Chapter 2
Tensor Fields and Differential Forms

Abstract After providing some definitions and results on tensor fields and differential forms, this chapter deals with some aspects of general vector bundles, including the 'cocycle approach'; other topics are: Tensors and tensor fields, exterior forms, Lie derivative and the interior product; calculus of differential forms and distributions. Some examples related to manifolds studied in the previous chapter are also present, such as the infinite Möbius strip, considered as a vector bundle, and the tautological bundle over the real Grassmannian. Certain problems intend to make the reader familiar with computations of vector fields, differential forms, Lie derivative, the interior product, the exterior differential, and their relationships. Other group of problems tries to develop practical abilities in computing integral distributions and differential ideals.

> L'algorithme du Calcul différentiel absolu, c'est à dire l'instrument matériel des méthodes (...) se trouve tout entier dans une remarque due a M. Christoffel (...) Mais les méthodes mêmes et les avantages, qu'ils présentent, ont leur raison d'être et leur source dans les rapports intimes, que les lient à la notion de variété à n dimensions, qui nous devons aux génies de Gauss et de Riemann. D'après cette notion une variété V_n est définie intrinséquement dans ses propriétés métriques par n variables indépendants et par toute une classe de formes quadratiques des différentielles de ces variables, dont deux quelconques son transformables l'une en l'autre par une transformation ponctuelle. Par conséquence une V_n reste invariée vis-à-vis de toute transformation de ses coordonnées. La Calcul differentiel absolu, en agissant sur des formes covariantes ou contrevariants au ds^2 de V_n pour en dériver d'autres de même nature, est lui aussi dans ses formules et dans ses résultats indépendent du choix des variables indépendantes. Étant de la sorte essentiellement attaché à V_n, il est l'instrument naturel de toutes les recherches, qui ont pour object une telle variété, ou dans lesquelles on rencontre comme élément caractéristique une forme quadratique positive des différentielles de n variables ou de leurs dérivées.[1]

[1]The algorithm of absolute differential Calculus, that is, the material instrument of the methods (...) is fully included in a remark by Mr. Christoffel (...) But the methods themselves and their

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*,
Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_2,

GREGORIO RICCI-CURBASTRO AND TULLIO LEVI-CIVITA, *Méthodes de calcul differentiel absolu et leurs applications*, Math. Annalen *54* (1900), no. 1–2, 127–128. (With kind permission from Springer.)

However Einstein realised his problems: '*If all accelerated systems are equivalent, then Euclidean geometry cannot hold in all of them.*' Einstein then remembered that he had studied Gauss' theory of surfaces as a student and suddenly realised that the foundations of geometry have physical significance. He consulted his friend [and mathematician] Grossmann who was able to tell Einstein of the important developments of Riemann, Ricci (Ricci-Curbastro) and Levi-Civita. Einstein wrote: '... *in all my life I have not laboured nearly so hard, and I have become imbued with great respect for mathematics, the subtler part of which I had in my simple-mindedness regarded as pure luxury until now.*' In 1913 Einstein and Grossmann published a joint paper where the tensor calculus of Ricci and Levi-Civita is employed to make further advances. Grossmann gave Einstein the Riemann–Christoffel tensor which, together with the Ricci tensor which can be derived from it, were to become the major tools in the future theory. Progress was being made in that gravitation was described for the first time by the metric tensor but still the theory was not right. (...) It was the second half of 1915 that saw Einstein finally put the theory in place. Before that however he had written a paper in October 1914 nearly half of which is a treatise on tensor analysis and differential geometry. This paper led to a correspondence between Einstein and Levi-Civita in which Levi-Civita pointed out technical errors in Einstein's work on tensors. Einstein was delighted to be able to exchange ideas with Levi-Civita whom he found much more sympathetic to his ideas on relativity than his other colleagues.

JOHN O'CONNOR AND EDMUND F. ROBERTSON, Article *General Relativity,* in 'The MacTutor History of Mathematics archive,' School of Mathematics and Statistics, University of St. Andrews, Scotland. (With kind permission from the authors.)

---

advantages have their foundation and their source in the intimate links they have with the notion of n-dimensional manifold, which we owe to the geniuses of Gauss and Riemann. According to this notion, a manifold V_n is intrinsically defined with respect to its metric properties by n independent variables and by a full class of quadratic forms of the differentials of these variables, such that any two may be mutually transformed by a pointwise transformation. Consequently, a V_n remains invariant under any transformation of its coordinates. The absolute differential Calculus, dealing with covariant or contravariant forms of the ds^2 of V_n, in order to obtain other ones of the same nature, is itself independent of the choice of independent variables inside its formulas and its results. Being so essentially linked to V_n, it is a natural tool of all the researches on such a manifold (...) or one meets positive quadratic differential forms and their derivatives."

2.1 Some Definitions and Theorems on Tensor Fields and Differential Forms

Definitions 2.1 Let $\xi = (E, \pi, M)$ be a locally trivial bundle with fibre F over M. A *chart* on ξ is a pair (U, Ψ) consisting of an open subset $U \subset M$ and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \Psi = \pi$, where $\mathrm{pr}_1 : U \times F \to U$ is the first projection map. Ψ is called a *trivialisation of* ξ *over* U.

Let V be real vector space of finite dimension n, and let $\xi = (E, \pi, M)$ be a locally trivial bundle of fibre V. A structure of *vector bundle on* ξ is given by a family $\mathscr{A} = \{(U_\alpha, \Psi_\alpha)\}$ of charts on ξ satisfying:

(i) U_α is an open covering of the base space M.
(ii) For each pair (α, β) such that $U_\alpha \cap U_\beta \neq \emptyset$, one has

$$\big(\Psi_\beta \circ \Psi_\alpha^{-1}\big)(p, v) = \big(p, g_{\beta\alpha}(p)v\big), \quad (p, v) \in (U_\alpha \cap U_\beta) \times V,$$

where $g_{\alpha\beta}$ is a C^∞ map from $U_\alpha \cap U_\beta$ to the group $\mathrm{GL}(V)$ of automorphisms of V.
(iii) If $\mathscr{A}' \supset \mathscr{A}$ is a family of charts on ξ satisfying properties (i), (ii) above, then $\mathscr{A}' = \mathscr{A}$.

Such a bundle $\xi = (E, \pi, M, \mathscr{A})$, or simply $\xi = (E, \pi, M)$, is called a (real) *vector bundle* of rank n. The C^∞ maps $g_{\alpha\beta} : M \to \mathrm{GL}(V)$ are called the *changes of charts* of the atlas $\mathscr{A}$.

Proposition 2.2 *The changes of charts of a vector bundle have the property* (*called the* cocycle condition)

$$g_{\alpha\gamma}(p) g_{\gamma\beta}(p) = g_{\alpha\beta}(p), \quad p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Definition 2.3 Two vector bundles of rank n are said to be *equivalent* if they are isomorphic and have the same base space B.

One has the following converse to Proposition 2.2:

Theorem 2.4 *Let* $\mathscr{U} = \{U_\alpha\}$ *be an open covering of a differentiable manifold* M, *and let* V *be a finite-dimensional real vector space. Let* $g_{\alpha\beta} : M \to \mathrm{GL}(V)$, $U_\alpha \cap U_\beta \neq \emptyset$, *be a family of* C^∞ *maps satisfying the cocycle condition in* Proposition 2.2. *Then there exists a real vector bundle* $\xi = (E, \pi, M, \mathscr{A})$, *unique up to equivalence, such that the maps* $g_{\alpha\beta}$ *are the changes of charts of the atlas* $\mathscr{A}$.

Definition 2.5 The family $(U_\alpha, g_{\alpha\beta})$ is said to be a $\mathrm{GL}(V)$-valued *cocycle on* M *subordinated to the open covering* $\mathscr{U}$.

Definitions 2.6 Let $\mathscr{T}_s^r(M)$ be the set of tensor fields of type (r, s) on a differentiable manifold M and write $\mathscr{T}(M) = \bigoplus_{r,s=0}^\infty \mathscr{T}_s^r(M)$. A *derivation* D of $\mathscr{T}(M)$ is a map of $\mathscr{T}(M)$ into itself satisfying:

(i) D is linear and satisfies

$$D_X(T_1 \otimes T_2) = D_X T_1 \otimes T_2 + T_1 \otimes D_X T_2, \quad X \in \mathfrak{X}(M),\ T_1, T_2 \in \mathscr{T}(M).$$

(ii) D_X is type-preserving: $D_X(\mathscr{T}_s^r(M)) \subset \mathscr{T}_s^r(M)$.
(iii) D_X commutes with every contraction of a tensor field.

Let $\Lambda^r M$ be the space of differential forms of degree r on the n-manifold M, that is, skew-symmetric covariant tensor fields of degree r. With respect to the exterior product, $\Lambda^* M = \bigoplus_{r=0}^n \Lambda^r M$ is an algebra over $\mathbb{R}$. A *derivation* (resp. *anti-derivation*) of $\Lambda^* M$ is a linear map of $\Lambda^* M$ into itself satisfying

$$D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + \omega_1 \wedge D\omega_2, \quad \omega_1, \omega_2 \in \Lambda^* M$$

$$\big(\text{resp. } D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge D\omega_2, \quad \omega_1 \in \Lambda^r M,\ \omega_2 \in \Lambda^* M.\big)$$

A derivation or anti-derivation D of $\Lambda^* M$ is said to be *of degree* k if it maps $\Lambda^r M$ into $\Lambda^{r+k} M$ for every r.

Theorem 2.7 (Exterior Differentiation) *There exists a unique anti-derivation*

$$\mathrm{d}\colon \Lambda^* M \to \Lambda^* M$$

of degree $+1$ *such that*:

(i) $\mathrm{d}^2 = 0$.
(ii) *Whenever* $f \in C^\infty M = \Lambda^0 M$, $\mathrm{d}f$ *is the differential of* f.

Definitions 2.8 Fix a vector field X on M and let φ_t be the local one-parameter group of transformations associated with X. Let Y be another vector field on M. The *Lie derivative of* Y *with respect to* X *at* $p \in M$ is the vector $(L_X Y)_p$ defined by

$$(L_X Y)_p = \lim_{t \to 0} \frac{Y_p - \varphi_{t*} Y_{\varphi_t^{-1}(p)}}{t} = -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_{t*} Y_{\varphi_t^{-1}(p)}).$$

The *Lie derivative of a differential form* ω *with respect to* X *at* p is defined by

$$(L_X \omega)_p = \lim_{t \to 0} \frac{\omega_p - \varphi_{-t}^*(\omega_{\varphi_t(p)})}{t}. \tag{2.1}$$

The *Lie derivative of a tensor field* T *of type* (r, s) *with respect to* X *at* p is defined by

$$(L_X T)_p = -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_t \cdot T)_p,$$

where the dot denotes, for an arbitrary diffeomorphism Φ of M,

$$\begin{aligned}&\Phi\cdot(X_1\otimes\cdots\otimes X_r\otimes\theta_1\otimes\cdots\otimes\theta_s)\\&\quad=\Phi\cdot X_1\otimes\cdots\otimes\Phi\cdot X_r\otimes\left(\Phi^{-1}\right)^*\theta_1\otimes\cdots\otimes\left(\Phi^{-1}\right)^*\theta_s,\end{aligned}$$

$X_i\in\mathfrak{X}(M)$, $\theta_j\in\Lambda^1M$.

In particular, the action of Φ on a differential form $\theta\in\Lambda^1M$ is given by

$$(\Phi\cdot\theta)_p=\theta_{\Phi^{-1}(p)}\circ\left(\Phi^{-1}\right)_*=\left(\left(\Phi^{-1}\right)^*\theta\right)_p,\quad p\in M.$$

For each $X\in\mathfrak{X}(M)$, the *interior product with respect to* X is the unique antiderivation i of degree -1 defined by $i_Xf=0$, $f\in C^\infty M$, and $i_X\theta=\theta(X)$, $\theta\in\Lambda^1M$. We shall use sometimes, to avoid confusion, ι instead of i to denote the interior product.

Theorem 2.9 *Let* $X\in\mathfrak{X}(M)$. *Then*:

(i) $L_Xf=Xf$, $f\in C^\infty M$.
(ii) $L_XY=[X,Y]$, $Y\in\mathfrak{X}(M)$.
(iii) L_X *maps* Λ^*M *to* Λ^*M, *and it is a derivation which commutes with the exterior differentiation* d.
(iv) *On* Λ^*M, *we have*

$$L_X=i_X\circ\mathrm{d}+\mathrm{d}\circ i_X,$$

where i_X *denotes the interior product with respect to* X.

Proposition 2.10 *Let* φ_t *be a local one-parameter group of local transformations generated by a vector field* X *on* M. *For any tensor field* T *on* M, *we have*

$$\varphi_s\cdot(L_XT)=-\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t\cdot T)\right)_{t=s}.$$

In particular, $L_XT=0$ *if and only if* $\varphi_t\cdot T=T$ *for all* t.

Definitions 2.11 Let m,n be integers, $1\leqslant m\leqslant n$. An *m-dimensional distribution* $\mathscr{D}$ on an n-dimensional manifold M is a choice of an m-dimensional subspace $\mathscr{D}_p$ of T_pM for each $p\in M$. $\mathscr{D}$ is C^∞ if for each $p\in M$, there are a neighbourhood U of p and m vector fields $X_1,\dots,X_m$ on U which span $\mathscr{D}$ at each point in U. A vector field is said to *belong to* (or *lie in*) the distribution $\mathscr{D}$ if $X_p\in\mathscr{D}_p$ for each $p\in M$. Then one writes $X\in\mathscr{D}$. A C^∞ distribution is called *involutive* (or *completely integrable*) if $[X,Y]\in\mathscr{D}$ whenever X and Y are vector fields lying in $\mathscr{D}$.

A submanifold (N,ψ) of M is an *integral manifold* of a distribution $\mathscr{D}$ on M if

$$\psi_*(T_qN)=\mathscr{D}_{\psi(q)},\quad q\in N.$$

Definitions 2.12 Let $\mathscr{D}$ be an r-dimensional C^∞ distribution on M. A differential s-form ω is said to *annihilate* $\mathscr{D}$ if, for each $p\in M$,

$$\omega_p(v_1,\dots,v_s)=0,\quad v_1,\dots,v_s\in\mathscr{D}_p.$$

A differential form $\omega \in \Lambda^* M$ is said to annihilate $\mathscr{D}$ if each of the homogeneous parts of ω annihilates $\mathscr{D}$. Let

$$\mathscr{I}(\mathscr{D}) = \{\omega \in \Lambda^* M : \omega \text{ annihilates } \mathscr{D}\}.$$

A function $f \in C^\infty M$ is said to be a *first integral of* $\mathscr{D}$ if $\mathrm{d}f$ annihilates $\mathscr{D}$. An ideal $\mathscr{I} \subset \Lambda^* M$ is called a *differential ideal* if it is closed under exterior differentiation d, that is, $\mathrm{d}\mathscr{I} \subset \mathscr{I}$.

