

Chapter 2

Sequences and Series

In this chapter we give the basic properties of monotone, strictly monotone, and Cauchy sequences in detail. In particular, we treat the convergence tests of Cauchy, D'Alembert, Leibniz, and Raabe. As examples, we consider the harmonic series, the geometric series, and products of series. The convergence or divergence of the hyperharmonic series with general term $1/n^p$ is determined, depending on p . We explain the concept of conditionally convergence and show that for any conditionally convergent series and any number $S \in [-\infty, \infty]$, there is a rearrangement of the given series such that the resulting series converges to S . But the rearrangement of an absolutely convergent series yields another absolutely convergent series having the same sum as the original one. Finally, it is pointed out that sequences and series with complex terms enjoy analogous properties.

2.1 The Basic Properties of Convergent Sequences

In the fifth century B.C., the Greek philosopher Zeno proposed the following paradox: in order for a runner to travel a given distance, the runner must first travel halfway, then half the remaining distance, then half the distance that still remains, and so on ad infinitum. But, Zeno argued, it is clearly impossible for a runner to accomplish these infinitely many tasks in a finite period of time, so motion from one point to another must be impossible. Zeno's paradox suggests the infinite subdivision of $[0, 1]$ into subintervals of length $1/2^n$ for each integer $n = 1, 2, 3, \dots$. If the length of the interval is the sum of the lengths of the subintervals into which it is divided, then it would appear that

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

Similarly, in order for a man to live his life he must first live half life, then half the remaining life, then half the life that still remains, and so on ad infinitum. But, if Zeno were correct, it would be impossible for a man to get from the time of birth to the time of death (T). This would imply that man is immortal. On the other hand, it is clear that:

$$T = \frac{T}{2} + \frac{T}{4} + \frac{T}{8} + \cdots + \frac{T}{2^n} + \cdots$$

Important examples of infinite sequences are the following.

- (1) The equilateral polygons inscribed in a circle.
- (2) Arithmetic and geometric series.
- (3) The sequence of rational numbers $x_1 = 0, 3, x_2 = 0, 33, x_3 = 0, 333, \dots$
- (4) The Fibonacci sequence defined as follows:

$F_1 = 1, F_2 = 1, F_{n+1} = F_{n-1} + F_n, n \geq 2$. This is a recursively defined sequence—after the initial values are given, each following term is calculated from two of its predecessors.

Informally speaking, the term “sequence” in mathematics is used to describe an unending succession of numbers. Before going into details, let us give some necessary definitions.

Definition 2.1. If to every natural number $n \in \{1, 2, 3, \dots\}$ there is associated real number x_n , then the set of real numbers

$$x_1, x_2, \dots, x_n, \dots \quad (2.1)$$

is called a *sequence* and x_n is the *n-th term of the sequence*. More precisely, a sequence of real numbers is a function defined on the set of all positive integers. Later on, we also write $\{x_n\}$ for the sequence.

Example 2.1. If $x_n = \frac{1}{n^3}$, then $\{x_n\} = \left\{1, \frac{1}{2^3}, \frac{1}{3^3}, \dots, \frac{1}{n^3}, \dots\right\}$ or, if $x_n = (-1)^n$, then $\{x_n\} = \{-1, 1, -1, \dots, (-1)^n, \dots\}$.

Remark 2.1. If the sequence

$$y_1, y_2, \dots, y_n, \dots \quad (2.2)$$

or, $\{y_n\}$ is another sequence, then from (2.1) and (2.2) can be formed the following sequences as a result of arithmetic operations: $\{x_n + y_n\}, \{x_n - y_n\}, \{x_n \cdot y_n\}, \left\{\frac{x_n}{y_n}\right\} (y_n \neq 0)$.

Definition 2.2. The sequence $\{x_n\}$ is said to have the limit a if, given any $\varepsilon > 0$, there is a positive integer $N(\varepsilon)$ such that

$$|x_n - a| < \varepsilon, \text{ for all } n \geq N(\varepsilon). \quad (2.3)$$

(The conclusion $|x_n - a| < \varepsilon$ can be reformulated as $a - \varepsilon < x_n < a + \varepsilon$, of course.) We say that the sequence $\{x_n\}$ converges to the real number a , and we write

$$\lim_{n \rightarrow \infty} x_n = a \text{ or } x_n \rightarrow a, n \rightarrow \infty.$$

Thus, if we choose any positive number ε , the terms in the sequence will eventually be within ε units of a .

A sequence $\{x_n\}$ that does not have a finite limit is said to diverge.

Example 2.2. Let $x_n = 0, \overbrace{3 \dots 3}^n$. We show that $\lim_{n \rightarrow \infty} x_n = a = \frac{1}{3}$. For a given $\varepsilon > 0$, by the definition of limits, we must find an integer N , which could depend on ε , such that $\left|x_n - \frac{1}{3}\right| < \varepsilon$ when $n \geq N = N(\varepsilon)$. Obviously (see 1.4 of Chapter 1), for all integers n

$$0, 3 \dots 3 \leq \frac{1}{3} \leq 0, 3 \dots 3 + \frac{1}{10^n}.$$

Thus,

$$\left|x_n - \frac{1}{3}\right| \leq \frac{1}{10^n}.$$

Since $\frac{1}{10^n} \leq \frac{1}{10^N}$ for all $n \geq N$, we can choose N satisfying the inequality $\frac{1}{10^N} < \varepsilon$. Here, $N = \left\lceil \log \frac{1}{\varepsilon} \right\rceil + 1$, if $0 < \varepsilon \leq 1$ and $N = 1$, if $\varepsilon > 1$. Here, $\lceil x \rceil$ is the integer part of the decimal fraction x .

Example 2.3. (a) If $x_n = \frac{1}{n}$, then we show that $\lim_{n \rightarrow \infty} x_n = 0$. We need to show this: To each positive number ε , there corresponds an integer N such that for all $n \geq N$,

$$|x_n - 0| = |x_n| = \frac{1}{n} < \varepsilon.$$

It suffices to pick any fixed integer $N = N(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$. Then $n \geq N$ implies $1/n \leq 1/N < \varepsilon$ as desired.

(b) Let $x_n = \frac{1}{n^2}$. We show $\lim_{n \rightarrow \infty} x_n = 0$. Since $|x_n - 0| = \frac{1}{n^2}$ for every $\varepsilon > 0$, we can take $N(\varepsilon) = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil + 1$. Then $|x_n - 0| < \varepsilon$ for all $n > N(\varepsilon)$.

Example 2.4. We show that the limit of the sequence $\{x_n\} = \{1 + (-1)^n\}$ does not exist, that is, we show that the sequence is divergent.

Suppose that the sequence is convergent. Then there exists a real number a and positive number $N = N(\varepsilon)$ such that for all $n > N(\varepsilon)$,

$$|x_n - a| < \varepsilon. \quad (2.4)$$

If we set $\varepsilon = 1$, then for all even integers $n > N(1)$,

$$|2 - a| < 1 \quad \text{or} \quad 1 < a < 3,$$

but for all odd numbers $n > N(1)$,

$$|a| < 1 \quad \text{or} \quad -1 < a < 1.$$

Obviously, there does not exist any such number a , simultaneously satisfying both these inequalities. This contradiction proves that the sequence diverges.

Theorem 2.1. *The limit of a convergent series $\{x_n\}$ is unique.*

□ On the contrary, let us suppose that the sequence $\{x_n\}$ has two limits a and b ($a \neq b$). Then for any given $\varepsilon > 0$, there are positive numbers $N_1(\varepsilon)$ and $N_2(\varepsilon)$ such that $|x_n - a| < \varepsilon$ for all $n > N_1(\varepsilon)$ and $|x_n - b| < \varepsilon$ for all $n > N_2(\varepsilon)$. Let us choose $\varepsilon = \frac{|a-b|}{2}$ and $N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. Then we have

$$|x_n - a| < \frac{|a-b|}{2}, \quad |x_n - b| < \frac{|a-b|}{2}, \quad n > N(\varepsilon).$$

These inequalities imply that

$$|a - b| = |a - x_n + x_n - b| \leq |x_n - a| + |x_n - b| < |a - b|,$$

and so $|a - b| < |a - b|$. By this contradiction, the proof of the theorem is completed. ■

Theorem 2.2. *A convergent sequence is bounded, that is, there is a constant $C > 0$ such that $|x_n| \leq C$, $n = 1, 2, 3, \dots$*

□ Assume that $x_n \rightarrow a$, $n \rightarrow \infty$, that is, given any number $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that $|x_n - a| < \varepsilon$ for all $n > N$. Then we have $|x_n| = |x_n - a + a| \leq |x_n - a| + |a| < \varepsilon + |a|$. Clearly, $|x_n| \leq C = \max\{|a| + \varepsilon, |x_1|, \dots, |x_N|\}$. ■

Remark 2.2. Obviously, the converse assertion to the theorem is not generally true. For example, the sequence $x_n = 1 + \{-1\}^n$ diverges (see Example 2.4), while $|x_n| \leq 2$. Consequently, bounded sequences may or may not converge.

Theorem 2.3. *Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ converge to limits a and b , respectively, that is $x_n \rightarrow a$, $y_n \rightarrow b$, $n \rightarrow \infty$. Then the sequences $\{cx_n\}$, $\{x_n y_n\}$, $\{x_n \pm y_n\}$, $\left\{\frac{x_n}{y_n}\right\}$ ($y_n \neq 0$) converge, for c any constant, and moreover:*

- (1) $\lim_{n \rightarrow \infty} cx_n = c \lim_{n \rightarrow \infty} x_n = c \cdot a$,
- (2) $\lim_{n \rightarrow \infty} (x_n \pm y_n) = \lim_{n \rightarrow \infty} x_n \pm \lim_{n \rightarrow \infty} y_n = a \pm b$,
- (3) $\lim_{n \rightarrow \infty} (x_n \cdot y_n) = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n = a \cdot b$,
- (4) $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n} = \frac{a}{b}$ ($b \neq 0$).

□ (1) Let $\varepsilon > 0$ be any positive number and $\varepsilon_0 = \frac{\varepsilon}{|c|}$. Then there is a positive integer $N = N(\varepsilon_0)$ such that $|x_n - a| < \varepsilon_0$ for all $n > N$. Consequently, $|cx_n - ca| = |c||x_n - a| < \varepsilon$, that is, $\lim_{n \rightarrow \infty} cx_n = ca$.

(2) For any $\varepsilon > 0$, there exist integers $N_1(\varepsilon/2)$ and $N_2(\varepsilon/2)$ such that $|x_n - a| < \varepsilon/2$ for all $n > N_1(\varepsilon/2)$ and $|y_n - b| < \varepsilon/2$ for all $n > N_2(\varepsilon/2)$.

Then,

$$|(x_n \pm y_n) - (a \pm b)| \leq |x_n - a| + |y_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad n > N(\varepsilon) = \max \left\{ N_1 \left(\frac{\varepsilon}{2} \right), N_2 \left(\frac{\varepsilon}{2} \right) \right\}$$

and the proof of part (2) is complete.

(3) By Theorem 2.2, there exists a constant $C > 0$ such that $|x_n| \leq C$ ($n = 1, 2, \dots$) and $|b| \leq C$. For any $\varepsilon > 0$ there exist positive integers $N_1 \left(\frac{\varepsilon}{2C} \right)$ and $N_2 \left(\frac{\varepsilon}{2C} \right)$ such that

$$|x_n - a| < \frac{\varepsilon}{2C} \text{ when } n > N_1 \left(\frac{\varepsilon}{2C} \right) \text{ and } |y_n - b| < \frac{\varepsilon}{2C} \text{ when } n > N_2 \left(\frac{\varepsilon}{2C} \right).$$

Let $N = N(\varepsilon)$ be the greater of N_1 and N_2 .

