



Truss structures constitute a special class of structures in which individual straight members are connected at joints. The members are assumed to be connected to the joints in a manner that permit rotation, and thereby it follows from equilibrium considerations, to be detailed in the following, that the individual structural members act as bars, i.e. structural members that can only carry an axial force in either tension or compression. Often the joints do not really permit free rotation, and the assumption of a truss structure then is an approximation. Even if this is the case the layout of a truss structure implies that it can carry its loads under the assumption that the individual members act as bars supporting only an axial force. This greatly simplifies the analysis of the forces in the structure by hand calculation and undoubtedly contributed to their popularity e.g. for bridges, towers, pavilions etc. up to the middle of the twentieth century. The layout of the structural members in the form of a truss structure also finds use with rigid or semi-rigid joints, e.g. space truss roofs, girders for suspension bridges, or steel offshore structures. The rigid joints introduce bending effects in the structural members, but these effects are easily included by use of numerically based computational methods.

In a statically determinate truss all the bar forces can be determined by the equilibrium equations, applied to the bars and joints of the truss. There are several strategies for carrying out the corresponding calculations, and three of these will be described in this chapter. The first and conceptually simplest method consists in considering each joint as an isolated body, for which the equilibrium equations must be satisfied. As there are no moment equations due to the hinge property of the joint this gives two equilibrium equations for a joint in a planar truss and three equilibrium equations for a joint in a space truss. The calculation of the bar forces proceeds by considering the individual joints sequentially as explained in Section 2.2. Alternatively, the bar forces can be calculated by using sections to separate larger parts of the structure and then applying suitable equilibrium equations for these larger parts. This method is dealt with in Section 2.3.

It is characteristic of the classic methods of joints and of sections, that they are arranged to determine the bar forces sequentially, and thus are convenient for calculation of the bar forces or a subset of these by hand. However, in their basic form these methods are limited to statically determinate trusses, and even for this class of structures the calculations may become quite elaborate for space trusses and larger planar trusses. Alternatively, a general systematic method can be developed for elastic trusses, irrespective of whether they are statically determinate or indeterminate. The method consists of setting up the equilibrium equations of all joints in a systematic way, using the elastic property of the bars. This method is a special case of the Finite Element Method, used in its general form for a wide range of problems within structural mechanics, see e.g. Cook et al. (2002) and Zienkiewicz and Taylor (2000). The formulation of the Finite Element Method takes on a particularly systematic form, when using the principle of virtual work, already mentioned for rigid bodies in Section 1.6. The extension of the principle of virtual work to truss structures is described in Section 2.4.3, and is used to formulate the Finite Element Method for elastic truss structures in Section 2.5.

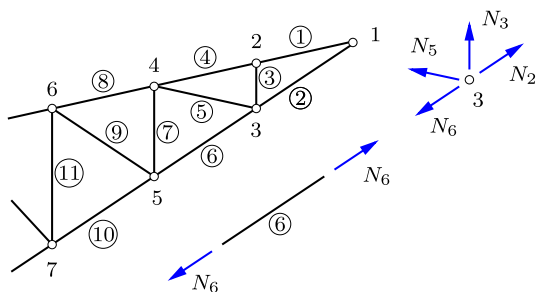


Fig. 2.1: Plane truss with joints and bars.

2.1 Basic principles

A truss structure consists of a number of joints, connected by bars. This is illustrated in Fig. 2.1 showing a truss structure consisting of the joints numbered as $1, 2, \dots, 7$, connected by bars indicated by numbers in a circle. The joints are assumed to act like hinges, permitting free rotation of the bars around the joint. It is furthermore assumed that the truss structure is only loaded by concentrated forces acting at the joints. As a consequence of the assumption of hinges the bar elements can only support an axial force. This is easily demonstrated by considering Fig. 2.2. Due to the hinge there can be no moment at the ends of the bar element. Furthermore, if the force N in the bar were not aligned along the direction of the bar, there would be a non-zero moment about the hinge at the other end of the bar. Thus, the bar can only support a force of magnitude N , aligned along the direction of the bar. By convention the force in the bars of a truss structure are defined as positive when corresponding to tension, and they are then negative when representing compression.

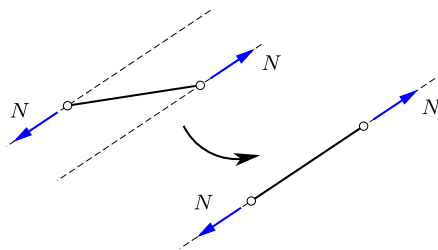


Fig. 2.2: Force in a bar along the bar axis.

A basic principle in the analysis of structures is the section. A section is used to represent a hypothetical separation of a part of the structure from the rest. This hypothetical separation enables a concise discussion of the forces exchanged between the parts on the two sides of the section. The situation is illustrated in simple form in Fig. 2.1. The figure shows the hypothetical situation in which the bar number 6 is separated from the structure by a section right next to the joints at the ends of the bar. The bar will be acted on by a force along the bar axis, and it follows from equilibrium, that the force at the sections at the ends of the bar must be of equal magnitude but opposite direction. The magnitude is denoted N_6 and is shown in the figure as positive corresponding to tension.

The figure also shows the effect of separating the joint 3 from the structure. In order to maintain the same state as in the structure the joint is acted upon by forces from each of the connected bars. The bar forces are defined as positive in tension, and positive bar forces therefore appear as forces N_2, N_3 ,

N_5 and N_6 pointing away from the joint. It is important to note that when the bar 6 has a tension force N_6 , the bar is acted on by a force of magnitude N_6 pointing away from the bar towards the connecting joint. The connecting joints will similarly be acted on by a force of magnitude N_6 pointing away from the node. By this sign convention positive forces will point away from the member – joint or bar – on which they act. When making a sketch of a structural part, i.e. a joint or a bar, the forces will always be shown corresponding to their positive direction, i.e. as tension forces. If a bar force is determined to be compressive, this corresponds to a negative value of the magnitude N , and the sign of the arrow in the figure will be retained in the direction corresponding to tension.

2.1.1 Building with triangles

The triangle plays an important role in the geometric layout of truss structures. The reason for this is illustrated by the three planar trusses shown in Fig. 2.3. To be specific they can be envisaged to have a simple fixed support at the left end, and a simple support permitting horizontal motion at the right end. At first glance they may look as ‘plausible’ candidates for a truss, but are they satisfactory structures?

The truss shown in Fig. 2.3a consists of two triangles, connected by a quadrilateral at the center. The original structure is shown in full line, while the dotted line shows a possible deformation mechanism, by which the central quadrilateral changes shape without need for changing the length of any of the bars in the structure. Thus, even for perfectly rigid bar members the structure has a deformation mechanism. The existence of one or more free deformation mechanisms within a structure is termed kinematic indeterminacy. In the present case the implication is that the structure can not be used with the prescribed support conditions, but will need an extra support preventing the mechanism.

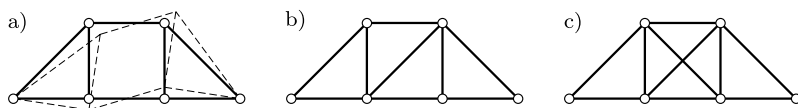


Fig. 2.3: From kinematic to static indeterminacy.

The mechanism can be locked by introducing a diagonal bar in the center quadrilateral as shown in Fig. 2.3b. It is seen that this prevents the free deformation mode, and also leads to a structure formed by triangles. The structure thereby becomes kinematically determinate. It is demonstrated below that this structure with supports providing three reaction components

permits determination of all bar forces by use of the equilibrium conditions only. This property is termed static determinacy.

The equilibrium conditions imply that the force at the two ends of each bar must be of identical magnitude but opposite direction. The remaining equilibrium conditions then express force equilibrium at each joint, illustrated e.g. as equilibrium of the four forces acting on the joint 3 of the truss in Fig. 2.1. It follows from this principle that introduction of an extra bar in a truss, as shown in Fig. 2.3c, will introduce a new undetermined force in this bar. However, for a statically determinate truss the equilibrium equations are precisely sufficient to determine the forces in all bars, and consequently the introduction of an extra bar will leave the number of equilibrium conditions one short. A truss structure, in which the number of equilibrium equations is insufficient to determine all bar forces, is termed statically indeterminate. In contrast to structures with deformation mechanisms, that are generally unsuitable, statical indeterminacy does not constitute a limitation of the potential usefulness of the structure. It just implies that the specific distribution of the forces between the bars, or some of the bars, depends on the deformation properties of these bars. In the present example the two crossing diagonals in Fig. 2.3c share in preventing the deformation mechanism of the quadrilateral of the original structure. However, due to the static indeterminacy the precise ratio in which they share depends on their relative stiffness. Thus, the analysis of statically indeterminate structures require additional information about the stiffness of the structural members. In this chapter hand calculation type methods are developed for statically determinate trusses, while statically indeterminate trusses are left as part of the Finite Element formulation developed in Section 2.5.

2.1.2 Counting joints and bars

Some typical planar trusses are shown in Figure 2.4. It is seen that they are formed by triangles, and this suggests that they are statically determinate, when supported appropriately by three independent reaction components. It is now demonstrated by a common method for planar trusses, that they are indeed statically determinate. The method leads to a necessary relation between the number of joints and the number of bars. However, and probably equally important, it identifies a rational way of thinking about a truss structure, in which a process is constructed by which the structure is extended joint by joint, simulating an actual construction of the truss from bar elements connected by joints.

For a planar truss the hypothetical construction process starts from a simple triangle, and in order to be specific this triangle is supported by a fixed and a movable support as shown in Fig. 2.5a. Equilibrium of the nodes can be established by two projection equations for the unsupported node, and

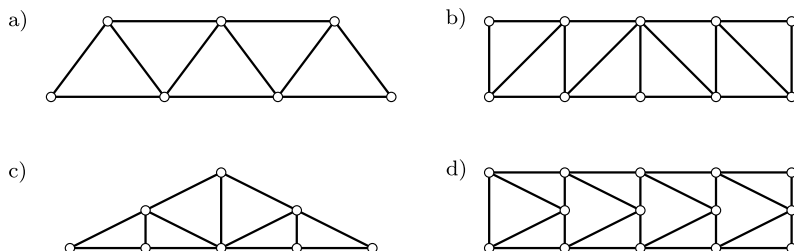


Fig. 2.4: a) V-truss, b) N-truss, c) Roof truss, d) K-truss.

a vertical projection equation of the forces on the node with the moveable support. This gives three equations, corresponding to the three bar forces to be determined. Thus, the initial triangle is statically determinate.

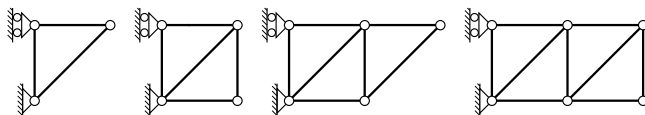


Fig. 2.5: Construction of plane truss girders by triangles.

The process is continued by attaching a new joint by two new bars as illustrated in Fig. 2.5. If the bars are not parallel, they will uniquely determine the position of the new joint, and two projection equations for the forces on the new joint will determine the bar forces. This step, in which a new joint is added and fastened by two new bars, can be continued as illustrated in the figure. The process defines a simple relation between the number of bars b and the number of joints j in a statically determinate planar truss:

$$b = 2j - 3. \quad (2.1)$$

This relation is easily verified by observing that it is correct for the original triangle with $b = 3$ and $j = 3$, and that inclusion of one new joint leads to two additional bars. The relation between the number of bars and the number of joints is necessary, but clearly not sufficient. This becomes obvious e.g. by considering removing one of the diagonals and rejoining it as a diagonal crossing the remaining diagonal. Hereby part of the truss becomes kinematically indeterminate, implying a mechanism, while the additional bar in the remaining structure makes this part statically indeterminate. Thus, the process, in which a gradual construction of the truss by statically determinate steps is imagined, is probably more valuable than the formula, if left alone.

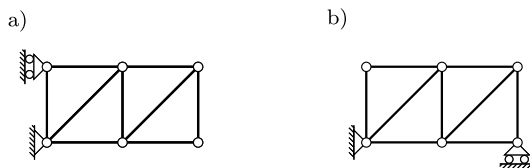


Fig. 2.6: Change of supports after 'constructing' the truss.

At first sight it may appear that the process is dependent on the supports being applied to the initial triangle. However, this is not the case. The result is independent of the specific support conditions as long as they provide three independent reaction components. After completing the truss structure, the supports can be moved as illustrated in Fig. 2.6.

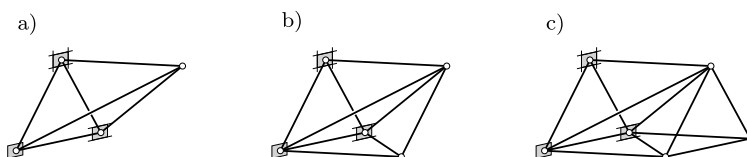


Fig. 2.7: Construction of space truss by addition of tetrahedra.

The results for planar trusses are easily extended to space trusses as illustrated in Fig. 2.7. The starting point is a tetrahedron (pyramid), formed by 4 joints and 6 connecting bars. The tetrahedron is supported by 6 independent reaction components. This leaves $4 \cdot 3 - 6 = 6$ equilibrium conditions from the 4 nodes for determination of the 6 bar forces. The process is continued in steps consisting in the addition of 1 new joint connected by 3 new bars. The three bars keep the joint fixed in space, and the three force projection equations associated with equilibrium of the new joint determine the three new bar forces. The figure shows the two first steps in this process leading to a truss girder of a type typically used for building cranes. This leads to the following relation between the number of bars b and the number of joints j of a statically determinate space truss:

$$b = 3j - 6. \quad (2.2)$$

Also in this case the relation is necessary but not sufficient, and the imaginary process of constructing the space truss constitutes an important part.

2.1.3 Qualitative tension-compression considerations

It is often possible to identify whether a bar member in a truss is loaded in tension or compression by a simple qualitative argument involving an esti-

mate of the actual deformation of the loaded truss, or by constructing the mechanism that would result if the member were removed from the truss. Figure 2.8a shows a simply supported N-truss girder consisting of the ‘head’, the ‘foot’, the ‘verticals’ and the ‘diagonals’. It is observed, that with $j = 10$ joints and $b = 17$ bars the truss satisfies the condition (2.1) for a statically determinate truss. Figure 2.8b shows a sketch of the deformed girder after loading by distributed downward forces. It is clearly seen that the bars in the foot are extended, indicating tension, while the bars in the head become shorter, indicating compression. However, it is more difficult to identify elongation or shortening of the verticals and the diagonals.

