Chapter IX.

Technical applications.

§1. The most important formula in the theory of the top. Generalities on stabilization by the gyroscopic effect.

We cannot regret, in the interest of the subject matter, that this concluding volume of our work has appeared so late. Indeed, the principal technical applications that we consider here have arisen only in the last ten years, during the printing of this book; we recall the high-speed railway, the gyroscopic ship stabilizer, and the gyroscopic compass.

The principle that we previously adopted as the basis of our entire presentation—the preeminence of the concept of the impulse—is particularly confirmed in the explanation of these applications. We arrive immediately from this principle to the formula upon which rests the theory of almost all technical applications.

Since we wish the developments of this chapter to prove fruitful in the hands of engineers and to be understandable without extensive study of the previous chapters, it appears appropriate to summarize here, in brief, the formulas and concepts that serve as their basis.

The impulse (or, more precisely, the "moment of the impulse") is the moment of the quantities of motion of the individual mass elements of the rotating rigid body, with respect to the fixed point of the body. If we consider a rapidly spinning rotor or wheel, then the rotation about the symmetry axis (the "figure axis") so exceeds the other rotations imparted to the body (the "top") that the plane of this moment stands perceptibly perpendicular to the figure axis, and thus falls perceptibly in the "equatorial plane of the top." The impulse vector that is erected perpendicular to this plane then falls perceptibly onto the figure axis. The component of the impulse with respect to this axis, compared to

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which the other components may generally be neglected, is called the "eigenimpulse" N. The eigenimpulse is the driving moment ("angular momentum") that we transfer to the top when starting it with a string, and which is maintained in the gyroscopic ship stabilizer by the turbine drive or in railway wheels by the pulling force of the locomotive.

Corresponding to the previous preference for left-handed coordinate systems and the clockwise sense of rotations (cf., for example, Fig. 3 on p. 18), we draw the impulse vector toward the side seen from which the rotation about the figure axis, the "eigenrotation," occurs in the clockwise sense.

The introduction of the impulse now provides the possibility of formulating the fundamental laws for the dynamics of the top just as simply, and with almost the same words, as we formulate the fundamental laws for the dynamics of a single mass particle: *the force-free top moves so that its impulse remains constant in magnitude and direction in space* (corresponding to the Galilean law of inertia for a single mass particle); *under the influence of external forces, the top moves in such a way that the rate of change of the impulse vector is equal in magnitude and direction to the moment of the external forces* (likewise represented by a vector) *with respect to the support point of the top* (Newtonian law of acceleration*) for a mass particle).

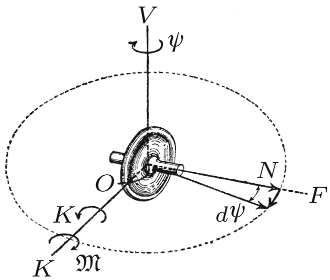

Fig. 113.

I. The most important formula in the theory of the top. We consider a rotor that has been given the eigenimpulse N (equal to the moment of inertia about the figure axis times the angular velocity about the same axis), and imagine that its figure axis is horizontal (Fig. 113). We turn the axis about the vertical through O (turning angle ψ, angular velocity $d\psi/dt$) without influencing the magnitude of the eigenimpulse. What is the required moment?

The answer is provided by our impulse theorem: in the time dt, the endpoint of the vector N describes the path $N\,d\psi$ perpendicular to the figure axis in the horizontal plane. Its rate of change is therefore

$$N\frac{d\psi}{dt}.$$

*) This designation is actually not historically correct. Newton speaks in his lex secunda not of the acceleration, but rather of the change of the "quantitas motus"; that is, the change of the impulse. The concept of the impulse is generally preeminent in the Newtonian principles.

This is, at the same time, the external moment $\mathfrak{M}$ that we must apply in order to effect the change of the impulse. The external moment acts about the axis OK that is perpendicular to the figure axis OF and the vertical OV; we previously called this axis the "line of nodes." The sense of the moment is seen in the figure. If we represent $\mathfrak{M}$ by a vector, then this vector has the same direction as the change of the impulse, and therefore points in the figure to the front. There corresponds a turning arrow that surrounds the semiaxis OK in the clockwise sense.

The moment $\mathfrak{M}$ is opposed by the *gyroscopic effect*; that is, the inertial effect of the spinning rotor that we must continuously overcome if we move the rotor in the described manner. We call this effect K, and have

$$K = N\frac{d\psi}{dt}. \tag{I}$$

The sense of the gyroscopic effect is opposite to that of $\mathfrak{M}$, and is thus counterclockwise.

