in continuum theories that involve *couple stresses* μ_{ij}. The reason is that in the description of materials with (micro)structure there is a need for a length scale in order to handle problems such as *strain localization*, sometimes in the form of *kinkbands* in wood, concrete and fiber reinforced epoxy.

[2.11] Collapse of the body would mean that $dV \not> 0$, which on physical grounds is not an acceptable idea.

[2.12] In all fairness, this is not the only useful stress tensor, which exhibits this nice property.

Again, this may not seem like a natural choice because the stress tensor t_{ij} is resolved in terms of $\mathbf{g}_j$—not $\mathbf{i}_j$—but in many cases the load keeps its direction throughout the deformation history. Should this not be the case, then it is a fairly easy task to take this into account and redo the following derivations. Now, (2.48), (2.49) and (2.55) give

$$\left(t_{ij}\mathbf{g}_j\right)_{,i} + \bar{q}_j\mathbf{i}_j = \mathbf{0} \tag{2.56}$$

Recall (2.16) and get

$$\big(t_{ij}(\delta_{mj} + u_{m,j})\mathbf{i}_m\big)_{,i} + \bar{q}_m\mathbf{i}_m = \mathbf{0} \tag{2.57}$$

which after some trivial manipulations provides

$$\Big(t_{im,i} + \left(t_{ij}u_{m,j}\right)_{,i} + \bar{q}_m\Big)\,\mathbf{i}_m = \mathbf{0} \tag{2.58}$$

Internal equilibrium. Static field equations

Take the inner product with $\mathbf{i}_k$ on both sides and note the expression for the scalar product of two base vectors in terms of the Kronecker delta (2.5) ($\delta_{ij} = \mathbf{i}_i \cdot \mathbf{i}_j$) to get[2.13]

$$t_{ik,i} + \left(t_{ij}u_{k,j}\right)_{,i} + \bar{q}_k = 0\ , \quad k \in [1, 2, 3] \tag{2.59}$$

These equations connect the components of the Piola-Kirchhoff stress with the body forces and the displacement gradients and thus they express internal equilibrium, i.e. (2.59) are the static field equations. In a way the term "static field equations" is somewhat misleading because (2.59) entails not only static quantities, but also kinematic ones. However, this is the usual nomenclature, and in the case of infinitesimal displacements and infinitesimal displacement gradients the second term vanishes and then the name is clearly justified, see Chapter 4, in particular (4.75).

2.3.2 Properties of the Stress Vector—Static Boundary Conditions

Static boundary conditions

In order to establish the static boundary conditions we consider an infinitesimal tetrahedron, see Fig. 2.5. Three of its faces are parallel to the coordinate planes, while the fourth is inclined at an angle given by its unit normal vector $\mathbf{n}^0$. It is our intention to establish the equilibrium equations of the tetrahedron *after* displacement and deformation, see Fig. 2.5. The total force on the face, which in the undeformed configuration had the normal $-\mathbf{i}_1$, is $-d\mathbf{F}_1$ with self-evident analogies for the two other faces. The force on the inclined face $P_1P_2P_3$ with the normal $\mathbf{n}^0$ is $d\mathbf{F}_n$. Equilibrium of the deformed tetrahedron requires that

$$-d\mathbf{F}_1 - d\mathbf{F}_2 - d\mathbf{F}_3 + d\mathbf{F}_n + d\bar{\mathbf{P}} = \mathbf{0} \tag{2.60}$$

where $d\bar{\mathbf{P}}$ is the total body force.

[2.13] We may also find (2.59) by appealing to the linear independence of the base vectors $\mathbf{i}_m$.

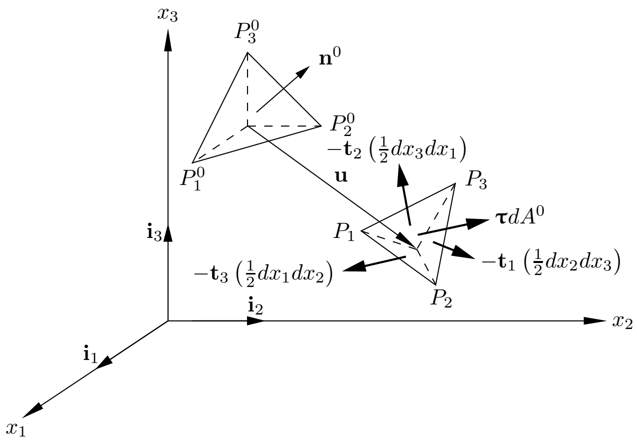

Fig. 2.5: Statics of an infinitesimal tetrahedron.