Then

$$|x_n y_n - ab| = |x_n y_n - x_n b + x_n b - ab| \leq |x_n| |y_n - b| + |b| |x_n - a| < \frac{C \cdot \varepsilon}{2C} + \frac{C \cdot \varepsilon}{2C} = \varepsilon$$

(4) To begin with, let us show that $\lim_{n \rightarrow \infty} \frac{1}{y_n} = \frac{1}{b}$. Again by the definition of the limit, given any number $\varepsilon = |b|/2$ there exists an integer $N_1(\varepsilon)$ such that $|y_n - b| < |b|/2$ for all $n > N_1(\varepsilon)$. Then, under the condition $n > N_1(\varepsilon)$,

$$|y_n| = |b + (y_n - b)| \geq |b| - |y_n - b| > |b| - \frac{|b|}{2} = \frac{|b|}{2}.$$

Moreover, let $N(\varepsilon)$ be the greater of $N_1(\varepsilon)$ and $N_2(\varepsilon)$. Then

$$\left| \frac{1}{y_n} - \frac{1}{b} \right| = \frac{|y_n - b|}{|y_n||b|} < \frac{2}{|b|^2} \varepsilon \frac{|b|^2}{2} = \varepsilon.$$

Now, by part (3) of the theorem, we have $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} \frac{1}{y_n} = \frac{a}{b}$. ■

Theorem 2.4 (Comparison Test). Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences. Then

(1) (Limit inequality) If $x_n \leq y_n$ for all $n > N$ and $\{x_n\}$, $\{y_n\}$ are convergent, then

$$\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n.$$

(2) (Sandwich or squeeze property) If $x_n \leq y_n \leq z_n$ for all $n > N$ and the sequences $\{x_n\}$, $\{z_n\}$ have the same limits, then the sequence $\{y_n\}$ is convergent and the equalities

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n \text{ hold.}$$

□ (1) Let $\lim_{n \rightarrow \infty} x_n = a$ and $\lim_{n \rightarrow \infty} y_n = b$ where $a > b$. The sequences are convergent, and so for any $\varepsilon = \frac{a-b}{2}$ there exist integers $N_1(\varepsilon)$ and $N_2(\varepsilon)$ such that $|x_n - a| < \frac{a-b}{2}$ for all $n > N_1(\varepsilon)$ and $|y_n - b| < \frac{a-b}{2}$ for all $n > N_2(\varepsilon)$. If $N_0(\varepsilon)$ is the greater of $N_1(\varepsilon)$ and $N_2(\varepsilon)$, then for all $n > N_0(\varepsilon)$

$$\begin{aligned} x_n - y_n &= (x_n - a) - (y_n - b) + (a - b) \\ &\geq -|x_n - a| - |y_n - b| + (a - b) \\ &> -\frac{a-b}{2} - \frac{a-b}{2} + (a - b) = 0. \end{aligned}$$

This contradiction proves that part (1) is true.

(2) If $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = a$, then for any $\varepsilon > 0$ there exist positive numbers $N_1 = N_1(\varepsilon)$, $N_2 = N_2(\varepsilon)$ such that $|x_n - a| < \varepsilon$ for all $n > N_1$ and $|z_n - a| < \varepsilon$ for all $n > N_2$. In particular, since $a - \varepsilon < x_n$ and

$$z_n < a + \varepsilon \text{ for all } n > N_0(\varepsilon) \quad (N_0(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}),$$

for all integers $n > \bar{N}(\varepsilon) = \max\{N, N_1, N_2\}$ the inequality $a - \varepsilon < x_n \leq y_n \leq z_n < a + \varepsilon$ is satisfied. In other words, $\lim_{n \rightarrow \infty} y_n = a$. ■

Remark 2.3. If in part (1) of the theorem the strict inequality $x_n < y_n$ holds, this does not necessarily imply the strict inequality of their limits, that is, in general $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$.

For example, if $x_n = \frac{1}{n^2 + 1}$, $y_n = \frac{1}{n}$, then $\frac{1}{n^2 + 1} < \frac{1}{n}$: but $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 0$.

The same is true for sequences $x_n = a$ or $y_n = b$, $n = 1, 2, 3, \dots$ (a, b are constants). Indeed, if $x_n = 0$ and $y_n = \frac{1}{n}$ then $\frac{1}{n} > 0$. By passing to the limit, we have

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 = \lim_{n \rightarrow \infty} x_n.$$

Example 2.5. Find $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sin^2 n$. At once it follows from the inequality $0 \leq \sin^2 n \leq 1$ that

$$0 \leq \frac{1}{n^2} \sin^2 n \leq \frac{1}{n^2}.$$

If we set $x_n = 0$, $z_n = \frac{1}{n^2}$, then by the Comparison Theorem we have $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = 0$, and as a result,

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sin^2 n = 0.$$

Example 2.6. Suppose that y_n satisfies the inequality $1 - \frac{1}{n^2} \leq y_n \leq \cos \frac{1}{n}$. Find the limit $\lim_{n \rightarrow \infty} y_n$. Clearly,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2}\right) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} \cos \frac{1}{n} = 1.$$

By the Comparison Test, we then have $\lim_{n \rightarrow \infty} y_n = 1$.

Example 2.7. We compute the limit of the well-known sequence $y_n = \sqrt[n]{n} - 1$ ($y_n \geq 0$). Using the Binomial Formula, it is easy to see that

$$n = (1 + y_n)^n \geq \frac{n(n-1)}{n} y_n^2.$$

Thus,

$$y_n \leq \sqrt{\frac{2}{n-1}} \quad (n \geq 2).$$

Consequently, we have

$$x_n \leq y_n \leq z_n, \quad x_n = 0, \quad z_n = \sqrt{\frac{2}{n-1}}.$$

By part (2) of Theorem 2.4, since $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = 0$, we have $\lim_{n \rightarrow \infty} y_n = 0$, or $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

Example 2.8. For the sequence $y_n = \frac{a^n}{n!}$ ($a > 0$), find $\lim_{n \rightarrow \infty} y_n$.

Let N be a positive integer satisfying the inequality $N \geq 2a$. Clearly, for $n > N$, i.e., for $n > 2a$, we have $\frac{a}{n} < \frac{1}{2}$. Thus,

$$0 < \frac{a^n}{n!} = \frac{a^N a^{n-N}}{N!(N+1) \cdots n} = \frac{a^N}{N!} \cdot \frac{a}{N+1} \cdot \frac{a}{N+2} \cdots \frac{a}{n} < \frac{a^N}{N!} \left(\frac{1}{2}\right)^{n-N}.$$

As a result,

$$0 = x_n < \frac{a^n}{n!} < z_n = \frac{a^N}{N!} \left(\frac{1}{2}\right)^{n-N}.$$

Here $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = 0$, and by the Comparison Test, the limit sought is zero:

$$\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0.$$

Definition 2.3. A sequence $\{x_n\}$ is called, respectively, *increasing*, *nondecreasing*, *decreasing*, or *nonincreasing* if

$$x_{n+1} > x_n, \quad x_{n+1} \geq x_n, \quad x_{n+1} < x_n, \quad \text{or} \quad x_{n+1} \leq x_n, \quad n = 1, 2, 3, \dots$$

A sequence that is either nondecreasing or nonincreasing is called *monotone*, and a sequence that is increasing or decreasing is called *strictly monotone*. It is clear that a strictly monotone sequence is monotone, but not conversely.

Theorem 2.5. *Every bounded monotone sequence converges.*

□ For definiteness, we prove the theorem for a nondecreasing sequence $\{x_n\}$ (the proofs for the other cases are analogous). By hypothesis, $A = \{x_n\}$ is bounded, and so the supremum $\sup A = a$ exists. Thus, there is a number $x_N \in A$ such that $x_n \leq a$, $a - \varepsilon < x_N \leq a$. Observe that $x_n \geq x_N$ for all integers $n > N$ (the sequence is nondecreasing), and consequently, $a - \varepsilon < x_n \leq a$ or $a - \varepsilon < x_n < a + \varepsilon$ for all $n > N$. Finally, $|x_n - a| < \varepsilon$, $n > N$.

■

By using this theorem, it can be shown that the limit of the sequence $x_n = \frac{a^n}{n!}$ ($a > 0$) is equal to zero. Obviously, for integers n satisfying the inequality $n > a - 1$, we have the relation $x_{n+1} = x_n \cdot \frac{a}{x_n + 1}$, and then $x_{n+1} > x_n$. Moreover, the sequence $\{x_n\}$ is bounded from below by zero. Finally, by Theorem 2.5, the sequence x_n is convergent, that is, $x_n \rightarrow x$, where x is a real number.

Thus, we have

$$x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} \frac{a}{n+1} = x \cdot 0 = 0.$$

Example 2.9. In the theory of numerical analysis, in order to calculate the square root of the positive number a , the following recurrence sequence is considered:

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right), \quad n = 1, 2, \dots \quad (2.5)$$

Here, x_1 is an initial approach and can be any positive number. It is easy to see from (2.5) that $x_n > 0$ ($n = 1, 2, \dots$), that is, the sequence $\{x_n\}$ is bounded below by zero. On the other hand, for a positive number $t > 0$ the inequality $(t - 1)^2 \geq 0$ gives us that $t + \frac{1}{t} \geq 2$. Now, let us put $t = \frac{x_n}{\sqrt{a}} > 0$. Then it follows from (2.5) that

$$x_{n+1} = \frac{\sqrt{a}}{2} \left(\frac{x_n}{\sqrt{a}} + \frac{\sqrt{a}}{x_n} \right) = \frac{\sqrt{a}}{2} \left(t + \frac{1}{t} \right) \geq \sqrt{a} \quad (n = 1, 2, \dots).$$

Thus, for integers $n \geq 2$, the inequality $x_n \geq \sqrt{a}$ holds. Next, let us show that the sequence $\{x_n\}$ ($n \geq 2$) is nonincreasing. We can rewrite (2.5) as follows:

$$\frac{x_{n+1}}{x_n} = \frac{1}{2} \left(1 + \frac{a}{x_n^2} \right). \quad (2.6)$$

Since $x_n \geq \sqrt{a}$ ($n \geq 2$), from equality (2.6) we can derive that $\frac{x_{n+1}}{x_n} \leq 1$ or $x_{n+1} \leq x_n$. Then from Theorem 2.5, the sequence $\{x_n\}$ is convergent. By Theorem 2.4, $\lim_{n \rightarrow \infty} x_n = x \geq \sqrt{a}$. Finally, by passing to the limit in (2.5), we deduce that

$$x = \frac{1}{2} \left(x + \frac{a}{x} \right),$$

which implies that $x = \sqrt{a}$.