Proposition 2.13 *A C^∞ distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.*

Theorem 2.14 (Frobenius' Theorem) *Let $\mathscr{D}$ be an $(n-q)$-dimensional, involutive, C^∞ distribution on the n-dimensional manifold M. Let $p \in M$. Then through p there passes a unique maximal connected integral manifold of $\mathscr{D}$, and every connected integral manifold of $\mathscr{D}$ through p is contained in the maximal one.*

Definitions 2.15 In the conditions of Theorem 2.14 it is said that the involutive distribution $\mathscr{D}$ defines a *foliation*, M is said to be *a foliated manifold*, the unique maximal connected integral manifold of $\mathscr{D}$ through each point is called a *leaf* of the foliation, and the foliation is said to be of *codimension q*.

Definition 2.16 A *codimension q foliation* $\mathfrak{F}$ on a differentiable manifold M of dimension n is a collection of disjoint, connected, $(n-q)$-dimensional submanifolds of M (the leaves of the foliation), whose union is M, and such that for each point $p \in M$, there is a chart (U, φ) containing p such that each leaf of the foliation intersects U in either the empty set or a countable union of $(n-q)$-dimensional slices of the form $x^{n-q+1} = c^{n-q+1}, \ldots, x^n = c^n$. More formally, a foliation $\mathfrak{F}$ consists of a covering $\mathfrak{U}$ of M by charts (U_i, φ_i) such that on each intersection $U_i \cap U_j$, the changes of charts $\Phi_{ij} = \varphi_j \circ \varphi_i^{-1}$ are of the form

$$\Phi_{ij}\big(x^1, \ldots, x^q, x^{q+1}, \ldots, x^n\big) = \big(\varphi_{ij}\big(x^1, \ldots, x^q\big), \psi_{ij}\big(x^1, \ldots, x^q, x^{q+1}, \ldots, x^n\big)\big)$$

with

$$\varphi_{ij} : \mathbb{R}^q \to \mathbb{R}^q, \qquad \psi_{ij} : \mathbb{R}^n \to \mathbb{R}^{n-q}.$$

2.2 Vector Bundles

Problem 2.17 Let (E, π, M) be a C^∞ vector bundle with fibre $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Prove that the homotheties

$$h : \mathbb{F} \times E \to E, \qquad (\lambda, y) \mapsto h(\lambda, y) = \lambda y,$$

are C^∞.

Solution Let U be an open subset of M. Let $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{F}$ be a trivialisation of (E, π, M), that is, a fibre-preserving diffeomorphism linear on the fibres, and φ a chart, that is, a diffeomorphism of the open subset $E_U = \pi^{-1}(U)$ of E onto $U \times \mathbb{F}^n$, linear on the fibres.

Then, $h|_U$ is the composition map

$$\begin{array}{ccccccc} \mathbb{F} \times E_U & \xrightarrow{\mathrm{id}_{\mathbb{F}} \times \varphi} & \mathbb{F} \times U \times \mathbb{F}^n & \xrightarrow{h'} & U \times \mathbb{F}^n & \xrightarrow{\varphi^{-1}} & E_U \\ (\lambda, y) & \longmapsto & (\lambda, p, x) & \longmapsto & (p, \lambda x) & \longmapsto & \lambda y. \end{array}$$

Since φ is a diffeomorphism and h' is C^∞, the map $h|_U$ is C^∞.

Problem 2.18 Show that for a C^∞ vector bundle $\xi = (E, \pi, M)$ with fibre $\mathbb{R}^n$, triviality is equivalent to the existence of n C^∞ global sections, linearly independent at each point.

Solution Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^n$. If we have a global trivialisation

$$\begin{array}{ccc} E & \xrightarrow{u} & M \times \mathbb{R}^n \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ M & \xrightarrow{\mathrm{id}} & M \end{array}$$

then we have sections $\tilde{e}_i$ of $M \times \mathbb{R}^n$ given by $\tilde{e}_i = (\mathrm{id}, e_i)$. Thus, we have sections ξ_i of E defined by $\xi_i = u^{-1} \circ \tilde{e}_i$, which are linearly independent because u^{-1} is an isomorphism on each fibre.

Conversely, if ξ_i are such linearly independent sections of E, we define the trivialisation u by $u(\alpha) = (\pi(\alpha), \alpha^1, \dots, \alpha^n)$ with $\alpha = \sum_i \alpha^i \xi_i(\pi(\alpha)) \in E$. Its inverse map is given by $u^{-1}(p, \alpha^1, \dots, \alpha^n) = \sum_i \alpha^i \xi_i|_p$, $p \in M$.

Problem 2.19 Prove that the infinite Möbius strip M (see Problem 1.31) can be considered as the total space of a vector bundle over S^1. Specifically:

(i) Determine the base space, the fibre and the projection map π.
(ii) Prove that the vector bundle (M, π, S^1) is locally trivial but not trivial.

Solution

(i) With the notations of Problem 1.31, we have that the base space is $S^1 \equiv ([0, 1] \times \{0\})/\sim \subset M$, the fibre is $\mathbb{R}$ (see Fig. 2.1), and the projection map is defined by

$$\pi\big([(x, y)]\big) = \begin{cases} [(x, 0)] & \text{if } 0 < x < 1, \\ [(0, 0)] = [(1, 0)] & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

(ii) The charts in Problem 1.31 are in fact trivialisations that cover S^1 entirely. Now suppose that there exists a non-vanishing global section $\sigma : S^1 \to M$, i.e. a continuous map such that $\pi \circ \sigma = \mathrm{id}_{S^1}$. This is equivalent to a continuous function

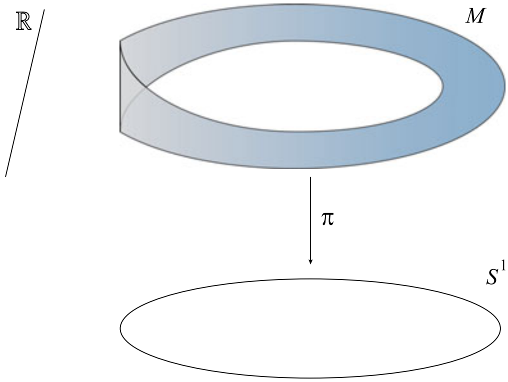

Fig. 2.1 The Möbius strip as the total space of a vector bundle

$s\colon [0,1] \to \mathbb{R}$ such that $s(0) = -s(1)$. Since s must vanish somewhere, σ must also vanish somewhere, so getting a contradiction.

Problem 2.20

(i) Consider

$$E = \left\{(u, v) = (x, y, z, a, b, c) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| = 1, \langle u, v\rangle = 0\right\}$$

and the projection map on the unit sphere S^2 given by $\pi\colon E \to S^2$, $\pi(u, v) = u$. Prove that $\xi = (E, \pi, S^2)$ is a locally trivial bundle over S^2 with fibre $\mathbb{R}^2$.

(ii) Let $\mathscr{A} = \{(U_i, \Phi_i)\}$, $i = 1, 2, 3$, be as in the solution of (i) below. Prove that $TS^2 = (E, \pi, S^2, \mathscr{A})$ is a vector bundle (see Definitions 2.1) with fibre $\mathbb{R}^2$.

Solution

(i) The open subsets U_1, U_2, U_3 of S^2 given by $|x| < 1$, $|y| < 1$, $|z| < 1$, respectively, are an open covering of S^2. Define local trivialisations by

$$\Phi_1\colon \pi^{-1}(U_1) \to U_1 \times \mathbb{R}^2, \qquad (x, y, z, a, b, c) \mapsto (x, y, z, bz - cy, a),$$

$$\Phi_2\colon \pi^{-1}(U_2) \to U_2 \times \mathbb{R}^2, \qquad (x, y, z, a, b, c) \mapsto (x, y, z, cx - az, b),$$

$$\Phi_3\colon \pi^{-1}(U_3) \to U_3 \times \mathbb{R}^2, \qquad (x, y, z, a, b, c) \mapsto (x, y, z, ay - bx, c).$$

It is immediate that they are diffeomorphisms.

(ii) As a computation shows, the changes of charts are given, for each $u = (x, y, z) \in S^2$, by

$$g_{21}(u) = \frac{-1}{y^2 + z^2}\begin{pmatrix} xy & z \\ -z & xy \end{pmatrix}, \qquad g_{32}(u) = \frac{-1}{z^2 + x^2}\begin{pmatrix} yz & x \\ -x & yz \end{pmatrix},$$

$$g_{13}(u) = \frac{-1}{x^2 + y^2}\begin{pmatrix} zx & y \\ -y & zx \end{pmatrix}.$$

The cocycle condition is thus satisfied. Indeed, one has

$$g_{21}(u)g_{13}(u) = \frac{1}{x^2+y^2}\begin{pmatrix} -yz & x \\ -x & -yz \end{pmatrix} = \bigl(g_{32}(u)\bigr)^{-1} = g_{23}(u)$$

and the similar identities for $g_{12}(u)g_{23}(u)$ and $g_{13}(u)g_{32}(u)$.

Moreover, for

$$\widehat{E} = \bigl\{\bigl((u,v),\bigl(u',v'\bigr)\bigr) \in E \times E : u = u',\, \langle u,v\rangle = \bigl\langle u,v'\bigr\rangle = 0\bigr\},$$

the maps

$$\begin{aligned} s\colon \widehat{E} \to E, &\qquad \bigl((u,v),\bigl(u',v'\bigr)\bigr) \mapsto \bigl(u, v+v'\bigr), \\ h\colon \mathbb{R}\times E \to E, &\qquad \bigl(\lambda,(u,v)\bigr) \mapsto (u, \lambda v), \end{aligned}$$

are C^∞ (as for h, see Problem 2.17), and they induce a structure of two-dimensional vector space on each fibre of TS^2.

Problem 2.21

(i) Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas on a manifold M, where $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$, $\varphi_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$, $n = \dim M$. Let $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{R})$ be the map

$$\bigl(g_{\alpha\beta}(p)\bigr)_i^h = \frac{\partial x_\alpha^h}{\partial x_\beta^i}(p), \qquad p \in U_\alpha \cap U_\beta.$$

Prove that $\{g_{\alpha\beta}\}$ is a cocycle on M whose associated vector bundle is the tangent bundle TM.

(ii) Similarly, if the map $g^*_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{R})$ is given by

$$\bigl(g^*_{\alpha\beta}(p)\bigr)_i^h = \frac{\partial x_\beta^i}{\partial x_\alpha^h}(p), \qquad p \in U_\alpha \cap U_\beta,$$

prove that $\{g^*_{\alpha\beta}\}$ is a cocycle on M whose associated vector bundle is the cotangent bundle T^*M.

Solution

(i) Let us define two linear frames at p:

$$u_\alpha = \left(\frac{\partial}{\partial x_\alpha^1}\bigg|_p, \dots, \frac{\partial}{\partial x_\alpha^n}\bigg|_p\right), \qquad u_\beta = \left(\frac{\partial}{\partial x_\beta^1}\bigg|_p, \dots, \frac{\partial}{\partial x_\beta^n}\bigg|_p\right).$$

According to the definition of $g_{\alpha\beta}(p)$, we have

$$\frac{\partial}{\partial x_\beta^i}\bigg|_p = \sum_{h=1}^n \bigl(g_{\alpha\beta}(p)\bigr)_i^h \frac{\partial}{\partial x_\alpha^h}\bigg|_p.$$

Hence $u_\beta = u_\alpha \cdot g_{\alpha\beta}(p)$, where the dot on the right-hand side stands for the right action of $\mathrm{GL}(n,\mathbb{R})$ on the bundle of linear frames FM (see Definitions 5.3). Accordingly,

$$u_\beta = u_\gamma \cdot g_{\gamma\beta}(p) = \big(u_\alpha \cdot g_{\alpha\gamma}(p)\big) \cdot g_{\gamma\beta}(p) = u_\alpha \cdot \big(g_{\alpha\gamma}(p) g_{\gamma\beta}(p)\big).$$

As $\mathrm{GL}(n,\mathbb{R})$ acts freely on FM, we conclude that

$$g_{\alpha\beta}(p) = g_{\alpha\gamma}(p) g_{\gamma\beta}(p),$$

thus proving that $\{g_{\alpha\beta}\}$ is a cocycle.

Moreover, if $\pi : TM \to M$ is the tangent bundle, for every index α, we have a trivialisation

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, \qquad \Phi_\alpha(X) = \big(p, \lambda^1, \dots, \lambda^n\big),$$

$X = \sum_{i=1}^n \lambda^i (\partial/\partial x^i_\alpha)_p \in T_p U_\alpha$, or in other words,

$$\Phi_\alpha(X) = \big(p, u_\alpha^{-1}(X)\big),$$

where u_α is understood as a linear isomorphism $u_\alpha : \mathbb{R}^n \to T_p M$.

In order to prove that the cocycle $\{g_{\alpha\beta}\}$ defines TM, it suffices to see that the cocycle associated to these trivialisations is $\{g_{\alpha\beta}\}$. In fact, if $\{e_i\}$ is the standard basis of $\mathbb{R}^n$, for $v = \sum_i \lambda^i e_i$, $p \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned}
\big(\Phi_\alpha \circ \Phi_\beta^{-1}\big)(p, v) &= \Phi_\alpha\big(u_\beta(v)\big) = \big(p, u_\alpha^{-1}\big(u_\beta(v)\big)\big) \\
&= \left(p, u_\alpha^{-1}\left(\sum_{i=1}^n \lambda^i \frac{\partial}{\partial x^i_\beta}\bigg|_p\right)\right) \\
&= \left(p, u_\alpha^{-1}\left(\sum_{i,h=1}^n \lambda^i \frac{\partial x^h_\alpha}{\partial x^i_\beta}(p) \frac{\partial}{\partial x^h_\alpha}\bigg|_p\right)\right) \\
&= \left(p, \sum_{i,h=1}^n \lambda^i \frac{\partial x^h_\alpha}{\partial x^i_\beta}(p) u_\alpha^{-1}\left(\frac{\partial}{\partial x^h_\alpha}\bigg|_p\right)\right) \\
&= \left(p, \sum_{i,h} \lambda^i \big(g_{\alpha\beta}(p)\big)^h_i e_h\right) = \big(p, g_{\alpha\beta}(p) \cdot v\big).
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\sum_{j=1}^n \big(g^*_{\alpha\beta}(p)\big)^h_j \big({}^t g_{\alpha\beta}(p)\big)^j_i &= \sum_{j=1}^n \frac{\partial x^j_\beta}{\partial x^h_\alpha}(p) \frac{\partial x^i_\alpha}{\partial x^j_\beta}(p) = \sum_{j=1}^n \frac{\partial x^j_\beta}{\partial x^h_\alpha}(p) \frac{\partial}{\partial x^j_\beta}\bigg|_p \big(x^i_\alpha\big) \\
&= \frac{\partial}{\partial x^h_\alpha}\bigg|_p \big(x^i_\alpha\big) = \delta^i_h.
\end{aligned}$$

Hence $g^*_{\alpha\beta}(p) = ({}^t g_{\alpha\beta})^{-1}(p)$, and then

$$g^*_{\alpha\gamma}(p) g^*_{\gamma\beta}(p) = \left({}^t g_{\alpha\gamma}\right)^{-1}(p)\left({}^t g_{\gamma\beta}\right)^{-1}(p) = \left({}^t g_{\alpha\beta}\right)^{-1}(p) = g^*_{\alpha\beta}(p),$$

thus proving that $\{g^*_{\alpha\beta}\}$ is a cocycle.