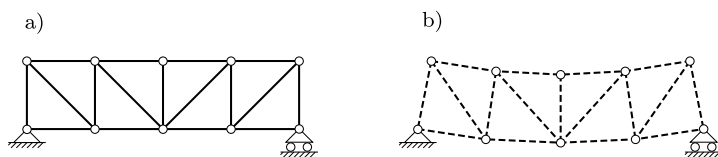


Fig. 2.8: Tension and compression members in truss girder.

A somewhat different and more precise way of estimating whether a bar is in tension or compression consists in imagining that the bar were removed from the truss. For a statically determinate structure this would create a mechanism. Figure 2.9a illustrates the mechanism generated by removing the second bar in the head, while Fig. 2.9b illustrates the mechanism associated with removal of the third bar in the foot. The mechanisms are shown corresponding to a downward load. It is clearly seen that the distance between the two joints constituting the end points of the removed bar approach each other in the case of the bar in the head, while they become further separated in the case of the bar in the foot. Thus, the bar in the head will experience compression, while the bar in the foot will experience tension, when the structure carries a downward load.

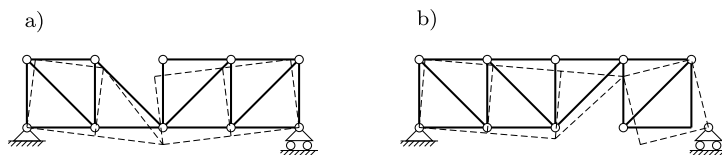


Fig. 2.9: Mechanisms by removing a bar in the head or in the foot.

A similar geometric argument can be used to identify the sign of the force in the diagonals and the verticals. Figure 2.10a illustrates the deformation mechanism generated when removing the second diagonal from the left. It is seen that diagonal would be extended by the illustrated mechanism. Thus there will be tension in the diagonal when the loads perform positive work

through the mechanism. this would be the case for a downward load at the central or right nodes of the head or the foot. However, a downward load in the first set of node to the right of the left support would create negative work and therefore contribute a compressive force. Figure 2.10b illustrates the mechanism generated by removing the second vertical from the left. A vertical downward load at any of the three inner nodes of the head would lead to compression in this vertical.

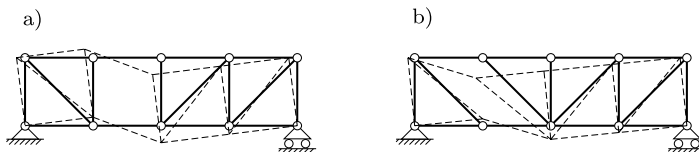


Fig. 2.10: Mechanisms by removing a diagonal or a vertical bar.

The qualitative arguments used to explain the implication of the mechanisms generated by removing a single bar from the truss can be made precise if the geometry of the infinitesimal motion of the mechanism is described exactly and used within the context of virtual work, described in Section 2.4.

2.2 Method of joints

The magnitude of the forces in the bars of a statically determinate truss structure can be determined by the method of joints. The idea of the method of joints is to consider each joint as separated from the rest of the truss structure by the introduction of a virtual section. The parts on the two sides of the section will exchange identical but opposite forces, and by introducing the section and identifying these forces explicitly, they can be analyzed by the equilibrium equations. The principle is illustrated in its simplest form in Fig. 2.11. The left part of the figure shows a joint C in a planar truss loaded by the vertical force P and connected to the rest of the truss by the two bars AC and BC . A section is now introduced, separating the joint from the rest of the structure. The forces N_{AC} and N_{BC} , by which the bars act on the joint, are indicated as acting on the joint together with the load P . Thus, the joint C is acted on by three forces. The forces in the bars are considered as positive, when representing tension in the bar. Thus, the effect on the joint is a force directed away from the joint. By the law of action and reaction equal but opposite forces act on the bars. As seen, these forces represent tension in the bars. It is noted that the forces N_{AC} and N_{BC} are uniquely defined as being positive in tension. A representation in terms of vectors is less direct, as it would require identification of the part on which the force acts.

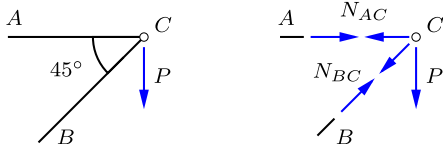


Fig. 2.11: Node C with load P and bar forces N_{AC} and N_{BC} .

Equilibrium of the joint C requires that two force projection equations are satisfied. Vertical projection gives

$$\downarrow \quad N_{BC} \sin 45^\circ + P = 0 \quad \Rightarrow \quad N_{BC} = \frac{-P}{\sin 45^\circ} = -\sqrt{2}P.$$

By taking a vertical projection, the force N_{AC} in the horizontal bar AC does not contribute to the equilibrium equation.

The remaining bar force N_{AC} can be determined by projection on the direction orthogonal to BC . The present case is simple due to the angle 45° , and gives $N_{AC} = P$ directly. In many cases it will be more convenient to use a horizontal projection, whereby

$$\leftarrow \quad N_{AC} + N_{BC} \cos 45^\circ = 0 \quad \Rightarrow \quad N_{AC} = -N_{BC} \cos 45^\circ = P.$$

Thus, there is compression in the inclined bar BC , while the horizontal bar AC is in tension to ensure horizontal equilibrium.

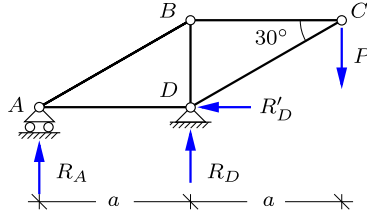
In this simple illustration there were only two bar forces, and thus they could be determined directly by the two equilibrium equations available for the planar joint C . Most joints in truss structures are connected by more bars than there are equilibrium equations available for the particular joint. The bar forces can therefore only be determined sequentially, if the joints are considered in a certain order. This is illustrated in the following examples.

2.2.1 Planar truss structures

Many truss structures can conceptually be broken down into planar parts, and this section illustrates the calculation of bar forces for some simple planar trusses.

Example 2.1. Double triangle. Figure 2.12 shows a planar truss consisting of two triangles. There are 4 joints, providing $4 \times 2 = 8$ equilibrium equations, that determine the three reaction components R_A , R_D and R'_D , plus the forces in the five bars in the truss.

In principle the analysis could be carried out completely on a joint by joint basis, starting from C and then proceeding through B , A and D . At each node there would be two

Fig. 2.12: Double triangle truss with load P .

unknown forces, and at the end all bar forces and reactions will be determined. However, for many truss structures it is not possible to start from a node with only two unknown forces, unless appropriate reaction components are determined first. Therefore, the analysis of a statically determinate truss usually starts with determination of the reactions, using equilibrium of the full truss or parts of the truss as discussed in Section 1.5 on reactions. In the present case the reaction components R'_D , R_D and R_A are determined by horizontal projection and moment about A and D , respectively:

$$R'_D = 0, \quad R_D = 2P, \quad R_A = -P.$$

Note the negative reaction in A , indicating downward direction of the force.

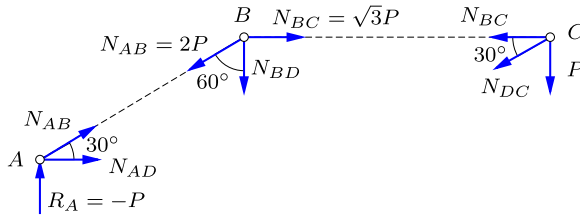
Fig. 2.13: Joints A , B and C of double triangle truss.

Figure 2.13 shows all the nodes with the forces indicated in their positive direction as tension. If a compression force is found in the analysis, this shows up as a negative force. The bar forces are determined from equilibrium of the forces acting on the individual nodes. Thus, vertical and horizontal projection of the forces acting on joint C lead to

$$\begin{aligned} \downarrow \quad N_{DC} \cos 60^\circ + P &= 0 & \Rightarrow & N_{DC} = -2P. \\ \leftarrow \quad N_{BC} + N_{DC} \cos 30^\circ &= 0 & \Rightarrow & N_{BC} = \sqrt{3}P. \end{aligned}$$

At joint A vertical projection gives

$$\uparrow \quad N_{AB} \cos 60^\circ + R_A = 0 \quad \Rightarrow \quad N_{AB} = 2P,$$

while horizontal projection then leads to

$$\rightarrow \quad N_{AD} + N_{AB} \cos 30^\circ = 0 \quad \Rightarrow \quad N_{AD} = -\sqrt{3}P.$$

Finally, vertical projection of the forces acting on joint B gives

$$\downarrow \quad N_{AB} \cos 60^\circ + N_{BD} = 0 \quad \Rightarrow \quad N_{BD} = -P.$$

This completes the calculation of the bar forces. There are still an unused projection equation at joint B and two projection equations for joint D . This corresponds to the three reaction components that were determined initially from total equilibrium. \square

Example 2.2. V-truss by the method of joints. Figure 2.14 shows a simply supported V-truss, loaded by a concentrated force at B . The support conditions permit the vertical reaction components R_A and R_C plus the horizontal reaction R'_A . They are determined from the equilibrium conditions for the total truss structures as discussed in detail in Chapter 1. The reactions follow from horizontal projection, moment about C and moment about A as

$$R'_A = 0, \quad R_A = \frac{1}{2}P, \quad R_C = \frac{1}{2}P.$$

Control by vertical projection gives $R_A + R_C = P$, corresponding to the load P .

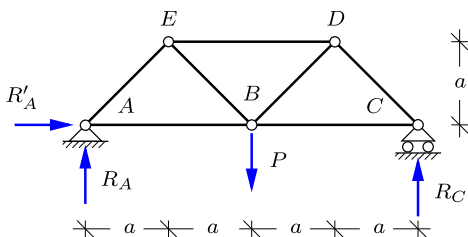


Fig. 2.14: Simply supported V-truss with load P .

The loading is symmetric, and because the horizontal reaction component R'_A vanishes, so are the reactions. Thus, the structure and its bar forces are symmetric with respect to a vertical line through B , and only the right half with the nodes B , C and D need to be considered to determine all the bar forces. These nodes and the corresponding forces from loads, reactions and bars are shown in Fig. 2.15.

It is seen from the figure that joint C only contains two unknown forces. Once they are determined, the joint D only contains two unknown forces. These forces, and their symmetric counterparts, determine equilibrium at joint B which can be used to check the previous calculations.

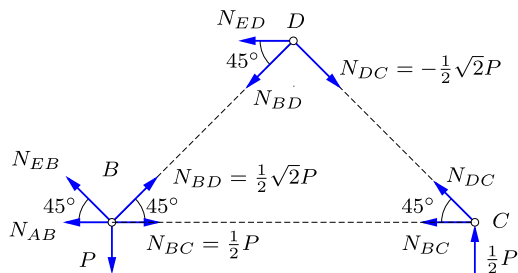


Fig. 2.15: Joints B , C and D of V-truss.

Equilibrium of joint C determines the forces N_{DC} and N_{BC} by vertical and horizontal projection, respectively:

$$\uparrow \quad N_{DC} \sin 45^\circ + \frac{1}{2}P = 0 \quad \Rightarrow \quad N_{DC} = -\frac{1}{2}\sqrt{2}P.$$

With N_{DC} known, horizontal projection then gives

$$\leftarrow \quad N_{BC} + N_{DC} \cos 45^\circ = 0 \quad \Rightarrow \quad N_{BC} = \frac{1}{2}P.$$

At joint D vertical projection gives

$$\downarrow \quad N_{BD} \sin 45^\circ + N_{DC} \sin 45^\circ = 0 \quad \Rightarrow \quad N_{BD} = \frac{1}{2}\sqrt{2}P.$$

The remaining bar force N_{ED} then follows from horizontal projection:

$$\leftarrow \quad N_{ED} + N_{BD} \cos 45^\circ - N_{DC} \cos 45^\circ = 0 \quad \Rightarrow \quad N_{ED} = -P.$$

This completes the calculation of bar forces, because the remaining bar forces now follow from their symmetric counterparts, e.g. $N_{EB} = N_{BD}$ and $N_{AB} = N_{BC}$. Using these forces from the left side of the structure, the equilibrium conditions for joint B may be used as control of the calculations, with vertical projection

$$\uparrow \quad N_{EB} \sin 45^\circ + N_{BD} \sin 45^\circ - P = 0$$

and horizontal projection

$$\leftarrow \quad N_{AB} - N_{BC} + (N_{EB} - N_{BD}) \cos 45^\circ = 0,$$

demonstrating equilibrium. □

Example 2.3. Roof truss by method of joints. For some loads one or more of the bars in a truss structure may have zero force, thus being essentially inactive in this load case. The identification of these bars is illustrated in this example by considering the roof truss in Fig. 2.16.

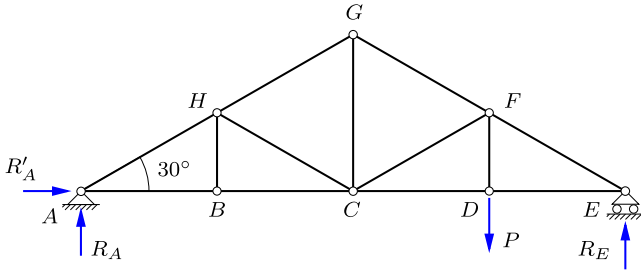


Fig. 2.16: Roof truss with load P at truss foot.

Bars with zero force may occur, when two bars at an unloaded joint are aligned. A transverse projection of the forces at this node will then not contain these forces. Thus, if there is only a single transverse bar, the force in this bar must vanish if there is no transverse load. This applies to the joint B as shown in Fig. 2.17a. The two bars AB and BC are horizontal, and thus only the force N_{BH} in the vertical bar BH contributes to vertical equilibrium. When there is no load at the node, this implies that $N_{BH} = 0$. In plain words, the argument is that when two bars share a common joint and follow the same direction, then they can not support a transverse force.