Example 2.10. (a) Find the limit of the sequence $x_n = \sqrt{a + \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}}$, where $a > 0$ and the number of radical signs is n . Here, as a recursively defined sequence, we can set

$$x_{n+1} = \sqrt{a + x_n}, \text{ for } (n = 1, 2, \dots), \quad x_1 = \sqrt{a}. \quad (2.7)$$

Firstly, let us show that the sequence (2.7) is increasing and bounded. We do this by mathematical induction. Observe that $x_1 = \sqrt{a} < \sqrt{a + \sqrt{a}} = x_2$. Suppose that $x_n < x_{n+1}$. The inductive step is then to prove that $x_{n+1} < x_{n+2}$. By (2.7), the assertion $x_{n+1} < x_{n+2}$ is equivalent to

$$x_{n+1} = \sqrt{a + x_n} < \sqrt{a + x_{n+1}} = x_{n+2}.$$

But $x_n < x_{n+1}$ by hypothesis, and so $\sqrt{a + x_n} < \sqrt{a + x_{n+1}}$ as required for the inductive step. We next prove that it is bounded (for example, by $\sqrt{a+1}$). Clearly, $x_1 = \sqrt{a} < \sqrt{a+1}$. We proceed by mathematical induction and so assume that for some integer n the inequality

$$x_n < \sqrt{a+1} \quad (2.8)$$

holds, whereas $x_{n+1} = \sqrt{a + x_n}$, it follows from (2.8) that

$$x_{n+1} < \sqrt{a + \sqrt{a+1}} < \sqrt{a + 2\sqrt{a+1}} = \sqrt{(\sqrt{a+1})^2} = \sqrt{a+1}.$$

Then, by Theorem 2.5, the sequence $\{x_n\}$ approaches some limit point, x . In order to find x , we square both sides of (2.7). By passing to the limit as n approaches infinity, we have

$$\begin{aligned} x_{n+1}^2 &= a + x_n, \\ \lim_{n \rightarrow \infty} x_{n+1}^2 &= x^2 = a + x = a + \lim_{n \rightarrow \infty} x_n. \end{aligned}$$

The quadratic equation here, $x^2 - x - a = 0$, has a unique positive root ($x > 0$) $x = \frac{1}{2}(1 + \sqrt{1+4a})$, which is the required expression for the limit.

(b) Let $\{F_n\}$ be the Fibonacci sequence. Assume that $a = \lim_{n \rightarrow \infty} \left(\frac{F_{n+1}}{F_n} \right)$ exists. We show that $a = \frac{1}{2}(1 + \sqrt{5})$. Taking into account that $F_{n+1} = F_n + F_{n-1}$, the equation

$$a = \lim_{n \rightarrow \infty} \left(\frac{F_{n+1}}{F_n} \right)$$

becomes

$$a = \lim_{n \rightarrow \infty} \left(1 + \frac{F_{n-1}}{F_n} \right) = 1 + \frac{1}{a}.$$

Thus $a^2 = a + 1$ and so $a = \frac{1}{2}(1 + \sqrt{5})$ (note $a > 0$).

2.2 Infinitely Small and Infinitely Large Sequences; Subsequences

Definition 2.4. If, for a given sequence $\{x_n\}$, $\lim_{n \rightarrow \infty} x_n = 0$, then the sequence is said to be an *infinitely small* (or *infinitesimal*) sequence.

Observe that if $x_n \rightarrow a$, then $y_n = x_n - a$ is an infinitely small sequence. Conversely, if $x_n = a + y_n$ and y_n is an infinitely small sequence, then $x_n \rightarrow a$.

Moreover, the limit of a sequence $\{x_n\}$ is said to be $+\infty$ ($-\infty$) if for any positive real number M there is an integer $N = N(M)$ such that $|x_n| > M$ ($x_n < -M$) for all $n > N$, and is then written

$$\lim_{n \rightarrow \infty} y_n = +\infty \quad \left(\lim_{n \rightarrow \infty} x_n = -\infty \right) \quad \text{or} \quad x_n \rightarrow +\infty \quad (x_n \rightarrow -\infty).$$

Sometimes the sequences such that $\lim_{n \rightarrow \infty} |x_n| = +\infty$ are called *infinitely large sequences*.

Theorem 2.6. If $\{x_n\}$ and $\{y_n\}$ are two infinitely small sequences, then their sum and product are infinitely small sequences.

□ By the definition of infinitely small sequences, $\lim_{n \rightarrow \infty} x_n = 0$ and $\lim_{n \rightarrow \infty} y_n = 0$. Then, by Theorem 2.3, $\lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n = 0$ and $\lim_{n \rightarrow \infty} x_n \cdot y_n = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n = 0$. ■

Theorem 2.7. The product of a bounded and an infinitely small sequence is an infinitely small sequence.

□ Let $\{x_n\}$ be bounded and let $\{y_n\}$ be infinitely small. By hypothesis, for some real number $C > 0$, the inequality $|x_n| \leq C$ ($n = 1, 2, \dots$) holds and $\lim_{n \rightarrow \infty} y_n = 0$. Because $\{y_n\}$ is infinitely small, given any number $\varepsilon > 0$ there is an integer $N = N(\varepsilon)$ such that $|y_n| < \frac{\varepsilon}{C}$ for all $n > N$. For such integers,

$$|x_n y_n| = |x_n| |y_n| < C \cdot \frac{\varepsilon}{C}.$$

In other words, $\lim_{n \rightarrow \infty} x_n y_n = 0$. ■

Theorem 2.8. If $\lim_{n \rightarrow \infty} |x_n| = +\infty$ and $x_n \neq 0$ ($n = 1, 2, \dots$), then $\{y_n\} = \left\{ \frac{1}{x_n} \right\}$ is an infinitely small sequence.

□ By hypothesis, for any real number $\varepsilon > 0$ there is an integer N such that $|x_n| > 1/\varepsilon$ for all $n > N$. This implies $|y_n| < \varepsilon$ for all $n > N$, that is, $\lim_{n \rightarrow \infty} y_n = 0$. ■

Example 2.11. Suppose $x_n = n^2$, $y_n = 1/n$. Obviously, $\{x_n\}$ and $\{y_n\}$ are infinitely large and infinitesimal sequences, respectively. Observe that $\left\{ \frac{1}{x_n} \right\} = \left\{ \frac{1}{n^2} \right\}$ and $\left\{ \frac{1}{y_n} \right\} = n$. The first is an infinitesimal sequence, but the second is an infinitely large sequence.

Definition 2.5. Let $\{x_n\}$ be a sequence and let $k_1 < k_2 < \cdots < k_n < \cdots$ be an increasing sequence of natural numbers. Then $\{x_{k_n}\} = \{x_{k_1}, x_{k_2}, \dots, x_{k_n}, \dots\}$ is said to be a *subsequence* of the sequence.

Theorem 2.9. If $\lim_{n \rightarrow \infty} x_n = a$, then the limit of an arbitrary subsequence $\{x_{k_n}\}$ is the same a .

□ By hypothesis, for any given $\varepsilon > 0$ there is an integer $N = N(\varepsilon)$ such that $|x_n - a| < \varepsilon$ for any $n > N$. Then from the inequality $k_n \geq n$, it follows that $|x_{k_n} - a| < \varepsilon, n > N$, that is, $x_{k_n} \rightarrow a$. ■

Theorem 2.10 (Bolzano¹–Weierstrass²). Every infinite and bounded set in Euclidean space has at least one limit point.

□ Let for simplicity M be an infinite and bounded set of the real axis. Let I_1 be a closed interval including M , i.e., $I_1 = [a, b] \supset M$. Divide I_1 into two equal subintervals. Suppose I_2 is one of them, containing infinitely many points of M . Then again we can divide the interval I_2 into two equal subintervals such that one of them, say I_3 , contains infinitely many points belonging to M . For $k = 2, 3, 4, \dots$, repeat this bisection on I_k and choose I_{k+1} the same way. We have $I_j = [a_j, b_j]$ and $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$. By Cantor's Theorem (Theorem 1.7), since $\lim_{j \rightarrow \infty} (b_j - a_j) = \lim_{j \rightarrow \infty} \frac{b_j - a_j}{2^j} = 0$, the intersection $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point, say x . At the same time, x is a limit point of the set M : we see this as follows: since $\lim_{n \rightarrow \infty} I_n = 0$ for any small number $\varepsilon > 0$, there is an integer n such that $I_n \subset (x - \varepsilon, x + \varepsilon)$. In other words, any ε -neighborhood of the point x contains infinitely many points of M . ■

Corollary 2.1. If $\{x_n\}$ is bounded, then $\{x_n\}$ has a convergent subsequence.

□ At first, assume that $\{x_n\}$ has only a finite number of different points. Then in our constructed subsequence, at least one of those points can be repeated:

$$x = x_k = x_{k_1} = \cdots = x_{k_n} = \cdots$$

Therefore, the point x is a limit of such a subsequence $\{x_{k_n}\}$.

Now, suppose that the sequence $\{x_n\}$ has infinitely many different points. By hypothesis, the sequence is bounded, hence there is at least one limit point, say x . For any positive real number ε_1 ($\varepsilon_1 > 0$), the ε_1 -neighborhood of x contains infinitely many different points of the sequence. Let us denote by x_{k_1} one such point, and form the ε_2 ($\varepsilon_2 < \varepsilon_1$)-neighborhood of x that does not contain x_{k_1} . Similarly, in this ε_2 -neighborhood of x we can

¹B. Bolzano (1781-1848), Czech mathematician.

²K. Weierstrass (1815-1897), German mathematician.

take another point $x_{k_2} \neq x$, $k_2 > k_1$. Continuing this process, we form ε_i -neighborhoods of the point x , where $\varepsilon_i \rightarrow 0$. Finally, we take the subsequence $x_{k_1}, x_{k_2}, \dots, x_{k_n}, \dots$ of $\{x_n\}$, the limit point of which is the point x . ■

Remark 2.4. As is seen from the proof, this assertion is true for an infinite bounded sequence in the Euclidean space $\mathbb{R}^n$.

Let us denote by E the set of limit points of subsequences of the bounded sequence $\{x_n\}$. Obviously the set E is nonempty and bounded. In fact, if $x_{k_n} \rightarrow x$ ($x \in E$), then from the inequality $|x_{k_n}| \leq C$ we have $|x| \leq C$, where C is a positive constant. Since E is bounded, $\sup E$ and $\inf E$ exist.

Definition 2.6. The numbers $x^* = \sup E$ and $x_* = \inf E$ are called the *superior* and *inferior limits* of the sequence $\{x_n\}$, respectively. Further, we use the symbols

$$x^* = \overline{\lim}_{n \rightarrow \infty} x_n \quad \text{and} \quad x_* = \underline{\lim}_{n \rightarrow \infty} x_n.$$

If $\{x_n\}$ and $\{y_n\}$ are two sequences, we have

$$\underline{\lim}_{n \rightarrow \infty} x_n + \underline{\lim}_{n \rightarrow \infty} y_n \leq \underline{\lim}_{n \rightarrow \infty} (x_n + y_n) \leq \overline{\lim}_{n \rightarrow \infty} x_n + \underline{\lim}_{n \rightarrow \infty} y_n \leq \overline{\lim}_{n \rightarrow \infty} (x_n + y_n) \leq \overline{\lim}_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n$$

Example 2.12. Find the inferior and superior limits of the sequence $\{x_n\} = \{1 + (-1)^n\}$.

Obviously, for even ($n = 2k$) and odd ($n = 2k + 1$) numbers $\{x_{2k}\} = \{2\}$ and $\{x_{2k+1}\} = \{0\}$, respectively. Clearly, $E = \{2, 0\}$. Here, $\overline{\lim}_{n \rightarrow \infty} x_n = 2$, $\underline{\lim}_{n \rightarrow \infty} x_n = 0$ because $\sup E = 2$ and $\inf E = 0$.

2.3 Cauchy Sequences

Definition 2.7. A sequence $\{x_n\}$ is a *Cauchy*³ sequence if for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ for all $m, n > N(\varepsilon)$.

Let $x_n = \frac{1}{n^2}$. For $m, n > N = N(\varepsilon)$, say $m > n$, then we have

$$|x_m - x_n| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} \leq \frac{1}{N^2}.$$

For every $\varepsilon > 0$, we can take an integer $N > \frac{1}{\sqrt{\varepsilon}}$. Then $m, n > N$ implies $|x_m - x_n| \leq \frac{1}{N^2} < \varepsilon$, so $\{x_n\}$ is a Cauchy sequence.

³A. Cauchy (1789–1857), French mathematician.

Warning. This is stronger than claiming that, beyond a certain point in the sequence, consecutive terms are within ε of one another.