Finally, by proceeding as in (i) above, it is easily checked that $\{g^*_{\alpha\beta}\}$ is the cocycle attached to the trivialisations of the cotangent bundle $\pi : T^*M \to M$ defined as follows:

$$\begin{aligned}
&\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n,\\
&\Psi_\alpha(\omega) = \left(p, u^*_\alpha(\omega)\right) = \left(p, \omega\left(\left.\frac{\partial}{\partial x^1_\alpha}\right|_p\right), \dots, \omega\left(\left.\frac{\partial}{\partial x^n_\alpha}\right|_p\right)\right),\\
&\quad \omega \in T^*_p M,\ \ p \in U_\alpha \cap U_\beta,
\end{aligned}$$

where $u^*_\alpha : T^*_p M \to (\mathbb{R}^n)^*$ is the dual map to $u_\alpha : \mathbb{R}^n \to T_p M$.

Problem 2.22 (The Tautological Bundle Over the Real Grassmannian) Denote by $\gamma^k(\mathbb{R}^n)$ the subset of pairs $(V, v) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n$ such that $v \in V$ and let $\pi : \gamma^k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ be the projection $\pi(V, v) = V$. Prove that $\gamma^k(\mathbb{R}^n)$ is a C^∞ vector bundle of rank k.

Solution The fibres of π are endowed with a natural structure of vector space as $\pi^{-1}(V) = V$. Hence $\operatorname{rank}\pi^{-1}(V) = k$ for all $V \in G_k(\mathbb{R}^n)$. The maps

$$\begin{aligned}
\gamma^k\left(\mathbb{R}^n\right) \times_{G_k(\mathbb{R}^n)} \gamma^k\left(\mathbb{R}^n\right) \to \gamma^k\left(\mathbb{R}^n\right), &\qquad \left((V, v), (V, w)\right) \mapsto (V, v + w),\\
\mathbb{R} \times \gamma^k\left(\mathbb{R}^n\right) \to \gamma^k\left(\mathbb{R}^n\right), &\qquad \left(\lambda, (V, v)\right) \mapsto (V, \lambda v),
\end{aligned}$$

are differentiable as they are induced by the corresponding operations in $\mathbb{R}^n$. It remains to prove that $\gamma^k(\mathbb{R}^n)$ is locally trivial. Let us fix a point $V_0 \in G_k(\mathbb{R}^n)$, and let $\mathscr{U}$ be the set of k-planes V such that $\ker p|_V = 0$, where p is the orthogonal projection onto V_0 relative to the decomposition $\mathbb{R}^n = V_0 \oplus V_0^\perp$. Certainly, $V_0 \in \mathscr{U}$ as $p|_{V_0} = \mathrm{id}$.

If $\{v^0_1, \dots, v^0_k\}$ is an orthonormal basis of V_0 and $\{v_1, \dots, v_k\}$ is a basis of V, then $V \in \mathscr{U}$ if and only if

$$\det\left(\langle v^0_i, v_j\rangle\right)_{i,j=1,\dots,k} \neq 0,$$

thus proving that $\mathscr{U}$ is an open neighbourhood of V_0. For every $V \in \mathscr{U}$, the restriction $p|_V : V \to V_0$ is an isomorphism as $\ker p|_V = 0$ and $\dim V = \dim V_0$. Hence we can define a C^∞ trivialisation

$$\begin{aligned}
\mathscr{U} \times V_0 &\xrightarrow{\tau} \pi^{-1}(\mathscr{U}) \subset \gamma^k\left(\mathbb{R}^n\right)\\
(V, v_0) &\mapsto \left(V, (p|_V)^{-1}(v_0)\right).
\end{aligned}$$

Problem 2.23 Let $\varPhi\colon E \to E'$ be a homomorphism of vector bundles over M with constant rank. Prove that $\ker\varPhi$ and $\operatorname{im}\varPhi$ are vector sub-bundles of E and E', respectively.

Solution As the problem is local, we can assume that E, E' are trivial: $E = M \times \mathbb{R}^n$, $E' = M \times \mathbb{R}^m$. Then $\varPhi$ is given by

$$\varPhi(p, v) = \bigl(p, A(p)v\bigr),$$

where $A = (a^i_j)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $a^i_j \in C^\infty M$, is a C^∞ $m \times n$ matrix. Set $r = \operatorname{rank}_p \varPhi$ for all $p \in M$. Given $p_0 \in M$, by permuting rows and columns in A, we can suppose that

$$\det\begin{pmatrix} a^1_1(p_0) & \dots & a^1_r(p_0) \\ \vdots & & \vdots \\ a^r_1(p_0) & \dots & a^r_r(p_0) \end{pmatrix} \neq 0.$$

Hence there exists an open neighbourhood U of p_0 such that

$$\det\begin{pmatrix} a^1_1(p) & \dots & a^1_r(p) \\ \vdots & & \vdots \\ a^r_1(p) & \dots & a^r_r(p) \end{pmatrix} \neq 0, \qquad p \in U.$$

As $\operatorname{rank} A(p) = r$ for all $p \in U$, it is clear that $\ker(\varPhi|_U)$ is defined by the equations

$$\sum_{j=1}^{n} a^i_j(p) v^j = 0, \qquad 1 \leqslant i \leqslant r,$$

where $v = \sum_j v^j e_j$, $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$. By using Cramer's formulas we conclude that the previous system is equivalent to

$$v^h = \sum_{k=r+1}^{n} b^h_k(p) v^k, \qquad 1 \leqslant h \leqslant r.$$

Hence $(p, v) \in \ker\varPhi$ if and only if

$$v = \sum_{k=r+1}^{n} v^k \left(e_k + \sum_{h=1}^{r} b^h_k(p) e_h \right).$$

Define sections of E over U by

$$\sigma_k(p) = \begin{cases} e_k, & 1 \leqslant k \leqslant r, \\ e_k + \sum_{h=1}^{r} b^h_k(p) e_h, & r+1 \leqslant k \leqslant n. \end{cases}$$

Then, $\{\sigma_1(p), \dots, \sigma_n(p)\}$ is a basis of E_p, and $\{\sigma_{r+1}(p), \dots, \sigma_n(p)\}$ is a basis of $(\ker\varPhi)_p$ for all $p \in U$, thus proving that $\ker\varPhi$ is a sub-bundle of E.

Moreover, if $F \subset E$ is a sub-bundle, then $F^0 = \{w \in E^* : w|_F = 0\}$ is a sub-bundle of E^*, as if $\{\sigma_1, \dots, \sigma_n\}$ is a basis of sections of E over U and $\{\sigma_{r+1}, \dots, \sigma_n\}$ is a basis of sections of F, then the dual basis $\{\sigma_1^*, \dots, \sigma_n^*\}$ is a basis of sections of $E^*|_U$, and $\{\sigma_1^*, \dots, \sigma_r^*\}$ is a basis of sections of F^0. Furthermore, as Φ has constant rank, then the same holds for $\Phi^* \colon E'^* \to E^*$, as a matrix and its transpose have the same rank. We can conclude by remarking that $\operatorname{im}\Phi = (\ker \Phi^*)^0$.

Finally, we give the following counterexample. Let $E = E' = \mathbb{R} \times \mathbb{R}$ be the trivial bundle over $\mathbb{R}$ with fibre $\mathbb{R}$, and let $\Phi \colon E \to E'$ be defined by $\Phi(p, \lambda) = (p, \lambda p)$. Then

$$(\ker \Phi)_p = \begin{cases} 0 & \text{if } p \neq 0, \\ \mathbb{R} & \text{if } p = 0. \end{cases}$$

2.3 Tensor and Exterior Algebras. Tensor Fields

Problem 2.24 Let V be a finite-dimensional vector space. An element $\theta \in \Lambda^* V^*$ is said to be *homogeneous* of degree k if $\theta \in \Lambda^k V^*$, and a homogeneous element of degree $k \geqslant 1$ is said to be *decomposable* if there exist $\theta^1, \dots, \theta^k \in \Lambda^1 V^*$ such that $\theta = \theta^1 \wedge \cdots \wedge \theta^k$.

(i) Assume that $\theta \in \Lambda^k V^*$ is decomposable. Calculate $\theta \wedge \theta$.
(ii) If $\dim V > 3$ and $\theta^1, \theta^2, \theta^3, \theta^4$ are linearly independent, is $\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4$ decomposable?
(iii) Prove that if $\dim V = n \leqslant 3$, then every homogeneous element of degree $k \geqslant 1$ is decomposable.
(iv) If $\dim V = 4$, give an example of a non-decomposable homogeneous element of $\Lambda^* V^*$.

Solution

(i) It is immediate that $\theta \wedge \theta = 0$.
(ii) No, since

$$\left(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4\right) \wedge \left(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4\right) = 2\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \neq 0,$$

so by virtue of (i) it is not decomposable.
(iii) If $\dim V = 1$ or 2, the result is trivial. Suppose then that $\dim V = 3$, and let $\{\alpha^1, \alpha^2, \alpha^3\}$ be a basis of V^*. If $\theta \in \Lambda^1 V^*$, the result follows trivially. If $\theta \in \Lambda^3 V^*$, then $\theta = a\alpha^1 \wedge \alpha^2 \wedge \alpha^3$, and hence it is decomposable. Then suppose that $\theta \in \Lambda^2 V^*$, so that $\theta = a\alpha^1 \wedge \alpha^2 + b\alpha^1 \wedge \alpha^3 + c\alpha^2 \wedge \alpha^3$. Assume that $a \neq 0$. Then

$$\theta = a\alpha^1 \wedge \left(\alpha^2 + \frac{b}{a}\alpha^3\right) + c\alpha^2 \wedge \alpha^3 = \left(a\alpha^1 - c\alpha^3\right) \wedge \left(\alpha^2 + \frac{b}{a}\alpha^3\right).$$

If $a = 0$, then $\theta = (b\alpha^1 + c\alpha^2) \wedge \alpha^3$.
(iv) The one given in (ii) in the statement is such an example.

Problem 2.25

1. Let A, B be two $(1,1)$ tensor fields on a C^∞ manifold M. Define S by
$$S(X,Y) = [AX,BY] + [BX,AY] + AB[X,Y] + BA[X,Y] - A[X,BY] - A[BX,Y] - B[X,AY] - B[AX,Y], \quad X,Y \in \mathfrak{X}(M).$$
Prove that S is a $(1,2)$ skew-symmetric tensor field on M, called the *Nijenhuis torsion* of A and B.
2. Let J be a tensor field of type $(1,1)$ on the C^∞ manifold M. The *Nijenhuis tensor* of J is defined by
$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] + J^2[X,Y], \quad X,Y \in \mathfrak{X}(M).$$
   (a) Prove that N_J is a tensor field of type $(1,2)$ on M.
   (b) Find its local expression in terms of that of J.

The relevant theory is developed, for instance, in Kobayashi and Nomizu [2, vol. 2, Chap. IX]. However, for the sake of simplicity we omit the factor 2 of the Nijenhuis tensor in that reference.

Solution

1. From the formula
$$[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$
it follows that $S(fX,gY) = fgS(X,Y)$, $f,g \in C^\infty M$. Since the Lie bracket is skew-symmetric, so is S.
2. (a) The proof is similar to the one in the case 1.
   (b) Let $x^1,\dots,x^n$ be local coordinates in which $J = \sum_{i,j} J^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ and $N_J = \sum_{i,j,k} N^i_{jk} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$, so
$$J\frac{\partial}{\partial x^k} = \sum_{i=1}^n J^i_k \frac{\partial}{\partial x^i}, \qquad N_J\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{k=1}^n N^k_{ij} \frac{\partial}{\partial x^k}.$$
   From the definition of the Nijenhuis tensor we obtain
$$N^i_{jk} = \sum_{l=1}^n \left(J^l_j \frac{\partial J^i_k}{\partial x^l} - J^l_k \frac{\partial J^i_j}{\partial x^l} + J^i_l \frac{\partial J^l_j}{\partial x^k} - J^i_l \frac{\partial J^l_k}{\partial x^j} \right).$$

Problem 2.26 Write the tensor field $J \in \mathscr{T}_1^1\mathbb{R}^3$ given by
$$J = \mathrm{d}x \otimes \frac{\partial}{\partial x} + \mathrm{d}y \otimes \frac{\partial}{\partial y} + \mathrm{d}z \otimes \frac{\partial}{\partial z}$$
in the system of spherical coordinates given by
$$x = r\cos\varphi\cos\theta, \qquad y = r\cos\varphi\sin\theta, \qquad z = \sin\varphi,$$
$$r > 0,\ \varphi \in (-\pi/2, \pi/2),\ \theta \in (0, 2\pi).$$

Solution We have

$$J = \mathrm{d}r \otimes \frac{\partial}{\partial r} + \mathrm{d}\varphi \otimes \frac{\partial}{\partial \varphi} + \mathrm{d}\theta \otimes \frac{\partial}{\partial \theta},$$

as J represents the identity map in the natural isomorphism $T^*\mathbb{R}^3 \otimes T\mathbb{R}^3 \cong \operatorname{End} T\mathbb{R}^3$, and hence it has the same expression in any coordinate system.

2.4 Differential Forms. Exterior Product

Problem 2.27 Consider on $\mathbb{R}^2$:

$$X = \left(x^2 + y\right)\frac{\partial}{\partial x} + \left(y^2 + 1\right)\frac{\partial}{\partial y}, \qquad Y = (y-1)\frac{\partial}{\partial x},$$
$$\theta = \left(2xy + x^2 + 1\right)\mathrm{d}x + \left(x^2 - y\right)\mathrm{d}y,$$

and let f be the map

$$f\colon \mathbb{R}^3 \to \mathbb{R}^2, \qquad (u, v, w) \mapsto (x, y) = \left(u - v, v^2 + w\right).$$

Compute:

(i) $[X, Y]_{(0,0)}$.
(ii) $\theta(X)(0,0)$.
(iii) $f^*\theta$.