As before, we measure the intensity of all forces on the *undeformed* configuration. The forces $-d\mathbf{F}_i, i = [1, 2, 3]$, are simply

$$-d\mathbf{F}_1 = -\mathbf{t}_1\left(\tfrac{1}{2}dx_2dx_3\right) \tag{2.61}$$

etc., by analogy with (2.44). The force intensity of the inclined face is $\boldsymbol{\tau}$, and thus

$$d\mathbf{F}_n = +\boldsymbol{\tau}dA^0 \tag{2.62}$$

The following expressions, which are associated with the *undeformed* configuration, prove to be useful

$$\begin{aligned} dA^0\mathbf{n}^0\cdot\mathbf{i}_1 &= \tfrac{1}{2}dx_2dx_3 \\ dA^0\mathbf{n}^0\cdot\mathbf{i}_2 &= \tfrac{1}{2}dx_3dx_1 \\ dA^0\mathbf{n}^0\cdot\mathbf{i}_3 &= \tfrac{1}{2}dx_1dx_2 \end{aligned} \tag{2.63}$$

As in Section 2.3.1, let $\bar{\mathbf{q}}$ denote the body force intensity, and recall that the volume of the undeformed tetrahedron is $\frac{1}{6}dx_1dx_2dx_3$, and rewrite (2.60) and get

$$\begin{aligned} \mathbf{0} = &-\mathbf{t}_1\tfrac{1}{2}dx_2dx_3 - \mathbf{t}_2\tfrac{1}{2}dx_3dx_1 - \mathbf{t}_3\tfrac{1}{2}dx_1dx_2 \\ &+\boldsymbol{\tau}dA^0 + \bar{\mathbf{q}}\tfrac{1}{6}dx_1dx_2dx_3 \end{aligned} \tag{2.64}$$

or

$$\mathbf{0} = -\mathbf{t}_j \mathbf{n}^0 \cdot \mathbf{i}_j dA^0 + \boldsymbol{\tau} dA^0 + \bar{\mathbf{q}} \tfrac{1}{3} dA^0 \mathbf{n}^0 \cdot \mathbf{i}_1 dx_1 \tag{2.65}$$

The load term is clearly of order 3 in the differentials dx_j because dA^0 is of order 2, while the other terms are of order 2. Therefore, the last term vanishes in the limit and may be omitted. Then, if we resolve $\boldsymbol{\tau}$ in terms of the *undeformed* base vectors $\mathbf{i}_j$

$$\boldsymbol{\tau} = \tau_j \mathbf{i}_j \tag{2.66}$$

and recall (2.49), we may get

$$-t_{jk} \mathbf{g}_k \left(\mathbf{n}^0 \cdot \mathbf{i}_j\right) + \tau_j \mathbf{i}_j = \mathbf{0} \tag{2.67}$$

In the same way as leads to (2.59) we take the inner product with $\mathbf{i}_m$ on both sides and get

$$-t_{jk} \left(\mathbf{g}_k \cdot \mathbf{i}_m\right) \left(\mathbf{n}^0 \cdot \mathbf{i}_j\right) + \tau_j \delta_{jm} = 0 \tag{2.68}$$

and arrive at

$$\tau_m = \left(\mathbf{g}_k \cdot \mathbf{i}_m\right) \left(\mathbf{n}^0 \cdot \mathbf{i}_j\right) t_{jk} \tag{2.69}$$

Utilize (2.16) and resolve $\mathbf{n}^0$ in terms of $\mathbf{i}_j$ and get

$$\tau_m = \left(\left(\delta_{nk} + u_{n,k}\right) \mathbf{i}_n \cdot \mathbf{i}_m\right) n_j^0 t_{jk} \tag{2.70}$$

which with (2.4) yields the following expression for the *surface tractions* τ_m

Surface tractions τ_m

$$\tau_m = \left(\delta_{mk} + u_{m,k}\right) n_j^0 t_{jk} \tag{2.71}$$

This equation expresses the stress τ_m, the *Surface traction*, on any surface with the normal n_j^0 in the *undeformed* geometry in terms of the components of the Piola-Kirchhoff stress tensor t_{jk} and the displacement gradients $u_{m,k}$. As is the case for the static field equations (2.59) the displacement gradients enter the static boundary conditions (2.71), which makes the relations nonlinear. On the static boundary S_T^0 the surface tractions are prescribed, i.e. $\tau_m = \bar{\tau}_m$, and the static boundary conditions become

Static boundary conditions

$$\left(\delta_{mk} + u_{m,k}\right) n_j^0 t_{jk} = \bar{\tau}_m \ , \quad x_n \in S_T^0 \tag{2.72}$$

In the kinematically linear case, see Chapter 4, the static equations are linear and we shall study them in more detail.

2.4 Principle of Virtual Displacements

Principle of Virtual Work.
Special case:
Principle of Virtual Displacements

The *Principle of Virtual Work* takes different forms depending on the purpose. In the following we concentrate on the particular version, which is known as the *Principle of Virtual Displacements* because this is the most convenient one for kinematically nonlinear problems.

As hinted at in the Introduction of the present chapter, the principle of virtual work plays a central role in continuum mechanics. This may, however, not become obvious until Parts II and IV–VI where we exploit the principle of virtual work over and over for various purposes.