For example, consider the sequence $x_n = \sqrt{n}$. It is not hard to see that

$$|x_{n+1} - x_n| = (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Therefore $|x_{n+1} - x_n| \rightarrow 0$ as $n \rightarrow \infty$. But $|x_m - x_n| = \sqrt{m} - \sqrt{n}$ if $m > n$, and so for any integer $N = N(\varepsilon)$ we can choose $n = N$ and $m > N$ such that $|x_m - x_n|$ is as large as desired. Thus the sequence $\{x_n\}$ is not a Cauchy sequence.

Theorem 2.11. *If a sequence $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ is bounded.*

□ Let $\varepsilon > 0$ be any positive number and suppose $\{x_n\}$ is a Cauchy sequence, that is, for any there is an integer such that $|x_n - x_m| < \varepsilon$ for all. Let us take. Then $|x_n| = |x_{N+1} + (x_n - x_{N+1})| \leq |x_{N+1}| + |x_n - x_{N+1}| < |x_{N+1}| + \varepsilon$ for all $n > N$. Let $C = \max\{|x_{N+1}| + \varepsilon, |x_1|, \dots, |x_N|\}$. Then for all $n > N$, the inequality $|x_n| < |x_{N+1}| + \varepsilon$ implies the desired inequality $|x_n| \leq C$ ($n = 1, 2, \dots$). ■

Theorem 2.12. *A sequence $\{x_n\}$ converges if and only if $\{x_n\}$ is a Cauchy sequence.*

□ At first, let $\{x_n\}$ be convergent and $x_n \rightarrow x$. We show that $\{x_n\}$ is a Cauchy sequence. For a given $\varepsilon > 0$, we can choose an integer $N = N(\varepsilon)$, such that $|x_n - x| < \varepsilon/2$ whenever $n > N$. Then for all $n, m > N$, we have

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e., $\{x_n\}$ is a Cauchy sequence.

Conversely, let $\{x_n\}$ be a Cauchy sequence. We prove that $\{x_n\}$ is convergent.

Observe that, by Theorem 2.11, the sequence $\{x_n\}$ is bounded. Moreover, by Corollary 2.1, it has a subsequence $\{x_{k_n}\}$ that converges, say to x . Next, we show that $\lim_{n \rightarrow \infty} x_n = x$.

Let $\varepsilon > 0$ be arbitrary. Since $\{x_n\}$ is a Cauchy sequence, there is an integer n_0 such that $m, n > n_0$ yields $|x_n - x_m| < \varepsilon/2$. Moreover, $k_n \geq n$ and so from the last inequality, setting $m = k_n$ for all $n > n_0$, we have $|x_n - x_{k_n}| < \varepsilon/2$. Since the sequence $\{x_{k_n}\}$ is convergent, i.e., $x_{k_n} \rightarrow x$, there is an integer N_1 such that $|x_{k_n} - x| < \varepsilon/2$ for all $n > N_1$. Let $N = \max\{n_0, N_1\}$. Then for all $n > N$ we have

$$|x_n - x| \leq |x_n - x_{k_n}| + |x_{k_n} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is, $x_n \rightarrow x$. ■

Let us apply this theorem to show the divergence of the sequence $\{x_n\}$, where $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Let us take, in Definition 2.7, $m = 2n$. Then

$$|x_m - x_n| = |x_{2n} - x_n| = \frac{1}{n+1} + \cdots + \frac{1}{2n} \geq n \cdot \frac{1}{2n} = \frac{1}{2}$$

and so if we take $\varepsilon = \frac{1}{2}$, then there is no integer N such that for all $m, n > N$ ($m = 2n$) $|x_m - x_n| < \frac{1}{2}$. This means that $\{x_n\}$ is divergent.

Definition 2.8. The notion of a Cauchy sequence makes sense in the more general context of a metric space (X, ρ) . At first note that if $\{x_n\} \subset X$ and $x \in X$, then $x_n \rightarrow x$ if and only if $\rho(x_n, x) \rightarrow 0$. Then, by analogy, $\{x_n\}$ is a Cauchy sequence if $\rho(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

A metric space (X, ρ) is complete if every Cauchy sequence in X has a limit in X . At once we note that the Euclidean space $\mathbb{R}^n$ is complete.

Below we will see that the set of rational numbers with the corresponding metric is not a complete space.

Remark 2.5. We remark that Theorem 2.12 is sometimes valid in the more general case of a metric space (X, ρ) . In this case, in Definition 2.7, the distance function $\rho(x_n, x_m) < \varepsilon$ is used. For example, Theorem 2.12 is valid for the Euclidean space $\mathbb{R}^n$. But it is not true for arbitrary metric spaces, i.e., it happens that a Cauchy sequence might not converge to an element of the space in question. For example, let X be the set of all rational numbers. The distance between two points $r_1, r_2 \in X$ is given by $\rho(r_1, r_2) = |r_1 - r_2|$. Clearly X is a metric space, and $r_n = \left(1 + \frac{1}{n}\right)^n$ ($n = 1, 2, \dots$) are rational numbers. Observe that the sequence $\{r_n\}$ is a Cauchy sequence, but its limit point e ($e = 2,71828\dots$) is an irrational number. Finally, this sequence $\{r_n\}$ can not be considered as a convergent sequence within the space X . In order to prove the convergence $r_n \rightarrow e$, we need Theorem 2.5. Let $x_n = \left(1 + \frac{1}{n}\right)^{n+1}$.

We show that $\{x_n\}$ is a monotone decreasing sequence. We have

$$\begin{aligned} \frac{x_n}{x_{n-1}} &= \left(\frac{1 + \frac{1}{n}}{1 + \frac{1}{n-1}} \right)^n \left(1 + \frac{1}{n} \right) = \left(1 - \frac{1}{n^2} \right)^n \left(1 + \frac{1}{n} \right) \\ &< \left(1 - \frac{1}{n^2} \right)^n \left(1 + \frac{1}{n^2} \right)^n = \left(1 - \frac{1}{n^4} \right)^n < 1 \end{aligned}$$

that is, $x_n < x_{n-1}$.

Moreover, $x_n > 1$ ($n = 1, 2, \dots$) and so, by Theorem 2.5, $x_n \rightarrow e$ converges:

$$\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^{n+1}}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1}}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)} = e.$$

Note that in terms of sequences there is another definition of compactness than the one given in Section 1.6; a subset M of a metric space (X, ρ) is compact if every sequence in M has a subsequence which converges to an element of M . Compactness turns out to be a stronger condition than completeness, though in some arguments one notion can be used in place of the other.

2.4 Numerical Series; Absolute and Conditional Convergence

Suppose $\{x_n\}$ is an infinite sequence of real numbers. Then an infinite numerical series is an expression of the form,

$$\sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots + x_n + \cdots \quad (2.9)$$

The number x_n is called the n -th term of the series. An example of an infinite series is the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

This was mentioned in Section 2.1. The n -th term of this infinite series is $x_n = \frac{1}{2^n}$.

To say what is meant by such an infinite sum, let us introduce the partial sums of the series (2.9). The n -th partial sum S_n of the series (2.9) is the sum of its first n terms

$$S_n = x_1 + \cdots + x_n.$$

Thus, each infinite series has associated with it an infinite sequence of partial sums

$$S_1, S_2, S_3, \dots, S_n, \dots$$

Definition 2.9 (Sum of an infinite series). We say that the infinite series (2.9) *converges* (or *is convergent*), with sum S , provided that the limit of its sequence of partial sums $\{S_n\}$

$$\lim_{n \rightarrow \infty} S_n = S$$

exists (and of course, is finite). Otherwise, we say that the series (2.9) *diverges* (or *is divergent*). If a series diverges then it has no sum.

Thus an infinite sum is a limit of finite sums

$$\lim_{n \rightarrow \infty} S_n = S = \sum_{k=1}^{\infty} x_k$$

provided that this limit exists.

Theorem 2.13 (The n -th term test for divergence). *If the infinite series $\sum_{k=1}^{\infty} x_k$ is convergent, then the limit of the n -th term is equal to zero, i.e., $\lim_{n \rightarrow \infty} x_n = 0$.*

□ Observe that the n -th term x_n is the difference

$$x_n = S_n - S_{n-1}. \quad (2.10)$$

Under the assumption that the series is convergent,

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} S_{n-1} = S. \quad (2.11)$$

Taking into account (2.11), and then passing to the limit in (2.10), we have:

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = S - S = 0.$$

■

Consequently, if $\lim_{n \rightarrow \infty} x_n \neq 0$, then the series $\sum_{k=1}^{\infty} x_k$ diverges. In other words, the condition $x_n \rightarrow 0$ is necessary but not sufficient for convergence. That is, a series may satisfy the condition $\lim_{n \rightarrow \infty} x_n = 0$ and yet diverge. An important example of a divergent series with terms that approach zero is the harmonic series below.

Example 2.13. The following series is called the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots. \quad (2.12)$$

Since each term of the harmonic series is positive, its sequence of partial sums $\{S_n\}$ is monotone increasing. We shall prove that the harmonic series diverges, by showing that there are arbitrarily large partial sums. We have

$$\begin{aligned} S_2 &= 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}, \\ S_4 &= S_2 + \frac{1}{3} + \frac{1}{4} > S_2 + \left(\frac{1}{4} + \frac{1}{4}\right) = S_2 + \frac{1}{2} > \frac{3}{2}, \\ S_8 &= S_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > S_4 + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = S_4 + \frac{1}{2} > \frac{4}{2}, \\ &\dots \\ &\dots \\ S_{2^n} &> \frac{n+1}{2}. \end{aligned}$$

Now, if C is an arbitrarily large number and n is such that $\frac{n+1}{2} > C$, then $S_{2^n} > C$. This means that the sequence of partial sums is unbounded, i.e., $S_n \rightarrow \infty$.

Note that if the sequence of partial sums of the series (2.9) diverges to infinity, then we say that the series itself diverges to infinity, and we write $\sum_{n=1}^{\infty} x_n = \infty$.

Theorem 2.14 (Sum of the geometric series). *If $|x| \geq 1$ then the geometric series $\sum_{k=1}^{\infty} ax^{k-1}$ ($a \neq 0$) diverges and if $|x| < 1$ then the geometric series is convergent and its sum is $\frac{a}{1-x}$.*

□ The elementary identity

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

follows after multiplying left and right hand sides by $1 - x$.

Therefore, for $x \neq 1$, we have $S_n = \frac{a(1 - x^n)}{1 - x} = \frac{a}{1 - x} - \frac{a}{1 - x} x^n$, and if $|x| < 1$, then

$$S = \lim_{n \rightarrow \infty} S_n = \frac{a}{1 - x}.$$

If $|x| = 1$, that is, $x = 1$ or $x = -1$, then the n -th partial sums of the geometric series are $S_n = n \cdot a$ and $S_n = a - a + \cdots + (-1)^{n-1}a$, respectively. In the first case, $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} na = \pm\infty$ (the sign depending on whether a is positive or negative). This proves divergence. In the second case, the sequence of partial sums is

$$a, 0, a, 0, a, 0, \dots$$

which diverges. Finally, if $|x| > 1$ then $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} ax^{k-1} = \infty$, and so the series diverges. ■

Theorem 2.15. *If $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ are convergent series, that is, $\sum_{k=1}^{\infty} x_k = A$ and $\sum_{k=1}^{\infty} y_k = B$ with A, B real numbers, then the sum of these series, $\sum_{k=1}^{\infty} (x_k + y_k)$, is convergent and equal to $C = A + B$. Furthermore, if α is a nonzero constant, then the series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} \alpha x_k$ either both converge or both diverge. In the case of convergence, the sums are related by*

$$\sum_{k=1}^{\infty} \alpha x_k = \alpha \sum_{k=1}^{\infty} x_k = \alpha \cdot A.$$

□ Let A_n , B_n , and C_n be the n -th partial sums of the series $\sum_{k=1}^{\infty} x_k$, $\sum_{k=1}^{\infty} y_k$, and $\sum_{k=1}^{\infty} (x_k + y_k)$, respectively. Then by part (2) of Theorem 2.3,

$$C = \lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} (A_n + B_n) = \lim_{n \rightarrow \infty} A_n + \lim_{n \rightarrow \infty} B_n = A + B$$

On the other hand, by part (1) of Theorem 2.3,

$$\lim_{n \rightarrow \infty} \alpha A_n = \alpha \lim_{n \rightarrow \infty} A_n = \alpha A.$$

■

Further, we note that convergence or divergence is unaffected by deleting a finite number of terms from the beginning of a series, that is, for any positive integer N , the series

$$\sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots + x_n + \cdots$$

and

$$\sum_{k=N}^{\infty} x_k = x_N + x_{N+1} + x_{N+2} + \cdots$$

both converge or both diverge.

