Chapter 4
More on Complex Numbers and Geometry

4.1 The Real Product of Two Complex Numbers

The concept of the scalar product of two vectors is well known. In what follows, we will introduce this concept for complex numbers. We will see that the use of this product simplifies the solution to many problems considerably.

Let a and b be two complex numbers.

Definition. Given complex numbers a and b, we call the number given by

$$a \cdot b = \frac{1}{2}(\overline{a}b + a\overline{b})$$

the *real product* of the two numbers. It is easy to see that

$$\overline{a \cdot b} = \frac{1}{2}(a\overline{b} + \overline{a}b) = a \cdot b;$$

hence $a \cdot b$ is a real number, which justifies the name of this product.

Let $A(a), B(b)$ be points in the complex plane, and let $\theta = (\widehat{\overrightarrow{OA}, \overrightarrow{OB}})$ be the angle between the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$. The following formula holds:

$$a \cdot b = |a||b| \cos\theta = \overrightarrow{OA} \cdot \overrightarrow{OB}.$$

Indeed, considering the polar form of a and b, we have

$$a = |a|(\cos t_1 + i \sin t_1), \quad b = |b|(\cos t_2 + i \sin t_2),$$

and

$$a \cdot b = \frac{1}{2}(\overline{a}b + a\overline{b}) = \frac{1}{2}|a||b|[\cos(t_1 - t_2) - i\sin(t_1 - t_2) + \cos(t_1 - t_2) + i\sin(t_1 - t_2)]$$

$$= |a||b|\cos(t_1 - t_2) = |a||b|\cos\theta = \overrightarrow{OA} \cdot \overrightarrow{OB}.$$

The following properties are easy to verify.

T. Andreescu and D. Andrica, *Complex Numbers from A to ... Z*,
DOI 10.1007/978-0-8176-8415-0_4,

Proposition 1. *For all complex numbers a, b, c, z, the following relations hold:*

(1) $a \cdot a = |a|^2$.
(2) $a \cdot b = b \cdot a$ *(the real product is commutative).*
(3) $a \cdot (b + c) = a \cdot b + a \cdot c$ *(the real product is distributive with respect to addition).*
(4) $(\alpha a) \cdot b = \alpha(a \cdot b) = a \cdot (\alpha b)$ *for all* $\alpha \in \mathbb{R}$.
(5) $a \cdot b = 0$ *if and only if* $OA \perp OB$, *where* A *has coordinate* a *and* B *has coordinate* b.
(6) $(az) \cdot (bz) = |z|^2(a \cdot b)$.

Remark. Suppose that A and B are points with coordinates a and b. Then the real product $a \cdot b$ is equal to the power of the origin with respect to the circle of diameter AB.

Indeed, let $M\left(\frac{a+b}{2}\right)$ be the midpoint of $[AB]$, hence the center of this circle, and let $r = \frac{1}{2}AB = \frac{1}{2}|a-b|$ be the radius of this circle. The power of the origin with respect to the circle is

$$OM^2 - r^2 = \left|\frac{a+b}{2}\right|^2 - \left|\frac{a-b}{2}\right|^2$$

$$= \frac{(a+b)(\overline{a}+\overline{b})}{4} - \frac{(a-b)(\overline{a}-\overline{b})}{4} = \frac{a\overline{b}+b\overline{a}}{2} = a \cdot b,$$

as claimed.

Proposition 2. *Suppose that* $A(a)$, $B(b)$, $C(c)$, *and* $D(d)$ *are four distinct points. The following statements are equivalent:*

(1) $AB \perp CD$;
(2) $(b-a) \cdot (d-c) = 0$;
(3) $\frac{b-a}{d-c} \in i\mathbb{R}^*$ *(or equivalently,* $\operatorname{Re}\left(\frac{b-a}{d-c}\right) = 0$*).*

Proof. Take points $M(b-a)$ and $N(d-c)$ such that $OABM$ and $OCDN$ are parallelograms. Then we have $AB \perp CD$ if and only if $OM \perp ON$. That is, $m \cdot n = (b-a) \cdot (d-c) = 0$, using property (5) of the real product.

The equivalence (2) $\Leftrightarrow$ (3) follows immediately from the definition of the real product. □

Proposition 3. *The circumcenter of triangle* ABC *is at the origin of the complex plane. If* a, b, c *are the coordinates of vertices* A, B, C, *then the orthocenter* H *has the coordinate* $h = a + b + c$.

Proof. Using the real product of the complex numbers, the equations of the altitudes AA', BB', CC' of the triangle are

$AA' : (z-a)\cdot(b-c) = 0, \quad BB' : (z-b)\cdot(c-a) = 0, \quad CC' : (z-c)\cdot(a-b) = 0.$

We will show that the point with coordinate $h = a+b+c$ lies on all three altitudes. Indeed, we have $(h-a)\cdot(b-c) = 0$ if and only if $(b+c)\cdot(b-c) = 0$. The last relation is equivalent to $b \cdot b - c \cdot c = 0$, or $|b|^2 = |c|^2$. Similarly, $H \in BB'$ and $H \in CC'$, and we are done. □

Remark. If the numbers $a,\ b,\ c,\ o,\ h$ are the coordinates of the vertices of triangle ABC, the circumcenter O, and the orthocenter H of the triangle, then $h = a + b + c - 2o$.

Indeed, if we take A' diametrically opposite A in the circumcircle of triangle ABC, then the quadrilateral $HBA'C$ is a parallelogram. If $\{M\} = HA' \cap BC$, then

$$z_M = \frac{b+c}{2} = \frac{z_H + z_{A'}}{2} = \frac{z_H + 2o - a}{2}, \text{ i.e., } z_H = a + b + c - 2o.$$

Problem 1. *Let $ABCD$ be a convex quadrilateral. Prove that*

$$AB^2 + CD^2 = AD^2 + BC^2$$

if and only if $AC \perp BD$.

Solution. Using the properties of the real product of complex numbers, we have

$$AB^2 + CD^2 = BC^2 + DA^2$$

if and only if

$$(b-a)\cdot(b-a) + (d-c)\cdot(d-c) = (c-b)\cdot(c-b) + (a-d)\cdot(a-d).$$

That is,

$$a \cdot b + c \cdot d = b \cdot c + d \cdot a,$$

and finally,

$$(c-a)\cdot(d-b) = 0,$$

or equivalently, $AC \perp BD$, as required.

Problem 2. *Let M, N, P, Q, R, S be the midpoints of the sides AB, BC, CD, DE, EF, FA of a hexagon. Prove that*

$$RN^2 = MQ^2 + PS^2$$

if and only if $MQ \perp PS$.

(Romanian Mathematical Olympiad—Final Round, 1994)

Solution. Let $a,\ b,\ c,\ d,\ e,\ f$ be the coordinates of the vertices of the hexagon (Fig. 4.1). The points $M,\ N,\ P,\ Q,\ R,\ S$ have coordinates

$$m=\frac{a+b}{2},\ n=\frac{b+c}{2},\ p=\frac{c+d}{2},$$
$$q=\frac{d+e}{2},\ r=\frac{e+f}{2},\ s=\frac{f+a}{2},$$

respectively.

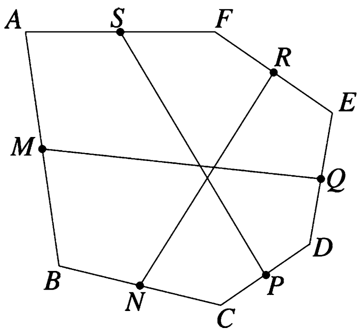

Figure 4.1.

Using the properties of the real product of complex numbers, we have

$$RN^2=MQ^2+PS^2$$

if and only if

$$(e+f-b-c)\cdot(e+f-b-c)=(d+e-a-b)\cdot(d+e-a-b)+(f+a-c-d)\cdot(f+a-c-d).$$

That is,

$$(d+e-a-b)\cdot(f+a-c-d)=0;$$

hence $MQ\perp PS$, as claimed.

Problem 3. *Let $A_1A_2\cdots A_n$ be a regular polygon inscribed in a circle with center O and radius R. Prove that for all points M in the plane, the following relation holds:*

$$\sum_{k=1}^{n} MA_k^2=n(OM^2+R^2).$$

Solution. Consider the complex plane with the origin at point O, with the x-axis containing the point A_1, and let $R\varepsilon_k$ be the coordinate of vertex A_k, where ε_k are the nth-roots of unity, $k=1,\ \ldots,\ n$. Let m be the coordinate of M.

Using the properties of the real product of the complex numbers, we have

$$\begin{aligned}\sum_{k=1}^{n} MA_k^2 &= \sum_{k=1}^{n} (m - R\varepsilon_k)\cdot\ (m - R\varepsilon_k)\\ &= \sum_{k=1}^{n}(m\cdot m - 2R\varepsilon_k\cdot m + R^2\varepsilon_k\cdot\varepsilon_k)\\ &= n|m|^2 - 2R\left(\sum_{k=1}^{n}\varepsilon_k\right)\cdot\ m + R^2\sum_{k=1}^{n}|\varepsilon_k|^2\\ &= n\cdot OM^2 + nR^2 = n(OM^2+R^2),\end{aligned}$$

since $\sum\limits_{k=1}^{n}\varepsilon_k = 0$.