Actually, if we stopped after having derived the principle of virtual displacements, the whole idea of establishing the principle could seem like an exercise in futility because we start out with three equilibrium equations and end up with only one equation instead, suggesting that information has been lost. This is, however, not the case, as we shall see below. Also, in the manipulations below the direction we are headed may be unclear until the final stage, so the reader must trust that something positive and useful eventually results.

For convenience repeat (2.48), which expresses the three equilibrium equations

Equilibrium equations

$$\mathbf{t}_{i,i} + \bar{\mathbf{q}} = \mathbf{0} \tag{2.73}$$

Since this statement holds everywhere in the body we may multiply by an arbitrary, smooth vector field[2.14] $\boldsymbol{\alpha}$ to get

$$(\mathbf{t}_{i,i} + \bar{\mathbf{q}}) \cdot \boldsymbol{\alpha} = 0 \;\forall\; \boldsymbol{\alpha} \tag{2.74}$$

Already here, we have transformed the three equilibrium equations into one scalar equation which, on the other hand, does not mean that we have lost information, because $\boldsymbol{\alpha}$ is arbitrary.

Integrate (2.74) over the (undeformed) volume V^0 with the result

$$\int_{V^0} (\mathbf{t}_{i,i} + \bar{\mathbf{q}}) \cdot \boldsymbol{\alpha} dV^0 = 0 \;\forall\; \boldsymbol{\alpha} \tag{2.75}$$

which again possesses as much information as (2.73) because of the arbitrariness of $\boldsymbol{\alpha}$. For reasons that become clear later rewrite (2.75)

$$\int_{V^0} \Big((\mathbf{t}_i \cdot \boldsymbol{\alpha})_{,i} - \mathbf{t}_i \cdot \boldsymbol{\alpha}_{,i} + \bar{\mathbf{q}} \cdot \boldsymbol{\alpha} \Big) dV^0 = 0 \;\forall\; \boldsymbol{\alpha} \tag{2.76}$$

By application of the *Divergence Theorem* the first term is transformed

[2.14] The field must not contain singularities, but we shall not state this explicitly in the following and by $\forall\, \boldsymbol{\alpha}$ imply "$\forall$ non-singular $\boldsymbol{\alpha}$."

into a surface integral instead of a volume integral[2.15]

$$\int_{S^0} \mathbf{t}_i \cdot \boldsymbol{\alpha} n_i^0 dS^0 + \int_{V^0} \bar{\mathbf{q}} \cdot \boldsymbol{\alpha} dV^0 = \int_{V^0} \mathbf{t}_i \cdot \boldsymbol{\alpha}_{,i} dV^0 \ \forall \ \boldsymbol{\alpha} \tag{2.77}$$

In principle, (2.77) could be useful as it stands, but we shall offer an interpretation of the vector field $\boldsymbol{\alpha}$. First, let us add a small *variation*,[2.16] $\epsilon\delta\mathbf{r}$ to the position vector $\mathbf{r}$ and let $\mathbf{r}_{\text{tot}}$ denote the total value

Variations $\epsilon\delta\mathbf{r}$

$$\mathbf{r}_{\text{tot}} = \mathbf{r} + \epsilon\delta\mathbf{r} \tag{2.78}$$

Shape $\delta\mathbf{r}$
Amplitude ϵ

where $\delta\mathbf{r}$ is the *shape* and ϵ is the *amplitude*[2.17] of the *variation* of $\mathbf{r}$. For reasons that, hopefully, will be clear later, we shall only concern ourselves with values of ϵ which observe

$$|\epsilon| \ll 1 \tag{2.79}$$

As an illustration, Fig. 2.6 shows some function, or a field, $f(x)$ and its

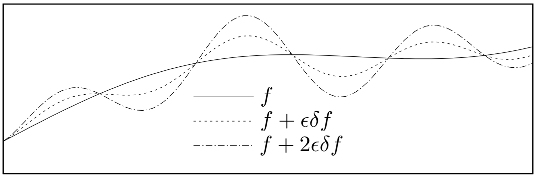

Fig. 2.6: A function and variations.

variations $f + \epsilon\delta f$ for two different values of ϵ. In order to make the drawing clear the magnitude of ϵ has been exaggerated. In this particular case the conditions enforced on the variation are that its value and first derivative at the left-hand end of the interval vanish, but many other conditions may be imposed.

Then, choose

$$\boldsymbol{\alpha} = \delta\mathbf{r} \ \Rightarrow \ \boldsymbol{\alpha}_{,i} = \delta\mathbf{r}_{,i} = \delta\mathbf{g}_i \tag{2.80}$$

[2.15] I expect you to know the divergence theorem—if you don't, go pick up your good old math books.