Solution

(i)

$$[X, Y] = \left(y^2 - 2xy + 2x + 1\right)\frac{\partial}{\partial x}, \quad \text{so} \quad [X, Y]_{(0,0)} = \left.\frac{\partial}{\partial x}\right|_{(0,0)}.$$

(ii)

$$\theta(X)(0,0) = \left(\left(2xy + x^2 + 1\right)\left(x^2 + y\right) + \left(x^2 - y\right)\left(y^2 + 1\right)\right)(0,0) = 0.$$

(iii)

$$\begin{aligned} f^*\theta = {} & \left\{2(u-v)\left(v^2 + w\right) + (u-v)^2 + 1\right\}\mathrm{d}u \\ & + \left\{2v\left((u-v)^2 - v^2 - w\right) - 2(u-v)\left(v^2 + w\right) - (u-v)^2 - 1\right\}\mathrm{d}v \\ & + \left\{(u-v)^2 - v^2 - w\right\}\mathrm{d}w. \end{aligned}$$

Problem 2.28 Consider the vector fields on $\mathbb{R}^2$:

$$X = x\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \qquad Y = y\frac{\partial}{\partial y},$$

and let ω be the differential form on $\mathbb{R}^2$ given by

$$\omega = \big(x^2 + 2y\big)\,\mathrm{d}x + \big(x + y^2\big)\,\mathrm{d}y.$$

Show that ω satisfies the relation

$$\mathrm{d}\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega\big([X, Y]\big),$$

between the bracket product and the exterior differential.

Solution We have $[X, Y] = 0$ and

$$\begin{aligned}\mathrm{d}\omega &= \left(\frac{\partial(x^2+2y)}{\partial x}\,\mathrm{d}x + \frac{\partial(x^2+2y)}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x \\ &\quad + \left(\frac{\partial(x+y^2)}{\partial x}\,\mathrm{d}x + \frac{\partial(x+y^2)}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\ &= -\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

From

$$X\omega(Y) = xy + 2x^2y + 6xy^3, \qquad Y\omega(X) = 2xy + 2x^2y + 6xy^3,$$
$$\mathrm{d}\omega(X, Y) = -(\mathrm{d}x \wedge \mathrm{d}y)\left(x\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\right) = -xy$$

one easily concludes.

Problem 2.29 Find the subset of $\mathbb{R}^2$ where the differential forms

$$\alpha = x\,\mathrm{d}x + y\,\mathrm{d}y, \qquad \beta = y\,\mathrm{d}x + x\,\mathrm{d}y$$

are linearly independent and determine the field of dual frames $\{X, Y\}$ on this set.

Solution We have $\det\left(\begin{smallmatrix} x & y \\ y & x\end{smallmatrix}\right) = x^2 - y^2 \neq 0$ on $\mathbb{R}^2 \setminus \{(x, y) : x = \pm y\}$. Thus α and β are linearly independent on the subset of $\mathbb{R}^2$ complementary to the diagonals $x + y = 0$ and $x - y = 0$.

The dual field of frames

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, \qquad Y = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}, \quad a, b, c, d \in C^\infty\mathbb{R}^2$$

must satisfy $X(\alpha) = Y(\beta) = 1$, $X(\beta) = Y(\alpha) = 0$. Hence,

$$\begin{cases} ax + by = 1 \\ ay + bx = 0 \end{cases} \quad \text{and} \quad \begin{cases} cx + dy = 0 \\ cy + dx = 1. \end{cases}$$

Solving these systems, we obtain

$$X = \frac{x}{x^2-y^2}\frac{\partial}{\partial x} - \frac{y}{x^2-y^2}\frac{\partial}{\partial y}, \qquad Y = -\frac{y}{x^2-y^2}\frac{\partial}{\partial x} + \frac{x}{x^2-y^2}\frac{\partial}{\partial y}.$$

Remark The result also follows (here and in other problems below) from the general fact that, $\{e_i = \sum_k \lambda_i^k \frac{\partial}{\partial x^k}\}$ being a basis of vector fields on a manifold and $\{\theta^j = \sum_l \mu_l^j \,\mathrm{d}x^l\}$ its dual basis, from $(\sum_l \mu_l^j \,\mathrm{d}x^l)(\sum_k \lambda_i^k \frac{\partial}{\partial x^k}) = \delta_{ij}$ one has $(\mu_j^i) = {}^t(\lambda_j^i)^{-1}$.

Problem 2.30 Consider the three vector fields on $\mathbb{R}^3$:

$$e_1 = (2+y^2)\mathrm{e}^z\frac{\partial}{\partial x}, \qquad e_2 = 2xy\frac{\partial}{\partial x} + (2+y^2)\frac{\partial}{\partial y},$$
$$e_3 = -2xy^2\frac{\partial}{\partial x} - y(2+y^2)\frac{\partial}{\partial y} + (2+y^2)\frac{\partial}{\partial z}.$$

(i) Show that these vector fields are a basis of the module of C^∞ vector fields on $\mathbb{R}^3$.
(ii) Write the elements θ^i of its dual basis in terms of $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$.
(iii) Compute the Lie brackets $[e_i, e_j]$ and express them in the basis $\{e_i\}$.

Solution

(i) The determinant of the matrix of coefficients is $(2+y^2)^3\mathrm{e}^z$, which is never null; hence the three fields are indeed a basis of $\mathfrak{X}(\mathbb{R}^3)$.
(ii) We proceed by direct computation. One has $\theta^i(e_j) = \delta^i_j$, where δ^i_j is the Kronecker delta. Hence, if

$$\theta^1 = A(x,y,z)\,\mathrm{d}x + B(x,y,z)\,\mathrm{d}y + C(x,y,z)\,\mathrm{d}z,$$

we have

$$1 = \theta^1(e_1) = A(2+y^2)\mathrm{e}^z, \qquad 0 = \theta^1(e_2) = A2xy + B(2+y^2),$$
$$0 = \theta^1(e_3) = A(-2xy^2) + B(-y(2+y^2)) + C(2+y^2).$$

Solving the system, we have

$$A = \frac{1}{(2+y^2)\mathrm{e}^z}, \qquad B = -\frac{2xy}{(2+y^2)\mathrm{e}^z}, \qquad C = 0.$$

Similarly, if $\theta^2 = D(x,y,z)\,\mathrm{d}x + E(x,y,z)\,\mathrm{d}y + F(x,y,z)\,\mathrm{d}z$, we deduce

$$D = 0, \qquad E = \frac{1}{2+y^2}, \qquad F = \frac{y}{2+y^2}.$$

Finally, if $\theta^3 = G(x,y,z)\,\mathrm{d}x + H(x,y,z)\,\mathrm{d}y + I(x,y,z)\,\mathrm{d}z$, we similarly obtain

$$G = 0, \qquad H = 0, \qquad I = \frac{1}{2+y^2}.$$

Hence,

$$\theta^1 = \frac{1}{(2+y^2)\mathrm{e}^z}\,\mathrm{d}x - \frac{2xy}{(2+y^2)^2\mathrm{e}^z}\,\mathrm{d}y,$$

$$\theta^2 = \frac{1}{2+y^2}\,\mathrm{d}y + \frac{y}{2+y^2}\,\mathrm{d}z, \qquad \theta^3 = \frac{1}{2+y^2}\,\mathrm{d}z.$$

(iii) Applying the formula

$$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X,Y],$$

we deduce $[e_1, e_2] = 0$. Similarly, one gets

$$[e_1, e_3] = -(2+y^2)e_1, \qquad [e_2, e_3] = (y^2 - 2)e_2 + 2ye_3.$$

Problem 2.31 Consider the three vector fields on $\mathbb{R}^3$:

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (1+x^2)\frac{\partial}{\partial z}.$$

(i) Show that these vector fields are a basis of the module of C^∞ vector fields on $\mathbb{R}^3$.
(ii) Write the elements of the dual basis $\{\theta^i\}$ of $\{e_i\}$ in terms of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$.

Solution

(i)

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1+x^2 \end{pmatrix} = 1 + x^2 \neq 0.$$

(ii)

$$1 = \theta^1(e_1) = (A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z)(e_1) = A, \qquad 0 = \theta^1(e_2) = A + B,$$

$$0 = \theta^1(e_3) = A + B + (1+x^2)C.$$

Solving the system, we have $A = 1$, $B = -1$, $C = 0$. Hence $\theta^1 = \mathrm{d}x - \mathrm{d}y$. Similarly, we obtain $\theta^2 = \mathrm{d}y - \mathrm{d}z/(1+x^2)$ and $\theta^3 = \mathrm{d}z/(1+x^2)$.

Problem 2.32 Consider the vector fields

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

on $\mathbb{R}^2$, and let $\psi : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by

$$u = x^2 - y^2, \qquad v = x^2 + y^2, \qquad w = x + y, \qquad t = x - y.$$

(i) Compute $[X, Y]$.
(ii) Show that X, Y are linearly independent on the open subset $\mathbb{R}^2 \setminus \{(0,0)\}$ of $\mathbb{R}^2$ and write the basis $\{\alpha, \beta\}$ dual to $\{X, Y\}$ in terms of the standard basis $\{\mathrm{d}x, \mathrm{d}y\}$.
(iii) Find vector fields on $\mathbb{R}^4$, ψ-related to X and Y, respectively.

Solution

(i) $[X, Y] = 0$.
(ii)

$$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \neq 0, \quad (x, y) \in \mathbb{R}^2 \setminus \big\{(0,0)\big\}.$$

Let

$$\alpha = a(x, y)\,\mathrm{d}x + b(x, y)\,\mathrm{d}y, \qquad \beta = c(x, y)\,\mathrm{d}x + d(x, y)\,\mathrm{d}y.$$

We thus have

$$1 = \alpha(X) = a(x, y)\,\mathrm{d}x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + b(x, y)\,\mathrm{d}y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),$$

$$0 = \alpha(Y) = a(x, y)\,\mathrm{d}x\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + b(x, y)\,\mathrm{d}y\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right).$$

That is, $1 = a(x, y)x + b(x, y)y$ and $0 = a(x, y)(-y) + b(x, y)x$, and one has $a(x, y) = x/(x^2 + y^2)$, $b(x, y) = y/(x^2 + y^2)$. Hence,

$$\alpha = \frac{x}{x^2 + y^2}\,\mathrm{d}x + \frac{y}{x^2 + y^2}\,\mathrm{d}y.$$

Similarly, we obtain $\beta = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$.

(iii)

$$\begin{aligned} \psi_* X &\equiv \begin{pmatrix} 2x & -2y \\ 2x & 2y \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\equiv \big(2x^2 - 2y^2\big)\left(\frac{\partial}{\partial u} \circ \psi\right) + \big(2x^2 + 2y^2\big)\left(\frac{\partial}{\partial v} \circ \psi\right) \\ &\quad + (x + y)\left(\frac{\partial}{\partial w} \circ \psi\right) + (x - y)\left(\frac{\partial}{\partial t} \circ \psi\right), \end{aligned}$$

$$\psi_* Y = -4xy\left(\frac{\partial}{\partial u} \circ \psi\right) + (x - y)\left(\frac{\partial}{\partial w} \circ \psi\right) + (-y - x)\left(\frac{\partial}{\partial t} \circ \psi\right).$$

Taking

$$\widetilde{X} = 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} + t\frac{\partial}{\partial t}, \qquad \widetilde{Y} = \left(t^2 - w^2\right)\frac{\partial}{\partial u} + t\frac{\partial}{\partial w} - w\frac{\partial}{\partial t},$$

we have

$$\psi_* X = \widetilde{X} \circ \psi, \qquad \psi_* Y = \widetilde{Y} \circ \psi.$$

Problem 2.33 Prove that the differential 1-forms $\omega^1, \dots, \omega^k$ on an n-manifold M are linearly independent if and only if $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$.

Solution If $\omega^1, \dots, \omega^k$ are linearly independent, then each T_pM, $p \in M$, has a basis $\{v_1, \dots, v_k, \dots, v_n\}$ such that its dual basis $\{\varphi^1, \dots, \varphi^k, \dots, \varphi^n\}$ satisfies $\varphi^i = \omega^i|_p$, $1 \leqslant i \leqslant k$; hence $\omega^1 \wedge \cdots \wedge \omega^k$ is an element of a basis of $\Lambda^k M$, and so it does not vanish.

Conversely, suppose that such differential forms are linearly dependent. Then there exist a point $p \in M$ and $i \in \{1, \dots, n\}$ such that $\omega^i|_p = \sum_{j\neq i} a_j \omega^j|_p$, and thus, at the point p,

$$\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^i \wedge \cdots \wedge \omega^k = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \sum_{j\neq i} a_j \omega^j \wedge \cdots \wedge \omega^k = 0.$$

Problem 2.34 Prove that the restriction to the sphere S^3 of the differential form

$$\alpha = x\,\mathrm{d}y - y\,\mathrm{d}x + z\,\mathrm{d}t - t\,\mathrm{d}z$$

on $\mathbb{R}^4$, does not vanish.

Solution Given $p \in S^3$, $(\alpha|_{S^3})_p = 0$ if and only if $\alpha_p(X) = 0$ for all

$$X \in T_pS^3 = \left\{X \in T_p\mathbb{R}^4 : \langle X, N\rangle = 0\right\},$$

where $\langle\,,\rangle$ stands for the Euclidean metric of $\mathbb{R}^4$, and

$$N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}$$

is the outward-pointing unit normal vector field to S^3. Define the differential form β by $\beta(X) = \langle X, N\rangle$. Thus $\beta = x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z + t\,\mathrm{d}t$.

If $(\alpha|_{S^3})_p = 0$, then α_p and β_p vanish on T_pS^3. But two linear forms vanishing on the same hyperplane are proportional, and thus $\alpha_p = \lambda\beta_p$, $\lambda \in \mathbb{R}$, or equivalently,

$$\frac{-y}{x} = \frac{x}{y} = \frac{-t}{z} = \frac{z}{t} = \lambda.$$

We find $x^2 + y^2 = 0$, $z^2 + t^2 = 0$, and hence $x = y = z = t = 0$, which is not possible because $p \in S^3$.

Problem 2.35 Let $\omega^1, \dots, \omega^r$ be differential 1-forms on a C^∞ n-manifold M that are independent at each point. Prove that a differential form θ belongs to the ideal $\mathscr{I}$ generated by $\omega^1, \dots, \omega^r$ if and only if

$$\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0.$$

Solution If $\theta \in \mathscr{I}$, then θ is a linear combination of exterior products where those forms appear as factors, and hence $\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0$.

Conversely, given a fixed point, complete $\omega^1, \dots, \omega^r$ to a basis

$$\omega^1, \dots, \omega^r, \omega^{r+1}, \dots, \omega^n,$$

so

$$\theta = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f_{i_1 \dots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}.$$

If $\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0$, then for each $\{i_1, \dots, i_k\}$, we have

$$f_{i_1 \dots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0.$$

Then

$$\begin{aligned} \{1, \dots, r\} \cap \{i_1, \dots, i_k\} \neq \emptyset \quad &\Longrightarrow \quad \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0, \\ \{1, \dots, r\} \cap \{i_1, \dots, i_k\} = \emptyset \quad &\Longrightarrow \quad f_{i_1 \dots i_k} = 0. \end{aligned}$$

Hence,

$$\theta = \sum_{\{1, \dots, r\} \cap \{i_1, \dots, i_k\} \neq \emptyset} f_{i_1 \dots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}.$$

Problem 2.36 Let M be a C^∞ manifold. If $\{\omega^1, \dots, \omega^n\}$ is a basis of $T_p^* M$, $p \in M$, prove that there are coordinate functions $x^1, \dots, x^n$ around p such that $(\mathrm{d}x^i)_p = \omega^i$ for all i.

