

## Chapter 2

# Variable Elasticity Effects in Rotating Machinery

**Abstract** The effects of variable elasticity in rotating machinery occur with a large variety of mechanical, electrical, etc., systems, in the present case, geometrical and/or mechanical problems. Parameters affecting elastic behavior do not remain constant, but vary as functions of time. Systems with variable elasticity are governed by differential equations with periodic coefficients of the Mathieu-Hill type and exhibit important stability problems. In this chapter, analytical tools for the treatment of this kind of equations are given, including the classical Floquet theory, a matrix method of solution, solution by transition into an equivalent integral equation and the BWK procedure. The present analysis is useful for the solution of actual rotor problems, as, for example, in case of a transversely cracked rotor subjected to reciprocating axial forces. Axial forces can be used to control large-amplitude flexural vibrations. Flexural vibration problems can be encountered under similar formulation.

## 2.1 Introduction

Variable elasticity effects occur with systems in which the parameters affecting elastic behavior do not remain constant, but vary as functions of time. Equations of motion pertaining to such systems remain linear, but they possess time-dependent coefficients. Similar phenomena appear in all fields of physics and are generally associated with wave propagation in periodic media, a problem encountered as early as 1887 by Lord Rayleigh, and subsequently by other prominent physicists [1]. It was recognized that such phenomena are described by means of Hill and Mathieu differential equations. A detailed and comprehensive review on the work on waves in periodic structures was given by Brillouin [2].

Omitting a large number of non-mechanical phenomena mentioned in the above references, it is interesting to concentrate attention on the following vibrating mechanical systems [3–10]:

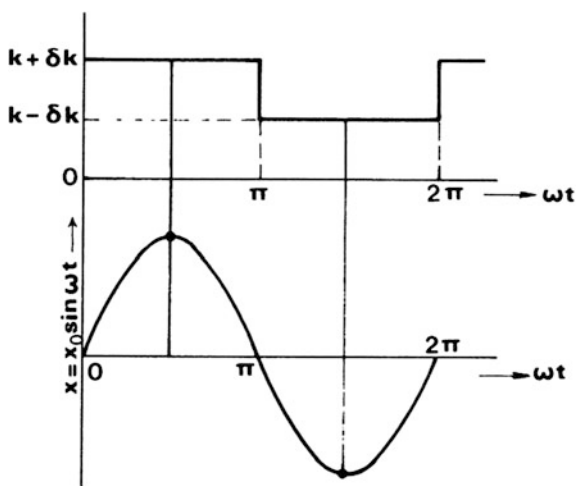
1. A rotating shaft with non-circular cross-section, i.e. non-uniform flexibility.
2. A mass suspended from a taut string with time-varying tension.
3. A pendulum with time-varying length.
4. An inverted pendulum attached to a vertically vibrating hinge.
5. The side-rod system of electric locomotives, exhibiting torsional vibrations.
6. The rotating parts of small motors, which are subject to the time-varying action of electromagnetic fields.
7. A rotating flywheel carrying radially moving masses.
8. A rotating flywheel eccentrically connected to reciprocating masses.

The list is by no means complete, but the examples are characteristic of the problems encountered.

If one of these structures is constrained to move at constant circular frequency, the respective parameters are periodic functions and under proper conditions large vibration amplitudes may develop. The reason is that, within each cycle, positive work is introduced into the system, which results in a gradually increasing amplitude, i.e. instability. This effect can be caused, for example, by a small accidental force producing an initial velocity and being enhanced, under certain conditions, by the action of gravity or static forces [3]. In Fig. 2.1, the response of a simple oscillator with a different spring constant in tension and in compression is presented. The oscillator is assumed to perform harmonic oscillations at constant circular frequency  $\omega$ .

By using the analysis given in Sect. 2.2, the total energy stored in the system during the first and second half-cycles can be evaluated. This may be positive, zero or negative. The second and third cases lead to stable situations; however, the first one leads to increasing vibration amplitudes, i.e. to instability.

**Fig. 2.1** Response of a simple oscillator with a different spring constant in tension and in compression



**Fig. 2.2** A beam on circular supports of finite radius exhibiting periodic length variation during flexural vibration



Consider now a rotating shaft, or the more general case of a beam executing flexural vibrations. The following cases of variable elasticity can be distinguished.

### 2.1.1 Variable Length $l$

A beam resting on circular supports with finite radius undergoes periodic changes of length during flexural vibrations. This case is obviously not related to rotating machinery; however, it is an interesting example of periodic changes in elasticity (Fig. 2.2).