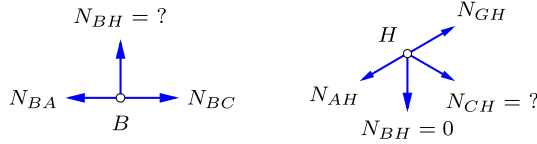
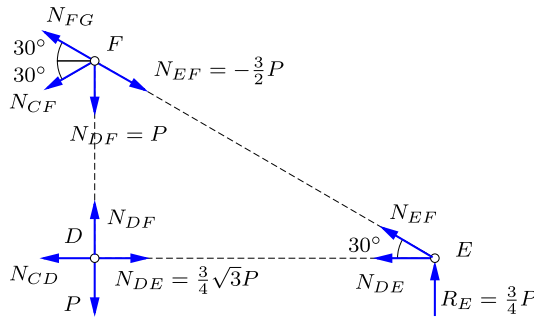


Fig. 2.17: Identification of zero force bars in roof truss.

This argument can be continued by now considering the joint H shown in Fig. 2.17b. There are two bar forces N_{BH} and N_{HC} that potentially could contribute to the transverse equilibrium of the joint H . However, as $N_{BH} = 0$ according to the previous calculation, the remaining transverse force must also vanish, whereby $N_{CH} = 0$. It should be noted that the argument does not depend on the angle between the two aligned bars and the bar providing the transverse component, as the latter is the only bar contributing to the transverse equilibrium. The presence of zero force bars simplify the computation of the remaining bar forces as illustrated below.

Fig. 2.18: Joints D , E and F of roof truss.

The computation of the non-zero bar forces conveniently starts with determination of the reactions from equilibrium of the complete truss structure as in the previous two examples. Horizontal projection and moments about E and A give

$$R'_A = 0, \quad R_A = \frac{1}{4}P, \quad R_E = \frac{3}{4}P.$$

The bar forces in the right half of the truss can then be calculated by considering equilibrium of the joints E , D and F , as illustrated in Fig. 2.18. For the joint E vertical equilibrium gives

$$\uparrow \quad N_{EF} \cos 60^\circ + R_E = 0 \quad \Rightarrow \quad N_{EF} = -\frac{3}{2}P,$$

while horizontal equilibrium gives

$$\leftarrow \quad N_{DE} + N_{EF} \cos 30^\circ = 0 \quad \Rightarrow \quad N_{DE} = \frac{3}{4}\sqrt{3}P.$$

It is now advantageous to move on to joint D due to the particularly simple form of the vertical and horizontal equilibrium conditions, each consisting of two opposing forces of equal magnitude:

$$N_{DF} = P, \quad N_{CD} = N_{DE} = \frac{3}{4}\sqrt{3}P.$$

At node F projection on the direction orthogonal to the truss head gives

$$N_{CF} \cos 30^\circ + N_{DF} \cos 30^\circ = 0 \quad \Rightarrow \quad N_{CF} = -N_{DF} = -P,$$

while N_{FG} follows from horizontal projection with the common angle 30° :

$$\begin{aligned} \leftarrow \quad N_{FG} \cos 30^\circ + N_{CF} \cos 30^\circ - N_{EF} \cos 30^\circ &= 0 \\ \Rightarrow \quad N_{FG} &= N_{EF} - N_{CF} = -\frac{1}{2}P. \end{aligned}$$

The remaining bar forces are left as Exercise 2.10. □

2.2.2 Space trusses

In the case of space trusses hand calculation methods typically make use of special features of the truss geometry, and rapidly become fairly impractical for larger structures. Most space structures are therefore analyzed by the numerical Finite Element Method described in Section 2.5. A glimpse of the hand calculation procedure is provided by the following example.

Example 2.4. Simple space truss. Figure 2.19 shows a simple cantilever space truss carrying a tip load P . The dimensions of the structure are given in terms of a , b and h as shown in the figure. The length of the inclined bars then is

$$\ell = \sqrt{a^2 + b^2 + h^2}.$$

The bar forces are determined by the method of joints, placing a section around the tip D . The truss and the load are symmetric about a vertical plane through AD , and thus the internal forces in the inclined bars are equal, $N_{BD} = N_{CD}$. Vertical projection for node D then gives

$$\downarrow \quad \frac{h}{\ell} N_{BD} + \frac{h}{\ell} N_{CD} + P = 0 \quad \Rightarrow \quad N_{BD} = N_{CD} = -\frac{\ell}{2h} P.$$

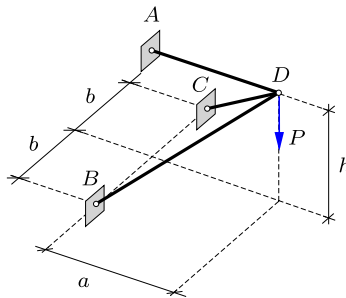


Fig. 2.19: Simple space truss carrying vertical tip load P .

The remaining bar force N_{AD} is obtained by horizontal projection, in the direction of AD ,

$$\leftarrow \quad 2 N_{BD} \frac{a}{\ell} + N_{AD} = 0 \quad \Rightarrow \quad N_{AD} = \frac{a}{h} P.$$

Note, that all bar forces increase for smaller depth h of the truss. Also observe that the force N_{AD} is independent of the width $2b$ of the truss, and therefore is the same if the inclined bars were collapsed in the vertical plane containing the horizontal bar AD . \square

2.3 Method of sections

The idea of introducing a section, whereby a structure is separated into two parts dates several hundred years back. The method of joints is a special case, in which the section is introduced in such a way that it separates precisely one joint. Hereby, all forces identified via the section pass through the released joint, and therefore are governed by force equilibrium of the joint. The idea of a section to identify the interaction of two parts of a structure is much more general and plays a central role in the theory of structures including beams and frames, but also in the general theory of a continuous bodies as discussed in Chapter 8.

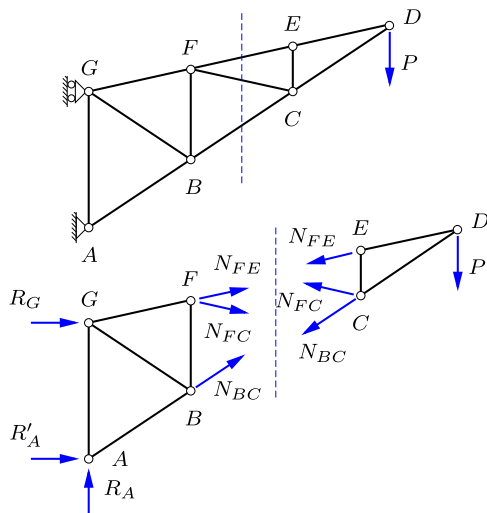


Fig. 2.20: Truss divided by vertical section.

A first step in the generalization of the idea of the section is to introduce a section that divides the truss structure into two parts. This is illustrated for the case of a planar truss in Fig. 2.20a. A section is introduced that intersects the bars BC , FC and FE and divides the structure into two parts shown in Fig. 2.20b. The section identifies three bar forces, acting on each part of the structure with different direction. Thus, three new forces N_{BC} , N_{FC} and N_{FE} have appeared. In the original structure three equilibrium equations were available for determining the reactions as discussed in Chapter 1. After separating the original structure into two parts, each part must satisfy three

equilibrium equations, and thus three new equations are now available for calculation of the bar forces N_{BC} , N_{FC} and N_{FE} . In the truss illustrated in Fig. 2.20 the three bar forces may be obtained from equilibrium of the right part of the truss, and the three reactions may then be determined from equilibrium of the left part.

2.3.1 Bar forces via the method of sections

The use of the method of sections to determine the bar forces in a planar truss is first illustrated by a simple example, and then summarized in concise form.

Example 2.5. V-truss by method of sections. Figure 2.21 shows a planar V-truss that has already been analyzed by the method of joints in Example 2.2. It is here analyzed by the method of sections to demonstrate the principles involved. First the reactions are determined by equilibrium of the full truss:

$$R'_A = 0, \quad R_A = \frac{1}{2}P, \quad R_C = \frac{1}{2}P.$$

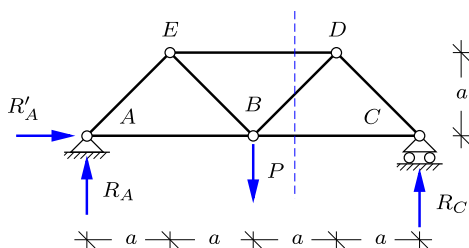


Fig. 2.21: Simply supported V-truss with load P .

In the method of joints the analysis would start from a node with two unknown bar forces – in the present case either of the joints A and C. This can also be used in the method of sections by introducing a vertical section, isolating the supported node. However, the method of sections also permits direct determination of the forces in the central bars. To illustrate the general procedure in the method of sections, a vertical section is introduced just to the right of the joint B as shown in the figure. This section intersects the bars BC , BD and DE and is used for calculating the corresponding bar forces N_{BC} , N_{BD} and N_{DE} .

Equilibrium of either the left or the right part of the structure is now used to determine the three bar forces N_{BC} , N_{BD} and N_{DE} . Note, that in contrast to the case of a single node the two parts have finite extent, and equilibrium therefore involves three equilibrium equations, and not just the two force projection equations associated with a single node. In the present problem equilibrium of the right part is the simpler, because it involves only one reaction component and not the load. The right part is shown in Fig. 2.22 together with all the forces acting on this part.

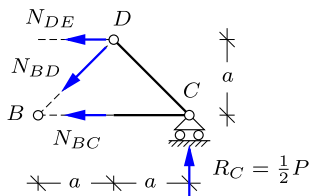


Fig. 2.22: Equilibrium in section.

The calculation of the bar forces proceeds in a systematic fashion by the following steps. First it is observed that two of the bar forces to be determined, namely N_{DE} and N_{BC} , are parallel. The remaining force N_{BD} in the inclined diagonal can then be determined by use of vertical equilibrium:

$$\downarrow \quad N_{BD} \cos 45^\circ - R_C = 0 \quad \Rightarrow \quad N_{BD} = \sqrt{2}R_C = \frac{1}{2}\sqrt{2}P.$$

This force intersects the two still unknown bar forces N_{BC} and N_{DE} in B and D , respectively. Thus, a moment equation about any of these two points will involve only one unknown bar force. The bar force N_{BC} is determined by moment about D :

$$\widehat{D} \quad a N_{BC} - a R_C = 0 \quad \Rightarrow \quad N_{BC} = R_C = \frac{1}{2}P.$$

Finally, the bar force N_{DE} is determined by moment about B :

$$\widehat{B} \quad a N_{DE} + 2a R_C = 0 \quad \Rightarrow \quad N_{DE} = -2R_C = -P.$$

It is seen that each of the three forces N_{BD} , N_{BC} and N_{DE} has been calculated from an equilibrium equation, that does not involve any of the other two forces. \square

The method of sections for planar trusses can be formalized by the following procedure.

- i) Determine the reactions on the truss structure.
- ii) Divide the truss structure into two parts by a section, intersecting two or three bars.
- iii) Consider each of the bar forces in turn and determine the bar force by: moment about the point of intersection of the other two forces, or projection on the transverse direction, if they are parallel.

It follows from the independence of the calculation of each bar force associated with a given section, that the order of the calculations can be changed, and indeed any of the bar forces can be calculated without calculating the others. As a consequence the method of sections can often be used to calculate isolated bar forces of a truss structure, without need for calculating the forces in adjoining bars. In addition to its computational simplicity, the method of sections often provides direct insight into the systematic variation of the forces in e.g. diagonals or verticals of regular truss structures.

2.3.2 Special types of planar trusses

Planar trusses appear in many contexts and often in the form of truss girders e.g. in bridges and cranes. Typical examples were illustrated in Fig. 2.4. The following examples illustrate the analysis for four types of trusses – three typical truss girders and a roof truss. The truss girder examples illustrate the method of analysis by introducing a typical section and calculating the bar forces associated with that section. The full analysis requires a sequence of similar sections, and omitting the repetitions associated with the typical section, the full results are summarized to illustrate how the girder layout determines the distribution of bar forces in the girder. The roof girder is more an individual type, where modification of the inner bars leads to modification of the analysis.

Example 2.6. N-truss girder. N-truss girders have constant or moderately changing height, filled with interchanging vertical and inclined bars. They find application in bridges, both traditional steel truss bridges and more recently in girders carrying the load on suspension bridges and cable stayed bridges. They also find use as supporting structure for roofs in industrial buildings where an important load component is a distributed vertical load. A simple illustration is shown as Fig. 2.23, where equal vertical forces P are applied to the 7 joints in the foot of a simply supported N-girder. For simplicity of analysis the horizontal spacing of the joints is taken equal to the height of the girder. While simplifying the expressions appearing in the analysis this has no principal impact on the procedure. The effect of the height of a regular truss girder under distributed load is discussed in the following example.

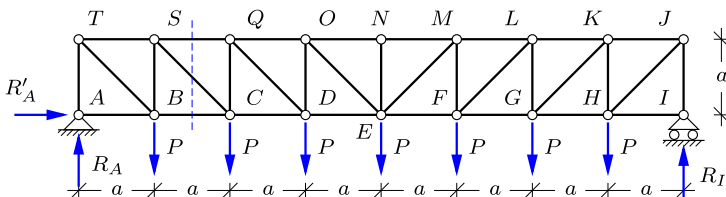


Fig. 2.23: Simply supported N-truss with distributed loads P .

The reactions are determined by horizontal projection and moments about nodes I and A :

$$R'_A = 0, \quad R_A = R_I = \frac{7}{2}P.$$

The forces in the bars are then determined by introducing vertical and inclined sections as illustrated in Fig. 2.24.

The vertical section shown in the figure intersects the diagonal SC and the corresponding bars SQ and BC in the head and the foot, respectively. The bar forces in the head and the foot are parallel, and the force N_{SC} in the diagonal is therefore determined by vertical projection of all forces on the left part of the truss girder:

$$\downarrow \quad N_{SC} \cos 45^\circ - R_A + P = 0 \quad \Rightarrow \quad N_{SC} = \frac{5}{2}\sqrt{2}P.$$

The force N_{BC} then follows from moment about S :

$$\overset{\curvearrowright}{S} \quad a N_{BC} - a R_A = 0 \Rightarrow N_{BC} = R_A = \frac{7}{2}P.$$

Finally, the force N_{SQ} in the head follows via moment about C :

$$\overset{\curvearrowright}{C} \quad a N_{SQ} + 2a R_A - aP = 0 \Rightarrow N_{SQ} = -6P.$$

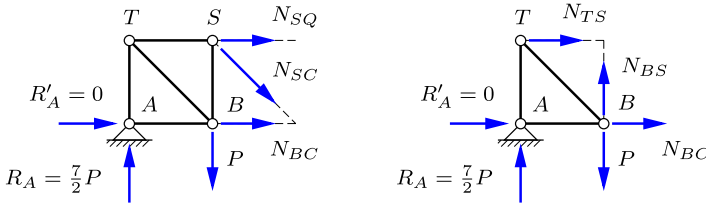


Fig. 2.24: Equilibrium at section.