Remark. If M lies on the circumcircle of the polygon, then

$$\sum_{k=1}^{n} MA_k^2 = 2nR^2.$$

Problem 4. *Let O be the circumcenter of the triangle ABC, let D be the midpoint of the segment AB, and let E is the centroid of triangle ACD. Prove that lines CD and OE are perpendicular if and only if $AB = AC$.*

(Balkan Mathematical Olympiad, 1985)

Solution. Let O be the origin of the complex plane and let $a,\ b,\ c,\ d,\ e$ be the coordinates of points $A,\ B,\ C,\ D,\ E$, respectively. Then

$$d = \frac{a+b}{2} \text{ and } e = \frac{a+c+d}{3} = \frac{3a+b+2c}{6}$$

Using the real product of complex numbers, if R is the circumradius of triangle ABC, then

$$a\cdot a = b\cdot b = c\cdot c = R^2.$$

Lines CD and OE are perpendicular if and only if $(d-c)\cdot e = 0$, that is,

$$(a+b-2c)\cdot(3a+b+2c) = 0.$$

The last relation is equivalent to

$$3a\cdot a + a\cdot b + 2a\cdot c + 3a\cdot b + b\cdot b + 2b\cdot c - 6a\cdot c - 2b\cdot c - 4c\cdot c = 0,$$

that is,

$$a\cdot b = a\cdot c. \tag{1}$$

On the other hand, $AB = AC$ is equivalent to

$$|b-a|^2 = |c-a|^2.$$

That is,

$$(b-a)\cdot(b-a) = (c-a)\cdot(c-a),$$

or

$$b\cdot b - 2a\cdot b + a\cdot a = c\cdot c - 2a\cdot c + a\cdot a,$$

whence

$$a\cdot b = a\cdot c. \tag{2}$$

The relations (1) and (2) show that $CD \perp OE$ if and only if $AB = AC$.

Problem 5. *Let a, b, c be distinct complex numbers such that $|a| = |b| = |c|$ and $|b+c-a| = |a|$. Prove that $b+c=0$.*

Solution. Let A, B, C be the geometric images of the complex numbers a, b, c, respectively. Choose the circumcenter of triangle ABC as the origin of the complex plane and denote by R the circumradius of triangle ABC. Then

$$a\overline{a} = b\overline{b} = c\overline{c} = R^2,$$

and using the real product of the complex numbers, we have

$$|b+c-a| = |a| \text{ if and only if } |b+c-a|^2 = |a|^2.$$

That is,

$$(b+c-a)\cdot(b+c-a) = |a|^2,$$

i.e.,

$$|a|^2 + |b|^2 + |c|^2 + 2b\cdot c - 2a\cdot c - 2a\cdot b = |a|^2.$$

We obtain

$$2(R^2 + b\cdot c - a\cdot c - a\cdot b) = 0,$$

i.e.,

$$a\cdot a + b\cdot c - a\cdot c - a\cdot b = 0.$$

It follows that $(a-b)\cdot(a-c) = 0$, and hence $AB \perp AC$, i.e., $\widehat{BAC} = 90^\circ$. Therefore, $[BC]$ is the diameter of the circumcircle of triangle ABC, so $b+c=0$.

Problem 6. *Let E, F, G, H be the midpoints of sides AB, BC, CD, DA of the convex quadrilateral $ABCD$. Prove that lines AB and CD are perpendicular if and only if*

$$BC^2 + AD^2 = 2(EG^2 + FH^2).$$

Solution. Denote by the corresponding lowercase letter the coordinate of a point denoted by an uppercase letter. Then

$$e = \frac{a+b}{2},\ f = \frac{b+c}{2},\ g = \frac{c+d}{2},\ h = \frac{d+a}{2}.$$

Using the real product of the complex numbers, the relation

$$BC^2 + AD^2 = 2(EG^2 + FH^2)$$

becomes

$$(c-b)\cdot(c-b) + (d-a)\cdot(d-a) = \frac{1}{2}(c+d-a-b)\cdot(c+d-a-b) + \frac{1}{2}(a+d-b-c)\cdot(a+d-b-c).$$

This is equivalent to

$$c\cdot c + b\cdot b + d\cdot d + a\cdot a - 2b\cdot c - 2a\cdot d$$

$$= a\cdot a + b\cdot b + c\cdot c + d\cdot d - 2a\cdot c - 2b\cdot d,$$

or

$$a\cdot d + b\cdot c = a\cdot c + b\cdot d.$$

The last relation shows that $(a-b)\cdot(d-c) = 0$ if and only if $AB \perp CD$, as desired.

Problem 7. *Let G be the centroid of triangle ABC and let $A_1,\ B_1,\ C_1$ be the midpoints of sides $BC,\ CA,\ AB$, respectively. Prove that*

$$MA^2 + MB^2 + MC^2 + 9MG^2 = 4(MA_1^2 + MB_1^2 + MC_1^2)$$

for all points M in the plane.

Solution. Denote by the corresponding lowercase letter the coordinate of a point denoted by an uppercase letter. Then

$$g = \frac{a+b+c}{3},\ a_1 = \frac{b+c}{2},\ b_1 = \frac{c+a}{2},\ c_1 = \frac{a+b}{2}.$$

Using the real product of the complex numbers, we have

$$MA^2 + MB^2 + MC^2 + 9MG^2$$

$$= (m-a)\cdot(m-a) + (m-b)\cdot(m-b) + (m-c)\cdot(m-c)$$
$$+9\left(m - \frac{a+b+c}{3}\right)\cdot\left(m - \frac{a+b+c}{3}\right)$$
$$= 12|m|^2 - 8(a+b+c)\cdot m + 2(|a|^2 + |b|^2 + |c|^2) + 2a\cdot b + 2b\cdot c + 2c\cdot a.$$

On the other hand,

$$4(MA_1^2 + MB_1^2 + MC_1^2)$$
$$= 4\left[\left(m - \frac{b+c}{2}\right) \cdot \left(m - \frac{b+c}{2}\right) + \left(m - \frac{c+a}{2}\right)\right.$$
$$\left.\cdot \left(m - \frac{c+a}{2}\right) + \left(m - \frac{a+b}{2}\right) \cdot \left(m - \frac{a+b}{2}\right)\right]$$
$$= 12|m|^2 - 8(a+b+c) \cdot m + 2(|a|^2 + |b|^2 + |c|^2) + 2a \cdot b + 2b \cdot c + 2c \cdot a,$$

so we are done.

Remark. The following generalization can be proved similarly.

Let $A_1A_2 \cdots A_n$ be a polygon with centroid G and let A_{ij} be the midpoint of the segment $[A_iA_j]$, $i<j$, i, $j \in \{1, 2, \ldots, n\}$.

Then

$$(n-2)\sum_{k=1}^{n} MA_k^2 + n^2MG^2 = 4\sum_{i<j} MA_{ij}^2,$$

for all points M in the plane. A nice generalization is given in Theorem 3 in Sect. 4.11.

4.2 The Complex Product of Two Complex Numbers

The cross product of two vectors is a central concept in vector algebra, with numerous applications in various branches of mathematics and science. In what follows, we adapt this product to complex numbers. The reader will see that this new interpretation has multiple advantages in solving problems involving area or collinearity.

Let a and b be two complex numbers.

Definition. The complex number

$$a \times b = \frac{1}{2}(\overline{a}b - a\overline{b})$$

is called the *complex product* of the numbers a and b.

Note that

$$a \times b + \overline{a \times b} = \frac{1}{2}(\overline{a}b - a\overline{b}) + \frac{1}{2}(a\overline{b} - \overline{a}b) = 0,$$

so $\operatorname{Re}(a \times b) = 0$, which justifies the definition of this product.

Let $A(a), B(b)$ be points in the complex plane, and let $\theta = (\widehat{\overrightarrow{OA}, \overrightarrow{OB}})$ be the angle between the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$. The following formula holds:

$$a \times b = \varepsilon i |a||b| \sin\theta,$$

where

$$\varepsilon = \begin{cases} -1, \text{ if triangle } OAB \text{ is positively oriented;} \\ +1, \text{ if triangle } OAB \text{ is negatively oriented.} \end{cases}$$

Indeed, if $a = |a|(\cos t_1 + i \sin t_1)$ and $b = |b|(\cos t_2 + i \sin t_2)$, then

$$a \times b = i|a||b|\sin(-t_1 + t_2) = \varepsilon i|a||b|\sin\theta.$$

The connection between the real product and the complex product is given by the following Lagrange-type formula:

$$|a \cdot b|^2 + |a \times b|^2 = |a|^2|b|^2.$$

The following properties are easy to verify:

Proposition 1. *Suppose that a, b, c are complex numbers. Then:*

(1) $a \times b = 0$ if and only if $a = 0$ or $b = 0$ or $a = \lambda b$, where λ is a real number.
(2) $a \times b = -b \times a$ (the complex product is anticommutative).
(3) $a \times (b + c) = a \times b + a \times c$ (the complex product is distributive with respect to addition).
(4) $\alpha(a \times b) = (\alpha a) \times b = a \times (\alpha b)$, for all real numbers α.
(5) If $A(a)$ and $B(b)$ are distinct points other than the origin, then $a \times b = 0$ if and only if O, A, B are collinear.

Remarks.