Solution Let $(U, y^1, \dots, y^n)$ be a coordinate system around p. Since the differentials $\{(\mathrm{d}y^1)_q, \dots, (\mathrm{d}y^n)_q\}$ are a basis of $T_q^* M$ for each $q \in U$, we can write $\omega^i = \sum_j f_j^i (\mathrm{d}y^j)_p$. Since $\{\omega^1, \dots, \omega^n\}$ is a basis of $T_p^* M$, we have $\det(f_j^i) \neq 0$. Thus the system $(U, x^1, \dots, x^n)$ defined by $x^i(q) = \sum_j f_j^i y^j(q)$ is a coordinate system, and one has $(\mathrm{d}x^i)_p = \sum_j f_j^i (\mathrm{d}y^j)_p = \omega^i$.

Problem 2.37 Determine which of the following differential forms on $\mathbb{R}^3$ are closed and which are exact:

(i) $\alpha = yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z$. (ii) $\beta = x\,\mathrm{d}x + x^2y^2\,\mathrm{d}y + yz\,\mathrm{d}z$.

(iii) $\gamma = 2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$.

Solution

(i) $\alpha = \mathrm{d}(xyz)$; thus α is exact and hence closed.
(ii) $\mathrm{d}\beta = 2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$; thus β is not closed, hence it is not exact.
(iii) $\gamma = \mathrm{d}\omega$, where $\omega = (x^2y^2 - \frac{1}{2}z^2)\,\mathrm{d}y$; thus γ is exact, hence closed.

Recall that, by the Poincaré lemma, every closed differential form on $\mathbb{R}^n$ is exact. Thus, another way to prove (i) and (iii) is:

(i) $\mathrm{d}\alpha = 0$, and thus α is closed and hence exact.
(iii) $\mathrm{d}\gamma = 0$, and thus γ is closed and hence exact.

Problem 2.38 Let $\pi : M \to M'$ be a surjective submersion of manifolds M and M'. Suppose that the set $\pi^{-1}(p')$ is connected for all $p' \in M'$. Let $\omega \in \Lambda^*(M)$.

Prove that there exists a unique differential form $\omega' \in \Lambda^*(M')$ such that $\omega = \pi^*(\omega')$ if and only if $i_Y\omega = 0$ and $L_Y\omega = 0$ for all vector fields Y belonging to the smooth distribution $\ker \pi_* \subset TM$ of vectors annihilated by π_*.

Solution The distribution $\ker \pi_*$ is an involutive smooth distribution (see Problem 2.57). Since the map $\pi : M \to M'$ is a submersion, by the Theorem of the Rank 1.11, for any point $p \in M$, there exist a connected neighbourhood U of p, coordinates $x^1, \dots, x^n$ on U and coordinates $x^1, \dots, x^{n'}$ ($n \geqslant n'$) on the open set $U' = \pi(U) \subset M'$ such that the restriction $\pi|_U$ in these coordinates has the form

$$\pi : \left(x^1, x^2, \dots, x^n\right) \to \left(x^1, x^2, \dots, x^{n'}\right),$$

i.e. in the neighbourhood U the restriction $\ker \pi_*|_U$ is spanned by the vector fields $\partial/\partial x^{n'+1}, \dots, \partial/\partial x^n$. Now let $\omega \in \Lambda^q M$. Let $i_Y\omega = 0$ and $L_Y\omega = 0$ for all vector fields $Y \in \ker \pi_*$. Since

$$L_Y = i_Y \circ \mathrm{d} + \mathrm{d} \circ i_Y$$

(see formula (7.3)), we obtain that $i_Y\,\mathrm{d}\omega = 0$. Then in the local coordinates $(x^1, \dots, x^n)$ on U we have

$$\mathrm{d}\omega|_U = \sum_{1 \leqslant j_1 < \dots < j_{q+1} \leqslant n'} b_{j_1 \dots j_{q+1}}\left(x^1, \dots, x^{n'}, \dots, x^n\right) \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_{q+1}}$$

and, consequently,

$$\omega|_U = \sum_{1 \leq i_1 < \dots < i_q \leq n'} a_{i_1 \dots i_q}\left(x^1, \dots, x^{n'}\right) \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_q},$$

where $a_{i_1 \dots i_q}$ are functions only of the variables $x^1, \dots, x^{n'}$. Hence, there is a unique local differential q-form $\omega' \in \Lambda^q U'$ such that $\omega|_U = p^*\omega'$.

Let $p_1 \in U$ and $p_2 \in \pi^{-1}(\pi(p_1))$, i.e. $\pi(p_1) = \pi(p_2)$. Since the set $\pi^{-1}(\pi(p_1))$ is a connected closed submanifold of M (by the Implicit map Theorem for Submersions), the points p_1, p_2 belong to the same leaf of the distribution $\ker \pi_*$. Then there exists a smooth vector field $Z \in \ker \pi_*$ such that for the corresponding (local) one-parameter group φ_t we have $\varphi_T(p_1) = p_2$, $T \in \mathbb{R}$. But $L_Z\omega = 0$, and therefore $\varphi_t^*\omega = \omega$ for all t (see Proposition 2.10). Thus, for $\varphi_{-T} = \varphi_T^{-1}$,

$$\omega_{p_2} = \varphi_{-T}^*\omega_{p_1} = \varphi_{-T}^*\big(\pi^*\omega'_{\pi(p_1)}\big) = (\pi \circ \varphi_{-T})^*\omega'_{\pi(p_2)} = \pi^*\omega'_{\pi(p_2)},$$

i.e. $\omega|_{\pi^{-1}(U')} = \pi^*(\omega')$. From the uniqueness of the local form $\omega' \in \Lambda^q U'$ it follows that there is a smooth global differential q-form $\omega' \in \Lambda^q M'$ such that $\omega = \pi^*\omega'$. Since the map π is a surjective submersion, such a form ω' is unique.

Problem 2.39 Let α be a closed differential 2-form of constant rank 2ℓ on a manifold M. Denote by $\ker\alpha$ the kernel of α, i.e. the distribution on M which is formed by the set of all vector fields $X \in \mathfrak{X}(M)$ satisfying $i_X\alpha = 0$.

Prove that the distribution $\ker\alpha$ is a smooth involutive distribution.

Solution Let $p_0 \in M$ be an arbitrary point. Locally, in a coordinate system $(U, x^1, \dots, x^n)$, where $U \subset M$ is an open subset containing p_0, the form α is determined by the expression

$$\sum_{1 \leqslant i < j \leqslant n} a_{ij}\big(x^1, \dots, x^n\big)\,\mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Since the two-form α is smooth, the map $p \mapsto A(p) = (a_{ij}(x^1(p), \dots, x^n(p)))$ $(a_{ij} = -a_{ji})$ determines a smooth matrix function on the set U. Moreover, there exists some $2\ell \times 2\ell$ minor of the matrix $A(p)$ nowhere vanishing on some open subset $O \subset U$ containing p_0. Therefore in O the kernel of the form α_p, which coincides with the kernel of $A(p)$, is generated by $n - 2\ell$ smooth vector fields. Thus $\ker\alpha$ is a smooth distribution of dimension $n - 2\ell$.

By the definition of $\mathrm{d}\alpha$ (see formula (7.2)), for arbitrary vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned}\mathrm{d}\alpha(X, Y, Z) = {} & X\big(\alpha(Y, Z)\big) - Y\big(\alpha(X, Z)\big) + Z\big(\alpha(X, Y)\big)\\ & - \alpha\big([X, Y], Z\big) + \alpha\big([X, Z], Y\big) - \alpha\big([Y, Z], X\big).\end{aligned}$$

Suppose now, in addition, that $X, Y \in \ker\alpha$. Then in the right-hand side of the expression above all terms vanish with the exception of the fourth term. Since $\mathrm{d}\alpha = 0$, we obtain that $\alpha([X, Y], Z) = 0$. Thus

$$X, Y \in \ker\alpha \quad \Rightarrow \quad [X, Y] \in \ker\alpha,$$

i.e. $\ker\alpha$ is involutive.

Problem 2.40 Let $\widetilde{M}$ be a submanifold of a manifold M. Suppose that X, Y are smooth vector fields on M which are tangent to $\widetilde{M}$ at each point belonging to $\widetilde{M}$, i.e. $X_p, Y_p \in T_p\widetilde{M} \subset T_pM$ if $p \in \widetilde{M}$.

Prove:

(i) The map $\widetilde{M} \to T\widetilde{M}$, $p \mapsto X_p$ (resp. $p \mapsto Y_p$), defines a smooth vector field $\widetilde{X}$ (resp. $\widetilde{Y}$) on $\widetilde{M}$.
(ii) The bracket $[X, Y]$ of the vector fields X, Y has the same property as X and Y: $[X, Y]_p \in T_p\widetilde{M} \subset T_pM$ if $p \in \widetilde{M}$.
(iii) We have $[\widetilde{X}, \widetilde{Y}]_p = [X, Y]_p$ for each $p \in \widetilde{M} \subset M$.

Solution It is clear that it is only necessary to prove all three assertions (i), (ii), (iii) locally.

(i) Fix some point $p_0 \in \widetilde{M} \subset M$. Since $\widetilde{M}$ is a submanifold of M, by the Theorem of the Rank 1.11 there exist neighbourhoods $O \subset M$ and $\widetilde{O} \subset \widetilde{M} \cap O$ of the point p_0, and coordinates $(x^1, \dots, x^n)$ in O such that a point $p \in O$ is an element of the subset $\widetilde{O}$ if and only if $x^i(p) = 0$ for all $i > \dim \widetilde{M}$, $i \leqslant n$. In particular, $x^1(p), \dots, x^l(p)$, where $l = \dim \widetilde{M}$, are coordinate functions in the open subset $\widetilde{O}$. We have

$$(X|_O)_p = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x^i}.$$

But for each $\widetilde{p} \in \widetilde{O} \subset O$, the vector $X_{\widetilde{p}}$ is an element of $T_{\widetilde{p}}\widetilde{M}$, i.e.

$$a_i(\widetilde{p}) = a_i\big(x^1(\widetilde{p}), \dots, x^l(\widetilde{p}), 0, \dots, 0\big) = 0, \quad i > l, \tag{$\star$}$$

and, consequently,

$$\frac{\partial a_i}{\partial x^j}(\widetilde{p}) = \frac{\partial a_i}{\partial x^j}\big(x^1(\widetilde{p}), \dots, x^l(\widetilde{p}), 0, \dots, 0\big) = 0, \quad i > l,\ j \leqslant l. \tag{$\star\star$}$$

Thus,

$$(\widetilde{X}|_{\widetilde{O}})_{\widetilde{p}} = \sum_{i=1}^{l} \widetilde{a}_i(\widetilde{p}) \frac{\partial}{\partial x^i}$$

for all $\widetilde{p} \in \widetilde{O} \subset O$, where $\widetilde{a}_i = a_i|_{\widetilde{O}}$. Since $\widetilde{a}_i(x^1, \dots, x^l) = a_i(x^1, \dots, x^l, 0, \dots, 0)$, the vector field $\widetilde{X}|_{\widetilde{O}}$ is smooth.

Similarly,

$$(Y|_O)_p = \sum_{i=1}^{n} b_i(p) \frac{\partial}{\partial x^i}$$

for all $p \in O \subset M$, and the vector field $\widetilde{Y}|_{\widetilde{O}}$,

$$(\widetilde{Y}|_{\widetilde{O}})_{\widetilde{p}} = \sum_{i=1}^{l} \widetilde{b}_i(\widetilde{p}) \frac{\partial}{\partial x^i},$$

is smooth.

(ii) and (iii) We shall see (ii) and (iii) giving two proofs. The first proof using the local representations of the vector fields:

$$
\begin{aligned}
[X,Y]_{\widetilde{p}} &= \sum_{j=1}^{n}\sum_{i=1}^{n}\left(a_i(\widetilde{p})\frac{\partial b_j}{\partial x^i}(\widetilde{p}) - b_i(\widetilde{p})\frac{\partial a_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j} \quad \text{(by definition)}\\
&= \sum_{j=1}^{n}\sum_{i=1}^{l}\left(a_i(\widetilde{p})\frac{\partial b_j}{\partial x^i}(\widetilde{p}) - b_i(\widetilde{p})\frac{\partial a_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j} \quad \text{(by }(\star))\\
&= \sum_{j=1}^{l}\sum_{i=1}^{l}\left(a_i(\widetilde{p})\frac{\partial b_j}{\partial x^i}(\widetilde{p}) - b_i(\widetilde{p})\frac{\partial a_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j} \quad \text{(by }(\star\star))\\
&= \sum_{j=1}^{l}\sum_{i=1}^{l}\left(\widetilde{a}_i(\widetilde{p})\frac{\partial \widetilde{b}_j}{\partial x^i}(\widetilde{p}) - \widetilde{b}_i(\widetilde{p})\frac{\partial \widetilde{a}_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j}.
\end{aligned}
$$

Hence assertions (ii) and (iii) hold.

As to the second proof, consider the one-to-one immersion $\pi\colon \widetilde{M} \to M$, $p \mapsto p$, defining the submanifold $\widetilde{M} \subset M$. Then the vector fields $\widetilde{X}$ and X, $\widetilde{Y}$ and Y are π-related, i.e.

$$\pi_* \circ \widetilde{X} = X \circ \pi, \qquad \pi_* \circ \widetilde{Y} = Y \circ \pi.$$

By Theorem 1.21 the vector fields (brackets) $[\widetilde{X}, \widetilde{Y}]$ and $[X, Y]$ are also π-related. Thus,

$$[X,Y]_p = \pi_*\big([\widetilde{X},\widetilde{Y}]_p\big) = [\widetilde{X},\widetilde{Y}]_p, \quad p \in \widetilde{M},$$

and, in particular, $[X,Y]_p \in T_p\widetilde{M}$.

Problem 2.41 Let ω be a differential 1-form on a manifold M and consider a nowhere-vanishing function $f\colon M \to \mathbb{R}$ such that $\mathrm{d}(f\omega) = 0$. Prove that $\omega \wedge \mathrm{d}\omega = 0$.

Solution We have $\mathrm{d}(f\omega) = \mathrm{d}f \wedge \omega + f\mathrm{d}\omega$, and since $f(x) \neq 0$ for all $x \in M$, one has $\mathrm{d}\omega = -(1/f)\,\mathrm{d}f \wedge \omega$. As ω is a differential 1-form, we have $\omega \wedge \mathrm{d}\omega = -(1/f)\omega \wedge \mathrm{d}f \wedge \omega = 0$.