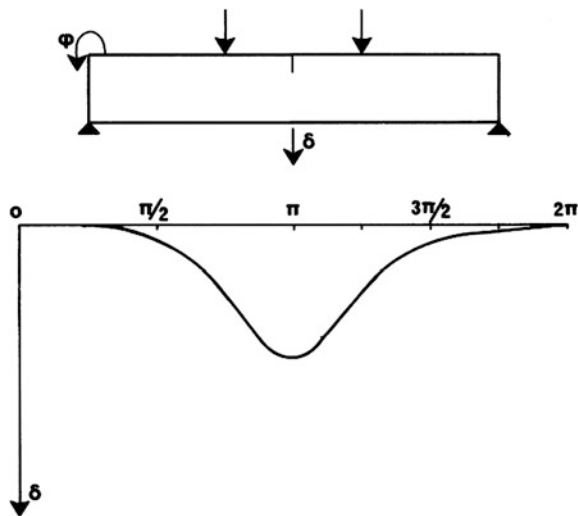
### 2.1.2 Variable Stiffness $EJ$

The flexural stiffness  $EJ$  may vary periodically if either elastic modulus  $E$  of the structure material or the second area moment  $J$  of the respective cross-sections or both vary accordingly. Rotor-to-stator rub often occurs in rotating machinery. Methods for detecting the rub-impact effect were developed based on the variable stiffness issue for the rotor system [11, 12]. Therefore, one can distinguish the following particular cases:

*Variable elastic modulus  $E$ :* Many materials, especially polymers and their composites, exhibit different elastic modulus in tension  $E_t$  and in compression  $E_c$  [12–15]. A beam performing longitudinal vibration would exhibit a variable stiffness of ripple form, as shown in Fig. 2.1. With flexural vibration, in the presence of an axial load, a respectively oscillating neutral axis would result, giving rise to periodically varying flexural stiffness.

*Variable second area moment  $J$ :* The second area moment  $J$  depends on the particular form of the respective cross-sections. With circular cross-sections,  $J$  remains constant in all directions and the flexural stiffness is not affected. However, with non-circular cross-sections, for example rectangular, if the beam rotates while the direction of the load or the couple vector remains constant, a periodically varying  $J$  results. With constant rotating speed, this variation is sinusoidal. Non-circular cross-sections are produced by keyways, etc., special formations of rotors, and also by longitudinal or transverse cracks. The latter may have a considerable effect and, alternatively, their presence can be detected by the dynamic behavior of the vibrating element. A quasi-sinusoidal experimental curve

**Fig. 2.3** Variation of flexural stiffness of a transversely cracked rotor



represents the periodic variation of the flexural stiffness [16] (Fig. 2.3). The simplest example of a system with periodically varying stiffness is a straight rotating shaft, whose cross-section has different principle moments of inertia. A classical example of systems with periodically varying stiffness was the drive system of an electric locomotive incorporating a coupling link for force transmission, applied during the first years of electric locomotion.

### 2.1.3 Variable Mass or Moment of Inertia

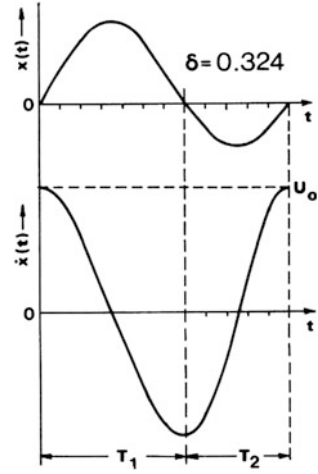
Such examples have already been mentioned. The case of a reciprocating engine is a very common one, by which the inertia of the piston mechanism varies periodically [17–19]. A heavy flywheel is necessary in order to maintain uniform rotating speed, a device important for all systems exhibiting variable elasticity.

In this chapter the various problems of such systems are discussed in detail, along with the respective mathematical techniques used for their treatment.