The forces in the verticals are determined by use of inclined sections as shown in Fig. 2.24b. Vertical equilibrium determines the force N_{BS} in the vertical

$$\uparrow \quad N_{BS} + R_A - P = 0 \Rightarrow N_{BS} = -\frac{5}{2}P.$$

The force N_{TS} in the head can be determined via moment about B , while the force N_{BC} in the foot has already been determined above by the vertical section. The procedure used for the four bar forces here is repeated along the left half of the girder, and the remaining bar forces then follow by symmetry. The resulting bar forces are shown in Fig. 2.25.

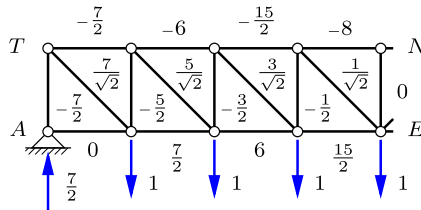


Fig. 2.25: Bar forces in N-truss.

A clear pattern can be seen in the magnitude of the forces in the verticals and diagonals of the truss. When starting at the left support A the vertical carries a compressive force $N_{AT} = -\frac{7}{2}P$. The force N_{BS} in the next vertical is P less due to the load P acting at B . This pattern continues towards the center, where the force in the vertical vanishes due to symmetry. The forces in the diagonals are closely related to those in the corresponding verticals. This is perhaps most easily seen by considering one of the unloaded joints in the head, e.g. S . Vertical equilibrium of this joint requires $N_{SC} = -\sqrt{2}N_{BS}$, and thus the pattern from the verticals is repeated in the diagonals, but with opposite sign. It is interesting to observe, that if the diagonals were turned the other way, the forces would retain their magnitude but become compression instead of tension. By the argument concerning equilibrium of the joints in the head of the girder it follows that the forces in the verticals

would then also change sign. The forces in head and foot increase towards the center. The pattern of this variation will become clear in connection with the analysis of beams in Chapter 3. \square

Example 2.7. V-truss. In a V-truss the diagonals are inclined to the right and to the left as illustrated in Fig. 2.26. The angle α with horizontal is given in terms of the truss height h and the length a of the horizontal bars by

$$\sin \alpha = \frac{2h}{\sqrt{4h^2 + a^2}}.$$

The truss of the present example is simply supported at nodes A and G , and loaded by vertical forces of magnitude P at all the nodes of the girder foot as shown in the figure.

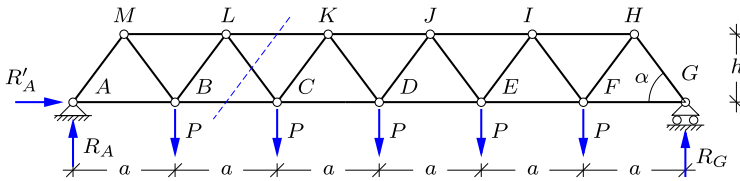


Fig. 2.26: Simply supported V-truss girder with distributed loads P .

The section shown in the figure intersects the diagonal CL . The analysis considers equilibrium of the left part of the structure as shown in Fig. 2.27a. The forces in the bars in the head and the foot are parallel and horizontal. Thus, the force N_{CL} in the diagonal is determined by vertical projection:

$$\downarrow \quad N_{CL} \sin \alpha - R_A + P = 0 \quad \Rightarrow \quad N_{CL} = \frac{3P}{2 \sin \alpha}.$$

The force N_{BC} in the foot is determined via moment about L :

$$\widehat{L} \quad hN_{BC} - \frac{3}{2}aR_A + \frac{1}{2}aP = 0 \quad \Rightarrow \quad N_{BC} = \frac{13}{4} \frac{a}{h} P.$$

Finally, the force N_{KL} in the head is found by moment about C :

$$\widehat{C} \quad hN_{KL} + 2aR_A - aP = 0 \quad \Rightarrow \quad N_{KL} = -4 \frac{a}{h} P.$$

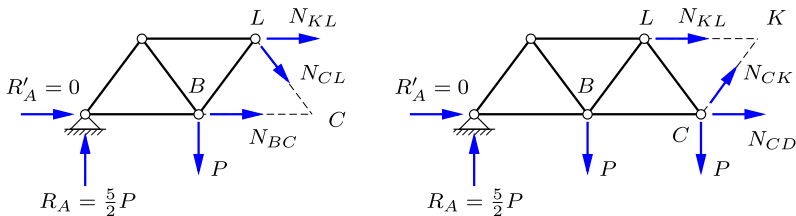


Fig. 2.27: Section equilibrium.

The calculation proceeds along the girder by next considering the section that intersects the diagonal CK as illustrated in Fig. 2.27b. The forces in the intersected bars now follow from equilibrium of the left part. The diagonal force N_{CK} is found by vertical projection:

$$\uparrow \quad N_{CK} \sin \alpha + R_A - 2P = 0 \quad \Rightarrow \quad N_{CK} = -\frac{P}{2 \sin \alpha}.$$

The force N_{CD} in the foot follows from moment about the node K in the head:

$$\hat{K} \quad hN_{CD} - \frac{5}{2}aR_A + \frac{3}{2}aP + \frac{1}{2}aP = 0 \quad \Rightarrow \quad N_{CD} = \frac{17}{4} \frac{a}{h} P.$$

Finally, the force N_{KL} in the head follows from moment about the node C in the foot:

$$\hat{C} \quad hN_{KL} + 2aR_A - aP = 0 \quad \Rightarrow \quad N_{KL} = -4 \frac{a}{h} P.$$

It should be noted that this force has already been calculated by the previous section.

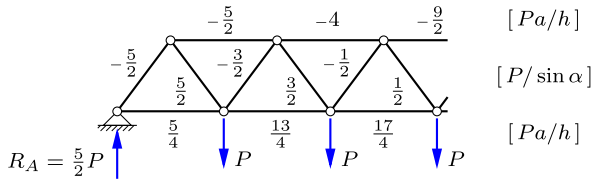


Fig. 2.28: Bar forces in V-truss girder.

By use of similar sections along the V-truss girder the member forces shown in Fig. 2.28 are found. It is noted that the forces in the diagonals are proportional with $P/\sin \alpha$, while the forces in the head and the foot are proportional with aP/h , i.e. smaller for larger girder height. \square

Example 2.8. K-truss. The so-called K-truss finds application e.g. in towers, masts, and in the legs of offshore jackup platforms. In spite of the fact that the K-truss is typically used in vertical orientation the following analysis will address the horizontal orientation and use the names head and foot for the two sides of the truss. In a K-truss the connection between head and foot is established by a combination of transverse and inclined bars, meeting at the center of the transversal bars as shown in Fig. 2.29. The angle α between the diagonals and direction of the truss girder is determined by

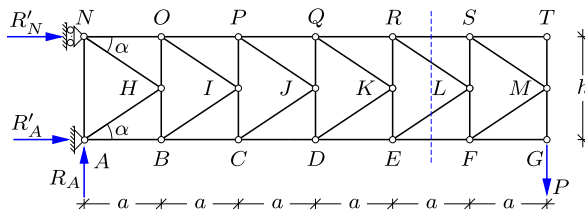


Fig. 2.29: Cantilever K-truss girder loaded by P at foot. ***

$$\sin \alpha = \frac{h}{\sqrt{h^2 + 4a^2}}, \quad \tan \alpha = \frac{h}{2a},$$

where h is the transverse girder dimension and a is the length of a single K -section as indicated in the figure.

Due to the double diagonals in the K -truss a general section will intersect four bars, and thus the bar forces can not be computed by use of a single section as in the case of N -trusses and V -trusses. However, this problem is easily overcome by considering equilibrium of the inner nodes of the K -truss. A typical inner node is shown in Fig. 2.30. The node is connected to two transverse bars with forces N_{LS} and N_{LF} , and to two diagonals with forces N_{LR} and N_{LE} . Horizontal projection only involves the forces in the diagonals, and thus

$$N_{LR} \cos \alpha + N_{LE} \cos \alpha = 0 \quad \Rightarrow \quad N_{LR} = -N_{LE}.$$

Thus, the resulting force from the two diagonals is a downward force of magnitude $2N_{LE} \sin \alpha$.

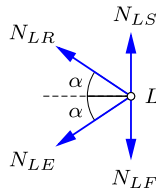


Fig. 2.30: Equilibrium of center node L .

The calculation now proceeds from a transverse section, here selected as shown in Fig. 2.31b. The two diagonal forces combine to a vertical downward force of magnitude $2N_{LE} \sin \alpha$, and thus vertical equilibrium of the right part of the structure gives

$$\downarrow \quad 2N_{LE} \sin \alpha + P = 0 \quad \Rightarrow \quad N_{LR} = -N_{LE} = \frac{P}{2 \sin \alpha}.$$

Note, that this is tension in the upper diagonal and compression in the lower diagonal. The force N_{EF} in the foot is determined by moment about the node S in the head:

$$\curvearrowright_S \quad hN_{EF} + aP = 0 \quad \Rightarrow \quad N_{EF} = -\frac{a}{h} P.$$

This corresponds to compression in the foot. Finally, the force N_{SR} follows from moment about node F :

$$\curvearrowright_F \quad hN_{SR} - aP = 0 \quad \Rightarrow \quad N_{SR} = \frac{a}{h} P.$$

corresponding to tension in the head.

The force in the transverse bars can be determined by equilibrium of the node in the foot or head, once the corresponding diagonal bar force has been determined. Thus, as shown in Fig. 2.31a, equilibrium of node E in the transverse direction gives

$$\uparrow \quad N_{EK} + N_{LE} \sin \alpha = 0 \quad \Rightarrow \quad N_{EK} = -N_{LE} \sin \alpha = \frac{1}{2} P.$$

The similar argument for node R in the head gives

$$\downarrow \quad N_{RK} + N_{LR} \sin \alpha = 0 \quad \Rightarrow \quad N_{RK} = -N_{LR} \sin \alpha = -\frac{1}{2} P.$$

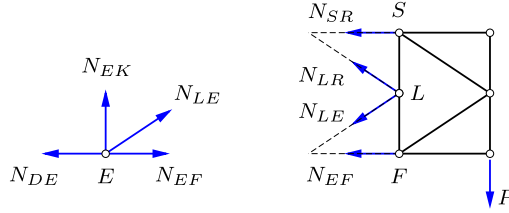


Fig. 2.31: Equilibrium of section.

The forces in the diagonals are found by projection, and in the present problem this involves only the end load P . Thus, all upper diagonals have the same tension force $\frac{1}{2}P/\sin\alpha$, while all lower diagonals have the compression force $-\frac{1}{2}P/\sin\alpha$. Similarly all upper transverse bars have the compression force $-\frac{1}{2}P$, while the lower transverse bars have the tension force $\frac{1}{2}P$.

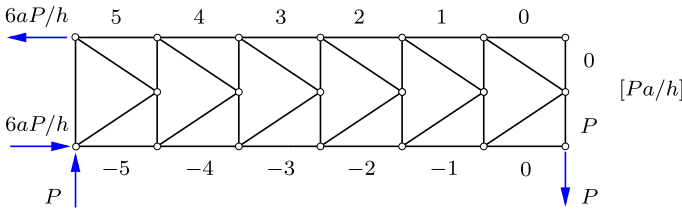


Fig. 2.32: Bar forces in K-truss.

The forces in the girder head and foot are determined by moment equilibrium about a node at distance h . The moment arm of the load increases by a when moving one step towards the support. The bar forces in the full K-truss girder are shown in Fig. 2.32. They are seen to follow a simple pattern with identical forces in similarly placed transverse and diagonal bars, while the forces in head and foot increase towards the support. This corresponds closely to the distribution of shear force and moment in a cantilever beam treated in the following chapter. \square

Example 2.9. Roof truss. Roofs of houses are often supported by truss structures, and the particular W-type shown Fig. 2.33 is quite common. The geometry is determined by the width $12a$ and the height $2h$ of the truss, together with the information that the inner bars are connected to the mid-points of the head, and divides the foot into three equal parts as shown in the figure. It follows from this definition of the geometry that all four inner bars form the same angle β with horizontal. This can be seen by observing that the longer internal bars have vertical projection $2h$ and horizontal projection $2a$, while the shorter internal bars have vertical projection h and horizontal projection a . The two angles α and β are then defined from the dimensions a and h by

$$\sin\alpha = \frac{h}{\sqrt{h^2 + (3a)^2}}, \quad \sin\beta = \frac{h}{\sqrt{h^2 + a^2}}.$$

In the present case the load consists of a single vertical force P acting at node E . The reactions are determined by horizontal projection, moment about D , and moment about A :

$$R'_A = 0, \quad R_A = \frac{1}{4}P, \quad R_D = \frac{3}{4}P.$$

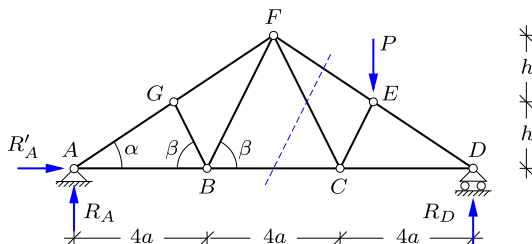


Fig. 2.33: W roof truss.

First, a vertical section is made to the right of the top node F as shown in Fig. 2.33. The right part of the structure and the exposed bar forces are shown in Fig. 2.34a. None of the exposed bar forces are parallel, and thus they are determined by a sequence of three independent moment equations. The first equation is moment about the supported node D . The load P and the bar force N_{CF} contribute to moment equilibrium. The contribution from the inclined force N_{CF} is most conveniently found by resolving it into a horizontal and a vertical component through node C . Of these only the vertical component $N_{CF} \sin \beta$ contributes to moment equilibrium, expressed by

$$\widehat{D} \quad 4a N_{CF} \sin \beta - 3a P = 0 \quad \Rightarrow \quad N_{CF} = \frac{3P}{4 \sin \beta}.$$

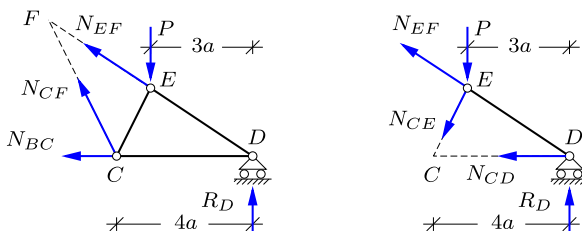


Fig. 2.34: Section forces in W-truss.

