

Preface

I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me

Isaac Newton (1643–1727)

*Nature and Nature's laws lay hid in night;
God said, Let Newton be! and all was light.*

Alexander Pope (1688–1744)

In an era of improving science and technology, the training of competent experts becomes increasingly important for a rising standard of living. Therefore, undergraduate and post-graduate students as well as scientists and engineers should improve their mastery of the various subsections of mathematics, especially of mathematical analysis, the base of mathematics.

In general, mathematical analysis consists of the function theory of real and complex variables, the theory of sequences and series, differential or integral equations, and the variational calculus. It provides general methods for the solution of various problems of both pure and applied mathematics.

Briefly, this book covers real and complex numbers, vector spaces, sets and some of their topological properties, series and sequences of functions (including complex-valued functions and functions of a complex variable), polynomials and interpolation, differential and integral calculus, extrema of functions, and applications of these topics.

The book differs from conventional studies in its content and style. Special attention is given to the right placement of subjects and problems. For instance, the properties of the

Newton, Riemann, Stieltjes, and Lebesgue integrals are handled and explained together. As is well known, mathematical analysis is also referred to as infinitesimal (incalculably small) analysis; but statements involving phrases such as “infinite numbers” may lead to serious misunderstandings. To prevent such ambiguity, we extend the idea of the number of elements in a set from finite sets to infinite sets.

In this book, approximate calculation (which plays an ever-increasing role in science) is mentioned and some care is devoted to interpolation methods and the calculation of integrals, elementary functions, and the other calculations of mathematical analysis.

Complex numbers and the corresponding formulas of De Moivre, logarithms of complex numbers, antilogarithms of complex variables, etc., are ordinary tasks for any teacher of school-mathematics. On the other hand, especially for the theory of analytic functions, there are important practical applications of the theory of complex variables to hydro- and aero-mechanics, integral calculus, algebra, and stability problems. Therefore, the arguments in this book are elaborated for complex variables.

One of the eventual objectives of this book is the application of these methods of mathematical analysis. Applied problems are presented in almost every chapter after the third, and they consist in finding the constrained or unconstrained extreme value of a function of a single variable. For instance, one of the historically important problems is the problem of Euclid, where in a given triangle it is required to inscribe the parallelogram with greatest area. Another is Johannes Kepler’s problem: in a ball with radius R , inscribe the cylinder with the greatest volume. These are both examples of extremum problems of a function of one variable. It should be noted that the required parallelogram in Euclid’s problem is the parallelogram with the midpoints of the triangle’s sides as three of its vertices. And the volume in Kepler’s problem is $4\pi R^3/3\sqrt{3}$.

The famous brachistochrone problem was posed by the illustrious Swiss mathematician Johann Bernoulli in 1696: this problem consists in finding the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time. The solution of this problem is a cycloid: this requires finding an extremum of a function of infinitely many variables. The history of mathematics made a sudden jump: from dimension one it passed at once to infinite dimensions, or from the theory of extrema of functions of one variable to the theory of problems of the type of the brachistochrone problem, i.e., to the calculus of variations, as this branch of mathematics was called in the 18th century. Historically, the year 1696 marks the birth of the calculus of variations.

This book consists of ten chapters, which include 181 theorems, 125 definitions, 261 solved examples, 575 answered problems, 41 figures and 87 remarks. Generally, these problems have the purpose of determining whether or not the concepts have been understood both theoretically and intuitively. The theorems, remarks, results, examples, and formulas are numbered separately in each chapter. For example, Theorem 8.3 indicates the third theorem within Chapter 8. The proofs begin with the empty square “ $\square$ ” and end with the Halmos box, “ $\blacksquare$ ”. They are presented in such a way that they hold good for the corresponding multi-dimensional generalization almost without any changes.

The major features of our book which, from a methodological and conceptual point of view, make it unique are the following:

- The power conception of sets, countable and uncountable sets, the cardinality of the continuum; Dedekind completeness theorem for the set of real numbers;
- Products of series;
- Polynomials and interpolations, Lagrange and Newton interpolation formulas;
- Functions with bounded variation;
- Newton and Stieltjes integrals;
- Lebesgue measure theory in line and Lebesgue integrals;
- Additional applications: potential and kinetic energy, a body in the earth’s gravitational field, escape velocity, nuclear reactions;
- The uncountable Cantor set with zero measure. Existence of non-measurable sets in line;
- The existence, at the end of each chapter, a plenty of problems with answers.

In the first chapter are given the main definitions and concepts of sets, some topological properties of such sets as metric spaces, the concepts of surjective, injective, and bijective functions, countability and uncountability of sets, cardinal number, and the cardinality of the continuum. These concepts are the very basis of mathematical analysis and in other fields of the mathematical sciences. For this reason, different theorems connected with the concepts of countability and uncountability of sets are proved, and it is shown that the cardinality of the set of functions defined on a closed interval is greater than the cardinality of the continuum. Furthermore, the Dedekind completeness theorem for the set of real numbers is formulated.