The gyroscopic effect thus strives to upright the axis of the rotor and place it in equi-orientational parallelism with the axis of the added rotation, in such a manner that the sense of the eigenrotation would coincide with that of the added rotation. The magnitude of this effort is determined by (I); *it is held in equilibrium by the external moment* $\mathfrak{M}$.

It is to be noted that a vertical impulse $A\,d\psi/dt$ is indeed required for the initiation, but not for the maintenance, of the rotation about the vertical ($A =$ the moment of inertia of the top about an equatorial axis). The total impulse of the top thus consists of the resultant of this vertical component and the horizontal eigenimpulse N. In that we neglect the former component and identify the total impulse with the eigenimpulse in the application of our impulse theorem, we commit an error, and implicitly assume that the rotational velocity $d\psi/dt$ is small in comparison with the eigenrotation. Our formula (I) is therefore approximate (cf. the rigorous formula (III)), but is well suited for application to practical cases.

II. Generalization of this formula. We again imagine that the top turns about the vertical, but now in such a way that the figure axis does not sweep the horizontal plane, but rather is inclined to the vertical at an arbitrary angle ϑ. If its length is constant, the eigenimpulse N describes a circular cone, and its endpoint describes a circle of radius $N \sin\vartheta$ (cf. Fig. 114). During the time dt, the path of the impulse endpoint has length $N \sin\vartheta\, d\psi$, and its velocity is $N \sin\vartheta\, d\psi/dt$, parallel to the line of nodes. The magnitude and the axis of the external moment $\mathfrak{M}$ that is to be applied for the maintenance of the motion are thus determined, as are the magnitude and axis of the *gyroscopic effect*

$$\text{(II)} \qquad K = N \sin\vartheta \frac{d\psi}{dt}$$

with which the top resists the rotation about the vertical. The sense of the gyroscopic effect is again (cf. Fig. 114) described by the tendency to equi-orientational parallelism. Formula (II) is only approximately valid; it assumes that the eigenrotation is large compared with the added rotation.

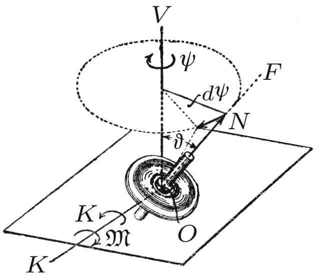

Fig. 114.

III. *Rigorous expression for the inertial effect of the top.* In spite of its limited importance, the general and rigorously valid value of the inertial effect, which is composed of the previously considered gyroscopic effect and an additional centrifugal effect, is also derived under the assumptions that the figure axis is inclined by the angle ϑ with respect to the vertical and the rotation $d\psi/dt$ is uniform, so that the impulse moves without change of its magnitude on a circular cone about the vertical.

The path element of the impulse endpoint I (cf. Fig. 115) is $R\, d\psi$, where $R = CI$ is the radius of the circle described by I. The applied moment $\mathfrak{M}$ and the opposed inertial effect K are determined as above. They therefore depend merely on the calculation of the radius R.

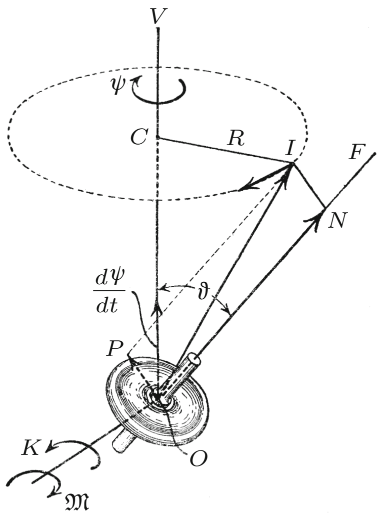

Fig. 115.

We draw the angular velocity $d\psi/dt$ as a rotation arrow in the vertical direction and decompose this arrow, since the vertical is not a principal axis for the mass distribution of the rotor, into the two components

$$\sin\vartheta \frac{d\psi}{dt} \quad \text{and} \quad \cos\vartheta \frac{d\psi}{dt}$$

with respect to the equatorial plane and the figure axis. The corresponding components of the impulse are obtained by multiplication with the respective moments of inertia. If the moment of inertia about an equatorial axis is denoted by A, then the component of the impulse perpendicular to the figure axis, which was neglected with respect to the eigenimpulse in the preceding approximate consideration, is

$$OP = A \sin\vartheta \, \frac{d\psi}{dt}.$$

The endpoint of the impulse vector I is obtained if one sets $OP = NI$ in the figure and adds this component that is perpendicular to the figure axis to the eigenimpulse $N = ON$. The radius R is now given, if we project the line segments ONI onto the direction of CI, by

$$R = ON \sin\vartheta - NI \cos\vartheta = N \sin\vartheta - A \sin\vartheta \cos\vartheta \, \frac{d\psi}{dt}.$$