(a) Suppose $A(a)$ and $B(b)$ are distinct points in the complex plane different from the origin (Fig. 4.2).
The complex product of the numbers a and b has the following useful geometric interpretation:

$$a \times b = \begin{cases} 2i. \text{ area } [AOB], & \text{if triangle } OAB \text{ is positively oriented;} \\ -2i. \text{ area } [AOB], & \text{if triangle } OAB \text{ is negatively oriented.} \end{cases}$$

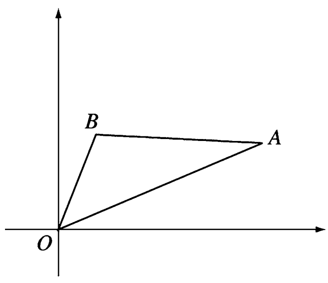

Figure 4.2.

Indeed, if triangle OAB is positively (directly) oriented, then

$$2i \cdot \text{area}\,[OAB] = i \cdot OA \cdot OB \cdot \sin(\widehat{AOB})$$

$$= i|a| \cdot |b| \cdot \sin\left(\arg\frac{b}{a}\right) = i \cdot |a| \cdot |b| \cdot \operatorname{Im}\left(\frac{b}{a}\right) \cdot \frac{|a|}{|b|}$$

$$= \frac{1}{2}|a|^2\left(\frac{b}{a} - \frac{\overline{b}}{\overline{a}}\right) = \frac{1}{2}(\overline{a}b - a\overline{b}) = a \times b.$$

In the other case, note that triangle OBA is positively oriented; hence

$$2i \cdot \text{area}[OBA] = b \times a = -a \times b.$$

(b) Suppose $A(a),\ B(b),\ C(c)$ are three points in the complex plane. The complex product allows us to obtain the following useful formula for the area of the triangle ABC:

$$\text{area}\,[ABC] = \begin{cases} \dfrac{1}{2i}(a \times b + b \times c + c \times a) \\ \text{if triangle } ABC \text{ is positively oriened;} \\ \\ -\dfrac{1}{2i}(a \times b + b \times c + c \times a) \\ \text{if triangle } ABC \text{ is negatively oriented.} \end{cases}$$

Moreover, simple algebraic manipulation shows that

$$\text{area}\,[ABC] = \frac{1}{2}\operatorname{Im}(\overline{a}b + \overline{b}c + \overline{c}a)$$

if triangle ABC is directly (positively) oriented.

To prove the above formula, translate points $A,\ B,\ C$ by the vector $-c$. The images of $A,\ B,\ C$ are the points $A',\ B',\ O$ with coordinates $a-c,\ b-c,\ 0$, respectively. Triangles ABC and $A'B'O$ are congruent with the same orientation. If ABC is positively oriented, then

$$\text{area}\,[ABC] = \text{area}\,[OA'B'] = \frac{1}{2i}((a-c) \times (b-c))$$

$$= \frac{1}{2i}((a-c) \times b - (a-c) \times c) = \frac{1}{2i}(c \times (a-c) - b \times (a-c))$$

$$= \frac{1}{2i}(c \times a - c \times c - b \times a + b \times c) = \frac{1}{2i}(a \times b + b \times c + c \times a),$$

as claimed.

The other situation can be handled similarly.

Proposition 2. *Suppose $A(a)$, $B(b)$, and $C(c)$ are distinct points. The following statements are equivalent:*

(1) Points A, B, C are collinear.
(2) $(b-a)\times(c-a)=0$.
(3) $a\times b+b\times c+c\times a=0$.

Proof. Points A, B, C are collinear if and only if area$[ABC]=0$, i.e., $a\times b+b\times c+c\times a=0$. The last equation can be written in the form $(b-a)\times(c-a)=0$. □

Proposition 3. *Let $A(a)$, $B(b)$, $C(c)$, $D(d)$ be four points, no three of which are collinear. Then $AB\|CD$ if and only if $(b-a)\times(d-c)=0$.*

Proof. Choose the points $M(m)$ and $N(n)$ such that $OABM$ and $OCDN$ are parallelograms; then $m=b-a$ and $n=d-c$.

Lines AB and CD are parallel if and only if points O, M, N are collinear. Using property 5, this is equivalent to $0=m\times n=(b-a)\times(d-c)$. □

Problem 1. *Points D and E lie on sides AB and AC of the triangle ABC such that*

$$\frac{AD}{AB}=\frac{AE}{AC}=\frac{3}{4}.$$

Consider points E' and D' on the rays $(BE$ and $(CD$ such that $EE'=3BE$ and $DD'=3CD$. Prove the following:

(1) *points D', A, E' are collinear.*
(2) *$AD'=AE'$.*

Solution. The points D, E, D', E' have coordinates: $d=\dfrac{a+3b}{4}$, $e=\dfrac{a+3c}{4}$,

$$e'=4e-3b=a+3c-3b,\ \text{and}\ d'=4d-3c=a+3b-3c,$$

respectively.

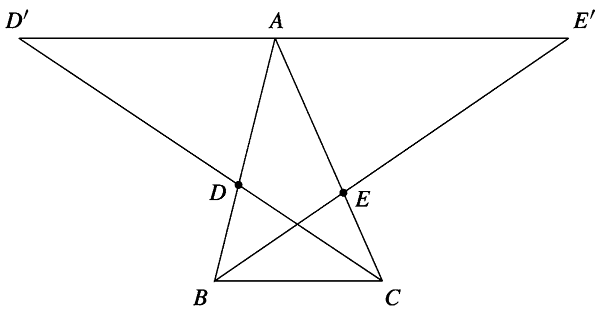

Figure 4.3.

(1) Since

$$(a - d') \times (e' - d') = (3c - 3b) \times (6c - 6b) = 18(c - b) \times (c - b) = 0,$$

it follows from Proposition 2 in Sect. 4.2 that the points D', A, E' are collinear (Fig. 4.3).

(2) Note that

$$\frac{AD'}{D'E'} = \left|\frac{a - d'}{e' - d'}\right| = \frac{1}{2},$$

so A is the midpoint of segment $D'E'$.

Problem 2. *Let $ABCDE$ be a convex pentagon and let M, N, P, Q, X, Y be the midpoints of the segments BC, CD, DE, EA, MP, NQ, respectively. Prove that $XY \| AB$.*

Solution. Let a, b, c, d, e be the coordinates of vertices A, B, C, D, E, respectively (Fig. 4.4).

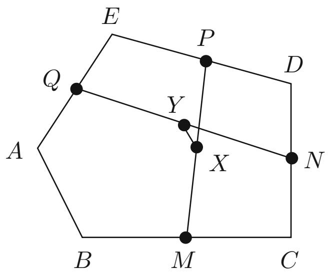

Figure 4.4.

Points M, N, P, Q, X, Y have coordinates

$$m = \frac{b+c}{2},\ n = \frac{c+d}{2},\ p = \frac{d+e}{2},$$

$$q = \frac{e+a}{2},\ x = \frac{b+c+d+e}{4},\ y = \frac{c+d+e+a}{4},$$

respectively. Then

$$\frac{y-x}{b-a} = \frac{\frac{a-b}{4}}{b-a} = -\frac{1}{4} \in \mathbb{R},$$

whence

$$(y - x) \times (b - a) = -\frac{1}{4}(b - a) \times (b - a) = 0.$$

From Proposition 3 in Sect. 4.2, it follows that $XY \| AB$.

4.3 The Area of a Convex Polygon

We say that the convex polygon $A_1A_2\cdots A_n$ is *directly* (or *positively*) *oriented* if for every point M situated in the interior of the polygon, the triangles $MA_kA_{k+1},\ k=1,2,\ \ldots,\ n$, are directly oriented, where $A_{n+1}=A_1$.

Theorem. *Consider a directly oriented convex polygon* $A_1A_2\cdots A_n$ *with vertices with coordinates* $a_1,\ a_2,\ \ldots,\ a_n$. *Then*

$$\text{area}\,[A_1A_2\cdots A_n]=\frac{1}{2}\mathrm{Im}(\overline{a_1}a_2+\overline{a_2}a_3+\cdots+\overline{a_{n-1}}a_n+\overline{a_n}a_1).$$

Proof. We use induction on n. The base case $n=3$ was proved above using the complex product. Suppose that the claim holds for $n=k$, and note that

$$\text{area}\,[A_1A_2\cdots A_kA_{k+1}]=\text{area}\,[A_1A_2\cdots A_k]+\text{area}\,[A_kA_{k+1}A_1]$$

$$=\frac{1}{2}\mathrm{Im}(\overline{a}_1a_2+\overline{a_2}a_3+\cdots+\overline{a_{k-1}}a_k+\overline{a_k1}a)+\frac{1}{2}\mathrm{Im}(\overline{a_k}a_{k+1}+\overline{a_{k+1}}a_1+\overline{a_1}a_k)$$

$$=\frac{1}{2}\mathrm{Im}(\overline{a}_1a_2+\overline{a_2}a_3+\cdots+\overline{a_{k-1}}a_k+\overline{a_k}a_{k+1}+\overline{a_{k+1}}a_1)$$

$$+\frac{1}{2}\mathrm{Im}(\overline{a_k}a_1+\overline{a_1}a_k)=\frac{1}{2}\mathrm{Im}(\overline{a_1}a_2+\overline{a_2}a_3+\cdots+\overline{a_k}a_{k+1}+\overline{a_{k+1}}a_1),$$

since

$$\mathrm{Im}(\overline{a_k}a_1+\overline{a_1}a_k)=0.$$

Alternative proof. Choose a point M in the interior of the polygon. Applying the formula (2) in Sect. 3.5.3, we have

$$\text{area}\,[A_1A_2\cdots A_n]=\sum_{k=1}^{n}\ \text{area}\,[MA_kA_{k+1}]$$

$$=\frac{1}{2}\sum_{k=1}^{n}\mathrm{Im}(\overline{z}a_k+\overline{a_k}a_{k+1}+\overline{a_{k+1}}z)$$

$$=\frac{1}{2}\sum_{k=1}^{n}\mathrm{Im}(\overline{a_k}a_{k+1})+\frac{1}{2}\sum_{k=1}^{n}\mathrm{Im}(\overline{z}a_k+\overline{a_{k+1}}z)$$

$$=\frac{1}{2}\mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right)+\frac{1}{2}\mathrm{Im}\left(\overline{z}\sum_{k=1}^{n}a_k+z\sum_{j=1}^{n}\overline{a_j}\right)=\frac{1}{2}\mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right),$$

since for any complex numbers $z,\ w$ the relation $\mathrm{Im}(\overline{z}w+z\overline{w})=0$ holds. $\square$