The second equation is moment equilibrium about node C . The contribution from the inclined force N_{EF} is most conveniently found by sliding the force along its line of action until it has origin in node D . It is then resolved into a horizontal component, and a vertical component $N_{EF} \sin \alpha$. Only the vertical component contributes to the moment equation, which takes the form

$$\widehat{C} \quad 4a N_{EF} \sin \alpha - aP + 4a R_D = 0 \quad \Rightarrow \quad N_{EF} = -\frac{P}{2 \sin \alpha}.$$

Node F is the intersection point of the two forces just determined, and thus moment about node F gives

$$\widehat{F} \quad 2h N_{BC} + 3a P - 6a R_D = 0 \quad \Rightarrow \quad N_{BC} = \frac{3a}{4h} P.$$

This completes the computation of the three bar forces from the first section.

The section is now moved to the left of node C , whereby the right part of the structure is as shown in Fig. 2.34b. Two of the forces pass through node D , and thus only the load and the bar force N_{CE} contribute to moment equilibrium about D . By sliding the force N_{CE} along its line of action to node C , and then resolving it in a horizontal component and the vertical component $N_{CE} \sin \beta$, the following moment equation is obtained:

$$\widehat{D} \quad 4a N_{CE} \sin \beta + 3a P = 0 \quad \Rightarrow \quad N_{CE} = \frac{-3P}{4 \sin \beta}.$$

Moment about node E determines the bar force N_{CD} :

$$\widehat{E} \quad h N_{CD} - 3a R_D = 0 \quad \Rightarrow \quad N_{CD} = \frac{9a}{4h} P.$$

Finally, a vertical section through ED isolates the supported node D , and vertical equilibrium gives

$$\uparrow \quad N_{DE} \sin \alpha + R_D = 0 \quad \Rightarrow \quad N_{DE} = -\frac{3P}{4 \sin \alpha}.$$

This completes the analysis of the right half of the W-truss. With the present loading equilibrium of nodes G and B leads to the conclusion that the forces in the inner bars GB and BF of the left half of the truss vanish. Thus, the remaining forces are found by the two projection equations for the supported corner node A . \square

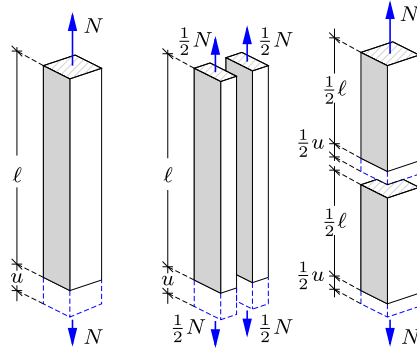
2.4 Stiffness and deformation of truss structures

In most structures strength and stiffness play important roles, even if it implies just having ‘enough strength’ and ‘sufficient stiffness’. Basically the concepts of strength and stiffness are material properties, and their effect in a structure depends on how the materials are used to form the structure. The concepts of strength and stiffness will be introduced gradually, when needed. Thus, the present section is devoted to stiffness of bars – the so-called uni-axial stiffness – while a general description of material stiffness and strength is given in Chapter 8.

2.4.1 Axial stress and strain

The stiffness of a bar relates the elongation u to the axial force N in the bar. The problem of relating these properties of the bar to material properties was discussed by GALILEO GALILEI in 1638. The essential part of this discussion is given here in a more modern form with reference to Fig. 2.35a. The figure shows a homogeneous bar of length ℓ and cross-section area A . The bar is loaded by application of an axial force of magnitude N , which is considered positive in tension. This force leads to an elongation of the bar of magnitude u .

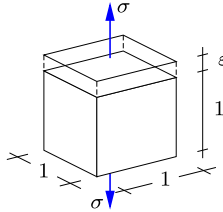
Now, a thought experiment is conducted, in which the bar is split lengthwise into two parts, each of area $\frac{1}{2}A$ as shown in Fig. 2.35b. Each of these parts support half the load, while maintaining the original elongation. Therefore

Fig. 2.35: Bar of length ℓ and cross-section area A .

the elongation must depend on the force normalized by the cross-section area. This normalized force is called the stress and is expressed as

$$\sigma = \frac{N}{A} = \frac{\text{Force}}{\text{Area}}, \quad \left[\frac{\text{N}}{\text{m}^2} \right] = [\text{Pa}]. \quad (2.3)$$

In this formula the units are shown in square brackets. When the force is expressed in Newton [N] and the area in square meters [m^2] the resulting unit for the stress is Pascal [Pa]. The stress is an expression of the magnitude of the loading of the material.

Fig. 2.36: Axial stress σ and axial strain ε .

A suitable measure of deformation is identified by cutting the original bar through the mid cross-section as shown in Fig. 2.35c. The length of each part is $\frac{1}{2}\ell$, and for a homogeneous bar each part contributes half of the full elongation. Thus, the extension experienced at the material level is the relative elongation. This is called the strain, defined by

$$\varepsilon = \frac{u}{\ell} = \frac{\text{Elongation}}{\text{Length}}, \quad \left[\frac{\text{m}}{\text{m}} \right] = [-]. \quad (2.4)$$

It follows from this definition of strain as the relative elongation that strain is non-dimensional.

The concept of stress and strain introduced here is just a special case, often denoted as axial stress σ and axial strain ε , illustrated in Fig. 2.36. A general discussion of stress and strain is given in Chapter 8. When considered as part of a general state of stress and strain, the stress and strain defined here are called normal stress and normal strain, because they act normal to the surface area defined by the section.

2.4.2 Linear elastic bars

The operating stress level of a structure is normally considerably below the stress level that would lead to irreversible processes and failure. For many materials used in structures this implies that there is proportionality between the stress σ and the strain ε in any part of the structure. This behavior is called linear elasticity, and is described by the relation

$$\sigma = E\varepsilon, \quad E \text{ [Pa]}. \quad (2.5)$$

This relation is often called Hooke's law after ROBERT HOOKE (1635–1703), who proposed it in 1675 and demonstrated it experimentally for several mechanical systems in 1678. The parameter E is called the modulus of elasticity. It is the factor of proportionality between the axial stress and axial strain in an experiment, where the loading is purely axial. Generally such an experiment leads to transverse contraction in addition to the axial elongation. The transverse contraction is not central to the present use in connection with trusses and will be dealt with in Chapter 8 in connection with the general discussion of elastic materials. The value of the elastic modulus varies between different materials as illustrated in Table 2.1 – from Gordon (2003).

Table 2.1: Typical elastic constants.

Material	MPa
Rubber	7
Nylon	1400
Plywood	7000
Wood	14000
Concrete	30000
Aluminum	70000
Steel	210000

The elastic relation of a bar follows by multiplication of the stress-strain relation with the cross-section area A of the bar, whereby

$$N = A\sigma = AE\varepsilon, \quad AE \text{ [N]}. \quad (2.6)$$

It is seen that the elastic stiffness of the bar, AE , is the product of the material parameter E and the area A of the structural member. Thus, there are two contributing factors to the stiffness of a bar: its material stiffness, represented by E , and a geometric parameter of the structural element, here represented by the area A . This product form for the stiffness is general for beams and frames and plays an important role e.g. in design against column instability, discussed in Chapter 5.

2.4.3 Virtual work for truss structures

It was demonstrated in Section 1.3.1 that the force and moment equilibrium equations of a rigid body can be expressed in terms of the so-called virtual work. The idea of virtual work is that the structure in question is subjected to an infinitesimally small virtual displacement. The name ‘virtual motion’ indicates that it is a motion associated with a thought experiment, and the virtual motion need not be related to any real motion of the body. The principle of virtual work was used in Section 1.6 to calculate the reactions of statically determinate beam structures. A virtual motion was constructed where the constraint corresponding to the reaction in question was released, while all other support constraints were maintained. This produces a balance equation between the virtual work of the reaction and the virtual work of all external loads. The virtual displacement field was defined by letting the structure – or its individual parts – move as rigid bodies. It is of great importance in modern structural engineering that this simple form of the principle of virtual work can be extended to deformable bodies. The extension of the principle of virtual work to deformable bodies will here be illustrated for the fairly simple case of truss structures. It is demonstrated, how this principle can be used to determine the displacement of individual nodes in an elastic truss structure, and a general numerically oriented computational procedure in terms of finite elements is developed for truss structures in Section 2.5. The method of virtual work is extended to beam and frame structures in Chapter 4, and is used to formulate the finite element method for beams and frames in Chapter 7.

Vector algebra

The analysis of elastic truss structures and the stiffness relations for truss elements with general orientation is conveniently performed by use of vector analysis. Vectors will generally be described by boldface letters \mathbf{x} , \mathbf{a} etc. The default Cartesian component representation has the individual components arranged in column format. The following analysis requires that component arrays can be written either in column format \mathbf{a} or in row format indicated by the transpose \mathbf{a}^T ,

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \mathbf{a}^T = [a_x, a_y, a_z]. \quad (2.7)$$

The component arrays are considered as matrices. The scalar product of two vectors \mathbf{a} and \mathbf{b} , previously denoted by a dot as $\mathbf{a} \cdot \mathbf{b}$, can then be written as a matrix product,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = [a_x, a_y, a_z] \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z. \quad (2.8)$$

In the matrix product the components of the rows of the first factor are multiplied by the components of the columns of the second factor, and the terms are then added. In the present case of the scalar product this leads to the indicated summation, of which the result is a scalar, i.e. a number. A special case is the scalar product of a vector with itself,

$$|\mathbf{a}|^2 = \mathbf{a}^T \mathbf{a} = [a_x, a_y, a_z] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x^2 + a_y^2 + a_z^2. \quad (2.9)$$

The result of this operation is the square of the length of the vector, indicated as $|\mathbf{a}|$.

In matrix products the order of the factors is typically important. Thus, $\mathbf{a}^T \mathbf{a}$ is the scalar product, while $\mathbf{a} \mathbf{a}^T$ is the matrix

$$\mathbf{a} \mathbf{a}^T = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} [a_x, a_y, a_z] = \begin{bmatrix} a_x a_x & a_x a_y & a_x a_z \\ a_y a_x & a_y a_y & a_y a_z \\ a_z a_x & a_z a_y & a_z a_z \end{bmatrix}, \quad (2.10)$$

formed by products of the original vector components. Both the products $\mathbf{a}^T \mathbf{a}$ and $\mathbf{a} \mathbf{a}^T$ find application in the following theory of the elastic bar element.

Virtual work for a bar

Figure 2.37 shows a bar with end points A and B . The bar is described by the vector $\mathbf{a} = \overrightarrow{AB}$ with length $a = |\mathbf{a}|$. The bar supports the internal force N which is defined as positive for tension. Thus, the external forces \mathbf{f}_A and \mathbf{f}_B at the nodes A and B of the bar are given by

$$\mathbf{f}_B = -\mathbf{f}_A = \frac{1}{a} \mathbf{a} N. \quad (2.11)$$

This form secures equilibrium of the bar.

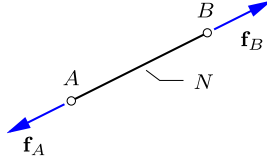


Fig. 2.37: Bar element with nodal loads.

Now, let the nodes A and B move by the virtual displacements $\delta \mathbf{u}_A$ and $\delta \mathbf{u}_B$, respectively. The nodal forces then perform the external virtual work

$$\delta V_{\text{ex}} = \delta \mathbf{u}_A^T \mathbf{f}_A + \delta \mathbf{u}_B^T \mathbf{f}_B. \quad (2.12)$$

Note, that the external virtual work is formed by the work of the external forces by the virtual displacement of the bar. The idea now is to demonstrate that this external virtual work can be reformulated to an internal virtual work, expressed in terms of the internal force N and a virtual strain of the bar. The key component in this reformulation is the equilibrium equation (2.11). When substituting the nodal forces \mathbf{f}_A and \mathbf{f}_B from the equilibrium equation the external virtual work takes the form

$$\delta V_{\text{ex}} = (\delta \mathbf{u}_B^T - \delta \mathbf{u}_A^T)(\mathbf{a}/a)N. \quad (2.13)$$

The task now is to show that the first factor can be expressed in the form of the virtual strain $\delta \varepsilon$, i.e. the strain that would arise in the bar, if the nodes were given the virtual displacements $\delta \mathbf{u}_A$ and $\delta \mathbf{u}_B$. First it is observed that the length of the bar can be expressed by the relation

$$a^2 = |\mathbf{a}|^2 = \mathbf{a}^T \mathbf{a}. \quad (2.14)$$

Differentiation of this relation gives the incremental relation

$$2a \delta a = 2\mathbf{a}^T \delta \mathbf{a}, \quad (2.15)$$

where the factor 2 enters because both factors contribute to the differentiation. From this relation the virtual elongation of the bar follows as

$$\delta a = \frac{1}{a} \mathbf{a}^T \delta \mathbf{a} = \frac{1}{a} \mathbf{a}^T (\delta \mathbf{u}_B - \delta \mathbf{u}_A). \quad (2.16)$$

The order of the two factors in a scalar vector product can be interchanged, and it is then seen that the factor to the internal force N in (2.13) is precisely the elongation of the bar as given in (2.16). Thus, the internal virtual work δV_{in} can be defined by

$$\delta V_{\text{in}} = \delta a N = a \delta \varepsilon N. \quad (2.17)$$

The last form uses the definition of the virtual strain $\delta\varepsilon = \delta a/a$ corresponding to the virtual elongation δa . This definition of the internal virtual work as the work of the internal force N through the virtual strain $\delta\varepsilon$ then gives the equality between external and internal virtual work,

$$\delta V_{\text{ex}} = \delta V_{\text{in}}. \quad (2.18)$$

The result, that the virtual work of the external loads is equal to the work of the internal forces through the strain is of general validity in structural mechanics. It is generalized to truss structures below, and to beams and frames in Chapter 4.