## 2.2 The Problem of Stability

Consider a simple oscillator, i.e. a mass  $m$  suspended from a massless spring, whose constant, however, assumes the value  $k(1 + \delta)$  for tension and the value  $k(1 - \delta)$  for compression. Moreover, assume that at  $t = 0$ , its deflection is  $x = 0$  and its velocity  $\dot{x} = v_0$ . The *free vibration* of the system is governed by the equations

**Fig. 2.4** Variation of displacement and velocity with time of a simple oscillator with a different spring constant in tension and compression



$$\left. \begin{aligned} \ddot{x} + \omega_0^2(1 + \delta)x &= 0 & \text{for } x > 0 \text{ (tension)} \\ \ddot{x} + \omega_0^2(1 - \delta)x &= 0 & \text{for } x < 0 \text{ (compression)} \end{aligned} \right\} \quad (2.1)$$

where  $\omega_0^2 = k/m$ . During the first half-cycle, the natural frequency of the system is

$$\omega_1 = \omega_0(1 + \delta)^{1/2} \quad (2.2a)$$

while during the second half-cycle

$$\omega_2 = \omega_0(1 - \delta)^{1/2}. \quad (2.2b)$$

The period  $T$  of a complete cycle is

$$T = T_1 + T_2 = \frac{\pi}{\omega_1} + \frac{\pi}{\omega_2} = \frac{2\pi}{\omega} \quad (2.3)$$

where  $\omega$  is the circular frequency of the free vibration, which, however, is not uniform over the two half-cycles. From Eq. (2.3) we obtain

$$\frac{1}{\omega} = \frac{1}{\omega_1} + \frac{1}{\omega_2} \quad (2.4a)$$

or

$$\omega = 0.5 \sqrt{\frac{k}{m}} \frac{(1 - \delta^2)^{1/2}}{(1 + \delta^2)^{1/2} + (1 - \delta^2)^{1/2}} \quad (2.4b)$$

In Fig. 2.4, the forms of the displacement function  $x(t)$  and the velocity function  $\dot{x}(t)$  are presented for  $\delta = 0.324$ . Amplitudes and velocities remain constant, as the system is conservative. However, the oscillation is clearly not simple harmonic, and can be analyzed as a Fourier series:

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \sin k\omega t + b_k \cos k\omega t) \quad (2.5)$$

Considering the odd (sine) functions, we obtain

$$a_k = \frac{2}{T} \int_0^T x(t) \sin k\omega t dt \quad (2.6)$$

where

$$x(t) = \begin{cases} x_1 \sin \omega_1 t & (0 < t < T_1) \\ x_2 \sin \omega_2 t & (T_1 < t < T) \end{cases} \quad (2.7)$$

As evident from Eq. (2.5), the system possesses in fact an infinite number of eigen frequencies  $\omega, 2\omega, 3\omega \dots$ , multiples of the circular frequency  $\omega$  of a full cycle, as defined in Eq. (2.4a).

Now, if the system is forced to oscillate at uniform circular frequency  $\omega$ , stability problems may arise. The respective analysis is due to van der Pol and Strutt [3, 4] and is based on the solution of Eq. (2.1) under the proper boundary conditions. Namely, Eq. (2.1) possess the following general solutions:

$$\begin{cases} x_1 = A_1 \sin \omega_1 t + A_2 \cos \omega_1 t \\ x_2 = B_1 \sin \omega_2 t + B_2 \cos \omega_2 t \end{cases} \quad (2.8)$$

Now, considering that at the end of the first half-cycle ( $t = \pi/\omega$ ), displacements and velocities for both solutions [Eq. (2.8)] must be equal and, moreover, that at the end of the first complete cycle ( $t = 2\pi/\omega$ ) displacements and velocities must be  $\lambda$  times as large as at  $t = 0$ , four homogeneous linear equations result, containing the integration constants as unknowns. The condition for these equations to give solutions other than zero for the latter leads to the quadratic

$$\lambda^2 - 2q\lambda + 1 = 0 \quad (2.9)$$

giving for  $\lambda$  the values

$$\lambda = q \pm \sqrt{q^2 - 1} \quad (2.10)$$

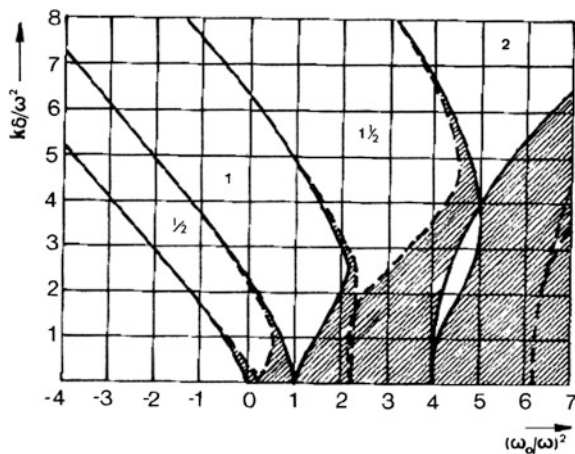
where

$$q = \cos \frac{\pi\omega_1}{\omega} \cos \frac{\pi\omega_2}{\omega} - \frac{\omega_1^2 + \omega_2^2}{2\omega_1\omega_2} \sin \frac{\pi\omega_1}{\omega} \sin \frac{\pi\omega_2}{\omega} \quad (2.11a)$$