Remark. From the above formula, it follows that the points $A_1(a_1)$, $A_2(a_2),\ldots,\ A_n(a_n)$ as in the theorem are collinear if and only if

$$\mathrm{Im}(\overline{a}_1a_2+\overline{a}_2a_3+\cdots+\overline{a}_{n-1}a_n+\overline{a}_na_1)=0.$$

For this result, the hypotheses in the theorem are essential, as we can see from the following counterexample.

Counterexample The points with respective complex coordinates $a_1 = 0$, $a_2 = 1$, $a_3 = i$, $a_4 = 1 + i$ are not collinear, but we have $\mathrm{Im}(\overline{a_1}a_2 + \overline{a_2}a_3 + \overline{a_3}a_4 + \overline{a_4}a_1) = \mathrm{Im}(-1) = 0$.

Problem 1. *Let $P_0P_1 \cdots P_{n-1}$ be the polygon whose vertices have coordinates $1, \varepsilon, \ldots, \varepsilon^{n-1}$, and let $Q_0Q_1 \cdots Q_{n-1}$ be the polygon whose vertices have coordinates $1, 1+\varepsilon, \ldots, 1+\varepsilon+\cdots+\varepsilon^{n-1}$, where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Find the ratio of the areas of these polygons.*

Solution. Consider $a_k = 1 + \varepsilon + \cdots + \varepsilon^k$, $k = 0, 1, \ldots, n-1$, and observe that

$$\begin{aligned}
\text{area }[Q_0Q_1\cdots Q_{n-1}] &= \frac{1}{2}\mathrm{Im}\left(\sum_{k=0}^{n-1}\overline{a_k}a_{k+1}\right)\\
&= \frac{1}{2}\mathrm{Im}\left(\sum_{k=0}^{n-1}\frac{(\overline{\varepsilon})^{k+1}-1}{\overline{\varepsilon}-1}\cdot\frac{\varepsilon^{k+2}-1}{\varepsilon-1}\right)\\
&= \frac{1}{2|\varepsilon-1|^2}\mathrm{Im}\left[\sum_{k=0}^{n-1}(\varepsilon-(\overline{\varepsilon})^{k+1}-\varepsilon^{k+2}+1)\right]\\
&= \frac{1}{2|\varepsilon-1|^2}\mathrm{Im}(n\varepsilon+n) = \frac{1}{2|\varepsilon-1|^2}n\sin\frac{2\pi}{n}\\
&= \frac{n}{8\sin^2\frac{\pi}{n}}2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{n}{4}\ \mathrm{cotan}\ \frac{\pi}{n},
\end{aligned}$$

since

$$\sum_{k=0}^{n-1}\overline{\varepsilon}^{k+1} = 0 \text{ and } \sum_{k=0}^{n-1}\varepsilon^{k+2} = 0.$$

On the other hand, it is clear that

$$\text{area }[P_0P_1\cdots P_{n-1}] = n \text{ area }[P_0OP_1] = \frac{n}{2}\sin\frac{2\pi}{n} = n\sin\frac{\pi}{n}\cos\frac{\pi}{n}.$$

We obtain

$$\frac{\text{area}[P_0P_1\cdots P_{n-1}]}{\text{area}[Q_0Q_1\cdots Q_{n-1}]} = \frac{n\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{\frac{n}{4}\mathrm{cotan}\frac{\pi}{n}} = 4\sin^2\frac{\pi}{n}. \tag{1}$$

Remark. We have $Q_kQ_{k+1} = |a_{k+1} - a_k| = |\varepsilon^{k+1}| = 1$ and $P_kP_{k+1} = |\varepsilon^{k+1} - \varepsilon^k| = |\varepsilon^k(\varepsilon-1)| = |\varepsilon^k||1-\varepsilon| = |1-\varepsilon| = 2\sin\frac{\pi}{n}$, $k = 0, 1, \ldots, n-1$. It follows that

$$\frac{P_kP_{k+1}}{Q_kQ_{k+1}} = 2\sin\frac{\pi}{n},\ k = 0, 1,\ \ldots,\ n-1.$$

That is, the polygons $P_0P_1\cdots P_{n-1}$ and $Q_0Q_1\cdots Q_{n-1}$ are similar, and the result in (1) follows.

Problem 2. *Let $A_1A_2\cdots A_n (n \geq 5)$ be a convex polygon and let B_k be the midpoint of the segment $[A_kA_{k+1}]$, $k = 1, 2,\ \ldots,\ n$, where $A_{n+1} = A_1$. Then the following inequality holds:*

$$\text{area}\ [B_1B_2\cdots B_n] \geq \frac{1}{2}\ \text{area}\ [A_1A_2\cdots A_n].$$

Solution. Let a_k and b_k be the coordinates of points A_k and B_k, $k = 1, 2,\ \ldots,\ n$. It is clear that the polygon $B_1B_2\cdots B_n$ is convex, and if we assume that $A_1A_2\cdots A_n$ is positively oriented, then $B_1B_2\cdots B_n$ also has this property. Choose as the origin O of the complex plane a point situated in the interior of polygon $A_1A_2\cdots A_n$.

We have $b_k = \frac{1}{2}(a_k + a_{k+1}),\ k = 1, 2,\ \ldots,\ n$, and

$$\text{area}\ [B_1B_2\cdots B_n] = \frac{1}{2}\mathrm{Im}\Big(\sum_{k=1}^{n}\overline{b_k}b_{k+1}\Big) = \frac{1}{8}\mathrm{Im}\sum_{k=1}^{n}(\overline{a_k} + \overline{a_{k+1}})(a_{k+1} + a_{k+2})$$

$$= \frac{1}{8}\mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right) + \frac{1}{8}\mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_{k+1}}a_{k+2}\right) + \frac{1}{8}\mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+2}\right)$$

$$= \frac{1}{2}\text{area}\ [A_1A_2\cdots A_n] + \frac{1}{8}\mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+2}\right)$$

$$= \frac{1}{2}\text{area}\ [A_1A_2\cdots A_n] + \frac{1}{8}\sum_{k=1}^{n}\mathrm{Im}(\overline{a_k}a_{k+2})$$

$$= \frac{1}{2}\text{area}\ [A_1A_2\cdots A_n] + \frac{1}{8}\sum_{k=1}^{n}OA_k\cdot OA_{k+2}\sin A_k\widehat{OA_{k+2}}$$

$$\geq \frac{1}{2}\text{area}\ [A_1A_2\cdots A_n],$$

where we have used the relations

$$\mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right) = \mathrm{Im}\left(\sum_{k=1}^{n}\overline{a_{k+1}}a_{k+2}\right) = 2\ \text{area}\ [A_1A_2\cdots A_n]$$

and $\sin A_k\widehat{OA_{k+2}} \geq 0,\ k = 1, 2,\ \ldots,\ n$, where $A_{n+2} = A_2$.

4.4 Intersecting Cevians and Some Important Points in a Triangle

Proposition. *Consider the points A', B', C' on the sides BC, CA, AB of the triangle ABC such that AA', BB', CC' intersect at point Q and let*

$$\frac{BA'}{A'C} = \frac{p}{n},\ \frac{CB'}{B'A} = \frac{m}{p},\ \frac{AC'}{C'B} = \frac{n}{m}.$$

If a, b, c are the coordinates of points A, B, C, respectively, then the coordinate of point Q is

$$q = \frac{ma + nb + pc}{m + n + p}.$$

Proof. The coordinates of A', B', C' are $a' = \dfrac{nb + pc}{n + p}$, $b' = \dfrac{ma + pc}{m + p}$, and $c' = \dfrac{ma + nb}{m + n}$, respectively. Let Q be the point with coordinate $q = \frac{ma+nb+pc}{m+n+p}$. We prove that AA', BB', CC' meet at Q.

The points A, Q, A' are collinear if and only if $(q - a) \times (a' - a) = 0$. This is equivalent to

$$\left(\frac{ma + nb + pc}{m + n + p} - a\right) \times \left(\frac{nb + pc}{n + p} - a\right) = 0,$$

or $(nb + pc - (n + p)a) \times (nb + pc - (n + p)a) = 0$, which is clear by definition of the complex product.