Theorem 2.16 (Cauchy's Test). *For the convergence of the series (2.9) it is necessary and sufficient that for any $\varepsilon > 0$ there is an integer N such that*

$$\left| \sum_{k=n+1}^{n+p} x_k \right| < \varepsilon \quad (2.13)$$

for all $n > N$ and all natural numbers p .

□ Let S_n be n -th partial sum of the series (2.9). By Theorem 2.12, the sequence $\{S_n\}$ converges if and only if $\{S_n\}$ is a Cauchy sequence, that is, for any number $\varepsilon > 0$, there is an integer N such that $|S_m - S_n| < \varepsilon$ for all $m, n > N$. Then by setting $m = n + p$ ($p \geq 1$) in the last inequality, we have $|S_{n+p} - S_n| = \left| \sum_{k=n+1}^{n+p} x_k \right| < \varepsilon$. ■

Definition 2.10. A series (2.9) is said to *converge absolutely* if the series of absolute values $\sum_{k=1}^{\infty} |x_k|$ converges.

Definition 2.11. A series (2.9) is said to be *conditionally convergent* if it is convergent, but not absolutely convergent, that is, the series $\sum_{k=1}^{\infty} |x_k|$ is divergent.

Theorem 2.17. *If the series $\sum_{k=1}^{\infty} |x_k|$ converges, so does the series $\sum_{k=1}^{\infty} x_k$.*

□ The series $\sum_{k=1}^{\infty} |x_k|$ is convergent and so, by Cauchy's test (Theorem 2.16), for any $\varepsilon > 0$ there is an integer N such that $\left| \sum_{k=n+1}^{n+p} x_k \right| < \varepsilon$ for all $n > N$ and all positive integers p .

Then, obviously,

$$\left| \sum_{k=n+1}^{n+p} x_k \right| \leq \sum_{k=n+1}^{n+p} |x_k| < \varepsilon.$$

Hence, the series (2.9) converges. ■

Theorem 2.18. (1) If (2.9) is a series with positive terms, and if there is a constant K such that $S_n \leq K$ for every n , then the series converges and the sum of the series S satisfies $S \leq K$. If no such K exists then the series diverges.

(2) If (2.9) is a convergent series with nonnegative terms, then the series $\sum_{k=1}^{\infty} y_k$ obtained from the series (2.9) by rearranging and renumbering its terms is also convergent having the same sum.

□ (1) If an infinite series has positive terms, $x_k \geq 0$ ($k = 1, 2, \dots$), then the partial sums S_n form an increasing sequence, that is $S_1 < S_2 < S_3 < \dots < S_n < \dots$. Then, since $S_n \leq K$, according to Theorem 2.4 the sequence of partial sums will converge to a limit S , satisfying $S \leq K$. If no such constant exists, then $\lim_{n \rightarrow \infty} S_n = \infty$.

(2) Let S be the sum of the series (2.9), that is $\lim_{n \rightarrow \infty} S_n = S$ and y_n be the partial sum of the series $\sum_{k=1}^{\infty} y_k$. Then for a fixed k we have

$$Y_k = y_1 + y_2 + \dots + y_k = x_{n_1} + x_{n_2} + \dots + x_{n_k}$$

Thus denoting $N = \max\{n_1, \dots, n_k\}$ we can write $Y_k \leq S_k \leq S$, which implies that the series $\sum_{k=1}^{\infty} y_k$ converges. So if Y is its sum we derive that $Y \leq S$. Reasoning vice versa, we conclude that $Y = S$ ■

Theorem 2.19 (Comparison Test). Suppose $|x_n| \leq y_n$ for all $n \geq n_0$, where n_0 is some natural number. Then the following assertions hold.

(1) If the “bigger series” $\sum_{k=1}^{\infty} y_n$ converges, then the “smaller series” $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(2) If the “smaller series” $\sum_{k=1}^{\infty} |x_k|$ diverges, then the “bigger series” $\sum_{k=1}^{\infty} y_n$ also diverges.

□ (1) If the series $\sum_{k=1}^{\infty} y_k$ is convergent, then the Cauchy test is satisfied and hence, for any $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that $0 \leq \left| \sum_{k=n+1}^{n+p} y_k \right| < \varepsilon$ for all $n > N(\varepsilon)$

and arbitrary natural number p . Then for all integers $n > \max\{N(\varepsilon), N_0\}$, the inequality $|x_n| \leq y_n$ holds, and so

$$\sum_{k=n+1}^{n+p} |x_k| \leq \sum_{k=n+1}^{n+p} y_k < \varepsilon.$$

Thus, the Cauchy test is satisfied by the series $\sum_{k=1}^{\infty} |x_k|$. Consequently, the series $\sum_{k=1}^{\infty} |x_k|$ converges and so $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(2) This part is really just an alternative phrasing of part (1). In fact, if the series $\sum_{k=1}^{\infty} |x_k|$ diverges, then $\sum_{k=1}^{\infty} y_k$ must diverge since the convergence of $\sum_{k=1}^{\infty} y_k$ would imply the convergence of $\sum_{k=1}^{\infty} |x_k|$, contrary to the hypothesis. ■

2.5 Alternating series; Convergence tests

Definition 2.12. A series whose terms are alternately positive and negative is called an *alternating series*. Such a series has the form:

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n + \cdots \quad (2.14)$$

where $a_k > 0$, $k = 1, 2, \dots$

Theorem 2.20 (Leibniz⁴). *An alternating series (2.14) converges if the following two conditions are satisfied:*

- (1) *The sequence $\{a_n\}$, $n = 1, 2, \dots$ is nonincreasing,*
- (2) $\lim_{n \rightarrow \infty} a_n = 0$.

□ Let $\{S_n\}$ be the sequence of n -th partial sums of (2.14). Consider the subsequences $\{S_{2m}\}$ and $\{S_{2m+1}\}$. Since $a_{2m+1} \geq a_{2m+2}$, the inequality $S_{2m+2} = S_{2m} + a_{2m+1} - a_{2m+2} \geq S_{2m}$ holds, which implies that the “even” subsequence $\{S_{2m}\}$ is nondecreasing. On the other hand, $S_{2m+1} = S_{2m-1} - a_{2m} + a_{2m+1} \leq S_{2m-1}$, and so the “odd” subsequence $\{S_{2m+1}\}$ is nonincreasing. Besides, $S_{2m+1} = S_{2m} + a_{2m+1} \geq S_{2m}$ and then, since $S_{2m} \leq S_{2m+1} \leq S_{2m-1} \leq \cdots \leq S_1 = a_1$, the “even” subsequence $\{S_{2m}\}$ is bounded above. According to Theorem 2.5, the subsequence $\{S_{2m}\}$ converges, say $S_{2m} \rightarrow S$. We next show that the sequence $\{S_{2m+1}\}$ also has the limit S . In fact, taking into account that $a_n \rightarrow 0$, we have $\lim_{n \rightarrow \infty} S_{2m+1} = \lim_{n \rightarrow \infty} S_{2m} + \lim_{n \rightarrow \infty} a_{2m+1} = S$. Then the convergence of both $S_{2m} \rightarrow S$ and $S_{2m+1} \rightarrow S$ implies that $S_n \rightarrow S$, so the series converges. ■

⁴G. Leibniz (1646-1716), German mathematician.

Remark 2.6. (a) It is not hard to see that Theorem 2.20 is valid in the nonincreasing case $a_n \geq a_{n+1} \geq 0$ ($n = 1, 2, \dots$), $n > N_0$, where n_0 is a positive integer.

(b) If an alternating series violates condition (2) of the alternating series test, then the series must diverge by the divergence test (Theorem 2.13). However, if condition (2) is satisfied, but (1) not, the series may either converge or diverge.

Example 2.14. (a) The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is called the alternating harmonic series. In the notation of Definition 2.12, $a_k = \frac{1}{k} > 0$. Clearly, $\frac{1}{k} = a_k > a_{k+1} = \frac{1}{k+1}$ ($k = 1, 2, \dots$), and $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{1}{k} = 0$. Thus the series converges by the alternating series (Leibniz) test. We emphasize that the series of absolute values is the harmonic series, which is diverges (Example 2.13). Hence, the alternating harmonic series is conditionally convergent (see Definition 2.11).

(b) The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ satisfies the hypothesis of Theorem 2.20, and so converges. In Section 5.6 we will see that the sum of this series is $\frac{\pi}{4}$.

Theorem 2.21 (The Ratio Test of D'Alembert⁵). *If, for all $n \geq n_0$ (n_0 an integer),*

$$\left| \frac{x_{n+1}}{x_n} \right| \leq \rho < 1 \quad (x_n \neq 0), \quad (2.15)$$

then the series (2.9) converges absolutely, and if

$$\left| \frac{x_{n+1}}{x_n} \right| \geq \rho > 1, \quad (2.16)$$

then the series (2.9) diverges.

□ It follows from condition (2.15) that for all $n > N_0$,

$$|x_n| \leq \rho |x_{n-1}| \leq \rho^2 |x_{n-2}| \leq \dots \leq \rho^{n-N_0} |x_{N_0}|.$$

Consider the series $\sum_{k=1}^{\infty} \rho^{k-N_0} |x_{N_0}|$ ($0 < \rho < 1$), which according to Theorem 2.14, is convergent. Then, taking into account the inequality $|x_n| \leq \rho^{n-N_0} |x_{N_0}|$, by Theorem 2.19 we conclude that the series $\sum_{k=1}^{\infty} |x_k|$, and so the series $\sum_{k=1}^{\infty} x_k$, are convergent. Thus, the first part of the theorem is proved. On the other hand, it follows from condition (2.16) that for all $n > N_0$,

$$|x_n| \geq \rho |x_{n-1}| \geq \dots \geq \rho^{n-N_0} |x_{N_0}| > |x_{N_0}| > 0,$$

and so $|x_n| > |x_{N_0}| > 0$. This implies that $\lim_{n \rightarrow \infty} x_n \neq 0$, and so, by Theorem 2.13, the series $\{x_n\}$ diverges. ■

⁵J. D'Alembert (1717-1783), French mathematician.

Corollary 2.2. The limiting case $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \rho$ of the D'Alembert ratio test consists of the following:

- (1) If $\rho < 1$, the series (2.9) converges absolutely.
- (2) If $\rho > 1$ or if $\rho = \infty$, then the series (2.9) diverges.
- (3) If $\rho = 1$, no conclusion about convergence can be drawn from this test.