or

$$\begin{aligned} q = & \cos \left[ \pi(1 + \delta)^{\frac{1}{2}} \frac{\omega_0}{\omega} \right] \cos \left[ \pi(1 - \delta)^{1/2} \frac{\omega_0}{\omega} \right] - \frac{1}{2(1 - \delta^2)^{\frac{1}{2}}} \\ & \times \sin \left[ \pi(1 + \delta)^{\frac{1}{2}} \frac{\omega_0}{\omega} \right] \sin \left[ \pi(1 - \delta)^{\frac{1}{2}} \frac{\omega_0}{\omega} \right] \end{aligned} \quad (2.11b)$$

**Fig. 2.5** Regions of stability and instability for a simple oscillator with a different spring constant in tension and in compression. (According to van der Pol and Strutt [3, 4])



Now, the system is stable for  $q = \pm 1$ , in which case  $\lambda = q$ , and for  $|q| < 1$ , when  $\lambda$  is complex, but its modulus equals one. Alternatively, instability occurs with  $|q| > 1$ . In this case, if  $q$  is positive, displacements and velocities at the end of each consecutive cycle of the spring fluctuation will increase, retaining the same sign. This means that the system vibrates at the same frequency  $\Omega$  of spring fluctuation or at a multiple of it:  $2\Omega, 3\Omega, \dots$ . If  $q$  is negative, displacements and velocities will also increase at the end of each consecutive cycle of the spring fluctuation, but will have alternating sign. This means that the system vibrates at frequencies  $\Omega/2, \Omega/3, \dots$ .

As  $q$  is a function of  $\delta$  and the ratio  $\omega_0/\omega$ , according to Eq. (2.11b) the above conditions of stability can be plotted in the classical diagram of Fig. 2.5. The shaded regions in Fig. 2.5 correspond to stable situations ( $-1 < q < 1$ ), blank regions to unstable ones  $|q| > 1$  and full and dotted lines to  $q = +1$  and  $q = -1$  respectively. Details of this representation can be found in the relevant literature [2–4]. In the presence of damping, the motion is again not simple harmonic, even for the free vibration, and a solution for  $x$  must be sought in the form of Fourier series [3].

## 2.3 The Mathieu-Hill Equation

The general form of an equation of motion, pertaining to a system with variable elasticity, is

$$\frac{d^2x}{dt^2} + f(t)x = 0 \quad (2.12)$$

where  $f(t)$  is a single-valued periodic function of fundamental period  $T$ , which can be represented by a general Fourier series of the form

$$f(t) = A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega t + B_k \sin k\omega t) \quad (2.13)$$

where  $\omega = 2\pi/T$ . Equation (2.12), if the function  $f(t)$  has the general form (2.13), is known as *Hill's equation*. If the Fourier series (2.13) can degenerate in the simple form

$$f(t) = A_0 + A_1 \cos \omega t \quad (2.14)$$

then Eq. (2.12) is known as *Mathieu's equation*. This equation can be solved by using the tabulated Mathieu functions [16].

An important property of a system expressed by the Mathieu equation is that instabilities are exhibited for certain ranges of the parameters  $A_0$  and  $A_1$ , i.e. if the system is displaced from the equilibrium,  $x = 0$ , a motion of increasing amplitude follows.

The classical method for the solution of Hill's equation, available in most textbooks [20], is as follows: Following the form (2.13) of the function  $f(t)$ , one can look for a similar expansion for the unknown periodic function  $x(t)$ :

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega t + b_k \sin k\omega t) \quad (2.15)$$

Equation (2.12) then leads to an infinite system of simultaneous linear homogeneous equations for the unknown coefficients  $a_k$  and  $b_k$ . In order to obtain a non-trivial solution, the corresponding infinite determinant must be zero. However, if the series of the coefficients  $A_k$  and  $B_k$  in Eq. (2.13) are absolutely converging, the latter can be easily evaluated or, alternatively, higher terms can be neglected, and the problem can be solved, by long computations in any case. The method is not very practical; however, it appears to be the only one operating with functions  $f(t)$ , for which no analytical expression is available, as occurs with experimentally determined forms.