Likewise, Q lies on lines BB' and CC', so the proof is complete. □

Some Important Points in a Triangle

(1) If $Q = G$, the centroid of the triangle ABC, we have $m = n = p$. Then we obtain again that the coordinate of G is

$$z_G = \frac{a + b + c}{3}.$$

(2) Suppose that the lengths of the sides of triangle ABC are $BC = \alpha$, $CA = \beta$, $AB = \gamma$. If $Q = I$, the incenter of triangle ABC, then using a known result concerning the angle bisector, it follows that $m = \alpha$, $n = \beta$, $p = \gamma$. Therefore, the coordinate of I is

$$z_I = \frac{\alpha a + \beta b + \gamma c}{\alpha + \beta + \gamma} = \frac{1}{2s}[(\alpha a + \beta b + \gamma c)],$$

where $s = \frac{1}{2}(\alpha + \beta + \gamma)$.

(3) If $Q = H$, the orthocenter of the triangle ABC, we easily obtain the relations

$$\frac{BA'}{A'C} = \frac{\tan C}{\tan B}, \ \frac{CB'}{B'A} = \frac{\tan A}{\tan C}, \ \frac{AC'}{C'B} = \frac{\tan B}{\tan A}.$$

It follows that $m = \tan A, \ n = \tan B, \ p = \tan C$, and the coordinate of H is given by

$$z_H = \frac{(\tan A)a + (\tan B)b + (\tan C)c}{\tan A + \tan B + \tan C}.$$

Remark. The above formula can also be extended to the limiting case in which the triangle ABC is a right triangle. Indeed, assume that $A \to \frac{\pi}{2}$. Then $\tan A \to \pm\infty$ and $\frac{(\tan B)b + (\tan C)c}{\tan A} \to 0, \ \frac{\tan B + \tan C}{\tan A} \to 0$. In this case, $z_H = a$, i.e., the orthocenter of triangle ABC is the vertex A.

(4) The Gergonne[1] point J is the intersection of the cevians $AA', \ BB', \ CC'$, where $A', \ B', \ C'$ are the points of tangency of the incircle to the sides $BC, \ CA, \ AB$, respectively. Then

$$\frac{BA'}{A'C} = \frac{\frac{1}{s-\gamma}}{\frac{1}{s-\beta}}, \ \frac{CB'}{B'A} = \frac{\frac{1}{s-\alpha}}{\frac{1}{s-\gamma}}, \ \frac{AC'}{C'B} = \frac{\frac{1}{s-\beta}}{\frac{1}{s-\alpha}},$$

and the coordinate z_J is obtained from the same proposition, where

$$z_J = \frac{r_\alpha a + r_\beta b + r_\gamma c}{r_\alpha + r_\beta + r_\gamma}.$$

Here $r_\alpha, \ r_\beta, \ r_\gamma$ denote the radii of the three excircles of triangle. It is not difficult to show that the following formulas hold:

$$r_\alpha = \frac{K}{s-\alpha}, \ r_\beta = \frac{K}{s-\beta}, \ r_\gamma = \frac{K}{s-\gamma},$$

where $K = \text{area}\ [ABC]$ and $s = \frac{1}{2}(\alpha + \beta + \gamma)$.

(5) The Lemoine[2] point K is the intersection of the symmedians of the triangle (the symmedian is the reflection of the bisector across the median). Using the notation from the proposition, we obtain

$$\frac{BA'}{A'C} = \frac{\gamma^2}{\beta^2}, \ \frac{CB'}{B'A} = \frac{\alpha^2}{\gamma^2}, \ \frac{AC'}{C'B} = \frac{\beta^2}{\alpha^2}.$$

---

[1] Joseph Diaz Gergonne (1771–1859), French mathematician, founded the journal *Annales de Mathématiques Pures et Appliquées* in 1810.

[2] Émile Michel Hyacinthe Lemoine (1840–1912), French mathematician, made important contributions to geometry.

It follows that

$$z_K = \frac{\alpha^2 a + \beta^2 b + \gamma^2 c}{\alpha^2 + \beta^2 + \gamma^2}.$$

(6) The Nagel[3] point N is the intersection of the cevian AA', BB', CC', where A', B', C' are the points of tangency of the excircles with respective sides BC, CA, AB. Then

$$\frac{BA'}{A'C} = \frac{s-\gamma}{s-\beta}, \ \frac{CB'}{B'A} = \frac{s-\alpha}{s-\gamma}, \ \frac{AC'}{C'B} = \frac{s-\beta}{s-\alpha},$$

and the proposition mentioned above gives the coordinate z_N of the Nagel point N:

$$z_N = \frac{(s-\alpha)a + (s-\beta)b + (s-\gamma)c}{(s-\alpha)+(s-\beta)+(s-\gamma)} = \frac{1}{s}[(s-\alpha)a + (s-\beta)b + (s-\gamma)c]$$
$$= \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Problem. *Let α, β, γ be the lengths of sides BC, CA, AB of triangle ABC and suppose $\alpha < \beta < \gamma$. If points O, I, H are the circumcenter, the incenter, and the orthocenter of triangle ABC, respectively, prove that*

$$\text{area } [OIH] = \frac{1}{8r}(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha),$$

where r is the inradius of ABC.

Solution. Consider triangle ABC, directly oriented in the complex plane centered at point O.

Using the complex product and the coordinates of I and H, we have

$$\text{area } [OIH] = \frac{1}{2i}(z_I \times z_H) = \frac{1}{2i}\left[\frac{\alpha a + \beta b + \gamma c}{\alpha+\beta+\gamma} \times (a+b+c)\right]$$
$$= \frac{1}{4si}[(\alpha-\beta)a \times b + (\beta-\gamma)b \times c + (\gamma-\alpha)c \times a]$$
$$= \frac{1}{2s}[(\alpha-\beta)\cdot \text{area } [OAB] + (\beta-\gamma)\cdot \text{area } [OBC] + (\gamma-\alpha)\cdot \text{area } [OCA]]$$
$$= \frac{1}{2s}\left[(\alpha-\beta)\frac{R^2 \sin 2C}{2} + (\beta-\gamma)\frac{R^2 \sin 2A}{2} + (\gamma-\alpha)\frac{R^2 \sin 2B}{2}\right]$$

[3] Christian Heinrich von Nagel (1803–1882), German mathematician. His contributions to triangle geometry were included in the book *The Development of Modern Triangle Geometry* [21].

$$= \frac{R^2}{4s}[(\alpha - \beta)\sin 2C + (\beta - \gamma)\sin 2A + (\gamma - \alpha)\sin 2B]$$

$$= \frac{1}{8r}(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha),$$

as desired.

4.5 The Nine-Point Circle of Euler

Given a triangle ABC, choose its circumcenter O to be the origin of the complex plane and let $a,\ b,\ c$ be the coordinates of the vertices $A,\ B,\ C$. We have seen in Sect. 4.1, Proposition 3, that the coordinate of the orthocenter H is $z_H = a + b + c$.

Let us denote by $A_1,\ B_1,\ C_1$ the midpoints of sides $BC,\ CA,\ AB$; by $A',\ B',\ C'$ the feet of the altitudes; and by $A'',\ B'',\ C''$ the midpoints of segments $AH,\ BH,\ CH$, respectively (Fig. 4.5).

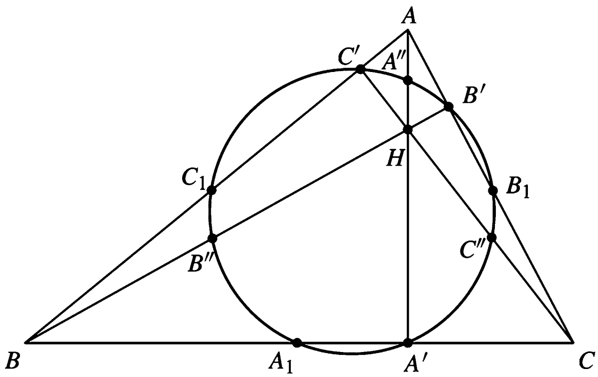

Figure 4.5.

It is clear that for the points $A_1,\ B_1,\ C_1,\ A'',\ B'',\ C''$, we have the following coordinates:

$$z_{A_1} = \frac{1}{2}(b+c),\ z_{B_1} = \frac{1}{2}(c+a),\ z_{C_1} = \frac{1}{2}(a+b),$$

$$z_{A''} = a + \frac{1}{2}(b+c),\ z_{B''} = b + \frac{1}{2}(c+a),\ z_{C''} = c + \frac{1}{2}(a+b).$$

It is not so easy to find the coordinates of $A',\ B',\ C'$.

Proposition. *Consider the point $X(x)$ on the circumcircle of triangle ABC. Let P be the projection of X onto line BC. Then the coordinate of P is given by*

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right),$$

where R is the circumradius of triangle ABC.