2.5 Lie Derivative. Interior Product

Problem 2.42 Let X and Y be vector fields on a C^∞ manifold M. Prove that if φ_t is the local 1-parameter group generated by X, we have for all $p \in M$:

$$\varphi_{s*}\big((L_X Y)_{\varphi_s^{-1}(p)}\big) = \lim_{t\to 0}\frac{1}{t}\big(\varphi_{s*}Y_{\varphi_s^{-1}(p)} - \varphi_{s+t*}Y_{\varphi_{s+t}^{-1}(p)}\big). \qquad (\star)$$

Solution Since φ_t is the local one-parameter group of X, one has $\varphi_s \cdot X = X$, where by definition $(\varphi_s \cdot X)_p = \varphi_{s*}(X_{\varphi_s^{-1}(p)})$. Then, applying Problem 1.107, we have

$$\varphi_s \cdot L_X Y = \varphi_s \cdot [X, Y] = [\varphi_s \cdot X, \varphi_s \cdot Y] = [X, \varphi_s \cdot Y] = L_X(\varphi_s \cdot Y).$$

Thus,

$$\begin{aligned}\varphi_{s*}\big((L_X Y)_{\varphi_s^{-1}(p)}\big) &= L_X(\varphi_{s*} Y_{\varphi_s^{-1}(p)}) \\ &= \lim_{t\to 0} \frac{1}{t}\big(\varphi_{s*} Y_{\varphi_s^{-1}(p)} - \varphi_{t*}\big((\varphi_{s*} Y_{\varphi_s^{-1}(p)})_{\varphi_t^{-1}(p)}\big)\big) \\ &= \lim_{t\to 0} \frac{1}{t}(\varphi_{s*} Y_{\varphi_s^{-1}(p)} - \varphi_{t*}\varphi_{s*} Y_{\varphi_s^{-1}(\varphi_t^{-1}(p))}) \\ &= \lim_{t\to 0} \frac{1}{t}(\varphi_{s*} Y_{\varphi_s^{-1}(p)} - \varphi_{s+t*} Y_{\varphi_{s+t}^{-1}(p)}).\end{aligned}$$

Problem 2.43 Let f denote a diffeomorphism of the C^∞ manifold M. Prove that

$$i_X\big(f^*\alpha\big) = f^*(i_{f\cdot X}\alpha), \quad X \in \mathfrak{X}(M),\ \alpha \in \Lambda^* M.$$

Solution If $\alpha \in \Lambda^r M$, then for $X_1, \dots, X_{r-1} \in \mathfrak{X}(M)$, one has

$$\begin{aligned}&\big(i_X\big(f^*\alpha\big)\big)_p(X_1|_p, \dots, X_{r-1}|_p) \\ &= \big(f^*\alpha\big)_p(X_p, X_1|_p, \dots, X_{r-1}|_p) \\ &= \alpha_{f(p)}\big(f_* X_p, f_*(X_1|_p), \dots, f_*(X_{r-1}|_p)\big) \\ &= \alpha_{f(p)}\big((f\cdot X)_{f(p)}, (f\cdot X_1)_{f(p)}, \dots, (f\cdot X_{r-1})_{f(p)}\big)\end{aligned}$$

and

$$\begin{aligned}&\big(f^*(i_{f\cdot X}\alpha)\big)_p(X_1|_p, \dots, X_{r-1}|_p) \\ &= (i_{f\cdot X}\alpha)_{f(p)}\big(f_*(X_1|_p), \dots, f_*(X_{r-1}|_p)\big) \\ &= \alpha_{f(p)}\big((f\cdot X)_{f(p)}, (f\cdot X_1)_{f(p)}, \dots, (f\cdot X_{r-1})_{f(p)}\big).\end{aligned}$$

Problem 2.44 Consider on an open subset of $\mathbb{R}^3$ the differential 1-form

$$\alpha = P_1(x)\,\mathrm{d}x^1 + P_2(x)\,\mathrm{d}x^2 + P_3(x)\,\mathrm{d}x^3,$$

where $x = (x^1, x^2, x^3)$.

(i) Find the conditions under which $i_X\,\mathrm{d}\alpha = 0$ for

$$X = X_1\partial/\partial x + X_2\partial/\partial y + X_3\partial/\partial z.$$

(ii) When do we have $i_X\alpha = 0$ and $i_X\,\mathrm{d}\alpha = 0$?

Solution

(i) Let us compute $\mathrm{d}\alpha$. If we write $P_{ij} = \partial P_i/\partial x^j$ and $Q_{ji} = P_{ji} - P_{ij}$, then

$$\begin{aligned}
\mathrm{d}\alpha &= (P_{21} - P_{12})\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + (P_{31} - P_{13})\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \\
&\quad + (P_{32} - P_{23})\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \sum_{i<j} Q_{ji}\,\mathrm{d}x^i \wedge \mathrm{d}x^j.
\end{aligned}$$

Hence,

$$\begin{aligned}
i_X\,\mathrm{d}\alpha = 0 \quad &\Leftrightarrow \quad i_X\,\mathrm{d}\alpha(Y) = 0, \quad Y \in \mathfrak{X}(\mathbb{R}^3) \\
&\Leftrightarrow \quad \mathrm{d}\alpha\left(X, \frac{\partial}{\partial x^k}\right) = 0, \quad k = 1, 2, 3 \\
&\Leftrightarrow \quad \sum_{i<j} Q_{ji}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \left(\sum_l X_l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right) \\
&\qquad = \sum_l \sum_{i<j} Q_{ji}\left(X_l \delta^i_l \delta^j_k - X_l \delta^i_k \delta^j_l\right) \\
&\qquad = \sum_l \left(\sum_{l<k} Q_{kl} X_l - \sum_{k<l} Q_{lk} X_l\right) \\
&\qquad = \sum_l Q_{kl} X_l = 0, \quad k = 1, 2, 3.
\end{aligned}$$

(ii) By (i),

$$i_X\,\mathrm{d}\alpha = 0 \quad \Leftrightarrow \quad \sum_{l=1}^{3} Q_{kl} X_l = 0, \quad k = 1, 2, 3,$$

and

$$i_X\alpha = \alpha(X) = 0 \quad \Leftrightarrow \quad \left(\sum_i P_i\,\mathrm{d}x^i\right)\left(\sum_j X^j \frac{\partial}{\partial x^j}\right) = 0 \quad \Leftrightarrow \quad \sum_i P_i X^i = 0.$$

2.6 Distributions and Integral Manifolds. Frobenius Theorem. Differential Ideals

Problem 2.45 Consider on the octant of $\mathbb{R}^3$ of positive coordinates the vector fields

$$X = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}, \qquad Y = xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z}.$$

(i) Prove that they span an involutive distribution on this octant of $\mathbb{R}^3$.
(ii) Find the integral surfaces.

Hint (to (ii)) Substitute Y by $x^{-1}Y$.

Solution

(i) $[X, Y] = Y$.
(ii) Since in the given domain x does not vanish, we can substitute $x^{-1}Y$ for Y, which, jointly with X, determines the same distribution. The integral curves of X are $(x_0 e^t, y_0 e^{-2t}, z_0)$, and those of $x^{-1}Y$ are $(x_0, y_0 e^s, z_0 e^{-s})$, so that the respective local flows are

$$\varphi_t(x, y, z) = \left(x e^t, y e^{-2t}, z\right), \qquad \psi_s(x, y, z) = \left(x, y e^s, z e^{-s}\right).$$

The map

$$\begin{aligned}(t, s) \in \mathbb{R}^2 \mapsto (\psi_s \circ \varphi_t)(x_0, y_0, z_0) &= \psi_s\left(x_0 e^t, y_0 e^{-2t}, z_0\right)\\ &= \left(x_0 e^t, y_0 e^{-2t+s}, z_0 e^{-s}\right)\end{aligned}$$

is the integral surface through (x_0, y_0, z_0). In fact, the point $(\psi_s \circ \varphi_t)(x_0, y_0, z_0)$ is obtained from (x_0, y_0, z_0) as follows: We first run an interval "t" from $p = (x_0, y_0, z_0)$ along the integral curve of X through p for $t = 0$ and then an interval "s" from $\varphi_t(p)$ along the integral curve of $x^{-1}Y$ through $\varphi_t(p)$ for $s = 0$. If we put

$$x(t, s) = x_0 e^t, \qquad y(t, s) = y_0 e^{-2t+s}, \qquad z(t, s) = z_0 e^{-s},$$

then we see that x^2yz is constant. Hence the integral surfaces are defined by $x^2yz = \text{const}$. As a verification, observe that $X(x^2yz) = Y(x^2yz) = 0$.

Problem 2.46 Consider on $\mathbb{R}^3$ the distribution $\mathscr{D}$ determined by

$$X = \frac{\partial}{\partial x} + \frac{2xz}{1 + x^2 + y^2}\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{2yz}{1 + x^2 + y^2}\frac{\partial}{\partial z}.$$

(i) Calculate $[X, Y]$ and find whether $\mathscr{D}$ is involutive or not.
(ii) Calculate the local flows of X and Y.
(iii) If $\mathscr{D}$ is involutive, find its integral surfaces.

Solution

(i) $[X, Y] = 0$, and thus $\mathscr{D}$ is involutive.
(ii) We have

$$\begin{cases} x' = 1 \\ y' = 0 \end{cases} \Leftrightarrow \quad \begin{cases} x = x_0 + t \\ y = y_0 \end{cases}$$

and

$$\frac{z'}{z} = \frac{2(x_0+t)}{1+(x_0+t)^2+y_0^2}$$

if and only if $\log z = \log A(1+(x_0+t)^2+y_0^2)$ if and only if $z = A(1+(x_0+t)^2+y_0^2)$. For $t=0$, $z_0 = A(1+x_0^2+y_0^2)$, so

$$z = z_0\frac{1+(x_0+t)^2+y_0^2}{1+x_0^2+y_0^2}.$$

Hence the local flow of X is

$$\varphi_t(x,y,z) = \left(x+t,\, y,\, z\frac{1+(x+t)^2+y^2}{1+x^2+y^2}\right).$$

Similarly, the local flow of Y is

$$\psi_s(x,y,z) = \left(x,\, y+s,\, z\frac{1+x^2+(y+s)^2}{1+x^2+y^2}\right).$$

(iii) The integral manifolds can be written as $\psi(t,s) \mapsto (\psi_s \circ \varphi_t)(x_0, y_0, z_0)$. But let us see a better solution. We are looking for a differential 1-form annihilating X and Y. For example, we have as a solution:

$$\begin{aligned}\alpha &= 2xz\,\mathrm{d}x + 2yz\,\mathrm{d}y - \left(1+x^2+y^2\right)\mathrm{d}z \\ &= z\,\mathrm{d}\left(1+x^2+y^2\right) - \left(1+x^2+y^2\right)\mathrm{d}z \\ &= -\left(1+x^2+y^2\right)^2\mathrm{d}\left(\frac{z}{1+x^2+y^2}\right).\end{aligned}$$

Hence, the integral manifolds are $\frac{z}{1+x^2+y^2} = \text{const.}$

Problem 2.47 The vector field $X = x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, defined on $x>0$, $y>0$, $z>0$ in $\mathbb{R}^3$, determines a two-dimensional distribution given by the vector fields orthogonal to X. Is this distribution involutive?

Solution The vector fields $U = -y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $V = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$ are orthogonal to X and linearly independent at each point. They span that distribution, but $[U,V] = -y\frac{\partial}{\partial z}$. Since

$$\begin{vmatrix} -y & 1 & 0 \\ -z & 0 & x \\ 0 & 0 & -y \end{vmatrix} = -yz$$

is not identically zero, we have $[U,V]_p \notin \langle U_p, V_p\rangle$. Hence the distribution is not involutive.

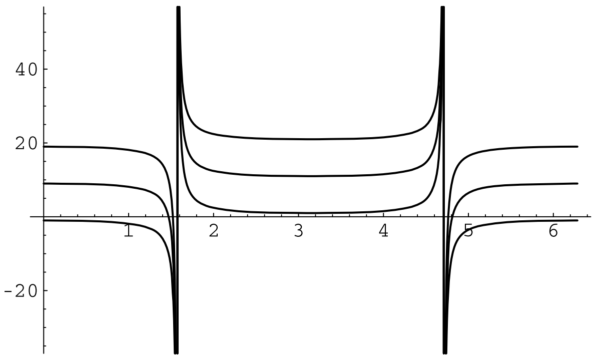

Fig. 2.2 An example of foliation with non-Hausdorff quotient manifold

Problem 2.48 Prove that

$$X = -\cos^2 x \frac{\partial}{\partial x} + \sin x \frac{\partial}{\partial y}$$

determines a foliation with non-Hausdorff quotient.

Solution This vector field determines an integrable distribution of codimension 1 of $\mathbb{R}^2$. We have two kind of solutions:

Integrating the equation that X determines, i.e.

$$\frac{\mathrm{d}x}{\cos^2 x} = -\frac{\mathrm{d}y}{\sin x},$$

we obtain the curves

$$y = -\sec x + A$$

(see Fig. 2.2) for $x \neq (2k+1)\pi/2$, $k \in \mathbb{Z}$.

Moreover, we have the solutions with initial conditions of the type $((2k+1)\pi/2, y_0)$, that is, the straight lines $t \mapsto ((2k+1)\pi/2, (-1)^k t)$. Actually, if p and q are two non-separable points of the quotient, then each of them corresponds to a solution of this kind.

Take, for instance, the integral curve $x = -\pi/2$; a point on it, say $(-\pi/2, y_0)$; and an open disk around this point. This open disk intersects all the integral curves intersecting the y-axis at the points with ordinate greater than or equal to $A_0 > 0$. This is also true for open disks around the point $(\pi/2, y_1)$. Such an open disk intersects all the integral curves that intersect the y-axis at points with ordinate greater than or equal to $A_1 > 0$. Now, the integral curves intersecting the y-axis at points with ordinate greater than $\max(A_0, A_1)$ intersect both open disks. Hence the projections of the two open disks on the quotient intersect, so that the projections of $x = -\pi/2$ and of $x = \pi/2$ cannot be separated. Consequently, the quotient manifold is not Hausdorff.

Problem 2.49 Consider on $\mathbb{R}^3$ the vector fields

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad Z = z\frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

(i) Prove that X, Y, Z define a C^∞ distribution $\mathscr{D}$ on $\mathbb{R}^3$. Which dimension is it? Is it involutive?

(ii) Compute the set $\mathscr{I}(\mathscr{D})$ of forms which annihilate $\mathscr{D}$. Is it a differential ideal? Is the ideal $\mathscr{I}$ generated by $\mathrm{e}^x\,\mathrm{d}y$ a differential ideal?