Apart from Hill's method of infinite determinants, further methods have been developed for the treatment of coupled Mathieu equations [17], based on the expansion of the converging infinite determinant, which also provides a criterion for stability of the solution. Moreover, a stability criterion for the general Hill equation can be developed in a similar manner [21].

## 2.4 The Classical Floquet Theory

Equations of the Mathieu-Hill type can be solved by means of the Floquet theory [2, 22–25]. This theory basically deals with equations of the form

$$\dot{y} = \mathbf{A}(t)y \quad (2.16)$$



where  $y = (y_1, y_2, \dots, y_n)$  and  $\mathbf{A}(t) = [a_{ij}(t)]$  is a continuous  $n \times n$  matrix defined in the range  $-\infty < t < \infty$ .  $\mathbf{A}(t)$  satisfies the relation

$$\mathbf{A}(t + T) = \mathbf{A}(t)$$

where the period  $T$  is a non-zero quantity. The complete Floquet theory, leading to the general form of solution of Eq. (2.16) can be found in the literature [24, 25]. A basic property of the solution, however, is described by the Floquet theorem stating that

*For the system expressed by the matrix Eq. (2.16), there exists a non-zero constant  $\lambda$ , real or complex, and at least one non-trivial solution  $y(t)$  having the property that*

$$y(t + T) = \lambda y(t)$$

The various values  $\lambda_1, \lambda_2, \dots, \lambda_m$  of the parameter  $\lambda$  are the distinct characteristic roots of a constant  $n \times n$  matrix  $\mathbf{C}$  defined by the relation

$$\Phi(t + T) = \Phi(t)\mathbf{C}$$

where  $\Phi(t) = [\varphi_{ij}(t)]$  is a fundamental system of solutions of Eq. (2.16). The above values are called *characteristic factors or multipliers* of Eq. (2.16). The following corollary is valid:

*The system (2.16) has periodic solution of period  $T$ , if and only if there is at least one characteristic factor equal to one [14].*

The quantities  $r_1, r_2, \dots, r_m$  defined by the relations

$$\lambda_j = \exp(r_j T) \quad j = 1, 2, \dots, m$$

are called the characteristic exponents of the system (2.16) and, according to the following theorem

*There are at least  $m$  solutions of the system (2.16) having the form*

$$x_i(t) = p_i(t) \exp(r_i t) \quad i = 1, 2, \dots, m$$

*where the functions  $p_i(t)$  are periodic with period  $T$ .*

Now, consider Hill's equation:

$$\ddot{x} + f(t)x = 0 \tag{2.12}$$

where  $f(t)$  is real-valued, continuous and periodic with period  $T$ , assumed traditionally as equal to  $\pi$ . Let a fundamental system of solutions be  $x_1(t)$  and  $x_2(t)$ , and

$$x_1(0) = 1 \quad \dot{x}_1(0) = 0 \quad x_2(0) = 0 \quad \dot{x}_2(0) = 1$$

whose Wronskian is equal to one. That is

$$\Phi(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{bmatrix}$$

and

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix}$$

when the matrix  $\mathbf{C}$  is equal to

$$\mathbf{C} = \begin{bmatrix} x_1(\pi) & x_2(\pi) \\ \dot{x}_1(\pi) & \dot{x}_2(\pi) \end{bmatrix}$$

whose characteristic roots  $\lambda_1, \lambda_2$  are roots of the equation

$$(\lambda \mathbf{I} - \mathbf{C}) = \lambda^2 - [x_1(\pi) + \dot{x}(\pi)]\lambda + 1 = 0$$

where  $\lambda_1 \lambda_2 = 1$  and  $r_1 + r_2 \equiv 0$ . Hence a number  $r$  can be defined satisfying the conditions

$$\exp(ir\pi) = \lambda_1$$

and

$$\exp(-ir\pi) = \lambda_2$$

The following cases are distinguished:

1.  $\lambda_1 \neq \lambda_2$ , leading to two linearly independent solutions:

$$\varphi_1(t) = \exp(irt)f_1(t)$$

$$\varphi_2(t) = \exp(-irt)f_2(t)$$

where  $f_1(t), f_2(t)$  are periodic with period  $\pi$ .

2.  $\lambda_1 = \lambda_2$ , leading to one non-trivial periodic solution with period  $\pi$  (with  $\lambda_1 - \lambda_2 = 1$ ) or  $2\pi$  (with  $\lambda_1 = \lambda_2 = -1$ ).