[2.16] I have chosen to attack the problem of variations directly here, but otherwise refer to Chapter 32 where the subject is dealt with in more detail. The derivation becomes a little more pedestrian this way, but I hope that it makes for easier reading.

[2.17] The "American" epsilon ϵ used here to denote a small quantity must not be confused with the other epsilon ε, which—with or without a number of subscripts—is employed as the symbol for strains.

When we recall (2.49)

$$\mathbf{t}_i = t_{ij}\mathbf{g}_j \tag{2.81}$$

and utilize (2.80) we may rewrite the right-hand side of (2.77)

$$\int_{S^0} t_{ij}\mathbf{g}_j \cdot \delta\mathbf{r} n_i^0 dS^0 + \int_{V^0} \bar{q}_i\mathbf{i}_i \cdot \delta\mathbf{r} dV^0 = \int_{V^0} t_{ij}\mathbf{g}_j \cdot (\delta\mathbf{r})_{,i} \, dV^0 \; \forall \; \delta\mathbf{r} \tag{2.82}$$

where we have resolved $\mathbf{t}_i$ in terms of the *deformed* base vectors $\mathbf{g}_j$ and the load $\bar{\mathbf{q}}$ in terms of the *undeformed* base vectors $\mathbf{i}_j$. Further, resolve $\mathbf{r}$ and $\delta\mathbf{r}$ in terms of the *undeformed* base vectors $\mathbf{i}_j$ and note that $(\delta\mathbf{r})_{,i} = \delta\mathbf{g}_i$ to get

$$\begin{aligned} &\int_{S^0} t_{ij}\mathbf{g}_j \cdot (\delta r_m \mathbf{i}_m)\, n_i^0 dS^0 + \int_{V^0} \bar{q}_i\mathbf{i}_i \cdot (\delta r_j \mathbf{i}_j)\, dV^0 \\ &= \int_{V^0} t_{ij}\mathbf{g}_j \cdot \delta\mathbf{g}_i dV^0 \; \forall \; \delta\mathbf{r} \end{aligned} \tag{2.83}$$

Before we proceed we need another expression for $\mathbf{g}_j \cdot \delta\mathbf{g}_i$. In order to get this, we compute

$$\begin{aligned} (\mathbf{g}_i \cdot \mathbf{g}_j)_{\text{tot}} &= (\mathbf{g}_i + \epsilon\delta\mathbf{g}_i) \cdot (\mathbf{g}_j + \epsilon\delta\mathbf{g}_j) \\ &= \mathbf{g}_i \cdot \mathbf{g}_j + \epsilon\,(\mathbf{g}_i \cdot \delta\mathbf{g}_j + \mathbf{g}_j \cdot \delta\mathbf{g}_i) + O(\epsilon^2) \end{aligned} \tag{2.84}$$

But, by (2.8b)

$$(\mathbf{g}_i \cdot \mathbf{g}_j)_{\text{tot}} = (g_{ij})_{\text{tot}} = g_{ij} + \epsilon\delta g_{ij} + O(\epsilon^2) \tag{2.85}$$

and therefore, under the assumption that ϵ is small, see (2.79)

$$\delta g_{ij} = \delta\,(\mathbf{g}_i \cdot \mathbf{g}_j) = \mathbf{g}_i \cdot \delta\mathbf{g}_j + \mathbf{g}_j \cdot \delta\mathbf{g}_i \tag{2.86}$$

Recall (2.14)

$$\mathbf{r} = \mathbf{r}^0 + \mathbf{u} = (x_j + u_j)\,\mathbf{i}_j \tag{2.87}$$

and the fact that $\mathbf{r}^0$ and x_j are given once the virgin state of the body is given. Then, all variations of $\mathbf{r}^0$ and x_j vanish with the result that (2.83) becomes

$$\int_{S^0} t_{ij}\mathbf{g}_j \cdot \mathbf{i}_m \delta u_m n_i^0 dS^0 + \int_{V^0} \bar{q}_j \delta u_j dV^0 = \int_{V^0} t_{ij}\tfrac{1}{2}\delta g_{ij} dV^0 \quad \forall \; \delta u_j \tag{2.88}$$

In order to get this, we have exploited that the *Kronecker delta* δ_{ij} is equal to $\mathbf{i}_i \cdot \mathbf{i}_j$, see (2.5), that t_{ij} and g_{ij} are symmetric, and that

$$\begin{aligned} t_{ij}\mathbf{g}_i \cdot \delta\mathbf{g}_j &= \tfrac{1}{2}\,(t_{ij}\mathbf{g}_i \cdot \delta\mathbf{g}_j + t_{ji}\mathbf{g}_j \cdot \delta\mathbf{g}_i) \\ &= \tfrac{1}{2}t_{ij}\,(\mathbf{g}_i \cdot \delta\mathbf{g}_j + \mathbf{g}_j \cdot \delta\mathbf{g}_i) \\ &= \tfrac{1}{2}t_{ij}\delta\,(\mathbf{g}_i \cdot \mathbf{g}_j) \\ &= \tfrac{1}{2}t_{ij}\delta g_{ij} \end{aligned} \tag{2.89}$$