In the second chapter, the basic properties of monotone, strictly monotone, and Cauchy sequences are given in detail. Moreover, different types of convergence as well as differ-

ent convergence tests (Cauchy, D'Alembert, Leibniz, Raabe) are given, and the harmonic series, the geometric series, and products of series are considered. By Riemann's Theorem, for any conditionally convergent series and an arbitrary number $S \in [-\infty, \infty]$ there is a rearrangement of the given series such that the resulting series converges to S . But the rearrangement of an absolutely convergent series yields another absolutely convergent series having the same sum as the original one. The convergence or divergence of the hyperharmonic series with general term $1/n^p$ is determined, depending on p . Lastly, it is also mentioned that the analogous properties of series with complex terms hold.

In the third chapter, the equivalency of the limit concepts of Cauchy and Heine is shown. Then continuous and uniformly continuous functions, the main properties of sequences of functions and series of functions, and the concepts of pointwise and uniform convergence are studied. It is shown that under some conditions a sequence of functions defined in a closed interval has a uniformly convergent subsequence of functions (Arzela's theorem). At the end of the chapter it is mentioned that some of the features of function sequences and series generalize to the case of complex-valued functions.

In the fourth chapter the important conceptions of mathematical analysis, such as the derivative and differential, with their mechanical and geometrical meanings, are given. Derivatives of product functions (Leibniz's formula), composite functions, and inverse functions are treated. Then the concept of the derivative is applied to study the free fall of a mass, linear motion at velocities near the speed of light, and applications in numerical analysis. It is also applied to curvature and evolvents.

In the fifth chapter, the theorems of Rolle, Lagrange, and Cauchy, the mean value theorem, and Fermat's theorem for local extreme values are presented. Taylor's formula, given with the remainder term in Lagrange's, Cauchy's, and Peano's forms, is given, and the reason is indicated why sometimes the Taylor series of a function does not converge to the function itself. Furthermore, the Maclaurin series expansions of some elementary functions are given and Euler's formulas are proved.

In the sixth chapter, the factorization problem for polynomials having repeated real and complex roots is studied, and Lagrange's and Newton's interpolation formulas are given. Then, a useful formula for differential calculus is derived by using Newton's interpolation formula. At the end of the chapter, the approximation by polynomials of a given function defined on a closed interval is studied.

In the seventh chapter, L'Hopital's rule and Taylor-Maclaurin's series are employed for the computation of limits of different indeterminate forms. By using the Taylor series

at critical points, sufficient conditions for a local extremum are obtained. Moreover, the determination of whether the curve is bending upward or downward depends on the sign of higher order derivatives. Then, necessary and sufficient conditions for the existence of straight line and parabolic asymptotes to a curve are proved.

In the eighth chapter, we give the properties of indefinite integrals which are the basis of mathematical analysis. It is shown that sometimes the integrals of the elementary functions cannot be evaluated in terms of finite combinations of the familiar algebraic and elementary transcendental functions. But the methods of substitution and integration by parts can often be used to transform complicated integration problems into simpler ones. Moreover, the method of integrating rational functions that are expressible as a sum of partial fractions of the first and second types is presented. In particular, the Ostrogradsky method of integration is studied. The evaluation of integrals involving irrational algebraic functions, quadratic polynomials, trigonometric functions, etc., is also considered.

In the ninth chapter, firstly, Newton's and Riemann's integrals, the conditions for their equality, and mean value theorems are studied. In term of a partition of a given interval and its mesh, necessary and sufficient conditions for existence of the Riemann integral are formulated, and the Fundamental Theorem of Calculus is proved. Furthermore, with the use of the concept of measure zero, the Lebesgue criterion is formulated. It is also shown that the measure of the Cantor set (which has the cardinality of the continuum) is zero. Moreover, the Lebesgue integral, the integrability of Dirichlet's and Riemann's functions, functional series, the improper integral (in the sense of Cauchy principal values), the Stieltjes integral (with the use of a function of bounded variation), and numerical integration are studied.

Finally, the tenth chapter begins by using the length, area, and volume concepts to compute arc lengths, surface areas of surfaces of revolution, and volumes by the method of cylindrical shells in Cartesian and polar coordinates. Then, centroids of plane regions and curves, and the moments of a curve and of a solid are calculated. It is shown that the definite integral can be applied to the law of conservation of mechanical energy, to Einstein's theory of relativity, and to the determination of the escape velocity from the earth. Finally, problems connected with atomic energy, critical mass, and atomic reactors are considered. In fact, the basic cause of the destabilization and explosion of the reactor of Chernobyl's nuclear power station in the Ukraine in 1986 is explained.

This book could be all the more useful for the disciplines of mathematics, physics, and engineering, for the logical punctiliousness of these features. For this reason, it can be

suggested as a text-book or a reference book for students and lecturers in faculties of mathematics, physics, and engineering, as well as engineers who want to extend their knowledge of mathematical analysis. References [1]–[36] may be consulted for discussed topics pertinent to calculus.

The book was written as a result of my experience lecturing at Azerbaijan State University, Istanbul University, and Istanbul Technical University. I express my gratitude to the Institute of Cybernetics of the National Academy of Sciences of Azerbaijan for providing a rich intellectual environment, and facilities indispensable for the writing of this textbook.

Professor Elimhan N. Mahmudov

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Mahmudov, E.

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