In order to apply the above results, information about the matrix  $\mathbf{C}$  is required, which can be derived from the conditions imposed on  $\mathbf{A}(t)$ , guaranteeing the existence of stable, unstable or oscillatory solutions.

Finally, for a non-homogeneous system

$$\dot{x} = \mathbf{A}(t)x + \mathbf{B}(t) \tag{2.17}$$

where  $\mathbf{A}(t)$  and  $\mathbf{B}(t)$  are periodic matrices with period  $T$ , the following theorem is valid:

*For the system (2.17), a period solution with period  $T$  exists for every  $\mathbf{B}(t)$ , if and only if the corresponding homogeneous system has no non-trivial solution of period  $T$ .*

## 2.5 Matrix Solution of Hill's Equation

Consider Eq. (2.12) again and two linearly independent solutions  $x_1(t)$  and  $x_2(t)$  in the interval  $0 \leq t \leq T$ . The general solution for displacement and velocity is given by the equations

$$\left. \begin{aligned} x(t) &= A_1 x_1(t) + A_2 x_2(t) \\ \dot{x}(t) &= A_1 \dot{x}_1(t) + A_2 \dot{x}_2(t) \end{aligned} \right\} \quad (2.18a)$$

or in matrix form

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad (2.18b)$$

The Wronskian of the two solutions  $x_1(t)$  and  $x_2(t)$ , given by the following equation

$$W = \begin{vmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{vmatrix} = x_1(t)\dot{x}_2(t) - \dot{x}_1(t)x_2(t) \quad (2.19)$$

is constant in the interval  $0 < t < T$  [21] and, since  $x_1(t)$  and  $x_2(t)$  are linearly independent, we have also  $W \neq 0$ ; hence the  $2 \times 2$  matrix in Eq. (2.18b) is non-singular and has the following inverse:

$$\begin{bmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{bmatrix}^{-1} = \frac{1}{W} \begin{bmatrix} \dot{x}_2(t) & -x_2(t) \\ -\dot{x}_1(t) & x_1(t) \end{bmatrix} \quad (2.20)$$

If the initial conditions are given:

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}_{t=0} = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \quad (2.21)$$

From Eq. (2.18b) we derive

$$\begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} x_1(0) & x_2(0) \\ \dot{x}_1(0) & \dot{x}_2(0) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad (2.22)$$

or

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} x_1(0) & x_2(0) \\ \dot{x}_1(0) & \dot{x}_2(0) \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \quad (2.23)$$

which determines the column of arbitrary constants, and, consequently, the final solution assumes the form

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \frac{1}{W} \begin{bmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{bmatrix} \begin{bmatrix} \dot{x}_2(0) & -x_2(0) \\ -\dot{x}_1(0) & x_1(0) \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \quad (2.24)$$

At the end of the first period, i.e. at  $t = T$ , Eq. (2.24) becomes

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}_{t=T} = \mathbf{M} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \quad (2.25)$$

where

$$\mathbf{M} = \frac{1}{W} \begin{bmatrix} x_1(T) & x_2(T) \\ \dot{x}_1(T) & \dot{x}_2(T) \end{bmatrix} \begin{bmatrix} \dot{x}_2(0) & -x_2(0) \\ -\dot{x}_1(0) & x_1(0) \end{bmatrix} \quad (2.26)$$

It can be easily shown that, at the end of the  $n$ th period of  $f(t)$ , i.e. at  $t = nT$ , we have

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}_{t=nT} = \mathbf{M}^n \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \quad (2.27)$$

which provides the complete solution of Hill's equation in terms of the initial conditions, and two linearly independent solutions of Hill's equation in the fundamental interval  $0 \leq t \leq T$ . The  $n$ th power of the matrix  $\mathbf{M}$  can be computed by means of Sylvester's theorem [26, 27]. Solutions have been given for various forms of the function  $f(t)$ , such as rectangular ripple, sum of step functions, exponential function, sawtooth variation, etc. [26].

## 2.6 Solution by Transition into an Equivalent Integral Equation

Equation (2.12), with the function  $f(t)$  developable in a Maclaurin series, can be solved by means of a transition to an equivalent Volterra integral equation of the second type. Again, by using the notation of Eq. (2.21), we obtain the following form [24]:

$$x(t) = x_0 t + x - \int_0^t (t - \tau) f(\tau) x(\tau) d\tau \quad (2.28)$$

The classical method of solution is by means of successive approximations. Problems of a practical nature may appear with the determination of the iterated kernels, depending on the function  $f(t)$ . Particular examples, such as for  $f(t) = t^n$  can be found in the literature [28].