Example 2.15. (a) For the series

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots,$$

the n -th partial sum is

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}.$$

Multiplying both sides of this equality by $1/10$, we have

$$\frac{1}{10} S_n = \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \frac{3}{10^{n+1}}.$$

Then, by subtracting the second equality from the first, we can define the n -th partial sum $S_n = \frac{1}{3} \left(1 - \frac{1}{10^n} \right)$. Thus, the sum of the series is $S = \lim_{n \rightarrow \infty} S_n = \frac{1}{3}$.

(b) Let us see whether the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges or diverges. Since the n -th partial sum

$$S_n = \sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1},$$

the limit $\lim_{n \rightarrow \infty} S_n = 1$. Notice that, since $x_n = \frac{1}{n(n+1)}$, the limit that occurs in D'Alembert's

test is $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+2} = 1$.

On the other hand we showed that (see Example 2.13) the Harmonic series is divergent and since $x_k = \frac{1}{k}$, the limit occurring in the D'Alembert ratio test is $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$, also. Consequently, in both cases $\rho = 1$, yet one of them converges, while the other diverges. Hence, no conclusion can be drawn from D'Alembert's test when $\rho = 1$.

Example 2.16. The series $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges by the D'Alembert ratio test since $\rho =$

$\lim_{k \rightarrow \infty} \frac{x_{k+1}}{x_k} = \lim_{k \rightarrow \infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k \rightarrow \infty} \frac{(k+1)^k}{k^k} = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k = e$. Thus $\rho = e > 1$ and the series diverges.

Theorem 2.22 (The Root Test of Cauchy). *If, for all $n \geq n_0$ (n_0 is some integer),*

$$\sqrt[n]{|x_n|} \leq \rho < 1, \quad (2.17)$$

then the series (2.9) converges absolutely.

On the other hand, if for all $n \geq n_0$,

$$\sqrt[n]{|x_n|} \geq \rho > 1, \quad (2.18)$$

then the series diverges.

□ Firstly, it follows from condition (2.17) that $|x_n| \leq \rho^n$. Thus the series $\sum_{k=1}^{\infty} |x_k|$ is eventually dominated by the convergent geometric series $\sum_{k=1}^{\infty} \rho^k$ ($0 < \rho < 1$). Hence, $\sum_{k=1}^{\infty} |x_k|$ converges, and so the series $\sum_{k=1}^{\infty} x_k$ converges absolutely. On the other hand, if (2.18) holds, then $|x_n| > \rho^n > 1$ ($n \geq n_0$) and hence $\lim_{n \rightarrow \infty} x_n \neq 0$. Therefore the n -th term divergence test (Theorem 2.13) implies that the series $\sum_{k=1}^{\infty} x_k$ diverges. ■

Corollary 2.3. *Suppose that $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \rho$. Then the infinite series $\sum_{k=1}^{\infty} x_k$*

(1) Converges absolutely if $\rho < 1$,

(2) Diverges if $\rho > 1$.

If $\rho = 1$, the root test is inconclusive.

Even though the ratio test is generally simpler to apply than the root test, there are certain series for which the root test succeeds while the ratio test fails.

Example 2.17. (a) In the Example 2.15 it was shown that the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges. Now, applying the root test, we can see that this test is inconclusive. In fact,

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n(n+1)}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n}} \cdot \sqrt[n]{\frac{1}{n+1}} = 1 \quad (\text{see Example 2.7}), \text{ that is, } \rho = 1.$$

(b) Consider the series $\sum_{k=0}^{\infty} \frac{1}{2^{k+(-1)^k}} = \frac{1}{2} + \frac{1}{1} + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \dots$. Then

$$\frac{x_{n+1}}{x_n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even,} \\ \frac{1}{8} & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, the limit required for the ratio test does not exist. On the other hand,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \lim_{n \rightarrow \infty} \left(\frac{1}{2^{n+(-1)^n}} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{2^{(-1)^n/n}} \right) = \frac{1}{2},$$

so the series converges by the root test.

Example 2.18. (a) The series $\sum_{k=1}^{\infty} \left(\frac{3k-2}{k+1} \right)^k$ diverges by the root test since

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{3n-2}{n+1} \right)^n} = \lim_{n \rightarrow \infty} \frac{3n-2}{n+1} = 3 > 1.$$

(b) Obviously the series $\sum_{k=1}^{\infty} k$ is divergent. Nevertheless, $\rho = \lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

By setting $x_n = n$, the proof of the next theorem can be obtained from Kummer's⁶ theorem⁷

Theorem 2.23 (Raabe⁸ test). Let $\sum_{k=1}^{\infty} x_k$ be a series with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \left[\frac{x_n}{x_{n+1}} - 1 \right] \cdot n = q.$$

Then:

- (1) if $q > 1$, the series converges;
- (2) if $q < 1$, the series diverges.

Example 2.19. Let us verify by the Raabe test that the series $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k}$ diverges. We can easily establish that the quotient occurring in the test is

$$\frac{x_k}{x_{k+1}} = \frac{2k+2}{2k+1}.$$

Then,

$$q = \lim_{n \rightarrow \infty} \left[\frac{2n+2}{2n+1} - 1 \right] n = \frac{1}{2}.$$

Since $q < 1$, the series diverges.

Theorem 2.24 (The Limit Comparison Test). Let $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ be series with positive terms and assume that $\rho = \lim_{k \rightarrow \infty} \frac{x_k}{y_k}$. If ρ is finite and $\rho \neq 0$, then either the series both converge or they both diverge.

□ By the definition of limit, for any $\varepsilon > 0$ there is a positive integer N such that

$$\left| \frac{x_k}{y_k} - \rho \right| < \varepsilon \text{ for all } k > N \text{ or } \rho - \varepsilon < \frac{x_k}{y_k} < \rho + \varepsilon \text{ for all } k > N.$$

Setting $\varepsilon = \rho/2$ (ρ a positive number) in the last inequality, we have

$$\frac{1}{2} \rho y_k < x_k < \frac{3}{2} \rho y_k, \quad k > N.$$

⁶E. Kummer (1810–1893), German mathematician

⁷See, Marian Nureshan, *A Concrete Approach to Classical Analysis*, Springer New, York, 2009.

⁸J. Raabe (1801–1859), German mathematician

Now, if $\sum_{k=1}^{\infty} y_k$ converges, then $\sum_{k=1}^{\infty} x_k$ is eventually dominated by the convergent series $\sum_{k=1}^{\infty} \frac{3}{2} \rho y_k$, so by Theorem 2.15, the series $\sum_{k=1}^{\infty} x_k$ converges. If $\sum_{k=1}^{\infty} y_k$ diverges, then $\sum_{k=1}^{\infty} x_k$ eventually dominates the divergent series $\sum_{k=1}^{\infty} \frac{1}{2} \rho y_k$, so Theorem 2.15 implies that $\sum_{k=1}^{\infty} x_k$ also diverges. Thus the convergence of either series implies the convergence of the other.

■

Example 2.20. Show that the series $\sum_{k=1}^{\infty} \frac{1}{k - \frac{1}{4}}$ diverges. Let us take $x_k = \frac{1}{k - \frac{1}{4}}$ and $y_k = \frac{1}{k}$. Then, by Theorem 2.24, since

$$\rho = \lim_{k \rightarrow \infty} \frac{k}{k - \frac{1}{4}} = \lim_{k \rightarrow \infty} \frac{1}{1 - \frac{1}{4k}} = 1,$$

the series is divergent.

■

Let $\sum_{k=1}^{\infty} x_k$ be a series and $\{m_k\}$ be an arbitrary sequence of natural integers such that every natural number take place only once. In other words $\{m_k\}$ is some rearrangement of the natural numbers. Setting $y_k = x_{m_k}$, consider the new numerical series $\sum_{k=1}^{\infty} y_k$. The series $\sum_{k=1}^{\infty} y_k$ is obtained as a result of rearrangement of the terms of the series $\sum_{k=1}^{\infty} x_k$, and contains each of the terms of the first series once only.

A very important property of a sum of a finite number of summands is the commutative property, that is, a rearrangement of the terms does not affect their sum. The following example shows that this is not the case for arbitrary series.

Example 2.21. Consider the alternating harmonic series (Example 2.14 (a)): denote its sum by S . Rearrange this series according to the rule: two negative terms after a positive one. Then we can write

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots \quad (2.19)$$

Denote by S_n the n -th partial sum of series (2.19). Then we have

$$\begin{aligned} S_{3n} &= \sum_{k=1}^n \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \right) = \sum_{k=1}^n \left(\frac{1}{4k-2} - \frac{1}{4k} \right) \\ &= \frac{1}{2} \sum_{k=1}^n \left(\frac{1}{2k-1} - \frac{1}{2k} \right) = \frac{1}{2} S_{2n} \end{aligned}$$

Finally, we can write

$$S_{3n} = \frac{S_{2n}}{2}, \quad S_{3n-1} = \frac{S_{2n}}{2} + \frac{1}{4n}, \quad S_{3n-2} = \frac{S_{2n}}{2} + \frac{1}{(4n-2)}. \quad (2.20)$$

and then from (2.20) we deduce that

$$\lim_{n \rightarrow \infty} S_{3n} = \lim_{n \rightarrow \infty} S_{3n-1} = \lim_{n \rightarrow \infty} S_{3n-2} = \frac{S}{2}.$$

Thus we have found that the series (2.19) converges and its sum is $\frac{S}{2}$. ■

But the next theorem of Cauchy shows that for absolutely convergent series, any series obtained from it by rearrangement has the same sum as the original series.

Theorem 2.25 (Cauchy). *The rearrangement of an absolutely convergent series supplies another absolutely convergent series having the same sum as the original one.*

□ Let $\sum_{k=1}^{\infty} x_k$ be an absolutely convergent series. Consider the positive and negative terms of this series:

$$x_k^+ = \begin{cases} x_k, & \text{if } x_k \geq 0, \\ 0, & \text{if } x_k < 0, \end{cases} \quad x_k^- = \begin{cases} -x_k, & \text{if } x_k \leq 0, \\ 0, & \text{if } x_k > 0. \end{cases} \quad (2.21)$$

Obviously, $x_k = x_k^+ - x_k^-$. Take the following two series with nonnegative terms $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$. Since $x_k^+ \leq |x_k|$ and $x_k^- \leq |x_k|$, these series are convergent. Let $\sum_{k=1}^{\infty} y_k$ be the rearranged series of $\sum_{k=1}^{\infty} x_k$. In view of (2.21) by analogy, for it construct the series positive and negative parts $\sum_{k=1}^{\infty} y_k^+$, $\sum_{k=1}^{\infty} y_k^-$, respectively. Then by using the Theorem 2.18 it is easy to see that

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} (x_k^+ - x_k^-) = \sum_{k=1}^{\infty} x_k^+ - \sum_{k=1}^{\infty} x_k^- = \sum_{k=1}^{\infty} y_k^+ - \sum_{k=1}^{\infty} y_k^- = \sum_{k=1}^{\infty} (y_k^+ - y_k^-) = \sum_{k=1}^{\infty} y_k.$$
■

Theorem 2.26 (Riemann⁹). *Let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series. Choose an $S \in [-\infty, \infty]$. Then there is a rearrangement of the series such that the resulting series converges to S .*

□ We split the series $\sum_{k=1}^{\infty} x_k$ into two series $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ as introduced in the proof of previous theorem (see (2.21)). Show that if $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series, both series $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ generated by it are divergent, where $x_n^+ \rightarrow 0$, $x_n^- \rightarrow 0$ as $n \rightarrow \infty$. By Theorem 2.25 it is not hard to see, that for series $\sum_{k=1}^{\infty} x_k$ to be absolutely convergent it

⁹G. Riemann (1826–1866), German mathematician

is necessary and sufficient that series $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ generated by it be convergent. By condition of theorem our series is a conditionally convergent series. Then it follows, that at least one of the series $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ generated by it is divergent, that is, $\sum_{k=1}^{\infty} x_k^+ = \infty$.