Proof. Using the complex product and the real product, we can write the equations of lines BC and XP as follows:

$$BC : (z - b) \times (c - b) = 0,$$
$$XP : (z - x) \cdot (c - b) = 0.$$

The coordinate p of P satisfies both equations; hence we have

$$(p - b) \times (c - b) = 0 \text{ and } (p - x) \cdot (c - b) = 0.$$

These equations are equivalent to

$$(p - b)(\overline{c} - \overline{b}) - (\overline{p} - \overline{b})(c - b) = 0$$

and

$$(p - x)(\overline{c} - \overline{b}) + (\overline{p} - \overline{x})(c - b) = 0.$$

Adding the above relations, we obtain

$$(2p - b - x)(\overline{c} - \overline{b}) + (\overline{b} - \overline{x})(c - b) = 0.$$

It follows that

$$p = \frac{1}{2}\left[b + x + \frac{c - b}{\overline{c} - \overline{b}}(\overline{x} - \overline{b})\right] = \frac{1}{2}\left[b + x + \frac{c - b}{\dfrac{R^2}{c} - \dfrac{R^2}{b}}(\overline{x} - \overline{b})\right]$$

$$= \frac{1}{2}\left[b + x - \frac{bc}{R^2}(\overline{x} - \overline{b})\right] = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right). \qquad \square$$

From the above proposition, we see that the coordinates of A', B', C' are

$$z_{A'} = \frac{1}{2}\left(a + b + c - \frac{bc\overline{a}}{R^2}\right),$$
$$z_{B'} = \frac{1}{2}\left(a + b + c - \frac{ca\overline{b}}{R^2}\right),$$
$$z_{C'} = \frac{1}{2}\left(a + b + c - \frac{ab\overline{c}}{R^2}\right).$$

Theorem 1 (The nine-point circle). *In every triangle ABC, the points A_1, B_1, C_1, A', B', C', A'', B'', C'' are all on the same circle, whose center is at the midpoint of the segment OH and whose radius is one-half the circumradius.*

Proof. Denote by O_9 the midpoint of the segment OH. Using our initial assumption, it follows that $z_{O_9} = \frac{1}{2}(a+b+c)$. Also, we have $|a| = |b| = |c| = R$, where R is the circumradius of triangle ABC.

Observe that $O_9A_1 = |z_{A_1} - z_{O_9}| = \dfrac{1}{2}|a| = \dfrac{1}{2}R$, and also $O_9B_1 = O_9C_1 = \dfrac{1}{2}R$.

We can write $O_9A'' = |z_{A''} - z_{O_9}| = \dfrac{1}{2}|a| = \dfrac{1}{2}R$, and also $O_9B'' = O_9C'' = \dfrac{1}{2}R$.

The distance O_9A' is also not difficult to compute:

$$
\begin{aligned}
O_9A' = |z_{A'} - z_{O_9}| &= \left|\frac{1}{2}\left(a+b+c-\frac{bc\overline{a}}{R^2}\right) - \frac{1}{2}(a+b+c)\right| \\
&= \frac{1}{2R^2}|bc\overline{a}| = \frac{1}{2R^2}|\overline{a}||b||c| = \frac{R^3}{2R^2} = \frac{1}{2}R.
\end{aligned}
$$

Similarly, we get $O_9B' = O_9C' = \dfrac{1}{2}R$. Therefore, $O_9A_1 = O_9B_1 = O_9C_1 = O_9A' = O_9B' = O_9C' = O_9A'' = O_9B'' = O_9C'' = \dfrac{1}{2}R$, and the desired property follows. □

Theorem 2.

(1) (Euler[4] *line of a triangle.) In any triangle ABC the points O, G, H are collinear.*

(2) (Nagel line of a triangle.) In any triangle ABC the points I, G, N are collinear.

Proof.

(1) If the circumcenter O is the origin of the complex plane, we have $z_O = 0$, $z_G = \dfrac{1}{3}(a+b+c)$, $z_H = a+b+c$. Hence these points are collinear by Proposition 2 in Sect. 3.2 or 4.2.

(2) We have $z_I = \dfrac{\alpha}{2s}a + \dfrac{\beta}{2s}b + \dfrac{\gamma}{2s}c$, $z_G = \dfrac{1}{3}(a+b+c)$, and $z_N = \left(1-\dfrac{\alpha}{s}\right)a + \left(1-\dfrac{\beta}{s}\right)b + \left(1-\dfrac{\gamma}{s}\right)c$, and we can write $z_N = 3z_G - 2z_I$.

Applying the result mentioned above and properties of the complex product, we obtain $(z_G - z_I) \times (z_N - z_I) = (z_G - z_I) \times [3(z_G - z_I)] = 0$; hence the points I, G, N are collinear. □

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[4] Leonhard Euler (1707–1783), one of the most important mathematicians of all time, created much of modern calculus and contributed significantly to almost every existing branch of pure mathematics, adding proofs and arranging the whole in a consistent form. Euler wrote an immense number of memoirs on a great variety of mathematical subjects. We recommend William Dunham's book *Euler: The Master of Us All* [33] for more details concerning Euler's contributions to mathematics.

Remark. Note that $NG = 2GI$, and hence the triangles OGI and HGN are similar. It follows that the lines OI and NH are parallel, and we have the basic configuration of triangle ABC shown in Fig. 4.6.

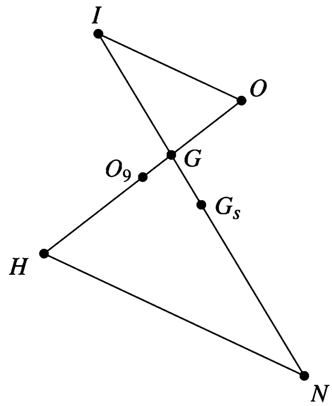

Figure 4.6.

If G_s is the midpoint of segment $[IN]$, then its coordinate is

$$z_{G_s} = \frac{1}{2}(z_I + z_N) = \frac{(\beta+\gamma)}{4s}a + \frac{(\gamma+\alpha)}{4s}b + \frac{(\alpha+\beta)}{4s}c.$$

The point G_s is called the *Spiecker point* of triangle ABC, and it is easy to verify that it is the incenter of the medial triangle $A_1B_1C_1$.

Problem 1. *Consider a point M on the circumcircle of triangle ABC. Prove that the nine-point centers of triangles MBC, MCA, MAB are the vertices of a triangle similar to triangle ABC.*

Solution. Let A', B', C' be the nine-point centers of the triangles MBC, MCD, MAB, respectively. Take the origin of the complex plane to be at the circumcenter of triangle ABC. Denote by the corresponding lowercase letter the coordinate of the point denoted by an uppercase letter. Then

$$a' = \frac{m+b+c}{2},\ b' = \frac{m+c+a}{2},\ c' = \frac{m+a+b}{2},$$

since M lies on the circumcircle of triangle ABC. Then

$$\frac{b'-a'}{c'-a'} = \frac{a-b}{a-c} = \frac{b-a}{c-a},$$

and hence triangles $A'B'C'$ and ABC are similar.

Problem 2. *Show that triangle ABC is a right triangle if and only if its circumcircle and its nine-point circle are tangent.*

Solution. Take the origin of the complex plane to be at the circumcenter O of triangle ABC, and denote by $a,\ b,\ c$ the coordinates of vertices $A,\ B,\ C$, respectively. Then the circumcircle of triangle ABC is tangent to the nine-point circle of triangle ABC if and only if $OO_9 = \dfrac{R}{2}$. This is equivalent to $OO_9^2 = \dfrac{R^2}{4}$, that is, $|a+b+c|^2 = R^2$.

Using properties of the real product, we have

$$|a+b+c|^2 = (a+b+c)\cdot(a+b+c) = |a|^2+|b|^2+|c|^2+2(a\cdot b+b\cdot c+c\cdot a)$$
$$= 3R^2 + 2(a\cdot b+b\cdot c+c\cdot a) = 3R^2 + (2R^2-\alpha^2+2R^2-\beta^2+2R^2-\gamma^2)$$
$$= 9R^2 - (\alpha^2+\beta^2+\gamma^2),$$

where $\alpha,\ \beta,\ \gamma$ are the lengths of the sides of triangle ABC. We have used the formulas $a\cdot b = R^2 - \dfrac{\gamma^2}{2}$, $b\cdot c = R^2 - \dfrac{\alpha^2}{2}$, $c\cdot a = R^2 - \dfrac{\beta^2}{2}$, which can be easily derived from the definition of the real product of complex numbers (see also the lemma in Sect. 4.6.2).

Therefore, $\alpha^2+\beta^2+\gamma^2 = 8R^2$, which is the same as $\sin^2 A + \sin^2 B + \sin^2 C = 2$. We can write the last relation as $1-\cos 2A + 1 - \cos 2B + 1 - \cos 2C = 4$. This is equivalent to $2\cos(A+B)\cos(A-B) + 2\cos^2 C = 0$, i.e., $4\cos A\cos B\cos C = 0$, and the desired conclusion follows.

Problem 3. *Let $ABCD$ be a cyclic quadrilateral and let $E_a,\ E_b,\ E_c,\ E_d$ be the nine-point centers of triangles $BCD,\ CDA,\ DAB,\ ABC$, respectively. Prove that the lines $AE_a,\ BE_b,\ CE_c,\ DE_d$ are concurrent.*

Solution. Take the origin of the complex plane to be the center O of the circumcircle of $ABCD$. Then the coordinates of the nine-point centers are

$$e_a = \frac{1}{2}(b+c+d),\ e_b = \frac{1}{2}(c+d+a),\ e_c = \frac{1}{2}(d+a+b),\ e_d = \frac{1}{2}(a+b+c).$$

We have $AE_a : z = ka + (1-k)e_a,\ k\in\mathbb{R}$, and the analogous equations for the lines $BE_b,\ CE_c,\ DE_d$. Observe that the point with coordinate $\frac{1}{3}(a+b+c+d)$ lies on all four lines $\left(k = \dfrac{1}{3}\right)$, and we are done.