## 2.7 The Bwk Procedure

The BWK<sup>1</sup> procedure, discovered while dealing with problems of wave mechanics [23, 29, 37] is suitable for the solution of Eq. (2.12) in the following manner: Let a function be considered, such that

$$u = g^{-1/2} \exp(-iS) S = \int_a^t G dx \quad (2.29)$$

where  $G(t)$  is any given function of  $t$ . The following equation can then be derived:

$$\ddot{u} + u \left[ G^2 - \frac{3}{4} \left( \frac{\dot{G}}{G} \right)^2 + \frac{1}{2} \frac{\ddot{G}}{G} \right] = 0 \quad (2.30)$$

which, when compared with Eq. (2.12), leads to the condition

$$f = G^2 - \frac{3}{4} \left( \frac{\dot{G}}{G} \right)^2 + \frac{1}{2} \frac{\ddot{G}}{G} \quad (2.31)$$

If Eq. (2.31) can be solved to a certain approximation, then the function  $u$  in Eq. (2.29) represents an approximate solution of Eq. (2.12). Moreover, in case that  $\dot{G}$  and  $\ddot{G}$  can be omitted in Eq. (2.31), an acceptable approximation can be obtained by taking

$$G = G_0 = [f(t)]^{1/2} \quad (2.32)$$

which is a zero-order approximation to the BWK procedure. As a particular case, with

$$G = A/(a+t)^2 \quad (2.33)$$

terms in  $\dot{G}$ ,  $\ddot{G}$  cancel out in Eq. (2.30), and a rigorous solution of Eq. (2.12) results [23].

An accurate approximation to  $G$  can be determined by means of a series

$$G = G_0 + G_1 + G_2 + \dots \quad (2.34)$$

under the assumption that

$$|\dot{G}_0/G^2| < \varepsilon \quad |\ddot{G}/G^3| < \varepsilon^2 \quad \varepsilon^2 \ll 1$$

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<sup>1</sup> Named after three authors: L. Brillouin (*J. Phys.*, **7**, 1926, 353); G. Wentzel (*Z. Phys.*, **38**, 1926, 518); and E.C. Kemble (*The Fundamental Principles of Quantum Mechanics*, McGraw-Hill, New York, 1937).

when the successive terms would be of the order  $\varepsilon^2$ ,  $\varepsilon^4$ ..... By substitution in Eq. (2.30) we obtain

$$\left. \begin{aligned} G_0 &= f^{1/2} \\ 2G_0G_1 &= \frac{3}{4} \left( \frac{\dot{G}_0}{G_0} \right)^2 - \frac{1}{2} \frac{\ddot{G}_0}{G_0} \\ 2G_0G_1 + G_1^2 &= \frac{3}{2} \left( \frac{\dot{G}_0}{G_0} \right)^2 \left( \frac{\dot{G}_1}{G_0} - \frac{\dot{G}_1}{G_0} \right) - \frac{1}{2} \frac{\ddot{G}_0}{G_0} \left( \frac{\dot{G}_1}{G_0} - \frac{G_1}{G_0} \right) \end{aligned} \right\} \quad (2.35)$$

and the expansion of  $G$  would yield a first solution from Eq. (2.29). A second independent solution  $v$  can be found either with  $f > 0$ ,  $G$  real and positive and  $S$  real, or with  $f < 0$ ,  $G$  and  $S$  purely imaginary and  $i^{1/2}u$  real. The first case corresponds to propagating waves, the second to attenuated waves without propagation. A detailed analysis can be found in the literature [23, 37].

## 2.8 Vibrations of Different-Modulus Media

As an example, consider the longitudinal vibrations of a bar with uniform cross-section made of a material with a different modulus in tension and in compression [37]. The one-dimensional equation of motion is

$$\frac{\partial^2 u}{\partial t^2} = a \frac{\partial^2 u}{\partial x^2} \quad (2.36)$$

where  $\alpha = (E/\rho)^{1/2}$ ,  $E$  is Young's modulus and  $\rho$  is density. Moreover

$$\begin{aligned} \alpha &= \alpha_T = \left( \frac{E_T}{\rho} \right)^{\frac{1}{2}} \quad \text{for } \sigma > 0 \text{ or } \frac{\partial u}{\partial x} > 0 \text{ (tension)} \\ \alpha &= \alpha_C = \left( \frac{E_C}{\rho} \right)^{\frac{1}{2}} \quad \text{for } \sigma < 0 \text{ or } \frac{\partial u}{\partial x} < 0 \text{ (compression)} \end{aligned}$$