Virtual work for a truss structure

The equality between external and internal virtual work for a bar only requires equilibrium of the bar. It must therefore apply to all bars of a truss structure, and therefore also to the sum of the contributions from each bar,

$$\sum_{\text{bars}} \delta V_{\text{ex}} = \sum_{\text{bars}} \delta V_{\text{in}}. \quad (2.19)$$

The external virtual work is now rewritten in terms of the external loads on the nodes. The basic principle is illustrated in Fig. 2.38 with reference to a two-dimensional truss structure. However, the principles are general and apply to three-dimensional trusses as well as to other structures.

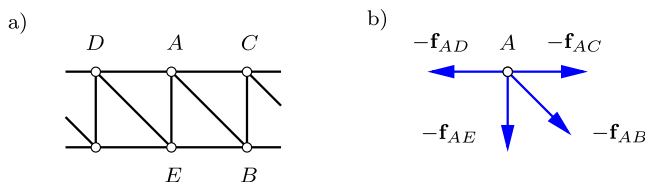


Fig. 2.38: Bar forces \mathbf{f}_{A*} acting on node A.

Node A is acted on by all forces $\mathbf{f}_{AB}, \mathbf{f}_{AC}, \dots$ from bars attached to this node. Thus, by the principle of action and reaction the similar forces but in opposite direction, $-\mathbf{f}_{AB}, -\mathbf{f}_{AC}, \dots$, are the forces acting on the node. In addition to these internal forces the node may also be acted on by an external force \mathbf{P}_A corresponding to a load. Equilibrium of the node requires the vector sum of all forces on the node to vanish. When arranging internal forces on the left side of the equation and the external force on the right, the equilibrium condition reads

$$\sum \mathbf{f}_{A*} = \mathbf{P}_A, \quad (2.20)$$

where \mathbf{f}_{A*} is the force in the bar $A*$, with $*$ denoting an arbitrary node connected to A by a bar – in the present example the nodes B, C, D, E .

This gives the virtual work equation for trusses in the form

$$\sum_{\text{nodes}} \delta \mathbf{u}_j^T \mathbf{P}_j = \sum_{\text{bars}} a_i \delta \varepsilon_i N_i, \quad (2.21)$$

where \mathbf{P}_j are the actual external forces acting at the nodes $j = 1, 2, \dots$ and N_i are the actual internal forces in the bars $i = 1, 2, \dots$. In contrast, the nodal displacements $\delta \mathbf{u}_j$ are virtual and define the virtual strains $\delta \varepsilon_i$ in the bars. The virtual displacements can be selected in a special way that enables explicit computation of the actual displacements of any node of an elastic truss structure as discussed in the next section.

2.4.4 Displacements of elastic truss structures

The procedure for calculation of the displacement of a node of an elastic truss structure is illustrated in Fig. 2.39. The top figure shows the structure with the actual loads – here consisting of vertical concentrated forces of magnitude P at all nodes in the girder foot. The bar forces corresponding to this load are calculated and denoted N_1^0, \dots, N_i^0 , where the superscript 0 indicates that these are the actual forces in the bars. The node displacements corresponding to the actual load are similarly denoted $\mathbf{u}_1^0, \dots, \mathbf{u}_j^0$.

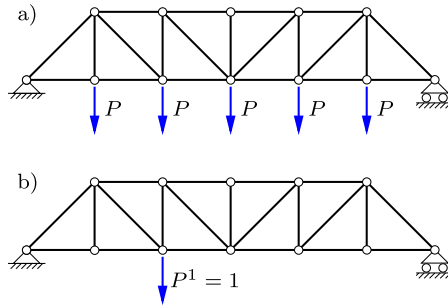


Fig. 2.39: Truss girder: a) Actual load, b) Unit test force.

The lower figure shows the same truss, but now loaded with only a single force $P^1 = 1$. This is an assumed load, used to determine the displacement component corresponding to the force P^1 . The assumed concentrated load generates the bar forces N_1^1, \dots, N_i^1 .

If all bars in the truss are elastic the displacement component corresponding to the assumed load P^1 can now be determined by the principle of virtual work. The principle of virtual work is an equality between the external and in-

ternal virtual work calculated as the work of a set of loads and corresponding internal forces, when exposed to a virtual displacement field. In the present case the roles are interchanged, such that the structure with the actual loads in Fig. 2.39a provides the displacement field, while the static forces are taken from the assumed load in Fig. 2.39b. The virtual equation (2.21) then takes the form

$$u^0 P^1 = \sum_{\text{bars}} a_i \varepsilon_i^0 N_i^1. \quad (2.22)$$

Due to the assumed load consisting of a single force, the external work consists of a single term $u^0 P^1$, where u^0 is the displacement in the direction of the assumed load P^1 , when the structure is loaded by the actual load.

When the truss is elastic, the strains ε_i^0 in the bars can be expressed in terms of the corresponding bar forces N_i^0 according to the elastic relation (2.6),

$$\varepsilon_i^0 = \frac{N_i^0}{(EA)_i}, \quad (2.23)$$

where $(EA)_i$ is the elastic stiffness of bar i . Substitution of this into the virtual work equation (2.22) together with the condition that the test load is of unit magnitude, $P^1 = 1$, gives the final result in the form

$$u^0 = u^0 P^1 = \sum_{\text{bars}} a_i \varepsilon_i^0 N_i^1 = \sum_{\text{bars}} \frac{a_i}{(EA)_i} N_i^0 N_i^1. \quad (2.24)$$

This is an explicit formula for the displacement u^0 in the direction of a unit test force in terms of a summation over all bars of the structure of the product of the bar force for the actual and the test load, respectively. It is notable, that the geometry of the truss does not appear directly in the formula. It has already been accounted for in the calculation of the bar forces.

Example 2.10. Displacement of node in a truss. The calculation of node displacements in elastic trusses is illustrated by considering the simple cantilever truss in Fig. 2.40a, supporting a single vertical force P at node C . All bars are assumed to have identical elastic stiffness parameter EA . In this example it is desired to calculate both the vertical and the horizontal displacement components of node C . This is done by considering two independent load cases shown in Fig. 2.40b: a vertical unit test force P^1 and a horizontal unit test force P^2 , both acting at node C .

The total computation consists of calculating three sets of bar forces: N_i^0 for the actual load, and N_i^1 and N_i^2 for the two test load cases. It is convenient to collect the bar lengths and forces in a table as illustrated by Table 2.2. The bar lengths are denoted by the symbol ℓ here, because a has been used for a specific dimension of the truss. If the bars had different elastic stiffness, the values $(EA)_i$ should also be included in the table.

The vertical displacement of node C is found from (2.24) as

$$u_C^1 = \sum_i \frac{\ell_i}{EA} N_i^0 N_i^1,$$

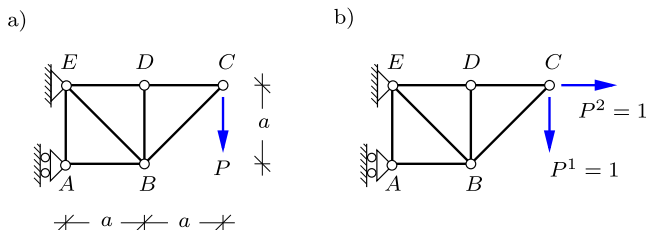
Fig. 2.40: Cantilever girder. a) actual load P , b) unit test loads P^1, P^2 .

Table 2.2: Tabular calculation of virtual work.

	ℓ	N^0	N^1	N^2
AB	a	$-2P$	-2	0
BC	$\sqrt{2}a$	$-\sqrt{2}P$	$-\sqrt{2}$	0
BD	a	0	0	0
CD	a	P	1	1
BE	$\sqrt{2}a$	$\sqrt{2}P$	$\sqrt{2}$	0
DE	a	P	1	1
AE	a	0	0	0

and substitution of the numerical values from the table gives

$$u_C^1 = \frac{aP}{EA} (4 + 2\sqrt{2} + 1 + 2\sqrt{2} + 1) = (6 + 4\sqrt{2}) \frac{aP}{EA} = 11.66 \frac{aP}{EA}.$$

Similarly the horizontal displacement of node C is

$$u_C^2 = \sum_i \frac{\ell_i}{EA} N_i^0 N_i^2 = \frac{aP}{EA} (1 + 1) = 2 \frac{aP}{EA}.$$

The vertical displacement associated with bending of the truss girder is much larger than the axial displacement associated directly with the elongation of the bars ED and DC . This behavior will also be seen in beams, where most of the displacement is usually associated with bending. \square

2.5 Finite element analysis of trusses

The analysis methods developed so far in this chapter for trusses have mainly been based on statics, i.e. use of equilibrium conditions for the full truss and the individual bars. This approach works well for smaller structures and analysis carried out by hand. For larger truss structures and analysis carried out by computer a systematic approach in which the individual bar elements and nodes are treated in a repetitive way is desirable. In order to isolate an individual bar element from the rest of the structure it is desirable to consider the structure as flexible and to use the displacements of the nodes

as the primary variables in the analysis. This represents a change in the point of view relative to the previous methods of nodes and sections, where the forces in the bars were the primary variables.

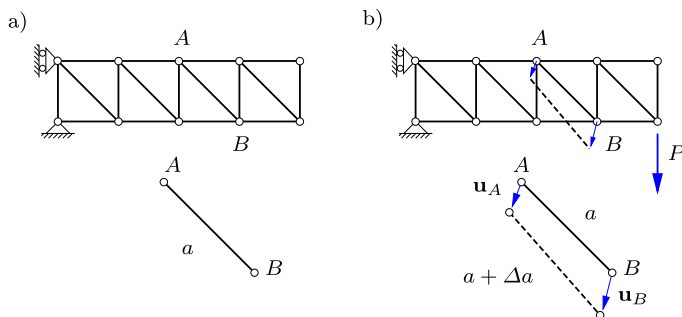


Fig. 2.41: Displacement of nodes A and B leads to elongation of the bar AB .

The basic idea of using the displacement of each of the nodes is illustrated in Fig. 2.41. Consider a flexible bar AB connecting the nodes A and B . Loading of the structure introduces a displacement \mathbf{u}_A of node A and \mathbf{u}_B of node B . These displacements may introduce a change of length of the bar AB from a to $a + \Delta a$. This change of length corresponds to a force N_{AB} in the bar.

An efficient analysis method, particularly suited for computer implementation, can be developed by expressing all bar forces in terms of the displacements $\mathbf{u}_A, \mathbf{u}_B, \dots$ of the nodes, and then formulating and solving the equilibrium conditions for all the nodes of the structure. This task requires:

- i) A constitutive relation between the elongation of each bar and the developed bar force.
- ii) A general formulation for the elongation of a bar with arbitrary orientation in space, expressed in terms of the displacement of the two nodes of the bar.
- iii) A systematic formulation of the equilibrium conditions for each node in terms of the relevant bar forces.
- iv) Introduction of suitable support conditions.

These tasks are described in the following subsections, leading to the development of a small computer program MINITRUS.

2.5.1 Elastic bar element

The derivation of the elastic bar element consists in first determining the strain in the element in terms of the displacements of the element nodes, and then expressing the forces in the nodes in terms of this strain.

Strain in bar element

The properties of a bar element are conveniently formulated by use of vector algebra as illustrated in Fig. 2.42.

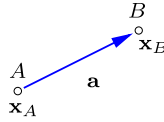


Fig. 2.42: Bar element represented as vector \mathbf{a} .

The bar element AB is described by the initial position of the nodes A and B with Cartesian coordinates

$$\mathbf{x}_A = [x_A, y_A, z_A]^T, \quad \mathbf{x}_B = [x_B, y_B, z_B]^T. \quad (2.25)$$

The bar element is given by the vector \mathbf{a} , represented in terms of the node coordinates as

$$\mathbf{a} = [x_B - x_A, y_B - y_A, z_B - z_A]^T. \quad (2.26)$$

The length a of the bar element in its initial position is therefore given as $a^2 = |\mathbf{a}|^2 = \mathbf{a}^T \mathbf{a}$.

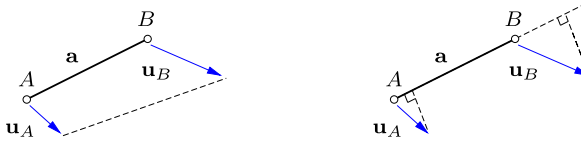


Fig. 2.43: Elongation of bar element by projection of node displacements.

When the structure is loaded, the nodes A and B move as described by the displacement vectors \mathbf{u}_A and \mathbf{u}_B , shown in Fig. 2.43. The nodes A and B are then located at $\mathbf{x}_A + \mathbf{u}_A$ and $\mathbf{x}_B + \mathbf{u}_B$, respectively. When the displacements of the nodes are small, e.g. relative to the length of the element, the elongation of the element Δa , and thereby the strain, can be calculated via projections as indicated in Fig. 2.43b. The projection of the displacement is obtained by scalar multiplication with the unit vector $a^{-1}\mathbf{a}$, and thus the elongation of the bar element is approximated by

$$\Delta a \simeq a^{-1} \mathbf{a}^T \mathbf{u}_B - a^{-1} \mathbf{a}^T \mathbf{u}_A. \quad (2.27)$$

The strain then follows from division by the bar length a , whereby

$$\varepsilon = \frac{1}{a} \mathbf{a}^T (\mathbf{u}_B - \mathbf{u}_A). \quad (2.28)$$

Note, that while use of projections to evaluate the elongation generally involves an approximation, the special case of a translation with identical displacements of the two nodes $\mathbf{u}_B = \mathbf{u}_A$ leads to zero strain, $\varepsilon = 0$.

Stiffness matrix of elastic bar element

Equilibrium of the bar element requires that the forces \mathbf{f}_A and \mathbf{f}_B at the nodes of the bar element are equal in magnitude but in opposite directions. The direction is described by the unit vector $a^{-1}\mathbf{a}$, and the magnitude of the force is denoted N with tension as positive. The force vectors in A and B are then similar to those previously given in (2.11),

$$\mathbf{f}_B = -\mathbf{f}_A = \frac{1}{a} \mathbf{a} N. \quad (2.29)$$

For an elastic bar with normal force $N = AE \varepsilon$ the force vectors \mathbf{f}_A and \mathbf{f}_B can then be expressed in terms of the displacements of the nodes via substitution of the expression (2.28) for the strain, whereby

$$\mathbf{f}_B = -\mathbf{f}_A = \frac{AE}{a^3} \mathbf{a} \mathbf{a}^T (\mathbf{u}_B - \mathbf{u}_A). \quad (2.30)$$

Note the occurrence of the product $\mathbf{a} \mathbf{a}^T$, forming the matrix shown in (2.10).