The desired inertial effect is thus

$$\text{(III)} \qquad K = R\,\frac{d\psi}{dt} = \left(N - A\cos\vartheta\,\frac{d\psi}{dt}\right)\sin\vartheta\,\frac{d\psi}{dt}.$$

The additional term

$$-A\sin\vartheta\cos\vartheta\left(\frac{d\psi}{dt}\right)^2$$

that is found here is known from the theory of the simple spherical pendulum; it is designated there as the moment of the centrifugal force. If, namely, the eigenimpulse of the top vanishes ($N = 0$), then the top swings like a spherical pendulum with moment of inertia A. We can imagine realizing this by a simple pendulum with length l and mass m, so that

$$ml^2 = A.$$

The horizontal centrifugal force is then

$$Z = ml\sin\vartheta\left(\frac{d\psi}{dt}\right)^2,$$

and its moment about the line of nodes is

$$ml\sin\vartheta\left(\frac{d\psi}{dt}\right)^2 l\cos\vartheta = A\sin\vartheta\cos\vartheta\left(\frac{d\psi}{dt}\right)^2,$$

which is exactly the expression above. The negative sign of our additional term also conforms with this consideration, in that the moment of the centrifugal force of the pendulum strives to distance it from the vertical, and thus has the opposite sense as the first term in (III) and the arrow K in Fig. 115.

It is to be remarked, however, that the separation of the inertial effect (III) into the gyroscopic effect and the centrifugal effect is not absolute, but rather depends on the fact that we have distinguished the figure axis in the computation of the impulse.

IV. Stabilization by the gyroscopic effect. One of the most striking and well-known consequences of the theory of the top is the possibility of stabilizing an unstable or neutral degree of freedom by the installation of a rotating mass. The schema of this procedure, for which the discussions of this chapter will provide many examples, may be represented on the basis of our formula (I) by an example in the following manner.

The position of the figure axis in the horizontal plane (cf. Fig. 116), which is measured by the angle ψ, is, in itself (that is, for a nonspinning rotor), indifferent: a turning-moment Ψ about the vertical causes an angular displacement ψ that is determined in terms of the equatorial moment of inertia A about the vertical by the acceleration equation

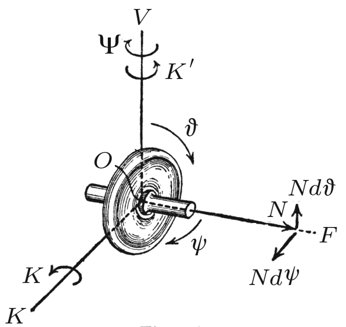

Fig. 116.

$$\text{(IVa)} \qquad A\,\frac{d^2\psi}{dt^2} = \Psi.$$

If the rotor spins, however, then this angular displacement is bound with a gyroscopic effect K that is determined by (I). *We assume that the figure axis is no longer fixed in the horizontal plane*, as we assumed in (I), *but rather that it can follow, by means of an appropriate suspension, its tendency to parallelism with the vertical rotation axis.* The inclination ϑ with respect to the vertical will then decrease according to the acceleration equation

$$\text{(IVb)} \qquad A\frac{d^2\vartheta}{dt^2} = -K = -N\,\frac{d\psi}{dt}.$$

The angular velocity of the elevation of the figure axis is thus

$$\text{(IVc)} \qquad \frac{d\vartheta}{dt} = -\frac{N}{A}\,\psi,$$

assuming that we measure the azimuth ψ from a position in which $d\vartheta/dt$ is equal to zero.

Now, however, the continuation of the rotation $d\vartheta/dt$ about OK would require an upward change of position of the impulse N, just as the rotation $d\psi/dt$ requires a change of position of N in the horizontal plane. Thus a gyroscopic effect again occurs, and indeed about the axis OV that is perpendicular to OK and OF. We call this gyroscopic effect K', and again determine it by formula (I) as

$$\text{(IVd)} \qquad K' = N\,\frac{d\vartheta}{dt}.$$

Corresponding to the tendency to equi-orientational parallelism, the gyroscopic effect K' acts in the opposite sense to that of the original rotation $\dfrac{d\psi}{dt}$, and is therefore counterclockwise in Fig. 116. The countermoment K' is therefore joined to the turning-moment Ψ.