Assuming that the bar is initially tensioned by a force producing displacement  $u_0$  at  $x = l$  and released at  $t = 0$ , the initial and boundary conditions become:

$$\left. \begin{aligned} t = 0 \quad \frac{\partial u}{\partial t} &= 0 \quad u = \frac{u_0 x}{l} \\ x = 0 \quad u &= 0 \\ x = l \quad \frac{\partial u}{\partial x} &= 0 \end{aligned} \right\} \quad (2.37)$$

By applying separation of variables, the general solution of Eq. (2.36) can be given the following form:

$$u(x, t) = \sum_{n=1}^x x_n(x) T_n(t) \quad (2.38)$$

where

$$\left. \begin{aligned} X_n(x) &= A_n \sin(\omega_n x/a) + B_n \cos(\omega_n x/a) \\ T_n(t) &= C_n \sin \omega_n t + D_n \cos \omega_n t \end{aligned} \right\} \quad (2.39)$$

where the constants  $A_n, B_n, C_n, D_n, \omega_n$  assume different values for tension and for compression (index c or T, respectively) and can be determined by the initial and boundary conditions. By virtue of the first boundary and the first initial condition [Eqs. (2.37), (2.38)] assumes the form

$$u = \sum_{n=1}^x A_n \sin \frac{(\omega_{Tn} x)}{a_T} \cos \omega_{Tn} t \quad (2.40)$$

and the second initial condition gives

$$\omega_{Tn} = (2n - 1) \pi \alpha_T / 2l \quad (2.41)$$

Moreover, we have

$$u_0 \frac{x}{l} = \sum_{n=1}^x A_n \sin \frac{(2n - 1)}{2l} \pi x$$

leading to

$$A_n = \frac{8u_0(-1)^{n-1}}{\pi^2(2n - 1)^2} \quad (2.42)$$

which gives Eq. (2.38) the following form:

$$u = \frac{8u_0}{\pi^2} \sum_{n=1}^x \frac{(-1)^{n-1}}{(2n - 1)^2} \sin \left( \frac{2n - 1}{2l} \pi x \right) \cos \frac{(2n - 1)}{2l} \pi \alpha_T t \quad (2.43)$$

valid for the interval  $0 \leq t \leq l/\alpha_T$ , during which the bar is in the tensile state, i.e. during the first half-cycle. For the second half-cycle, when the bar enters the compressive state, the initial conditions are

$$t = \frac{l}{a_T} \quad u = 0 \quad \frac{\partial u}{\partial t} = -u_0 a_T / l$$

as Eq. (2.43) yields, with the same boundary conditions. From Eq. (2.36), with  $a = a_c$ , we have

$$u = \sum_{n=1}^{\infty} B_n \sin \frac{\omega_{cn} x}{a_c} \sin \omega_{cn} \left( t - \frac{l}{a_T} \right) \quad (2.44)$$

where

$$\omega_{cn} = \frac{2n-1}{2l} \pi \alpha_c \quad (2.45)$$

$$B_n = -\frac{8u_0 a_T}{\pi^2 \alpha_c (2n-1)^2} \quad (2.46)$$

leading eventually to the equation

$$u = \frac{8u_0}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin \frac{2n-1}{2l} \pi x \sin \frac{2n-1}{2l} \pi \alpha_T \left( t - \frac{l}{a_T} - \frac{2l}{a_c} \right) \quad (2.47)$$

valid within the interval

$$\frac{l}{a_T} \leq t \leq \frac{l}{a_T} + \frac{2l}{a_c}$$

during which the bar is in the compressive state. The period of vibration for the bar particles is equal to

$$T = 2l \left( \frac{l}{a_T} + \frac{l}{a_c} \right)$$

or

$$T = 2l \sqrt{\rho} \frac{1 + \sqrt{\frac{E_c}{E_T}}}{\sqrt{1 + \frac{E_c}{E_T}}} \quad (2.48)$$

The present analysis is useful for the solution of actual rotor problems, as, for example, in the case of a transversely cracked rotor subjected to reciprocating axial forces. The problem of flexural vibrations can be encountered in a similar manner. Axial forces are very important in this case. The latter, even if not applied externally, develop during large deflections, and the problem assumes a rather complex form. An axial force can be used to control large-amplitude flexural vibrations [30, 31]. Other problems of instability including crack breathing, damping or parametric instabilities have to be treated with non-linear formulation [32–37].

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