When assembling all the nodal forces of a truss structure into a model for the full structure it is advantageous to combine the two vector equations from (2.30) in an explicit block matrix format. For this purpose the forces and displacements at the two element nodes A and B are arranged in an expanded vector format of double size,

$$[\mathbf{f}_A^T, \mathbf{f}_B^T] = [f_x^A, f_y^A, f_z^A, f_x^B, f_y^B, f_z^B], \quad (2.31)$$

and in the same way for the displacements $[\mathbf{u}_A^T, \mathbf{u}_B^T]$. With this notation the expressions (2.30) for the nodal forces can be expressed in the following block matrix format:

$$\begin{bmatrix} \mathbf{f}_A \\ \mathbf{f}_B \end{bmatrix} = \frac{AE}{a^3} \underbrace{\begin{bmatrix} \mathbf{a} \mathbf{a}^T & -\mathbf{a} \mathbf{a}^T \\ -\mathbf{a} \mathbf{a}^T & \mathbf{a} \mathbf{a}^T \end{bmatrix}}_{\mathbf{K}_{\text{bar}}} \begin{bmatrix} \mathbf{u}_A \\ \mathbf{u}_B \end{bmatrix}. \quad (2.32)$$

The matrix \mathbf{K}_{bar} in this relation is called the element stiffness matrix of the bar. The derivations have been illustrated for the three-dimensional case, where the individual vectors have 3 components, leading to a 6 by 6 element stiffness matrix. In the case of a plane truss the vector dimension is 2, and the element stiffness matrix has the dimension 4 by 4.

The forces acting on the bar element AB are conveniently expressed in the generic block matrix format

$$\begin{bmatrix} \mathbf{f}_A \\ \mathbf{f}_B \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{K}_{AA} & \mathbf{K}_{AB} \\ \mathbf{K}_{BA} & \mathbf{K}_{BB} \end{bmatrix}}_{\mathbf{K}_{\text{bar}}} \begin{bmatrix} \mathbf{u}_A \\ \mathbf{u}_B \end{bmatrix}. \quad (2.33)$$

In this format the block matrices \mathbf{K}_{AA} and \mathbf{K}_{AB} represent the force at node A from the displacement \mathbf{u}_A of node A and the displacement \mathbf{u}_B of node B , respectively.

2.5.2 Finite Element Method for trusses

The next step is to use the information about the individual bar elements to set up conditions for all the nodes of the truss structure. The principle was illustrated in Fig. 2.38, where it was demonstrated that equilibrium of a node A requires the sum of the forces \mathbf{f}_{A*} from all connecting bars to balance the external load \mathbf{f}_A^{ex} at node A ,

$$\sum \mathbf{f}_{A*} = \mathbf{f}_A^{\text{ex}}. \quad (2.34)$$

The forces in the individual bar elements are available from a element matrix relation of the form (2.32), and a central point in the formulation of the finite element method is the procedure used to assemble the contributions from the individual elements into a model for the structure.

Assembling the global stiffness matrix

The structure of the element stiffness matrix (2.33), where the force contribution at the element nodes is given in terms of the displacements of the nodes via a block matrix, leads to the following simple procedure to create a model of the complete truss structure.

- i) Identify all nodes of the structure by numbers $1, \dots, n$. Denote the corresponding coordinate set of the nodes by $\mathbf{x}_1, \dots, \mathbf{x}_n$.
- ii) Associate each bar element with two nodes, e.g. bar element AB with the nodes i and j of the structure, as illustrated in Fig. 2.44. This association between the element nodes A, B and the global structural nodes i, j is called the topology of the model.

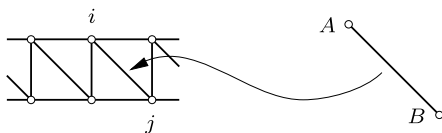


Fig. 2.44: Bar member AB as element ij in truss structure.

- iii) The contribution of the forces from the individual elements can now be obtained by placing the submatrices from the element stiffness relation (2.33) in the global format as shown here,

$$\begin{bmatrix} \vdots \\ \mathbf{f}_i \\ \vdots \\ \mathbf{f}_j \\ \vdots \end{bmatrix}_{AB} = \begin{bmatrix} \vdots & \vdots \\ \cdots \mathbf{K}_{AA} \cdot \mathbf{K}_{AB} \cdots \\ \vdots & \vdots \\ \cdots \mathbf{K}_{BA} \cdot \mathbf{K}_{BB} \cdots \\ \vdots & \vdots \end{bmatrix}_{AB} \begin{bmatrix} \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_j \\ \vdots \end{bmatrix} \begin{matrix} i \\ j \end{matrix} \quad (2.35)$$

When placed in this global format the displacements \mathbf{u}_i and \mathbf{u}_j contribute in the correct way to the internal forces \mathbf{f}_i and \mathbf{f}_j at nodes i and j . Adding the contributions from all elements to form the global stiffness matrix of the structure is seen to correspond to adding the internal forces at each of the nodes as prescribed in (2.33).

Support conditions

The model must also include provisions for supports, typically in the form of constraints on the displacements of certain nodes. Constraint of a node can typically be introduced by imposing one or more relations between the displacement components $\mathbf{u}_i = [u_x, u_y, u_z]^T_i$ at the corresponding node. The introduction of such a constraint reduces the number of unknown displacement components in the model. Before presenting the implementation of general constraints two simple alternative methods of implementing the support conditions are discussed.

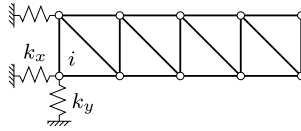


Fig. 2.45: Support springs attached to node i .

A simple method is to constrain the supported nodes by introducing stiff springs as illustrated in Fig. 2.45. The springs connect the node to a rigid support. They act essentially as bar elements, but because the ‘other end’ of the spring is fully constrained, the corresponding stiffness matrix to be included in the model is just a block matrix \mathbf{K}_s appearing in the diagonal position corresponding to the supported node as illustrated by the corresponding global force stiffness matrix contribution

$$\begin{bmatrix} \cdot \\ \mathbf{f}_i \\ \cdot \\ \vdots \end{bmatrix}_s = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \mathbf{K}_s & \cdot \\ \cdot & \vdots & \cdot \end{bmatrix}_s \begin{bmatrix} \cdot \\ \mathbf{u}_i \\ \cdot \\ \vdots \end{bmatrix}_i. \quad (2.36)$$

A support consisting of springs with stiffness constants k_x, k_y, k_z in the coordinate directions correspond to the diagonal block stiffness matrix

$$\mathbf{K}_s = \begin{bmatrix} k_x & & \\ & k_y & \\ & & k_z \end{bmatrix}. \quad (2.37)$$

This format permits some of the springs to have vanishing stiffness. The stiffness of an inclined spring with spring constant k_n along a direction described by the unit vector $\mathbf{n} = [n_x, n_y, n_z]$ follows from the bar element stiffness matrix (2.32) as

$$\mathbf{K}_s = k_n \mathbf{nn}^T = k_n \begin{bmatrix} n_x n_x & n_x n_y & n_x n_z \\ n_y n_x & n_y n_y & n_y n_z \\ n_z n_x & n_z n_y & n_z n_z \end{bmatrix}. \quad (2.38)$$

Several springs can be applied to a node, simply by adding their stiffness contributions.

The use of stiff springs to represent constraints involves a compromise. Ideally the springs should be infinitely stiff, relative to the stiffness components of the structure itself. This would lead to ill-conditioning of the equations, and numerical roundoff errors in the solution of the equation system sets a limit on the magnitude of the spring constants that can be used without compromising the accuracy of the solution procedure. A simple modification of the idea of springs can be used when the involved degrees of freedom are constrained to zero. In that case the rows and columns corresponding to the constrained degrees of freedom can be set to zero, except for the diagonal element that is retained. When removing any loads associated with these degrees of freedom, the equations for the constrained degrees of freedom are uncoupled from the unconstrained degrees of freedom, and the equations can be solved directly, retaining the original size and organization of the matrix and displacement components.

A third more general alternative exists, that has a particularly elegant implementation in MATLAB. The first step is to separate the displacement vector into two parts: a vector \mathbf{u}_c containing the constrained displacement components, and a vector \mathbf{u}_u containing the remaining unconstrained displacement components. Rearrangement of the equilibrium equations gives the block matrix equation format

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uc} \\ \mathbf{K}_{cu} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_c \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_c + \mathbf{r} \end{bmatrix}. \quad (2.39)$$

The stiffness sub-matrices follow from rearrangement of the original stiffness matrix \mathbf{K} . At the constrained degrees of freedom the total force consists of any imposed load \mathbf{f}_c plus the reaction force components \mathbf{r} produced by the support. In this format \mathbf{u}_c represents imposed displacements, that may be non-zero. The solution proceeds in two steps. First the unconstrained displacements \mathbf{u}_u are obtained from the top part of the equations,

$$\mathbf{K}_{uu}\mathbf{u}_u = \mathbf{f}_u - \mathbf{K}_{uc}\mathbf{u}_c, \quad (2.40)$$

and then the reaction forces \mathbf{r} follow from the lower part as

$$\mathbf{r} = \mathbf{K}_{cu}\mathbf{u}_u + \mathbf{K}_{cc}\mathbf{u}_c - \mathbf{f}_c. \quad (2.41)$$

In classic programming this procedure would imply the formation of the corresponding sub-matrices. However, in high-level programming languages like MATLAB the operations can be implemented via the corresponding index sets without rearranging the data as explained in connection with the MINITRUS program in the next section.

2.5.3 The *MiniTruss* program

The principles described in the previous sections have been implemented in a small Finite Element program MINITRUS using the high level programming language MATLAB. However, other high-level programming languages permit similar implementations. The main structure of the program and its data structure are explained in relation to the specific roof truss analysis described in Example 2.9. Figure 2.46 shows the roof truss with the nodes numbered from 1 to 7 and the bars from 1 to 11. The program receives information about the nodes via their coordinates $[x_1, y_1], \dots, [x_7, y_7]$, while the attachment of the elements to the nodes is defined by a topology matrix \mathbf{T} .

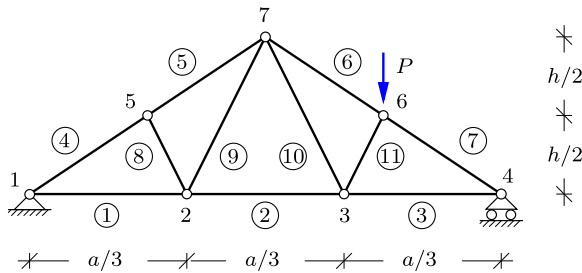


Fig. 2.46: Roof truss with concentrated vertical force at node 6.

The program is built as a script file `MiniTruss.m` that serves as a driver that reads a data file and activates subroutines that set up the model, form the global stiffness matrix, introduce the support conditions, apply the load, and finally solves for the displacement of all nodes and calculates the forces in all bars. The data of the present example is contained in the data file `W_Truss.m`. The structure and content of this file is described in the following.

Node data. The W-truss is described in an xy -coordinate system with origo at the center of the truss foot, and the y -axis vertical upwards. The width a and height h of the truss are given in parametric form in terms of variables a and h . The node coordinates are given in the form of an array X , with each node corresponding to one row. The first part of the data file then is

```
% Width 'a' and height 'h' of truss
a = 12.0; h = 4.0;

% Coordinates of nodes X = [x y (z)],
X = [ -a/2  0.00
      -a/6  0.00
       a/6  0.00
       a/2  0.00
      -a/4  h/2
       a/4  h/2
       0.00 h    ];
```

The node coordinates $[x, y]$ are given in the order of the node number, starting with node 1. Thus, the node number is not given explicitly, but implied by the row number in the node coordinate matrix X .

The program identifies the truss structure as being 2 or 3-dimensional by counting the number of columns in the node coordinate matrix X . If the truss is 3-dimensional the displacement vector for a node has 3 components, while a 2-dimensional structure has 2 displacement components per node.

Element data. The truss elements are defined in the topology matrix T . Each row of this matrix defines an element, by listing its two nodes by their node number, and by giving a third number identifying a set of element properties, area A and elasticity modulus E , given in a material property matrix H .

```
% Topology matrix T = [node1 node2 propno],
T = [ 1  2  1
      2  3  1
      3  4  1
      1  5  2
      5  7  2
      6  7  2
      4  6  2
      2  5  3
      2  7  3
      3  7  3
      3  6  3 ];

% Element property matrix H = [ A E ],
H = [ 1.0 100.0
      1.0 100.0
      0.8 100.0 ];
```

In this example there are three element types: type 1 for the foot, type 2 for the head, and type 3 for the diagonals.

Loads. The loads are specified in the load matrix P. This matrix contains a row for each loaded node. The data line specifies the node number and the force components. In the present example node 6 is loaded by a force with components $[f_x, f_y] = [0.000, -1.000]$.

```
% Prescribed loads P = [node fx fy (fz)]
P = [ 6   0.000 -1.000 ];
```

Support conditions. The support conditions are given in the constraint matrix C. The constraint matrix contains a row for each constrained displacement component. In the present example there are 3 constrained displacement components u_1 and v_1 at node 1 and v_4 at node 4.

```
% Constraints C = [node 'dof' (uc)]
C = [ 1  1
      1  2
      4  2 ];
```

The optional parameter uc indicates the magnitude of a prescribed displacement component. If not included as a third column in C, constrained displacement components are set to zero.

Graphics. The MINITRUS program produces two plots of the structure: a plot of the initial undeformed geometry including node numbers, and another plot of the deformed structure after application of the load. For most real structures it is necessary to scale the displacements to be able to see the deformation. The coordinate window used for the plots is controlled via definition of the plot axes, specified in the array

```
% Axes used for geometry plots [Xmin Xmax Ymin Ymax]
PlotAxes = [-0.55*a 0.55*a -0.5*h 1.2*h];
```

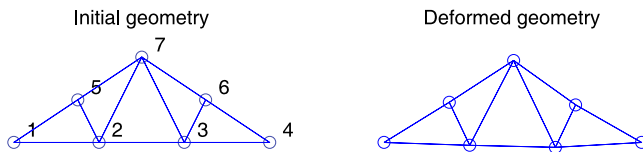


Fig. 2.47: Plots of initial and deformed geometry of W-truss.