As a consequence of the definition (2.11) of *Lagrange Strains*

$$\gamma_{ij} \equiv \tfrac{1}{2}\left(g_{ij} - \delta_{ij}\right) \tag{2.90}$$

we get

$$\delta\gamma_{ij} = \tfrac{1}{2}\delta\left(g_{ij} - \delta_{ij}\right) = \tfrac{1}{2}\delta g_{ij} \tag{2.91}$$

because the *Kronecker delta* is constant. By (2.19) this gives

$$\delta\gamma_{ij} = \tfrac{1}{2}\left(\delta u_{i,j} + \delta u_{j,i}\right) + \tfrac{1}{2}\left(u_{k,i}\delta u_{k,j} + \delta u_{k,i}u_{k,j}\right) \tag{2.92}$$

From (2.69)

$$\tau_m = \left(\mathbf{g}_k \cdot \mathbf{i}_m\right)\left(\mathbf{n}^0 \cdot \mathbf{i}_j\right) t_{jk} \tag{2.93}$$

it follows that

$$t_{ij}\mathbf{g}_j \cdot \mathbf{i}_m \delta u_m n_i^0 = \tau_m \delta u_m \tag{2.94}$$

and finally we arrive at the *Principle of Virtual Displacements*

Principle of Virtual Displacements

$$\int_{V^0} t_{ij}\delta\gamma_{ij}dV^0 = \int_{S^0} \tau_i \delta u_i dS^0 + \int_{V^0} \bar{q}_j \delta u_j dV^0 \ \forall\ \delta u_j \tag{2.95}$$

where we have interchanged the right-hand and left-hand sides.

Principle of virtual work: *Not* real work

When we investigate (2.95) we may see that the left-hand side is equal to the *virtual* work done by the *Piola-Kirchhoff Stresses* t_{ij} together with the *variation* $\delta\gamma_{ij}$ of the *Lagrange Strains* γ_{ij}, while the right-hand side expresses the *virtual* work done by the applied body force $\bar{q}$ and the surface tractions, i.e. applied loads $\bar{\tau}_i$ on S_T^0 and reactions τ_i on S_u^0 together with their associated displacement variations.[2.18]

In many cases we wish to[2.19]—and are able to—fulfill homogeneous kinematic boundary conditions on δu_i

$$\delta u_i = 0\ ,\ x_j \in S_u^0 \tag{2.96}$$

and then the principle is

Principle of Virtual Displacements for $\delta u_i = 0,\ x_j \in S_u^0$

$$\int_{V^0} t_{ij}\delta\gamma_{ij}dV^0 = \int_{S_T^0} \bar{\tau}_i \delta u_i dS^0 + \int_{V^0} \bar{q}_j \delta u_j dV^0 \quad \forall\ \delta u_j = 0,\ x_j \in S_u^0 \tag{2.97}$$

Actually, in most applications it is (2.97) rather than (2.95) we employ,

[2.18] The reason why I emphasize the word *virtual* here is that it is extremely important to note that the principle of virtual work does *not* entail *real* work.
[2.19] The reasons for this are probably not obvious at this point, so the reader will have to trust me on this.

but, since (2.97) is a special case of (2.95), you might want to focus on the latter only.

It is important to note that the assumptions behind the derivation of the *Principle of Virtual Displacements*, both (2.95) and (2.97), are[2.20]

- the loads and the stresses are in equilibrium and satisfy (2.59). Therefore, *no kinematic or constitutive* relations need apply to the stress field,
- the strain variations are derived from the displacements according to the strain-displacement relation, here given by (2.92), and that the displacement variation satisfies the *kinematic* boundary conditions. Thus, *no static or constitutive* conditions need apply to the variations of the strain-displacement field.

Kinematically admissible displacement field

A (variation of a) strain-displacement field which satisfies the strain-displacement relation and observes the appropriate[2.21] continuity and boundary conditions is called a *kinematically admissible displacement field*. Only such fields may be utilized in the *Principle of Virtual Displacements*.

2.4.1 The Budiansky-Hutchinson Dot Notation

Budiansky-Hutchinson Dot Notation

It is the property of (virtual) work inherent in the *Principle of Virtual Displacements*—and in other forms of the *Principle of Virtual Work*[2.22]—that makes the principle such a strong tool and foundation. I shall come back to this in Part II where we shall see how the principle serves as a useful and convenient basis for deriving theories for specialized continua such as beams and plates. In this connection, the so-called *Budiansky-Hutchinson (Dot) Notation*, see Chapter 33, proves to be a very convenient tool. When we utilize this, the principle may be expressed in the short form (33.14)

Principle of Virtual Displacements

$$\boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon} = \overline{\boldsymbol{T}} \cdot \delta \boldsymbol{u} \tag{2.98}$$

which is valid when

- the stress field $\boldsymbol{\sigma}$ is in equilibrium with the applied loads $\overline{\boldsymbol{T}}$,

[2.20] I emphasize the statements below so much because the experience of a long life as a teacher has proved to me that almost no student remembers the assumptions two days after I have gone through the derivations and told the class that they are very important.