The equation of motion (IVa) now becomes[279]

$$\text{(IVe)} \qquad A\,\frac{d^2\psi}{dt^2} = \Psi + K' = \Psi + N\,\frac{d\vartheta}{dt}.$$

The schema of stabilization by the gyroscopic effect is now given in equations (IVe) and (IVb). With the use of equation (IVc), namely, equation (IVe) becomes

$$\text{(IV)} \qquad A \cdot \frac{d^2\psi}{dt^2} = \Psi - \frac{N^2}{A}\psi.$$

Equation (IV) expresses the possibility of a stable oscillation in the coordinate ψ.

If, namely, the eigenimpulse N is sufficiently large, then the countermoment K' exceeds the original deflecting moment Ψ to such an extent that we can, for qualitative questions, neglect Ψ with respect to K'. The right-hand side of (IV) is then negative for positive ψ, and positive for negative ψ.

The axis of the rotor therefore strives to return to its initial position, exactly like a stable pendulum; it overruns this position, again approaches it, etc. It possesses "a specific capability for resistance against a change of direction, a kind of absolute orientation in space," as stated at the conclusion of Chapter III.

Our derivation of this possibility of stabilization reveals, however, an additional necessary condition of fundamental significance. The axis of the top must have the possibility to deflect in the vertical direction; the rotation $d\vartheta/dt$ must actually be made possible by the suspension of the rotor. If the axis of the top remains restricted to the horizontal plane, then $d\vartheta/dt$ and our stabilizing countermoment K' become zero, and only the gyroscopic effect K occurs, which is expressed as a pressure against the guide that restricts the freedom of motion of the top. The top must therefore have full freedom of motion (two degrees of freedom for the motion of the figure axis in the horizontal and vertical directions, and a third degree of freedom for rotation about the figure axis) if it is to act as a stabilizer. We thus state the following principle:

Stabilization is possible only by a top with t h r e e degrees of freedom. If one degree of freedom of the top is eliminated, then the gyro-

scopic effect produced by this degree of freedom ceases to act, and the "specific capability of resistance against a change of direction" is lost. A top with t w o *degrees of freedom thus follows the impulses that act on it without resistance. If one degree of freedom is impeded but not entirely eliminated, there remains a certain capability of resistance that is smaller than that for unimpeded freedom of motion.*

If, for example, we clamp the inner ring in Fig. 23 and thus eliminate the possibility of rotation about the axis ST, the handle may be turned, when the rotor is spinning, as if it did not contain a top. The reaction forces on the ring then produce the change of position of the impulse vector that is required by the guidance of the figure axis on the prescribed path. If, on the other hand, the axis ST rotates in its bearings with considerable friction, then the countermoment K' will be significantly smaller than what corresponds to equation (IV). In this case, namely, the available impulse change that is given by the right-hand side of equation (IVd) is applied only in small part for the generation of the angular velocity $d\psi/dt$, and in large part for the overcoming of the friction that opposes the rotation $d\vartheta/dt$.

Equations (IV), since they are derived on the basis of formula (I), are burdened with an imprecision to which we will immediately return, and hold only for the time interval in which the figure axis stands approximately perpendicular to the vertical. For now, a qualitative conclusion may be drawn with the use of our imprecise equations.

If we assume, for example, that the deflecting moment Ψ is constant, then equation (IV) can be integrated immediately, and yields

$$\psi = \frac{A\Psi}{N^2} + a\cos\frac{N}{A}t + b\sin\frac{N}{A}t;$$

if, moreover, we set ψ and $\dfrac{d\psi}{dt}$ equal to zero for $t = 0$, then $b = 0$ and $a = -A\Psi/N^2$, and therefore

$$\text{(IVf)} \qquad \psi = \frac{A\Psi}{N^2}\Big(1 - \cos\frac{N}{A}t\Big).$$

The figure axis thus yields somewhat to the moment Ψ*, and indeed in the mean, for large* N*, by the small angle* $\psi_m = A\Psi/N^2$*; the turning angle oscillates periodically between its original value* 0 *and the maximum value* $2\psi_m$*.*

At the same time, however, the figure axis is elevated; according to equation (IVc), namely,

$$\frac{d\vartheta}{dt} = -\frac{\Psi}{N}\Big(1 - \cos\frac{N}{A}t\Big),$$

so that if ϑ is initially equal to $\pi/2$, then

$$\vartheta = \frac{\pi}{2} - \frac{\Psi}{N}t + \frac{A\Psi}{N^2}\sin\frac{N}{A}t. \tag{IVg}$$

This motion is not purely periodic, but rather has a principal component given by the first two terms that increases with time and is overlaid with small oscillations represented by the last term. The amplitude and period of this oscillation coincide with those of the angle ψ.