Let us study the behavior of the right-hand side of sum $\sum_{k=1}^n x_k^- = \sum_{k=1}^n x_k^+ - \sum_{k=1}^n x_k$ as $n \rightarrow \infty$.

Since $\sum_{k=1}^{\infty} x_k$ is convergent, the second sum tends to infinite number. The first sum increases to ∞ . Consequently, the sum $\sum_{k=1}^{\infty} x_k^-$ increases to ∞ as $n \rightarrow \infty$. As a result if one of the series

$\sum_{k=1}^{\infty} x_k^+$, $\sum_{k=1}^{\infty} x_k^-$ is divergent, the other is divergent, too. We remark, that since the series $\sum_{k=1}^{\infty} x_k$ is convergent, the sequences $\{x_n^+\}$ and $\{x_n^-\}$ tend to zero. It remains to show, that for any $S \in [-\infty, \infty]$ we can construct a series

$$x_1^+ + x_2^+ + \cdots + x_{m_1}^+ - x_1^- - x_2^- - \cdots - x_{n_1}^- \\ + x_{m_1+1}^+ + x_{m_2+2}^+ + \cdots + x_{m_2}^+ - x_{n_1+1}^- - \cdots - x_{n_2}^- + \cdots \quad (2.22)$$

whose sum is equal to S . Series (2.22) contains all the terms of $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ only once. At first we assume, that S is a real finite number. In this case the indices $m_1, m_2, \dots$ and $n_1, n_2, \dots$ can be chosen as the smallest natural numbers for which the corresponding inequalities

$$\beta_1 = \sum_{k=1}^{m_1} x_k^+ > S, \quad \beta_2 = \beta_1 - \sum_{k=1}^{n_1} x_k^- < S, \\ \beta_3 = \beta_2 + \sum_{k=m_1+1}^{m_2} x_k^+ > S, \quad \beta_4 = \beta_3 - \sum_{k=n_1+1}^{n_2} x_k^- < S, \quad (2.23)$$

are fulfilled. Since the series $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ have positive terms and diverge, at the p -th step of this construction we indeed can choose the natural numbers m_p and n_p satisfying the p -th inequality. Then, taking into account the inequalities (2.23) and the fact that $x_n^+ \rightarrow 0$, $x_n^- \rightarrow 0$ as $n \rightarrow \infty$ it follows that the series thus constructed converges to S . Note that in the case $S = +\infty$ we can replace S in the right-hand side of the inequalities (2.23) by a divergent sequence of the form 2, 1, 4, 3, 5, $\dots$ ■

Theorem 2.27 (Dirichlet¹⁰). (1) Let a series $\sum_{k=0}^{\infty} x_k$ be given, and suppose that the sequence of n -th partial sums $\{A_n\}$ of this series is bounded.

(2) Suppose that the sequence $\{y_n\}$ ($n = 0, 1, 2, \dots$), is nondecreasing and $\lim_{n \rightarrow \infty} y_n = 0$.

Then the series $\sum_{k=0}^{\infty} x_k y_k$ converges.

¹⁰P.G. Le Jeune-Dirichlet (1805–1859), German mathematician

□ Let $A_n = \sum_{k=0}^n x_k$, $n \geq 0$ and $A_{-1} = 0$. Then for $0 \leq p \leq q$, we have

$$\begin{aligned} \sum_{n=p}^q x_n y_n &= \sum_{n=p}^q (A_n - A_{n-1}) y_n = \sum_{n=p}^q A_n y_n - \sum_{n=p-1}^{q-1} A_n y_{n+1} \\ &= \sum_{n=p}^{q-1} A_n (y_n - y_{n+1}) + A_q y_q - A_{p-1} y_p. \end{aligned} \quad (2.24)$$

On the other hand, by condition (1) of the theorem, there is a positive constant C such that $|A_n| \leq C$ ($n = 0, 1, 2, \dots$). For any number $\varepsilon > 0$ there is a positive integer $N = N(\varepsilon)$ such that $y_n \leq \frac{\varepsilon}{2C}$. Then, since $y_n - y_{n+1} \geq 0$ for $N \leq p \leq q$, it follows from (2.24) that

$$\left| \sum_{n=p}^q x_n y_n \right| = \left| \sum_{n=p}^{q-1} A_n (y_n - y_{n+1}) + A_q y_q - A_{p-1} y_p \right| \leq C \left| \sum_{n=p}^{q-1} (y_n - y_{n+1}) + y_q + y_p \right|.$$

But $\left| \sum_{n=p}^{q-1} (y_n - y_{n+1}) + y_q + y_p \right| = 2y_p$ and so $\left| \sum_{n=p}^q x_n y_n \right| \leq 2C y_p \leq 2C y_N \leq \varepsilon$.

Then by the Cauchy criterion (Theorem 2.16), the series $\sum_{k=0}^{\infty} x_k y_k$ converges. ■

Theorem 2.28 (Cauchy's Condensation Test). *Let $\{x_n\}$ be a nonincreasing series which is bounded below by 0. Then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k x_{2^k} = x_1 + 2x_2 + 4x_4 + \dots \quad (2.25)$$

converges.

□ By Theorem 2.5, it suffices to consider the boundedness of the sequences $\{A_n\}$. ($A_n = x_1 + \dots + x_n$). Let S_k be the n -th partial sum of the series (2.25):

$$S_k = x_1 + 2x_2 + \dots + 2^k x_{2^k}$$

For $n < 2^k$

$$A_n \leq x_1 + (x_2 + x_3) + \dots + (x_{2^k} + \dots + x_{2^{k+1}-1}) \leq x_1 + 2x_2 + \dots + 2^k x_{2^k} = S_k.$$

On the other hand, for $n > 2^k$,

$$\begin{aligned} A_n &\geq x_1 + x_2 + (x_3 + x_4) + \dots + (x_{2^{k-1}+1} + \dots + x_{2^k}) \\ &\geq \frac{1}{2} x_1 + x_2 + 2x_4 + \dots + 2^{k-1} x_{2^k} = \frac{1}{2} S_k \end{aligned}$$

From the obtained inequalities, $A_n \leq S_k$ and $2A_n \geq S_k$ we have that the sequences $\{A_n\}$ and $\{S_k\}$ are simultaneously either bounded or unbounded. ■

The harmonic series is a special case, with $p = 1$, of a class of series called the p -series, or, *hyperharmonic series*, $\sum_{n=1}^{\infty} \frac{1}{n^p}$. We will determine the convergence or divergence of the hyperharmonic series using the previous theorem. (Later, in Section 9.8, applying the integral test, we will again investigate the convergence of this series.)

Theorem 2.29. *The hyperharmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

□ If $p \leq 0$, its divergence follows by the n -th term test for convergence (Theorem 2.13). If $p > 0$, then by Theorem 2.28, setting $x_{2^k} = \frac{1}{2^{kp}}$, we are led to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{k(1-p)}.$$

Obviously, $2^{(1-p)} < 1$ if and only if $1 - p < 0$, and the result follows by comparison with the geometric series (Theorem 2.14), taking $a = 1$ and $x = 2^{1-p}$. ■

2.6 Products of Series

Definition 2.13. Let $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ be two series. Then the series $\sum_{k=1}^{\infty} z_k$, where

$$z_k = \sum_{i=0}^k x_i y_{k-i} = x_0 y_k + x_1 y_{k-1} + \cdots + x_k y_0, \quad (2.26)$$

is called the product of the two given series.

Theorem 2.30. *Suppose the series $\sum_{k=0}^{\infty} x_k$ and $\sum_{k=0}^{\infty} y_k$ are convergent and their sums are A and B , respectively, and one of them, say $\sum_{k=0}^{\infty} x_k$, converges absolutely. Then the series $\sum_{k=0}^{\infty} z_k$ converges and has the sum $C = AB$.*

□ Let us denote $A_n = \sum_{k=0}^n x_k$, $B_n = \sum_{k=0}^n y_k$, and $C_n = \sum_{k=0}^n z_k$. By Definition 2.13,

$$\begin{aligned} C_n &= x_0 y_0 + (x_0 y_1 + x_1 y_0) + \cdots + (x_0 y_n + x_1 y_{n-1} + \cdots + x_n y_0) \\ &= x_0 B_n + x_1 B_{n-1} + \cdots + x_n B_0 = x_0 (B + R_n) + x_1 (B + R_{n-1}) + \cdots + x_n (B + R_0) \\ &= A_n B + x_0 R_n + x_1 R_{n-1} + \cdots + x_n R_0, \end{aligned}$$

where $R_n = B_n - B \rightarrow 0$.

Therefore,

$$C_n - A_n B = x_0 R_n + \cdots + x_n R_0 = r_n. \quad (2.27)$$

Now, we should show that $\lim_{n \rightarrow \infty} r_n = 0$.

Let us denote $Q = \sum_{k=0}^{\infty} |x_k|$. Since $R_n \rightarrow 0$, for a given $\varepsilon > 0$ we can choose a positive integer N such that $|R_n| < \varepsilon/(2Q)$ for all $n > N$.

For $n > N$, we have

$$\begin{aligned} |r_n| &\leq |R_0x_n + \cdots + R_Nx_{n-N}| + |R_{N+1}x_{n-N-1} + \cdots + R_nx_0| \\ &\leq |R_0x_n + \cdots + R_Nx_{n-N}| + \frac{\varepsilon}{2Q}(|x_{n-N-1}| + \cdots + |x_0|) \\ &\leq |R_0x_n + \cdots + R_Nx_{n-N}| + \frac{\varepsilon}{2}. \end{aligned} \quad (2.28)$$

Since $x_n \rightarrow 0$, for fixed N we can find an integer n_0 such that for all $n > N_0$,

$$|R_0x_n + \cdots + R_Nx_{n-N}| \leq \frac{\varepsilon}{2}. \quad (2.29)$$

Then the inequalities (2.28) and (2.29) imply that $|r_n| < \varepsilon$ for all $n > \max\{N, N_0\}$. Thus $r_n \rightarrow 0$. Finally, by passing to the limit in (2.27) we have $C = A \cdot B$. ■

Remark 2.7. In the above proved Theorem 2.30, it is essential that one of the series converges absolutely. For example, the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$ converges by Theorem 2.20, since it is an alternating series, $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}} = 0$, and $a_{k+1} = \frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k}} = a_k$ ($k = 1, 2, \dots$). But the convergence is not absolute. In fact, $\frac{1}{k} < \frac{1}{\sqrt{k}}$ and by the Comparison Test (Theorem 2.19), the series $\sum_{k=0}^{\infty} \left| \frac{(-1)^k}{\sqrt{k+1}} \right|$ diverges. Setting $x_k = y_k = \frac{(-1)^k}{\sqrt{k+1}}$, we form the product series of the given series with itself,

$$\begin{aligned} \sum_{n=0}^{\infty} z_n &= 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}} \right) \\ &\quad - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}} \right) + \cdots \end{aligned}$$

so that

$$z_n = (-1)^n \sum_{k=0}^n \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

Since

$$(n-k+1)(k+1) = \left(\frac{n}{2} + 1 \right)^2 - \left(\frac{n}{2} - k \right)^2 \leq \left(\frac{n}{2} + 1 \right)^2,$$

we have

$$|z_n| \geq \sum_{k=0}^n \frac{2}{n+2} = \frac{2(n+2)}{n+2},$$

so that $\lim_{n \rightarrow \infty} z_n \neq 0$, and by the n -th term divergence test (Theorem 2.13), the series $\sum_{n=0}^{\infty} z_n$ diverges.