[2.21] The meaning of the term "appropriate" depends on the actual version of the *Principle of Virtual Displacements*, see for example the differences between the restrictions on δu_j in (2.95) and in (2.97).

Also, in the above derivations we have assumed that the variation of the displacement field is continuous everywhere inside the body, i.e. continuous where the real displacement field is continuous, but sometimes, in particular in connection with formulation of *Finite Element Equations*, it proves convenient to abandon this requirement and add the appropriate terms to the principle. Since the presentation here is introductory, I have decided to avoid such complications.

[2.22] There exist other kinds of *Principle of Virtual Work* such as the *Principle of Virtual Forces* which is sometimes used in the linear case. In the nonlinear cases its formulation presents so great problems that it is rarely applied.

- the displacements $\boldsymbol{u}$ are "sufficiently smooth"[2.23] and satisfy the kinematic boundary conditions $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on the kinematic boundary S_u,
- the displacement variations $\delta\boldsymbol{u}$ are "sufficiently smooth" and satisfy the homogeneous kinematic boundary conditions $\delta\boldsymbol{u} = \delta\bar{\boldsymbol{u}}$, i.e. vanish on the kinematic boundary S_u,
- the strain variations $\delta\boldsymbol{\varepsilon}$ are given by the strain-displacement relation, see e.g. (2.92), which is valid for the general three-dimensional case, or (33.18)

Strain variation $\delta\varepsilon$

$$\delta\boldsymbol{\varepsilon} = \mathbf{l}_1(\delta\boldsymbol{u}) + \mathbf{l}_{11}(\boldsymbol{u}, \delta\boldsymbol{u}) \tag{2.99}$$

which is written by use of the *Budiansky-Hutchinson Notation* and covers (2.92) as well as many other relevant strain-displacement relations.

Clearly, (2.98) does not display any information about the dimension of the body—it could just as well be a one-dimensional body such as a beam instead of the three-dimensional body treated here. Of course, in that case the different fields must be reinterpreted accordingly. In Part II we shall see how this is done and at this point merely note that the strength of the short notation employed in (2.98) is that it covers all sorts of bodies. Part VI, Section 33.8, contains a summary of interpretations of (2.98) as well as other relevant formulas for a number of different structures.

2.4.2 Generalized Strains and Stresses

Generalized strains and stresses

Work conjugate quantities

The strain and stress measures $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are called *Generalized* in the sense that they do work together—are *Work Conjugate*. In Part II we derive continuum theories for a number of specialized continua and our guideline in this connection will be the requirement that the strain and stress measures are each other's work conjugate —are generalized quantities. At this point it is probably not self-evident why it is important that the strain and stress measures possess this property, and I refer the reader to Part II. It should be mentioned that the validity of minimum principles such as the *Principle of Minimum Potential Energy* and the upper- and lower-bound theorems of the theory of plasticity hinges on the fact that the participating strain and stress measures are generalized.[2.24]

[2.23] By this term I mean so smooth that the continuity of the body is not violated. This requirement is particular to the different types of structures. As an example, for the three-dimensional body the displacements themselves must be differentiable, while for a plate also the first derivatives of the transverse displacement component must also be differentiable.

[2.24] If the stresses and strains are not generalized you may find upper bound solutions to problems of perfect plasticity which appear to be *lower* than the equivalent lower bound solutions which, obviously, is wrong. Actually, this happened to one of my former colleagues when he worked on his Master's Thesis. The error disappeared when he used appropriate stress and strain measures.

2.5 Principle of Virtual Forces

There is a duality between kinematic and static quantities, which sometimes may be exploited.[2.25] By duality I refer to the fact that kinematic and static quantities appear in pairs, e.g. as generalized strains and their work conjugate generalized stresses, see Sections 2.4.2 and 33.2, and that there exist dual principles such as the *Principle of Virtual Displacements* and the *Principle of Virtual Forces*.

While we were able to establish the *Principle of Virtual Displacements* quite easily, see above, there seems to be no universally accepted *Principle of Virtual Forces* for the present case of large displacements and large strains. I shall therefore postpone derivation of this principle to the chapter on infinitesimal displacements and infinitesimal strains.

For large displacements: Problems with Principle of Virtual Forces

2.6 Constitutive Relations

The purpose of *Constitutive Relations* is to connect the strain and stress measures. This connection may be as simple as a linear relation between the stresses t_{ij} and the strains γ_{kl}, the so-called *Generalized Hooke's Law*,[2.26] or it may be more complicated and for instance involve information about the loading history of the material point in question, e.g. whether the point is subjected to further loading or it is experiencing unloading. Such materials display *Plastic* properties.