Solution

(i) X, Y, Z are not linearly independent because $Z = X - Y$. Hence $\mathscr{D}$ is a two-dimensional C^∞ distribution spanned, for instance, by X and Y, which are linearly independent. $\mathscr{D}$ is not involutive, as $[X, Y] = -\frac{\partial}{\partial x}$ and $-\frac{\partial}{\partial x} \notin \mathscr{D}$, since if it were

$$-\frac{\partial}{\partial x} = az\frac{\partial}{\partial x} + a\frac{\partial}{\partial z} + b\frac{\partial}{\partial y} + b\frac{\partial}{\partial z},$$

we would have $az = -1$, $b = 0$, $b + a = 0$, which would lead us to a contradiction.

(ii) $\{X, Y, \partial/\partial x\}$ is a basis of $\mathfrak{X}(\mathbb{R}^3)$. Therefore, if $\{\alpha, \beta, \omega\}$ is its dual basis of 1-forms, then $\mathscr{I}(\mathscr{D}) = \langle\omega\rangle$, where $\langle\omega\rangle$ stands for the ideal generated by ω.

Let us determine $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$, $f, g, h \in C^\infty\mathbb{R}^3$. From

$$0 = \omega(X) = fz + h, \qquad 0 = \omega(Y) = g + h, \qquad 1 = \omega\left(\frac{\partial}{\partial x}\right) = f$$

it follows that $f = 1$. Thus $h = -z$, and hence $g = z$; that is, $\omega = \mathrm{d}x + z\,\mathrm{d}y - z\,\mathrm{d}z$. Since $\mathscr{D}$ is not involutive, $\mathscr{I}(\mathscr{D})$ cannot be a differential ideal.

We can also prove this directly. One has $\mathrm{d}\omega = \mathrm{d}z \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}z$. If it were, for $a, b, c \in C^\infty\mathbb{R}^3$,

$$\begin{aligned}\mathrm{d}\omega &= \omega \wedge (a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z)\\ &= (b - az)\,\mathrm{d}x \wedge \mathrm{d}y + (c + az)\,\mathrm{d}x \wedge \mathrm{d}z + (zc + zb)\,\mathrm{d}y \wedge \mathrm{d}z,\end{aligned}$$

we would have $b - az = 0$, $c + az = 0$, $zc + zb = -1$. From the first and second equations one has $b + c = 0$, in contradiction with the third equation. One can also conclude by applying Problem 2.35, as $\omega \wedge \mathrm{d}\omega = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0$. Finally, $\mathscr{I}$ is a differential ideal since

$$\mathrm{d}\big(\mathrm{e}^x\,\mathrm{d}y\big) = \mathrm{e}^x\,\mathrm{d}x \wedge \mathrm{d}y = \mathrm{e}^x\,\mathrm{d}y \wedge (-\mathrm{d}x).$$

Problem 2.50 Given on $\mathbb{R}^4 = \{(x, y, z, t)\}$ the 1-forms $\alpha = \mathrm{d}x + z\,\mathrm{d}t$ and $\beta = \mathrm{d}z + \mathrm{d}t$, let $\mathscr{I}$ be the ideal generated by α and β, and let $\mathscr{D}$ be the distribution associated to $\mathscr{I}$.

(i) Compute a basis for $\mathscr{D}$.
(ii) Is $\mathscr{D}$ involutive?
(iii) If $p = (1, 0, 1, 0) \in \mathbb{R}^4$, do we have

$$v_p = -3\left.\frac{\partial}{\partial y}\right|_p + z\left.\frac{\partial}{\partial x}\right|_p \in \mathscr{D}_p?$$

(iv) If $\omega = \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}z \wedge \mathrm{d}t$, is $\omega \in \mathscr{I}$?
(v) Is $y = \text{const}$, $z = \text{const}$ an integral manifold of $\mathscr{D}$?

Solution

(i) For $X, Y \in \mathscr{D}$ given by

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + d\frac{\partial}{\partial t}, \qquad Y = e\frac{\partial}{\partial x} + f\frac{\partial}{\partial y} + g\frac{\partial}{\partial z} + h\frac{\partial}{\partial t},$$

for $a, b, c, d, e, f, g, h \in C^\infty\mathbb{R}^4$, it must be

$$\alpha(X) = a + zd = 0, \qquad \alpha(Y) = e + zh = 0,$$
$$\beta(X) = c + d = 0, \qquad \beta(Y) = g + h = 0.$$

Thus, for instance, we can consider

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z} - \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y}.$$

(ii) $[X, Y] = 0$, and hence $\mathscr{D}$ is involutive.
(iii) No, as

$$\alpha_p(v_p) = (\mathrm{d}x + z\,\mathrm{d}t)_p\left(-3\frac{\partial}{\partial y} + z\frac{\partial}{\partial x}\right)_p = 1 \neq 0.$$

(iv) $\omega = \mathrm{d}x \wedge \beta + \mathrm{d}z \wedge \beta$, and hence $\omega \in \mathscr{I}$.
(v) The tangent space is $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \rangle$, but $\alpha(\frac{\partial}{\partial x}) = 1$, so $y = \text{const}$, $z = \text{const}$ is not an integral manifold of $\mathscr{D}$.

Problem 2.51 Prove that the 1-form $\alpha = (1 + y^2)(x\,\mathrm{d}y + y\,\mathrm{d}x)$, defined on $\mathbb{R}^2 \setminus \{0\}$, generates a rank-1 differential ideal and find the integral manifolds.

Solution Since $1 + y^2$ does not vanish, α generates the same annihilator ideal as

$$\frac{\alpha}{1 + y^2} = x\,\mathrm{d}y + y\,\mathrm{d}x = \mathrm{d}(xy).$$

As $\mathrm{d}(x\,\mathrm{d}y + y\,\mathrm{d}x) = 0$, the ideal is differential.

The integral manifolds are $xy = \text{const}$ (see Fig. 2.3).

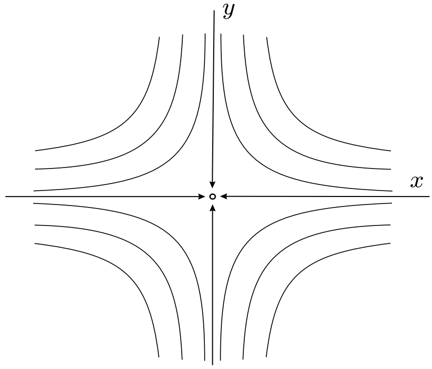

Fig. 2.3 Integral manifolds of $\alpha = (1 + y^2)(x\,\mathrm{d}y + y\,\mathrm{d}x)$

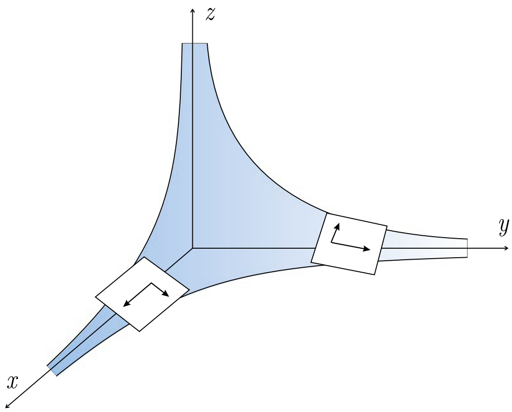

Fig. 2.4 The component in the first octant of an integral surface of the distribution $\alpha = yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z$

Problem 2.52 Let $U = \mathbb{R}^3 \setminus \{\text{axes}\}$. Compute the integral surfaces of the distribution determined by the ideal of $\Lambda^* U$ generated by

$$\alpha = yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z.$$

Solution We have $\alpha = \mathrm{d}(xyz)$. If X is annihilated by α, then we have $\alpha(X) = X(xyz) = 0$. Thus the integral surfaces are the surfaces $xyz = \text{const}$ (see Fig. 2.4).

Problem 2.53 Consider the $(1, 1)$ tensor field

$$J = \frac{1}{\cosh x}\frac{\partial}{\partial y} \otimes \mathrm{d}x + \cosh x \frac{\partial}{\partial x} \otimes \mathrm{d}y$$

on $\mathbb{R}^2$ and the distribution $\mathscr{D}$ defined by the condition: $X \in \mathscr{D}$ if and only if $JX = X$.

(i) Compute the integral curves of $\mathscr{D}$.
(ii) Compute the fields $X \in \mathscr{D}$ for which $L_X J = 0$.

Solution

(i) If $X = f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y} \in \mathscr{D}$, $f, h \in C^\infty\mathbb{R}^2$, then

$$\begin{aligned}&\left(\frac{1}{\cosh x}\frac{\partial}{\partial y} \otimes \mathrm{d}x + \cosh x\frac{\partial}{\partial x} \otimes \mathrm{d}y\right)\left(f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}\right)\\ &\quad = \frac{f}{\cosh x}\frac{\partial}{\partial y} + h\cosh x\frac{\partial}{\partial x} = f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}.\end{aligned}$$

Thus $f = h\cosh x$. Denoting by (x, y) the integral curves of $\mathscr{D}$, we have $\mathrm{d}x/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}t)\cosh x$. Hence $\mathrm{d}y = \mathrm{d}x/\cosh x$, and thus

$$y = \arctan\sinh x + A. \tag{$\star$}$$

That is, the integral curves of $\mathscr{D}$ are given by ($\star$).

(ii)

$$\begin{aligned}L_X J &= \left(h_x\cosh x - \frac{f_y}{\cosh x}\right)\left(\frac{\partial}{\partial x} \otimes \mathrm{d}x - \frac{\partial}{\partial y} \otimes \mathrm{d}y\right)\\ &\quad + (h_y\cosh x + f\sinh x - f_x\cosh x)\left(\frac{\partial}{\partial x} \otimes \mathrm{d}y - \frac{1}{\cosh^2 x}\frac{\partial}{\partial y} \otimes \mathrm{d}x\right)\\ &= 0.\end{aligned} \tag{$\star\star$}$$

Moreover, if $X \in \mathscr{D}$, then we have $f = g\cosh x$, and from this equation and from ($\star\star$) we conclude that we have to solve only the following equation:

$$\frac{\partial h}{\partial x}\cosh x = \frac{\partial h}{\partial y}.$$

Let $u = 2\arctan \mathrm{e}^x$. Then we have

$$\frac{\partial h}{\partial x} = \frac{1}{\cosh x}\frac{\partial h}{\partial u},$$

and hence $\frac{\partial h}{\partial u} = \frac{\partial h}{\partial y}$. Taking $t = u + y$, $w = u - y$, we obtain

$$0 = \frac{\partial h}{\partial u} - \frac{\partial h}{\partial y} = 2\frac{\partial h}{\partial w}.$$

Thus $h = h(u + y) = h(2\arctan \mathrm{e}^x + y)$, and we finally have

$$f = h\big(2\arctan \mathrm{e}^x + y\big)\cosh x,$$

where $h(2\arctan \mathrm{e}^x + y)$ is an arbitrary differentiable function in that argument.

Problem 2.54 (A Reeb Foliation of S^3) The three-sphere S^3 can be decomposed as two solid 2-tori joint along their common 2-torus boundary. In fact, if one removes

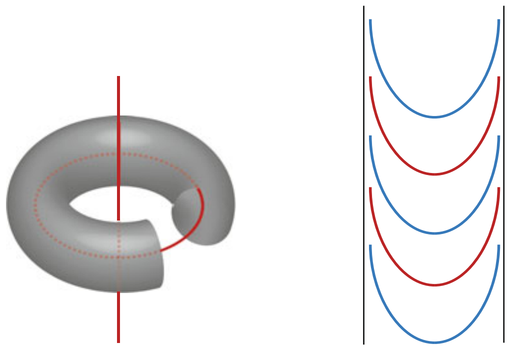

Fig. 2.5 *Left:* The two core circles of S^3 (here actually the part in $S^3 \setminus \{\infty\}$), this viewed as the union of two solid 2-tori. *Right:* Some curves $y = f(x) + c'$

the solid torus of rotation from $\mathbb{R}^3 = S^3 \setminus \{\infty\}$, what remains is homeomorphic to a solid torus minus an interior point. Consider the vertical coordinate axis as the core circle (see Fig. 2.5, left).

Find a foliation of the strip

$$\{(x, y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 1\}$$

originating a foliation in (each) solid torus and so a codimension 1 foliation of the three-sphere S^3.

For the development of the relevant theory, see Reeb [4] and Lawson [3].

Solution Consider the C^∞-foliation of the (x, y)-plane given by the lines $x = c$ for $|c| \geqslant 1$ together with the graphs of the functions

$$y = f(x) + c', \quad -1 < x < 1, \; c' \in \mathbb{R},$$

where f has the property that its derivatives $f^{(r)}$ satisfy $\lim_{|x| \to 1} f^{(r)} = \infty$ for all r (see Fig. 2.5, right).

Consider now the foliation of the solid cylinder obtained by rotating the strip given in the statement about the y-axis in $\mathbb{R}^3$. This foliation is invariant by vertical translations, and so we can obtain a foliation of the solid torus where each noncompact leaf has the form that one can see in Fig. 2.6, left. Gluing together two copies of the foliated solid torus gives a Reeb foliation of S^3 (see Fig. 2.6, right, showing part of a transversal cutting of two leaves). Note that both the "interior" and the "exterior" leaves approach their common 2-torus boundary after turning around it.

Problem 2.55 Let M be a C^∞ n-manifold, and let $\mathscr{D} \subset TM$ be an integrable distribution of rank p. By Frobenius' theorem, $\mathscr{D}$ is spanned by $\partial/\partial x^1, \dots, \partial/\partial x^p$ on an open subset U of M, for a certain coordinate system (U, x^i). We can consider local frames of M of the type

$$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}, X_1, \dots, X_q\right), \qquad p + q = n = \dim M,$$

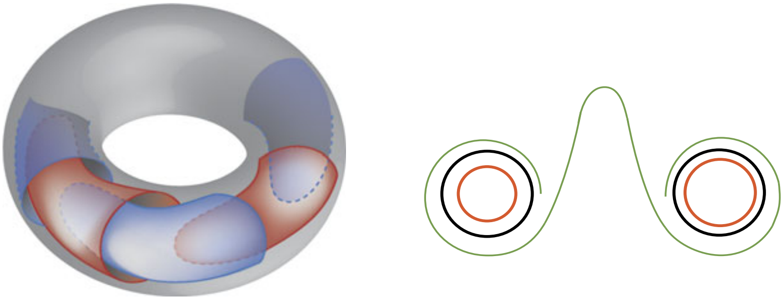

Fig. 2.6 *Left:* The foliation (generated by curves as the previous ones) of the "interior" solid torus in S^3. *Right:* Transversal cut of a Reeb foliation of S^3 showing two sections of an "interior" leaf and part of an "exterior" leaf

where

$$X_u = \frac{\partial}{\partial x^{p+u}} - \sum_a f_u^a \frac{\partial}{\partial x^a}, \quad 1 \leqslant a \leqslant p,\ 1 \leqslant u \leqslant q,\ f_u^a \in C^\infty M.$$

Write the integrability condition of the complementary distribution $\mathscr{H}$ generated by $X_1, \dots, X_q$ on the open subset where these vector fields are defined.