Definition 2.14. Let $\{x_n\}$ be a numerical series. Then an expression of the form

$$\prod_{k=1}^{\infty} x_k = x_1 x_2 \cdots x_k \cdots \quad (2.30)$$

is called the *infinite product* of the numbers x_n , $n = 1, 2, 3, \dots$. The number $P_n = \prod_{k=1}^n x_k = x_1 x_2 \cdots x_n$ is called the *n-th partial product* of the infinite product (2.30).

Definition 2.15. The infinite product (2.30) is said to be convergent if the sequence of n -th partial sums $\{P_n\}$ converges, i.e., $P_n \rightarrow P$ ($P \neq 0$). In this case we write

$$\prod_{k=1}^{\infty} x_k = P.$$

Theorem 2.31. If the infinite product (2.30) converges, then $\lim_{n \rightarrow \infty} x_n = 1$.

□ According to Definition 2.15, the infinite product converges if $P_n \rightarrow P$ ($P \neq 0$). Clearly, the limit of the subsequence is the same, P . But $\frac{P_n}{P_{n-1}} = x_n$, and so

$$\lim_{n \rightarrow \infty} x_n = \frac{\lim_{n \rightarrow \infty} P_n}{\lim_{n \rightarrow \infty} P_{n-1}} = 1.$$

■

Remark 2.8. It follows from the theorem that there is an integer N_0 such that $x_n > 0$ for all $n > N_0$. Hence, without loss of generality, we may assume $x_k > 0$ for all k .

Moreover, the convergence of the infinite product with $x_k > 0$ can be reduced to the convergence of the series

$$\sum_{k=1}^{\infty} y_k, \quad \text{where } y_k = \ln x_k. \quad (2.31)$$

In reality, we observe that if P_n and y_n are the n -th partial product and n -th partial sum of the infinite product (2.30) and (2.31), respectively, then

$$Y_n = \ln x_1 + \ln x_2 + \cdots + \ln x_n = \ln x_1 x_2 \cdots x_n = \ln P_n$$

and so $P_n = e^{Y_n}$. If now there is the limit $\lim_{n \rightarrow \infty} y_n = Y$, then $P = e^Y$. ■

The following example shows that, generally, the converse statement of Theorem 2.31 does not hold. It occurs because the condition $\lim_{n \rightarrow \infty} x_n = 1$ is only a necessary condition for convergence of infinite products, and is not a sufficient condition.

Example 2.22. (1) Consider the infinite product $\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)$.

The n -th partial product is

$$P_n = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) = \prod_{k=1}^{\infty} \frac{k+1}{k} = \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n+1}{n} = 1.$$

Evidently, $P_n \rightarrow +\infty$ when $n \rightarrow \infty$, and so the infinite product diverges. Nevertheless,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1.$$

(2) Test for convergence the infinite product $\prod_{k=1}^{\infty} \left(1 - \frac{1}{(k+1)^2}\right)$. The n -th partial product is

$$P_n = \prod_{k=1}^n \left(1 - \frac{1}{(k+1)^2}\right) = \prod_{k=1}^n \frac{k(k+2)}{(k+1)^2} = \frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdots \frac{n(n+2)}{(n+1)^2} = \frac{n+2}{2(n+1)}.$$

Obviously, $\lim_{n \rightarrow \infty} P_n = \frac{1}{2}$. Thus, the infinite product converges. Naturally, the limit of the n -th partial product is equal to one: $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{(n+1)^2}\right) = 1$.

Remark 2.9. Generally, the definitions, concepts, and properties of the sequences and series of real numbers are also valid for sequences and series of complex numbers (with the possible exception of some properties connected with comparison operations and the order relations on the real line). For example, it is known that necessary and sufficient conditions for the convergence of sequences of complex numbers $\{z_n\}$ ($z_n = a_n + ib_n$) consist of the same conditions as for the real series $\{a_n\}$ and $\{b_n\}$ ($n = 1, 2, \dots$). Since convergence of the series of real numbers is given by means of sequences, the convergence of series $\sum_{k=0}^{\infty} z_n$ is defined by analogy.

2.7 Problems

Using the definition of limits prove the assertions in (1)–(5).

- (1) $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$.
- (2) $\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$ ($a > 0$).
- (3) $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n!}} = 0$.
- (4) $\lim_{n \rightarrow \infty} \frac{n^k}{a^n} = 0$ ($a > 1$).
- (5) $\lim_{n \rightarrow \infty} \frac{2^n}{n!} = 0$.
- (6) (a) For all numerical sequences $\{x_n\}$ and $\{y_n\}$, prove that

$$\overline{\lim}_{n \rightarrow \infty} (x_n + y_n) \leq \overline{\lim}_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n.$$

- (b) Prove that if for a sequence $\{x_n\}$, any sequence whatever, $\{y_n\}$, would provide that

$$\overline{\lim}_{n \rightarrow \infty} (x_n + y_n) = \overline{\lim}_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n,$$

then the sequence $\{x_n\}$ must be convergent.

- (7) Prove that for the existence of the limit (finite or infinite) of a sequence $\{x_n\}$, the condition $\underline{\lim}_{n \rightarrow \infty} x_n = \overline{\lim}_{n \rightarrow \infty} x_n$ is necessary and sufficient.

- (8) Prove that $\lim_{n \rightarrow \infty} \left(\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \right) = 0$.

Suggestion: See Example 5 of Section 1.7.

- (9) (a) Prove that $\lim_{n \rightarrow \infty} \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \right) = e$.

Suggestion: Use the fact that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$.

- (b) Prove that for a positive integer p , $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \frac{1}{p+1}$.

- (10) Prove that if some subsequence of the monotone sequence $\{x_n\}$ converges, then the sequence $\{x_n\}$ is convergent.

- (11) If $x_n \rightarrow a$, then investigate the limit $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}$.

- (12) Find the limits $\underline{\lim}_{n \rightarrow \infty} x_n$ and $\overline{\lim}_{n \rightarrow \infty} x_n$ for the sequence $x_n = \frac{n}{n+1} \sin^2 \frac{n\pi}{4}$.

Answer: 0; 1.

- (13) Prove the Stolz theorem, if (a) $y_{n+1} > y_n$ ($n = 1, 2, \dots$), (b) $\lim_{n \rightarrow \infty} y_n = +\infty$, and

(c) $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$ exists, then $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$.

- (14) Prove that if $\{x_n\}$ converges, then the sequence $t_n = \frac{1}{n}(x_1 + \cdots + x_n)$ ($n = 1, 2, \dots$) converges and $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} x_n$.

Suggestion: Use Problem (13).

- (15) Let the sequence $\{x_n\}$ satisfies the condition $0 \leq x_{m+1} \leq x_m + x_n$ ($m, n = 1, 2, \dots$).

Prove that $\lim_{n \rightarrow \infty} \frac{x_n}{n}$ exists.

- (16) Prove that if $\lim_{n \rightarrow \infty} x_n = +\infty$, then $\lim_{n \rightarrow \infty} \frac{x_1 + \cdots + x_n}{n} = +\infty$.

- (17) Let the sequence $\{x_n\}$ be defined as follows:

$$x_1 = a, \quad x_2 = b, \quad x_n = \frac{x_{n-1} + x_{n-2}}{2} \quad (n = 3, 4, \dots).$$

Find $\lim_{n \rightarrow \infty} x_n$.

Answer: $\frac{1}{3}(a + 2b)$.

- (18) If the sequence $\{x_n\}$ converges and $x_n > 0$ ($n = 1, 2, \dots$), prove that

$$\lim_{n \rightarrow \infty} \sqrt[n]{x_1 \cdots x_n} = \lim_{n \rightarrow \infty} x_n.$$

(19) Find the limit $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right)$.

Answer: $\ln 2$.

(20) Let the sequence $\{x_n\}$ be defined as $x_0 > 0$, $x_{n+1} = \frac{1}{2} \left(x_n + \frac{9}{x_n} \right)$ ($n = 0, 1, 2, \dots$).

Prove that $\lim_{n \rightarrow \infty} x_n = 3$.

Find the sums of the series in problems (21)–(24).

(21) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^{n-1}}{2^{n-1}} + \cdots$.

Answer: $\frac{2}{3}$.

(22) $\left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{2^2} + \frac{1}{3^2} \right) + \cdots + \left(\frac{1}{2^n} + \frac{1}{3^n} \right) + \cdots$.

Answer: $\frac{3}{2}$.

(23) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$.

Answer: 1.

(24) $\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \cdots + \frac{2n-1}{2^n} + \cdots$.

Answer: 3.

(25) Determine whether the series $\sum_{n=1}^{\infty} \sin nx$ converges or diverges.

Suggestion: For the values $x \neq k\pi$ (k integer) $\lim_{n \rightarrow \infty} \sin nx \neq 0$.

Determine whether the series $\sum_{n=1}^{\infty} x_n$ converges or diverges.

(26) $x_n = \frac{1}{\sqrt{(2n-1)(2n+1)}}$.

Answer: Diverges.

(27) $x_n = \frac{n}{2n-1}$.

Answer: Diverges.

(28) $x_n = \ln \left(\frac{n+1}{n} \right)$.

Answer: Diverges.

(29) $x_n = \frac{1}{(2n-1)^2}$.

Answer: Converges.

Suppose that the series $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} y_n^2$ are convergent. Determine whether the following series converge or diverge.

(30) $\sum_{n=1}^{\infty} \frac{|x_n|}{n}$.

Answer: Converges.

$$(31) \sum_{n=1}^{\infty} |x_n y_n|.$$

Answer: Converges.

$$(32) \sum_{n=1}^{\infty} (x_n + y_n)^2.$$

Answer: Converges.

Use Cauchy's criterion to determine the convergence of each series $\sum_{n=1}^{\infty} x_n$ in (33)–(35).

$$(33) x_n = \frac{\sin nx}{2^n}.$$

$$(34) x_n = \frac{\cos x^n}{n^2}.$$

Suggestion: $\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \quad (n = 2, 3, \dots).$

$$(35) x_n = \frac{a^n}{10^n}, \quad |a| < 10.$$

Use the convergence tests to determine whether the series $\sum_{n=1}^{\infty} x_n$ converges or diverges, where:

$$(36) x_n = \frac{n!}{n^n}.$$

Answer: Converges.

$$(37) x_n = \frac{(n!)^2}{2^{n^2}}.$$

Answer: Converges.

$$(38) x_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_n.$$

Answer: Converges.

Suggestion: $\sqrt{2} = 2 \cos \frac{\pi}{4}.$

$$(39) x_n = \frac{1}{\sqrt[n]{\ln n}}.$$

Answer: Diverges.

$$(40) \sum_{n=1}^{\infty} \frac{(-1)^{[\ln n]}}{n}.$$

Answer: Diverges.

$$(41) \sum_{n=1}^{\infty} \sin n^2.$$

Answer: Diverges.

Suggestion: Show that $\lim_{n \rightarrow \infty} \sin n^2 \neq 0.$

Determine whether the series in problems (42)–(44) converge absolutely or converge conditionally.

$$(42) \sum_{n=2}^{\infty} \frac{\sin \frac{n\pi}{12}}{\ln n}.$$

Answer: Converges conditionally.

$$(43) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n \sin^{2n} x}{n}.$$

Answer: (a) Converges absolutely, if $|x - \pi k| < \frac{\pi}{4}$ (k integer).

(b) Converges conditionally, if $x = \pi k \pm \frac{\pi}{4}$.

$$(44) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}.$$

Answer: (a) Converges absolutely, if $p > 1$.

(b) Converges conditionally, if $0 < p \leq 1$.

(45) Show that the product of the divergent series

$$1 - \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \quad \text{and} \quad 1 + \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{n-1} \left(2^n + \frac{1}{2^{n+1}}\right)$$

converges absolutely.



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