Generalized Hooke's "Law"

2.6.1 Hyperelastic Materials

The term *elastic* implies that all deformation is reversible, e.g. no matter how hard we pull on a bar of an elastic material it will always recover its original shape when the loading is removed. *Hyperelastic* materials are defined by the assumption of the existence of a *Strain Energy Function* $W(\gamma_{ij})$, also called the *Strain Energy Density*, with the property that

Hyperelasticity
Strain energy function $W(\gamma_{ij})$ = Strain energy density $W(\gamma_{ij})$

$$t_{mn} = \frac{\partial W(\gamma_{ij})}{\partial \gamma_{mn}} \tag{2.100}$$

which presupposes that the stress is independent of the strain history. At this point we do not intend to proceed investigating the general case but limit ourselves to linear hyperelasticity. Omitting inconsequential constants, for linear hyperelasticity the form of $W(\gamma_{ij})$ must be

$$W(\gamma_{ij}) = \tfrac{1}{2} E_{ijkl}\gamma_{ij}\gamma_{kl} \tag{2.101}$$

Linear hyperelasticity

[2.25] Earlier this duality was often used to compute the displacements of beams by determining the bending moments in a so-called "conjugate beam," which was loaded by the curvature of the real beam. Such methods do not seem to be used anymore.

[2.26] It is important to note that *Hooke's Law* is a *material model* and not a law. No material obeys "Hooke's Law," but the relation (2.101) is a simple and useful material model which, by the way, is *the* most commonly used material model.

because differentiation of W given by (2.101) provides

$$t_{ij} = E_{ijkl}\gamma_{kl} \tag{2.102}$$

When we define linear hyperelasticity according to (2.101) it is obvious that E_{ijkl} may be assumed to possess *group symmetry* in the sense that

$$E_{ijkl} = E_{klij} \tag{2.103}$$

since any antisymmetric part of E_{ijkl} vanishes from the product on the right-hand side of (2.101). The linear (hyper)elastic model entails two more symmetry properties of E_{ijkl}, which we derive below. First, because t_{ij} is symmetric in its indices we must have

$$E_{ijkl} = E_{jikl} \tag{2.104}$$

and, secondly, since γ_{kl} is symmetric in its indices, without loss of generality, we may take

$$E_{ijkl} = E_{ijlk} \tag{2.105}$$

Isotropy

Anisotropy

These relations hold for all materials whether they are *Isotropic*, i.e. their properties are independent of direction,[2.27] or they are *Anisotropic*,[2.28] which means that their properties depend on direction. Because of the symmetry relations (2.104) and (2.105) the original 81 constants in E_{ijkl} are reduced to only 36, and the group symmetry further reduces the number to 21 independent constants in the general, anisotropic case. If the material is special, e.g. isotropic, then the number of different constants is reduced further, but except for this very short introduction to the subject we defer discussion of constitutive relations, i.e. material models, to the section on the kinematically linear theory, Section 5.

2.6.2 Plastic Materials

In Chapter 5 we discuss plastic material models and do not pursue the subject here.

2.7 Potential Energy

Principle of Virtual Displacements No information about material

The *Principle of Virtual Displacements*, see (2.97) or (2.98), does not entail any information about the material. This is not surprising since the principle was derived as an auxiliary way of expressing equilibrium. There exists a very important principle, which is valid for (hyper)elastic solids and structures and thus exploits the constitutive information. This principle is called the *Principle of Stationary Potential Energy*.

As usual in continuum mechanics there are several ways to arrive at

[2.27] To a good approximation, steel is such a material.
[2.28] Wood is such a material.

a result. Here, we postulate that the *Potential Energy*[2.29] of the three-dimensional *Hyperelastic* body is

Potential energy, hyperelasticity

$$\mathbf{\Pi}_{\mathrm{P}}(u_i) = \int_{V^0} W(\gamma_{ij}) dV^0 - \int_{V^0} \bar{q}_i u_i dV^0 - \int_{S_T^0} \bar{\tau}_i u_i dS^0 \tag{2.106}$$

and investigate its properties below. Observe that for a given structure with a given load $\mathbf{\Pi}_{\mathrm{P}}$ does not depend on the stresses, but only on kinematic quantities, namely the displacements u_m and the strains $\gamma_{ij}(u_m)$, which according to (2.19) are given by the displacements

$$\gamma_{mn} = \tfrac{1}{2}\left(u_{m,n} + u_{n,m}\right) + \tfrac{1}{2} u_{k,m} u_{k,n} \tag{2.107}$$

where the displacement field u_m obviously must satisfy the condition that its first derivatives are defined in V^0. This, however, is not sufficient because we shall appeal to the *Principle of Virtual Displacements*, which presupposes that the displacements satisfy the kinematic boundary conditions $u_m = \bar{u}_m$ on S_u^0.