Solution In order for $\mathscr{H}$ to be integrable, it must be $[X_u, X_v] \in \mathscr{H}$ for any $X_u, X_v \in \mathscr{H}$, $u, v = 1, \dots, q$. Then

$$\begin{aligned}[X_u, X_v] &= \left[\frac{\partial}{\partial x^{p+u}} - \sum_a f_u^a \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^{p+v}} - \sum_b f_v^b \frac{\partial}{\partial x^b}\right] \\ &= \sum_a \left(\frac{\partial f_u^a}{\partial x^{p+v}} - \frac{\partial f_v^a}{\partial x^{p+u}} + \sum_b \left(f_u^b \frac{\partial f_v^a}{\partial x^b} - f_v^b \frac{\partial f_u^a}{\partial x^b}\right)\right) \frac{\partial}{\partial x^a} \in \mathscr{D}.\end{aligned}$$

As $[X_u, X_v] \in \mathscr{H}$, the last expression in parentheses must be zero, that is, the condition is

$$\frac{\partial f_u^a}{\partial x^{p+v}} - \frac{\partial f_v^a}{\partial x^{p+u}} + \sum_b \left(f_u^b \frac{\partial f_v^a}{\partial x^b} - f_v^b \frac{\partial f_u^a}{\partial x^b}\right) = 0.$$

Problem 2.56 Let X be a vector field on a smooth manifold M, and let φ_t be its local one-parameter group (local flow) on M. Let $\mathscr{D} \subset TM$ be a smooth distribution.

Prove that the following conditions are equivalent:

(i) For any vector field Y lying in $\mathscr{D}$, the bracket $[X, Y]$ belongs to $\mathscr{D}$ (the distribution $\mathscr{D}$ is preserved by the vector field X).
(ii) For any vector field Y lying in $\mathscr{D}$, the local vector field $\varphi_t \cdot Y$ belongs to $\mathscr{D}$ (the distribution $\mathscr{D}$ is preserved by the local flow φ_t of X).

For a development of the relevant theory, see, for instance, Gawedzki [1].

Solution Let $p \in M$ and suppose that $\mathscr{D}$ is preserved by the vector field X. Let us choose a scalar product in T_pM. Let P_t be the orthogonal projection onto $\varphi_{t*}(\mathscr{D}_{\varphi_{-t}(p)}) \subset T_pM$. The operator function $P_t \colon T_pM \to T_pM$ smoothly depends on the parameter t. Let $Y \in \mathscr{D}$, and let

$$Y_t = (\varphi_t \cdot Y)_p, \quad \text{i.e. by definition} \quad Y_t = \varphi_{t*}(Y_{\phi_{-t}(p)}).$$

We have by Proposition 2.10

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = -\big(\varphi_t \cdot [X, Y]\big)_p \in P_t(T_pM),$$

because $[X, Y] \in \mathscr{D}$. Now $Y_t = P_t(Y_t)$ by the definition of P_t, and consequently,

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = \frac{\mathrm{d}(P_tY_t)}{\mathrm{d}t} = \frac{\mathrm{d}P_t}{\mathrm{d}t}Y_t + P_t\frac{\mathrm{d}Y_t}{\mathrm{d}t} = \frac{\mathrm{d}P_t}{\mathrm{d}t}Y_t + \frac{\mathrm{d}Y_t}{\mathrm{d}t}.$$

Thus,

$$\frac{\mathrm{d}P_t}{\mathrm{d}t}Y_t = 0.$$

Since varying Y, Y_t span the range of P_t, we get

$$\frac{\mathrm{d}P_t}{\mathrm{d}t}P_t = 0.$$

Let P_t^* denote the transpose operator of P_t (with respect to the scalar product in T_pM). From $P_t = P_t^*$ and $P_t^2 = P_t$ one obtains

$$\left(\frac{\mathrm{d}P_t}{\mathrm{d}t}\right)^* = \frac{\mathrm{d}P_t}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}P_t}{\mathrm{d}t}P_t + P_t\frac{\mathrm{d}P_t}{\mathrm{d}t} = \frac{\mathrm{d}P_t}{\mathrm{d}t}.$$

Hence,

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = P_t\frac{\mathrm{d}P_t}{\mathrm{d}t} = \left(\frac{\mathrm{d}P_t}{\mathrm{d}t}P_t\right)^* = 0.$$

Consequently, $P_t = P_0$, and φ_t preserves $\mathscr{D}$.

Clearly, from the definition of the Lie bracket (see also Proposition 2.10) we have that if $\varphi_t \cdot Y \in \mathscr{D}$, then $[X, Y] \in \mathscr{D}$.

Problem 2.57 Let $\pi \colon M \to M'$ be a surjective submersion of manifolds M and M'.

(i) Prove that $\ker \pi_* \subset TM$ (the set of vectors annihilated by π_*) is an involutive smooth distribution on M.

Let the set $\pi^{-1}(p')$ be connected for all $p' \in M'$, and let $\mathscr{D} \subset TM$ be a smooth distribution on M containing the distribution $\ker \pi_*$. Suppose that $\mathscr{D}$ is preserved by $\ker \pi_*$, i.e. $[Z, Y] \in \mathscr{D}$ for all vector fields $Z \in \ker \pi_*$ and $Y \in \mathscr{D}$.

Prove:

(ii) There exists a unique smooth distribution $\mathscr{D}'$ on M' such that $\mathscr{D}'_{\pi(p)} = \pi_* \mathscr{D}_p$ for all $p \in M$. Moreover, for any point $p \in M$, there exist a neighbourhood $U \subset M$ and vector fields $\{Y_l\}$ lying in $\mathscr{D}|_U$ such that the restriction $\mathscr{D}'|_{U'}$, where $U' = \pi(U)$, is spanned by vector fields $\{Y'_l\}$, and the vector fields Y_l, Y'_l are π-related for each l.

(iii) If the distribution $\mathscr{D}$ is involutive, then so is $\mathscr{D}'$.

Solution

(i) Since the map $\pi : M \to M'$ is a submersion, by the Theorem of the Rank 1.11, for any point $p \in M$, there exist a neighbourhood U of p, coordinates $x^1, \dots, x^n$ on U, $-1 < x^j < 1$, $j = 1, \dots, n$, and coordinates $x^1, \dots, x^{n'}$ ($n' \leqslant n$) on the open subset $U' = \pi(U) \subset M'$ such that the point p has coordinates $(0, \dots, 0)$ and the restriction $\pi|_U$ in these coordinates has the form

$$\pi : \left(x^1, x^2, \dots, x^n\right) \to \left(x^1, x^2, \dots, x^{n'}\right), \tag{$\star$}$$

i.e. in the neighbourhood U the restriction $\ker \pi_*|_U$ is spanned by the commuting vector fields $\partial/\partial x^{n'+1}, \dots, \partial/\partial x^n$. Therefore $\mathscr{D}$ is an involutive smooth distribution on M.

(ii) Let $p_1, p_2 \in \pi^{-1}(p') \subset M$ for some point $p' \in M'$. Since the set $\pi^{-1}(p')$ is connected, the points p_1, p_2 belong to the same leaf of the distribution $\ker \pi_*$. Then there exists a smooth vector field $Z \in \ker \pi_*$ such that for a corresponding (local) one-parameter group φ_t, we have $\varphi_{t_0}(p_1) = p_2$, $t_0 \in \mathbb{R}$ (we can use a partition of unity to construct such a field). But $\pi \circ \varphi_t = \pi$ for all t, and therefore it follows (see Problem 2.56) that

$$\pi_*(\mathscr{D}_{p_1}) = (\pi_* \circ \varphi_{t_0 *})(\mathscr{D}_{p_1}) = \pi_*(\mathscr{D}_{p_2}).$$

Hence the distribution $\mathscr{D}'$ is well defined. To prove the smoothness of $\mathscr{D}'$, choose a point $p \in M$ and neighbourhoods $U \subset M$, $U' \subset M'$, with the coordinates as above. Let Y be any local vector field belonging to $\mathscr{D}|_U$:

$$Y\left(x^1, \dots, x^n\right) = \sum_{j=1}^{n} a_j\left(x^1, \dots, x^n\right) \frac{\partial}{\partial x^j}.$$

The sub-bundle $\ker \pi_*$ is spanned on U by $\partial/\partial x^k$, $k = n' + 1, \dots, n$, and the distribution $\mathscr{D}|_U$ is preserved by these vector fields $\partial/\partial x^k$ and, consequently (see Problem 2.56), by the corresponding local flows

$$\varphi_t^k : \left(x^1, \dots, x^{k-1}, x^k, x^{k+1}, \dots, x^n\right) \mapsto \left(x^1, \dots, x^{k-1}, x^k + t, x^{k+1}, \dots, x^n\right).$$

Therefore the vector field

$$Y''\left(x^1, \dots, x^n\right) = \sum_{j=1}^{n} a_j\left(x^1, \dots, x^{n'}, 0, \dots, 0\right) \frac{\partial}{\partial x^j} = \sum_{j=1}^{n} b_j\left(x^1, \dots, x^{n'}\right) \frac{\partial}{\partial x^j}$$

(recall that $x^j(p)=0$, $j=1,\dots,n$) is a smooth vector field belonging to $\mathscr{D}|_U$. Thus,

$$Y'=\pi_* Y''\big(x^1,\dots,x^{n'}\big)=\sum_{j=1}^{n'} b_j\big(x^1,\dots,x^{n'}\big)\frac{\partial}{\partial x^j}$$

is a smooth vector field belonging to $\mathscr{D}'|_{U'}$, and the vector fields Y'', Y' are π-related. Thus there are vector fields $\{Y''_l\}$ and $\{Y'_l\}$ belonging to the restrictions $\mathscr{D}|_U$ and $\mathscr{D}'|_{U'}$, respectively, such that $\mathscr{D}'|_{U'}$ is spanned by the vector fields $\{Y'_l\}$ and the vector fields Y''_l, Y'_l are π-related for each l.

(iii) By Proposition 1.21, if the distribution $\mathscr{D}$ is involutive, then so is $\mathscr{D}'$.

Problem 2.58 Let $\pi: M\to M'$ be a surjective submersion of manifolds M and M'. Let the set $\pi^{-1}(p')$ be connected for all $p'\in M'$, and let $X\in\mathfrak{X}(M)$ be a smooth vector field which preserves the distribution $\ker\pi_*$.

Prove that there exists a unique smooth vector field X' on M' such that the vector fields X, X' are π-related.

Solution We will use the notation of the solution of the previous Problem 2.57. As above, consider the vector field Z (belonging to the distribution $\ker\pi_*$) with its local one-parametric group φ_t connecting points p_1, p_2 for which $\pi(p_1)=\pi(p_2)$. For the vector field X, we have (see Proposition 2.10)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t\cdot X)=-\varphi_t\cdot[Z,X].$$

Since the bracket $[Z,X]$ belongs to the distribution $\ker\pi_*$ and the local flow φ_t of $Z\in\ker\pi_*$ preserves the (involutive) distribution $\ker\pi_*$, the difference $\varphi_t\cdot X-X$ is a vector field belonging to $\ker\pi_*$ for all t. Thus $\pi_*X_{p_1}=\pi_*X_{p_2}$ and X', $X'_{\pi(p)}=\pi_*X_p$ is a well-defined vector field on the manifold M'. Therefore in the coordinate system $(U,x^1,\dots,x^n)$ around $p\in M$, the smooth vector field $X|_U$ has the following form (see the local expression ($\star$) for π in the solution of Problem 2.57):

$$\begin{aligned}(X|_U)\big(x^1,\dots,x^n\big)&=\sum_{j=1}^{n'} a_j\big(x^1,\dots,x^{n'}\big)\frac{\partial}{\partial x^j}\\&\quad+\sum_{j=n'+1}^{n} a_j\big(x^1,\dots,x^{n'},\dots,x^n\big)\frac{\partial}{\partial x^j}.\end{aligned}$$

Now it is clear that the vector field

$$X'\big|_{U'}=\sum_{j=1}^{n'} a_j\big(x^1,\dots,x^{n'}\big)\frac{\partial}{\partial x^j}$$

is also smooth. The vector fields X, X' are π-related.

References

1. Gawedzki, K.: Fourier-Like Kernels in Geometric Quantization. Dissertationes Math., vol. 1284 (1976), 83 pp.
2. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vols. I, II. Wiley Classics Library. Wiley, New York (1996)
3. Lawson, H.B.: Foliations. Bull. Am. Math. Soc. **80**(3), 369–418 (1974)
4. Reeb, G.: sur Certaines Propriétés Topologiques des Variétés Feuillétées. Actualités Sci. Indust., vol. 1183. Hermann, Paris (1952)

Further Reading

5. Bishop, R.L., Crittenden, R.J.: Geometry of Manifolds. AMS Chelsea Publishing, Providence (2001)
6. Boothby, W.M.: An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd revised edn. Academic Press, New York (2002)
7. Brickell, F., Clark, R.S.: Differentiable Manifolds. Van Nostrand Reinhold, London (1970)
8. Gallot, S., Hulin, D., Lafontaine, J.: Riemannian Geometry. Springer, Berlin (2004)
9. Godbillon, C.: Géométrie Différentielle et Mécanique Analytique. Hermann, Paris (1969)
10. Hicks, N.J.: Notes on Differential Geometry. Van Nostrand Reinhold, London (1965)
11. Lee, J.M.: Manifolds and Differential Geometry. Graduate Studies in Mathematics. Am. Math. Soc., Providence (2009)
12. Lee, J.M.: Introduction to Smooth Manifolds. Graduate Texts in Mathematics, vol. 218. Springer, New York (2012)
13. Lichnerowicz, A.: Global Theory of Connections and Holonomy Groups. Noordhoff, Leyden (1976)
14. Lichnerowicz, A.: Geometry of Groups of Transformations. Noordhoff, Leyden (1977)
15. O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)
16. Spivak, M.: Differential Geometry, vols. 1–5, 3nd edn. Publish or Perish, Wilmington (1999)
17. Sternberg, S.: Lectures on Differential Geometry, 2nd edn. AMS Chelsea Publishing, Providence (1999)
18. Tu, L.W.: An Introduction to Manifolds. Universitext. Springer, Berlin (2008)
19. Warner, F.W.: Foundations of Differentiable Manifolds and Lie Groups. Graduate Texts in Mathematics. Springer, Berlin (2010)

http://www.springer.com/978-94-007-5951-0

Analysis and Algebra on Differentiable Manifolds
A Workbook for Students and Teachers
Gadea, P.M.; Muñoz Masqué, J.; Mykytyuk, I.V.
2013, XXVI, 618 p., Hardcover
ISBN: 978-94-007-5951-0