For very large N, both oscillations become imperceptibly small (in the same manner as the nutation of the pseudoregular precession in Ch. V, §2), and the elevation of the axis of the top occurs very slowly. Only in this case are the assumptions of our calculation for the application of the formula (I) fulfilled with sufficient accuracy for a not too long observation time t.

If, on the other hand, we would proceed rigorously and consider the elevation of the figure axis that appears in our last example, we must determine the gyroscopic effect K not from equation (I), but rather from equation (III); further, we must take from the gyroscopic effect K', which has as its axis the common perpendicular to OF and OK, only the component that acts about OV. Then, however, the eigenimpulse N is also not constant, but rather will be changed by the turning-moment Ψ about the vertical as soon as the figure axis is no longer perpendicular to the vertical. Finally, the vertical is no longer a principal axis of the mass distribution as soon as the equatorial plane of the top no longer passes through the vertical; thus the acceleration effect of the turning-moment Ψ is no longer given by $A\,d^2\psi/dt^2$, as in equation (IVa), but rather must be determined according to the general impulse theorem and the general relation between the impulse vector and the rotation vector.

These exact equations of motion can indeed be written according to the schema of the Lagrange equations, but they no longer permit of further integration. Nevertheless, the general character of the phenomena can only be, without doubt, the following.

As the angle ϑ diminishes from $\pi/2$ to 0, the gyroscopic effect K and the relevant component of K' successively decrease. The stability of the top with respect to the external moment Ψ is correspondingly reduced, until it vanishes completely for $\vartheta = 0$. In this latter limiting case, where the eigenimpulse is vertical, a change of position of N and a a gyroscopic effect no longer occur. Experience with every model of the top completely confirms this conclusion:

For a progressive elevation of the figure axis, there is a progressive diminishment of stability with respect to a moment Ψ about the vertical.

The following references may serve to relate the preceding with the developments of the previous volumes.

The fundamental impulse theorems are established in Ch. II, §5.

To I. Formula (I) is discussed, for example, in the review of the popular top literature on p. 311, and is subsumed under the general concept of the "deviation resistance for regular precession" in Ch. III, §6. In fact, the motion considered in (I) is a regular precession about the vertical with the inclination angle $\vartheta = \pi/2$, and formula (I) is identical with equation (1) of p. 175 under this condition and within the limit of precision stated above. Concerning the rule of equi-orientational parallelism, see Ch. VIII, p. 734.

To III. Expression (III) coincides with equation (1) of p. 175 for the deviation resistance not only approximately for large N, but rather exactly, in that $N = C(\mu + \nu\cos\vartheta)$; we have presently suppressed the negative sign in that equation, since we now prefer to establish the sense of the gyroscopic effect by the rule of equi-orientational parallelism in a manner that is valid for all cases. Our formula (III) is also contained in the Lagrange equations (p. 154, equations (1)). Our gyroscopic effect K is indeed oppositely equal, according to its definition, to the external moment Θ that acts about the line of nodes, the moment that is required for the maintenance of the regular precession. The ϑ-component of the Lagrange equations for regular precession ($\vartheta =$ const.) thus becomes

$$-K = \frac{d[\Theta]}{dt} - \frac{\partial T}{\partial\vartheta}.$$

For $\vartheta =$ const., however, expression (6) on p. 156 gives

$$[\Theta] = \frac{\partial T}{\partial\vartheta'} = A\vartheta' = 0,$$

$$\frac{\partial T}{\partial\vartheta} = A\sin\vartheta\cos\vartheta\,\psi'^2 - C(\varphi' + \cos\vartheta\,\psi')\sin\vartheta\,\psi' = (A\cos\vartheta\,\psi' - N)\sin\vartheta\,\psi',$$

and thus

$$-K = (N - A\cos\vartheta\,\psi')\sin\vartheta\,\psi',$$

in agreement with equation (III) (up to the sign not written there).

In the general case that the motion is not a regular precession, and thus the external moment does not exactly maintain equilibrium with the gyroscopic effect, we have

$$[\Theta] = A\vartheta'; \quad \frac{d[\Theta]}{dt} = A\vartheta'',$$

and thus the Lagrange equation for the ϑ-component is

$$A\vartheta'' = (A\cos\vartheta\,\psi' - N)\sin\vartheta\,\psi' + \Theta,$$

$$\text{or} \quad A\vartheta'' = -K + \Theta,$$

in agreement with equation (IVb) (in which the external moment Θ was assumed to vanish).

To IV. Our present expression for the countermoment of the gyroscopic effect in (IV) naturally agrees, apart from the notation, with the previous equation (3) of p. 195.

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