Furthermore, the variation $\delta\gamma_{mn}$ of the strain must be derived from the displacements u_m according to (2.92)

Strain variation $\delta\gamma_{mn}$ derived from displacement u_m and displacement variation δu_m

$$\delta\gamma_{mn} = \tfrac{1}{2}\left(\delta u_{m,n} + \delta u_{n,m}\right) + \tfrac{1}{2}\left(\delta u_{k,m} u_{k,n} + u_{k,m} \delta u_{k,n}\right) \tag{2.108}$$

According to Chapter 33, (33.20), the first variation of $\mathbf{\Pi}_{\mathrm{P}}$ is

$$\delta\mathbf{\Pi}_{\mathrm{P}}(u_i) = \int_{V^0} \frac{\partial W(\gamma_{ij})}{\partial \gamma_{kl}} \delta\gamma_{kl} dV^0 - \int_{V^0} \bar{q}_i \delta u_i dV^0 - \int_{S_T^0} \bar{\tau}_i \delta u_i dS^0 \tag{2.109}$$

which, when we utilize (2.100), gives

$$\delta\mathbf{\Pi}_{\mathrm{P}}(u_i) = \int_{V^0} t_{kl} \delta\gamma_{kl} dV^0 - \int_{V^0} \bar{q}_i \delta u_i dV^0 - \int_{S_T^0} \bar{\tau}_i \delta u_i dS^0 \tag{2.110}$$

When we require that $\delta\mathbf{\Pi}_{\mathrm{P}}(u_i)$ vanishes we arrive at the *Principle of Virtual Displacements* (2.97)

$$\begin{gathered} \int_{V^0} t_{ij} \delta\gamma_{ij} dV^0 = \int_{S_T^0} \bar{\tau}_i \delta u_i dS^0 + \int_{V^0} \bar{q}_j \delta u_j dV^0 \\ \forall \left(\delta u_j = 0 \,, x_j \in S_u^0\right) \end{gathered} \tag{2.111}$$

Note that in the derivation of (2.111) we have assumed hyperelasticity to be valid, while (2.97) is only based on equilibrium and is therefore valid for *all* material models.

[2.29] Potentials are discussed in some detail in Chapter 32 and in Chapter 33.

2.7.1 Linear Elasticity

For linear (hyper)elasticity, i.e. for *Hooke's "Law"* the expression for the potential energy becomes

Potential energy, linear hyperelasticity

$$\mathbf{\Pi}_{\mathrm{P}}(u_i) = \frac{1}{2}\int_{V^0} E_{ijkm}\gamma_{ij}\gamma_{km}dV^0 - \int_{V^0}\bar{q}_i u_i dV^0 - \int_{S_T^0}\bar{\tau}_i u_i dS^0 \quad (2.112)$$

This is an expression which is very often used in various connections, e.g. as a foundation for study of elastic buckling and other nonlinear problems.

2.8 Complementary Energy

We shall not attempt to establish a *Complementary Energy* for the present case of large displacements and large strains, see the comment in Section 2.5, and once more refer to the chapter on infinitesimal displacements and infinitesimal strains.

2.9 Static Equations by the Principle of Virtual Displacements

Derivation of equilibrium equations for generalized strains and stresses by the principle of virtual displacements

Deriving static equations in the spirit of Section 2.3 is not always a straightforward task, in particular if the type of continuum is a specialized one such as a plate or shell. Fortunately, there exists another way of getting the equilibrium equations, namely via the principle of virtual displacements. This approach is discussed below and utilized in Part II.

We may derive the equilibrium equations by use of the principle of virtual displacements observing the kinematic relations, i.e. the strain-displacement relation, the compatibility conditions, the kinematic (dis)continuity conditions, and the kinematic boundary conditions, rather than attempt to establish them in a more direct way. One reason for doing this is that in this way we insure that the stresses and strains are *generalized* in the sense that these quantities work together in producing the correct internal virtual work.

Difficult static plate boundary condition easily established by use of the principle of virtual displacements

Earlier, particularly in the twentieth century, when this approach was not *en vogue*—not known then, to be exact—there was much discussion about how to derive the static boundary conditions, in particular boundary conditions involving the shear force and the torsional moment in plates, see (9.25c). Another reason for the newer approach is that in specialized, kinematically nonlinear continuum theories, e.g. nonlinear theories for beams, plates, shells, etc., it is often extremely difficult to choose the static quantities in a meaningful way. If, however, they are defined through the *Principle of Virtual Displacements* they will always have a sound interpretation, albeit sometimes not very evident.

At a more philosophical level I mention that nobody has been able to measure continuum mechanical stresses, while sometimes it is possible to measure strains quite accurately. The reason for this is that the stress components are obtained as the limit of force per area, whereas the the strain components are given as the limit of changes in distance and direction, which is much easier to measure.

Stresses are figments of our imagination—strains are more real

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