4.6 Some Important Distances in a Triangle

4.6.1 Fundamental Invariants of a Triangle

Consider the triangle ABC with sides $\alpha,\ \beta,\ \gamma$; semiperimeter

$$s = \frac{1}{2}(\alpha+\beta+\gamma);$$

inradius r; and circumradius R. The numbers $s,\ r,\ R$ are called the *fundamental invariants* of triangle ABC.

Theorem. *The sides $\alpha,\ \beta,\ \gamma$ are the roots of the cubic equation*

$$t^3 - 2st^2 + (s^2+r^2+4Rr)t \ - 4sRr = 0.$$

Proof. Let us prove that α satisfies the equation. We have

$$\alpha = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2} \text{ and } s-\alpha = r\text{cotan}\frac{A}{2} = r\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}},$$

whence

$$\cos^2\frac{A}{2} = \frac{\alpha(s-\alpha)}{4Rr} \text{ and } \sin^2\frac{A}{2} = \frac{\alpha r}{4R(s-\alpha)}.$$

From the formula $\cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1$, it follows that

$$\frac{\alpha(s-\alpha)}{4Rr} + \frac{\alpha r}{4R(s-\alpha)} = 1.$$

That is, $\alpha^3 - 2s\alpha^2 + (s^2+r^2+4Rr)\alpha - 4sRr = 0$. We can show analogously that β and γ are roots of the above equation. □

From the above theorem, using the relations between the roots and the coefficients, it follows that

$$\alpha+\beta+\gamma = 2\text{s},$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = s^2+r^2+4Rr,$$

$$\alpha\beta\gamma = 4sRr.$$

Corollary. *The following formulas hold in every triangle ABC:*

$$\alpha^2+\beta^2+\gamma^2 = 2(s^2-r^2-4Rr),$$

$$\alpha^3+\beta^3+\gamma^3 = 2s(s^2-3r^2-6Rr).$$

Proof. We have

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2 &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha) = 4s^2-2(s^2+r^2+4Rr)\\ &= 2s^2-2r^2-8Rr = 2(s^2-r^2-4Rr).\end{aligned}$$

In order to prove the second identity, we can write

$$\alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)+3\alpha\beta\gamma$$

$$= 2s(2s^2-2r^2-8Rr-s^2-r^2-4Rr)+12sRr = 2s(s^2-3r^2-6Rr). \quad \square$$

4.6.2 The Distance OI

Assume that the circumcenter O of the triangle ABC is the origin of the complex plane, and let $a,\ b,\ c$ be the coordinates of the vertices $A,\ B,\ C$, respectively.

Lemma. *The real products $a\cdot b,\ b\cdot c,\ c\cdot a$ are given by*

$$a\cdot b = R^2-\frac{\gamma^2}{2},\ b\cdot c = R^2-\frac{\alpha^2}{2},\ c\cdot a = R^2-\frac{\beta^2}{2}.$$

Proof. Using the properties of the real product, we have

$$\gamma^2 = |a-b|^2 = (a-b)\cdot(a-b) = a\cdot a-2a\cdot b+b\cdot b = |a|^2-2a\cdot b+|b|^2 = 2R^2-2a\cdot b,$$

and the first formula follows. $\square$

In order to simplify the formulas, we will use the symbol $\sum_{\text{cyc}}$, called the *cyclic sum*:

$$\sum_{\text{cyc}} f(x_1,x_2,x_3) = f(x_1,x_2,x_3)+f(x_2,x_3,x_1)+f(x_3,x_1,x_2),$$

where the sum is taken over all cyclic permutations of the variables.

Theorem (Euler). *The following formula holds*:

$$OI^2 = R^2-2Rr.$$

Proof. The coordinate of the incenter is given by

$$z_I = \frac{\alpha}{2s}a+\frac{\beta}{2s}b+\frac{\gamma}{2s}c,$$

so we can write

$$OI^2 = |z_I|^2 = \left(\frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c\right) \cdot \left(\frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c\right)$$
$$= \frac{1}{4s^2}(\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\frac{1}{4s^2}\sum_{\text{cyc}}(\alpha\beta)a \cdot b.$$

Using the above lemma, we find that

$$OI^2 = \frac{1}{4s^2}(\alpha^2 + \beta^2 + \gamma^2)R^2 + \frac{2}{4s^2}\sum_{\text{cyc}}\alpha\beta\left(R^2 - \frac{\gamma^2}{2}\right)$$

$$= \frac{1}{4s^2}(\alpha + \beta + \gamma)^2R^2 - \frac{1}{4s^2}\sum_{\text{cyc}}\alpha\beta\gamma^2 = R^2 - \frac{1}{4s^2}\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= R^2 - \frac{1}{2s}\alpha\beta\gamma = R^2 - 2\frac{\alpha\beta\gamma}{4K} \cdot \frac{K}{s} = R^2 - 2\mathrm{Rr},$$

where the well-known formulas

$$R = \frac{\alpha\beta\gamma}{4K},\ r = \frac{K}{s},$$

are used. Here K is the area of triangle ABC. □

Corollary (Euler's inequality). *In every triangle ABC, the following inequality holds*:

$$R \geq 2r.$$

We have equality if and only if triangle ABC is equilateral.

Proof. From the above theorem. we have $OI^2 = R(R-2r) \geq 0$, hence $R \geq 2r$. The equality $R - 2r = 0$ holds if and only if $OI^2 = 0$, i.e., $O = I$. Therefore, triangle ABC is equilateral. □

4.6.3 The Distance ON

Theorem 1. *If N is the Nagel point of triangle ABC, then*

$$ON = R - 2r.$$

Proof. The coordinate of the Nagel point of the triangle is given by

$$z_N = \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Therefore,

$$ON^2 = |z_N|^2 = z_N \cdot z_N = R^2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)^2 + 2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right) \left(1 - \frac{\beta}{s}\right) a \cdot b$$

$$= R^2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)^2 + 2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right) \left(1 - \frac{\beta}{s}\right) \left(R^2 - \frac{\gamma^2}{2}\right)$$

$$= R^2 \left(3 - \frac{\alpha + \beta + \gamma}{s}\right)^2 - \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right) \left(1 - \frac{\beta}{s}\right) \gamma^2$$

$$= R^2 - \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right) \left(1 - \frac{\beta}{s}\right) \gamma^2 = R^2 - E.$$

To calculate E, we note that

$$E = \sum_{\text{cyc}} \left(1 - \frac{\alpha + \beta}{s} + \frac{\alpha\beta}{s^2}\right) \gamma^2 = \sum_{\text{cyc}} \gamma^2 - \frac{1}{s} \sum_{\text{cyc}} (\alpha + \beta)\gamma^2 + \frac{1}{s^2} \sum_{\text{cyc}} \alpha\beta\gamma^2$$

$$= \sum_{\text{cyc}} \gamma^2 - \frac{1}{s} \sum_{\text{cyc}} (2s - \gamma)\gamma^2 + \frac{2\alpha\beta\gamma}{s} = -\sum_{\text{cyc}} \alpha^2 + \frac{1}{s} \sum_{\text{cyc}} \alpha^3 + 8\frac{\alpha\beta\gamma}{4K} \cdot \frac{K}{s}$$

$$= -\sum_{\text{cyc}} \alpha^2 + \frac{1}{s} \sum_{\text{cyc}} \alpha^3 + 8Rr.$$

Applying the formula in the corollary of Sect. 4.6.1, we conclude that

$$E = -2(s^2 - r^2 - 4Rr) + 2(s^2 - 3r^2 - 6Rr) + 8Rr = -4r^2 + 4Rr.$$

Hence $ON^2 = R^2 - E = R^2 - 4Rr + 4r^2 = (R - 2r)^2$, and the desired formula is proved by Euler's inequality. □

Theorem 2 (Feuerbach[5]). *In any triangle the incircle and the nine-point circle of Euler are tangent.*

Proof. Using the configuration in Sect. 4.5 we observe that

$$\frac{1}{2} = \frac{GI}{GN} = \frac{GO_9}{GO}.$$

Therefore, triangles GIO_9 and GNO are similar. It follows that the lines IO_9 and ON are parallel and $IO_9 = \frac{1}{2}ON$. Applying Theorem 1 in Sect. 4.6.3, we get $IO_9 = \frac{1}{2}(R - 2r) = \frac{R}{2} - r = R_9 - r$, and hence the incircle is tangent to the nine-point circle. □

[5] Karl Wilhelm Feuerbach (1800–1834), German geometer, published the result of Theorem 2 in 1822.

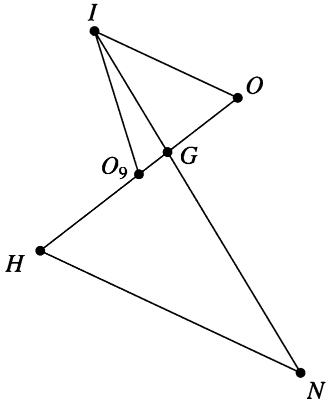

Figure 4.7.

The point of tangency of these two circles is denoted by φ and is called the *Feuerbach point* of the triangle (Fig. 4.7).