The default in the program MINITRUS is to use the two top subplots in a 2×2 plot layout. For long trusses, as e.g. bridges etc., a 2×1 plot format can be introduced by changing to `subplot(2,1)`.

Analysis process. The first step in an analysis with the MINITRUS program is to read the appropriate data file into memory. This is done either by writing `W_Truss` in the command window, or by uploading `W_Truss` to the MATLAB editor and pressing the F5 key from the editor. The data is now available in active memory and the analysis is carried out by activating the script file `MiniTruss.m`, either by writing `MiniTruss` in the command window or by pressing the F5 key with `MiniTruss.m` in the editor.

```
% Nodal loads into load vector
if exist('P','var')
    f = loadnode(f,P,dof);
end

% Global stiffness matrix
K = kbar(K,T,X,H);

% Solve stiffness equation
[u,r,ic] = solveeq(K,f,C,dof);

% Nodal displacements
Un = reshape(u,dof,size(X,1))';

% Calculate element forces and strains
[s,e] = Nbar(T,X,H,u);
```

The program activates the following processes. First the global load vector `f` is formed. Then the global stiffness matrix of the structure `K` is formed by the function `kbar` by collecting the stiffness contributions from all bar-elements. The function `solveeq` then solves the constrained equations, accounting for the support conditions. Finally the displacements are reshaped into vector components for each node in the matrix `Un`, and the internal forces `s` and strains `e` are obtained by post-processing.

The solution of the constrained equations in `solveeq` is obtained by a reformulation of the block equations (2.39)–(2.41) in terms of index sets. The structure in Fig. 2.46 has 7 nodes and thereby 14 degrees of freedom. Thus the index set containing all degrees of freedom is $\mathbf{ii} = [1, 2, \dots, 13, 14]$. At node 1 both displacement components are constrained, and at node 4 the second displacement component is constrained. Thus, the index set of the constraints is $\mathbf{ic} = [1, 2, 8]$. In MATLAB the index set of the unconstrained degrees of freedom \mathbf{iu} can then be obtained by the function call

$$\mathbf{iu} = \text{setdiff}(\mathbf{ii}, \mathbf{ic}) = [3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14] .$$

In MATLAB the sub-matrices are obtained by using the full matrices with the appropriate index set as index. Thus, $\mathbf{K}(\mathbf{iu}, \mathbf{iu})$ extracts the sub-matrix \mathbf{K}_{uu} etc. The equation (2.40) for the unconstrained displacement components then takes the form

$$\mathbf{K}(\mathbf{iu}, \mathbf{iu}) * \mathbf{u}(\mathbf{iu}) = \mathbf{f}(\mathbf{iu}) - \mathbf{K}(\mathbf{iu}, \mathbf{ic}) * \mathbf{u}(\mathbf{ic})$$

and the reaction is found from the expression

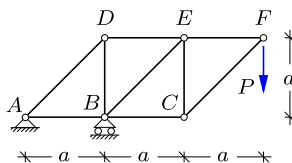
$$\mathbf{r} = \mathbf{K}(\mathbf{ic}, \mathbf{ii}) * \mathbf{u}(\mathbf{ii}) - \mathbf{f}(\mathbf{ic}) .$$

In the MATLAB syntax this gives the reactions \mathbf{r} as a 3-component vector corresponding to the global degrees of freedom $\mathbf{ic} = [1, 2, 8]$.

2.6 Exercises

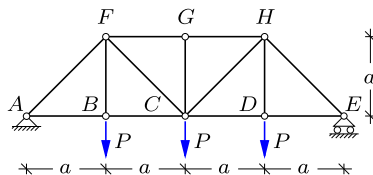
Exercise 2.1. The figure shows a cantilever N-truss supporting a vertical load P acting at node F .

- Determine the reaction forces.
- Determine all bar forces by the method of joints.



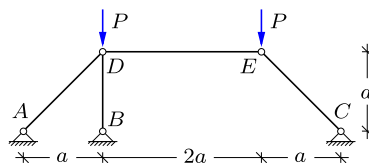
Exercise 2.2. The figure shows a symmetric and simply supported truss girder with vertical loads P acting at nodes B , C and D , respectively.

- Determine the reaction forces.
- Determine all bar forces by the method of joints.



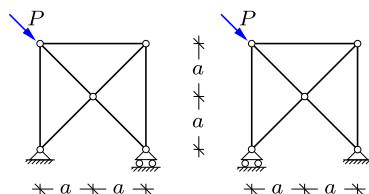
Exercise 2.3. The figure shows a truss structure with fixed simple supports at nodes A , B and C and a vertical load P at nodes D and E , respectively.

- Illustrate the direction of the resulting reaction forces at the three supports.
- Determine the reaction force in B by a single equilibrium equation and subsequently the bar force in BD .
- Determine the all bar forces and the remaining reactions



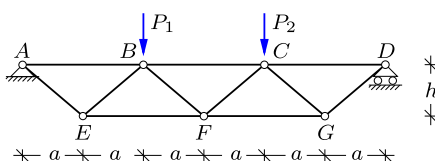
Exercise 2.4. The behavior of structures often depends on the support conditions. The two truss structures in the figure are identical, except that the supports with and without rollers have been interchanged.

- Determine the reactions and bar forces for both structures.



Exercise 2.5. The figure shows a simply supported V-truss, loaded by two vertical forces P_1 and P_2 .

- Determine the reactions and the bar forces for $P_1 = P_2 = P$.
- Explain the influence of the relative height h/a of the truss girder on the bar forces in head and foot, and in the diagonals.

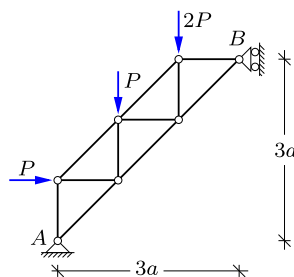


Exercise 2.6. Consider the simply supported V-truss from Exercise 2.5 with the load $P_1 = P$ and $P_2 = 0$.

- Determine the reactions and the bar forces.

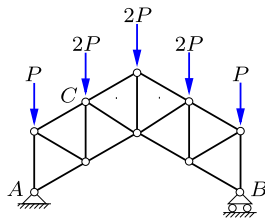
Exercise 2.7. The truss structure shown in the figure has a fixed simple support at A and a simple support permitting vertical motion at B . The truss is loaded by the vertical forces P and $2P$ and by the horizontal force P .

- Determine the reaction forces.
- Determine the bar forces.



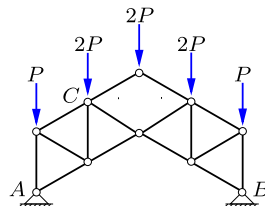
Exercise 2.8. The figure shows a truss structure in which all bars have the length a . The structure has a fixed simple support at A and a simple support permitting horizontal motion at B . The truss is loaded by vertical forces of magnitude P and $2P$.

- Determine the reaction forces.
- Determine the forces in all the bars connected to node C .



Exercise 2.9. The figure shows a truss structure similar to that in Exercise 2.8. In the present truss the vertical middle bar has been removed, and the support in B is fixed in both directions.

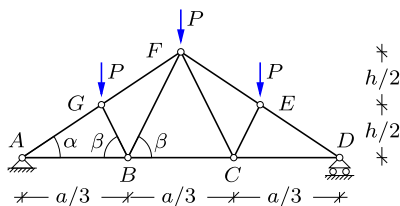
- Determine the reaction forces.
- Determine the forces in all the bars connected to node C .



Exercise 2.10. Determine the remaining bar forces in the roof truss from Example 2.3, and indicate all bar forces in a figure.

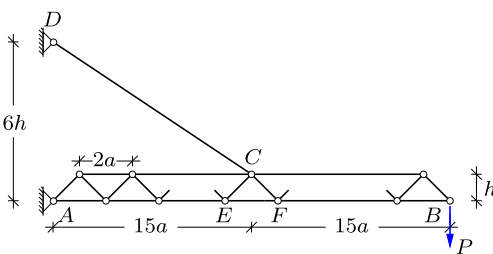
Exercise 2.11. The figure shows a simply supported roof truss of length a and height h with vertical loads P at nodes E , F and G , respectively. The symmetric structure of the interior bars is determined by the angles α and β .

- Determine the reaction forces.
- Determine all bar forces by the method of joints.



Exercise 2.12. The figure shows a typical V-truss $A-B$ from a building crane. In this exercise the truss is considered as plane. The truss is supported by a fixed hinge at A and an inclined tension bar DC , connected to the head of the truss at C . It carries the vertical load P at B .

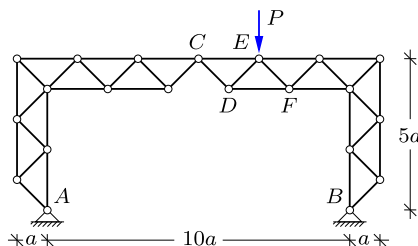
- Determine all reaction components at A and D .
- Determine the force in the inclined bar DC supporting the truss.
- Find the bar forces N_{EF} , N_{CE} and N_{CF} .



Exercise 2.13. The figure shows a symmetric three-hinge frame formed by two identical truss frames AC and BC that are connected by a hinge at node C . The truss frames are

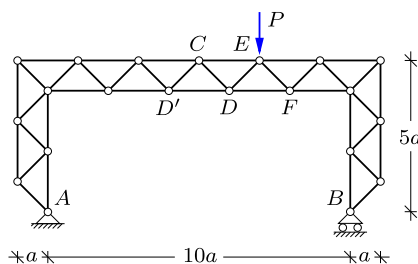
supported by fixed simple supports at A and B , respectively. The load consists of a single vertical force P acting at E .

- Determine all reaction components at A and B .
- Determine the forces in the bars CD , CE , DE , DF and EF by the method of sections.
- Check the result by e.g. isolating node D and verify that the node is in equilibrium.



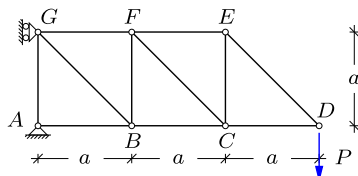
Exercise 2.14. A modified version of the truss frame from Exercise 2.13 is shown in the figure. A horizontal bar DD' is introduced to close the hinge at C , and rollers are added to the simple support in B , permitting horizontal motion and retaining the static determinacy of the structure.

- Repeat questions a) to c) of Exercise 2.13 for the modified truss frame and determine also the force in the new bar DD' .



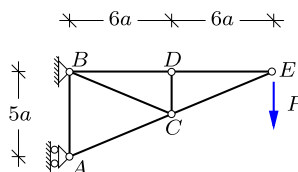
Exercise 2.15. The figure shows an N-type cantilever truss, loaded by concentrated vertical force of magnitude P at the tip joint D .

- Find and show the reaction components in a figure.
- Determine all the bar forces.
- Find both vertical and horizontal displacement of the node D .



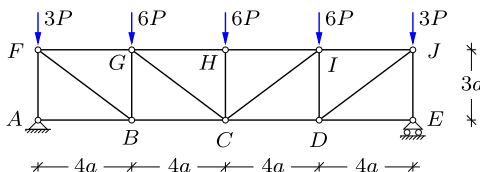
Exercise 2.16. The figure shows a simple cantilever truss, loaded by a concentrated force P at the joint E .

- Find and show the reaction components in a figure.
- Determine all the bar forces.
- Find the vertical and horizontal displacement of the nodes E and D .



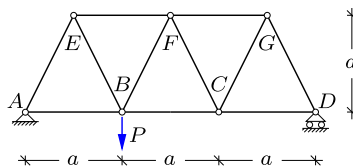
Exercise 2.17. The figure shows an N-truss girder, loaded by concentrated vertical forces of magnitude $3P$ at the joints F and J , and of magnitude $6P$ at the joints G , H and I . All bars are elastic with stiffness EA .

- Find and show the reaction components on a drawing.
- Determine all the bar forces.
- Find the vertical displacement of the nodes F , G and H .



Exercise 2.18. The figure shows a V-truss girder with a vertical load in B . All bars are elastic with stiffness EA .

- Determine the reaction and bar forces.
- Determine the vertical displacement of nodes B and C .
- Determine the horizontal displacement of node D .



Exercise 2.19. Consider the W roof-truss shown in Fig. 2.46 and used in the description of the MINITRUS program in Section 2.5.3. Let the dimensions be $a = 12.0$ and $h = 4.0$, choose the properties so that $EA = 1$ and apply $P = 1$ at node 6.

- Use the data file for the MINITRUS program and determine the reactions and bar forces. Compare with the results from Example 2.9.

Exercise 2.20. Consider the W roof-truss used in the description of the MINITRUS program in Section 2.5.3. Choose the dimensions and properties as in the Exercise 2.19, but let the load be symmetric as in Exercise 2.11.

- Introduce the modified load case in a new data file **W.Truss02.m**.
- Determine the reactions and bar forces by MINITRUS and compare with the results obtained in Exercise 2.11.

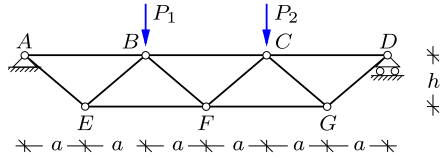
Exercise 2.21. Consider the simple triangular cantilever truss in Exercise 2.16, and choose $a = 1$, $EA = 1$ and $P = 1$.

- Create a data file **Canti.Truss.m** to be used in MINITRUS.
- Determine the vertical displacement of nodes D and E and compare with Exercise 2.16.
- Determine the horizontal displacement of nodes D and E and compare the magnitude with the vertical displacements found in b).
- Determine the reactions and bar forces and compare with the results obtained in Exercise 2.16.

Exercise 2.22. The figure shows a V-truss girder, loaded by two concentrated vertical forces $P_1 = P_2$.

- Create a data file **V.Truss.m** for the V-truss to be used in MINITRUS. Introduce a and h as parameters in the data file. Choose the properties so that $EA = 1$ and let $P = 1$.

- b) Use the MINITRUSS program to determine the forces in the bars for $h = \sqrt{3}a$, i) when all bars have the same length, and ii) for $h = a$, respectively.



Exercise 2.23. Consider the simply supported N-truss girder in Example 2.6, where $a = 1$, $EA = 1$ and $P = 1$.

- Create a data file `N.Truss.m` to be used in MINITRUSS.
- Determine the vertical displacement of node E .
- Determine the bars with the largest tension and compression force, respectively.
- Consider the load case where the only load is a vertical force of magnitude $7P$ acting in node E . Create a new data file and repeat questions b) and c).

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