4.6.4 The Distance OH

Theorem. *If H is the orthocenter of triangle ABC, then*

$$OH^2 = 9R^2 + 2r^2 + 8Rr - 2s^2.$$

Proof. Assuming that the circumcenter O is the origin of the complex plane, the coordinate of H is

$$z_H = a + b + c.$$

Using the real product, we can write

$$\begin{aligned} OH^2 &= |z_H|^2 = z_H \cdot z_H = (a+b+c) \cdot (a+b+c) \\ &= \sum_{\text{cyc}} |a|^2 + 2 \sum \ a \cdot b = 3R^2 + 2 \sum_{\text{cyc}} a \cdot b. \end{aligned}$$

Applying the formulas in the lemma and then the first formula in Corollary 4.6.1, we obtain

$$OH^2 = 3R^2 + 2 \sum_{\text{cyc}} \left(R^2 - \frac{\gamma^2}{2} \right) = 9R^2 - (\alpha^2 + \beta^2 + \gamma^2)$$

$$= 9R^2 - 2(s^2 - r^2 - 4Rr) = 9R^2 + 2r^2 + 8Rr - 2s^2. \qquad \square$$

Corollary 1. *The following formulas hold:*

(1) $OG^2 = R^2 + \frac{2}{9}r^2 + \frac{8}{9}Rr - \frac{2}{9}s^2$;
(2) $OO_9^2 = \frac{9}{4}R^2 + \frac{1}{2}r^2 + 2Rr - \frac{1}{2}s^2$.

Corollary 2. *In every triangle ABC, the inequality*

$$\alpha^2 + \beta^2 + \gamma^2 \leq 9R^2$$

is valid. Equality holds if and only if the triangle is equilateral.

4.6.5 Blundon's Inequalities

Given a triangle ABC, denote by O its circumcenter, I the incenter, G the centroid, N the Nagel point, s the semiperimeter, R the circumradius, and r the inradius. In what follows, we present a geometric proof to the so-called fundamental triangle inequality. This relation contains, in fact, two inequalities, and it was first proved by E. Rouché in 1851, answering a question of Ramus concerning necessary and sufficient conditions for three positive real numbers s, R, r to be the semiperimeter, circumradius, and inradius of a triangle. The standard simple proof was first given by W.J. Blundon, and it is based on the following algebraic property of the roots of a cubic equation: The roots x_1, x_2, x_3 of the equation

$$x^3 + a_1x^2 + a_2x + a_3 = 0$$

are the side lengths of a (nondegenerate) triangle if and only if the following three conditions are satisfied:

(i) $18a_1a_2a_3 + a_1^2a_2^2 - 27a_3^2 - 4a_2^3 - 4a_1^3a_3 > 0$;
(ii) $-a_1 > 0,\ a_2 > 0,\ -a_3 > 0$;
(iii) $a_1^3 - 4a_1a_4 + 8a_3 > 0$.

The following result contains a simple geometric proof of the fundamental inequality of a triangle, as presented in the article [15].

Theorem 1. *Assume that the triangle ABC is not equilateral. The following relation holds:*

$$\cos\widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R-2r)\sqrt{R^2 - 2Rr}}.$$

Proof. It is known (see Theorem 2 in Sect. 4.5) that the points N, G, and I are collinear on a line called Nagel's line of the triangle, and we have $NI = 3GI$. If we use Stewart's theorem in the triangle ION, then we get

$$ON^2 \cdot GI + OI^2 \cdot NG - OG^2 \cdot NI = GI \cdot GN \cdot NI,$$

and it follows that

$$ON^2 \cdot GI + OI^2 \cdot 2GI - OG^2 - 3GI = 6GI^3.$$

This relation is equivalent to

$$ON^2 + 2OI^2 - 3OG^2 = 6GI^2.$$

Now, using formulas for ON, OI, and OG, we obtain

$$GI^2 = \frac{1}{6}\left(\frac{a^2+b^2+c^2}{3} - 8Rr + 4r^2\right) = \frac{1}{6}\left(\frac{2(s^2-r^2-4Rr)}{3} - 8Rr + 4r^2\right).$$

So we get

$$NI^2 = 9GI^2 = 5r^2 + s^2 - 16Rr.$$

We use the law of cosines in the triangle ION to obtain

$$\cos\widehat{ION} = \frac{ON^2 + OI^2 - NI^2}{2ON \cdot OI}$$

$$= \frac{(R-2r)^2 + (R^2 - 2Rr) - (5r^2 + s^2 - 16Rr)}{2(R-2r)\sqrt{R^2-2Rr}} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R-2r)\sqrt{R^2-2Rr}},$$

and we are done.

If the triangle ABC is equilateral, then the points I, O, N coincide, i.e., triangle ION degenerates to a single point. In this case, we extend the formula by $\cos\widehat{ION} = 1$. □

Theorem 2 (Blundon's inequalities). *A necessary and sufficient condition for the existence of a triangle with elements s, R, and r is*

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}$$

$$\leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R^2 - 2r)\sqrt{R^2 - 2Rr}.$$

Proof. If we have $R = 2r$, then the triangle must be equilateral, and we are done. If we assume that $R - 2r \neq 0$, then the desired inequalities are direct consequences of the fact that $-1 \leq \cos\widehat{ION} \leq 1$. □

Equilateral triangles give the trivial situation in which we have equality. Suppose that we are not working with equilateral triangles, i.e., we have $R - 2r \neq 0$. Denote by $\mathcal{T}(R, r)$ the family of all triangles with circumradius R and inradius r. Blundon's inequalities give, in terms of R and r, the exact interval for the semiperimeter s of triangles in the family $\mathcal{T}(R, r)$. We have

$$s^2_{\min} = 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}$$

and

$$s_{\max}^2 = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.$$

If we fix the circumcenter O and the incenter I such that $OI = \sqrt{R^2 - 2Rr}$, then the triangle in the family $\mathcal{T}(R, r)$ with minimal semiperimeter corresponds to the case $\cos\widehat{ION} = 1$ of equality, i.e., points I, O, N are collinear, and I and N belong to the same ray with the origin O. Taking into account the well-known property that points O, G, H belong to Euler's line of the triangle, we see that O, I, G must be collinear, and hence in this case, triangle ABC is isosceles. In Fig. 4.8, this triangle is denoted by $A_{\min}B_{\min}C_{\min}$. Also, the triangle in the family $\mathcal{T}(R, r)$ with maximal semiperimeter corresponds to the case of equality $\cos\widehat{ION} = -1$, i.e., points I, O, N are collinear, and O is situated between I and N. Using again the Euler line of the triangle, we see that triangle ABC is isosceles. In Fig. 4.8, this triangle is denoted by $A_{\max}B_{\max}C_{\max}$.

Note that we have $B_{\min}C_{\min} > B_{\max}C_{\max}$. The triangles in the family $\mathcal{T}(R, r)$ are "between" these two extremal triangles (see Fig. 4.8). According to Poncelet's closure theorem, they are inscribed in the circle $\mathcal{C}(O; R)$, and their sides are externally tangent to the circle $\mathcal{C}(I; r)$.

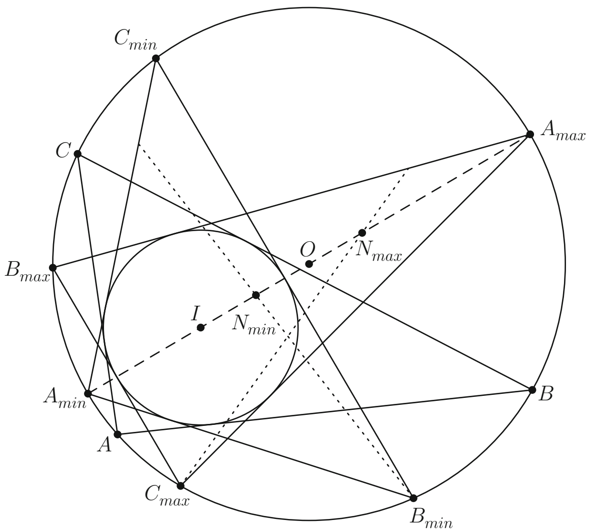

Figure 4.8.

4.7 Distance Between Two Points in the Plane of a Triangle

4.7.1 Barycentric Coordinates

Consider a triangle ABC and let $\alpha,\ \beta,\ \gamma$ be the lengths of sides $BC,\ CA,\ AB$, respectively.

Proposition. *Let $a,\ b,\ c$ be the coordinates of vertices $A,\ B,\ C$ and let P be a point in the plane of the triangle. If z_P is the coordinate of P, then there exist unique real numbers $\mu_a,\ \mu_b,\ \mu_c$ such that*

$$z_P = \mu_a a + \mu_b b + \mu_c c \text{ and } \mu_a + \mu_b + \mu_c = 1.$$

Proof. Assume that P is in the interior of triangle ABC and consider the point A' such that $AP \cap BC = \{A'\}$. Let $k_1 = \dfrac{PA}{PA'}$, $k_2 = \dfrac{A'B}{A'C}$, and observe that

$$z_P = \frac{a + k_1 z_{A'}}{1+k_1},\ z_{A'} = \frac{b + k_2 c}{1+k_2}.$$

Hence in this case, we can write

$$z_P = \frac{1}{1+k_1}a + \frac{k_1}{(1+k_1)(1+k_2)}b + \frac{k_1 k_2}{(1+k_1)(1+k_2)}c.$$

Moreover, if we consider

$$\mu_a = \frac{1}{1+k_1},\ \mu_b = \frac{k_1}{(1+k_1)(1+k_2)},\ \mu_c = \frac{k_1 k_2}{(1+k_1)(1+k_2)},$$

we have

$$\begin{aligned}\mu_a + \mu_b + \mu_c &= \frac{1}{1+k_1} + \frac{k_1}{(1+k_1)(1+k_2)} + \frac{k_1 k_2}{(1+k_1)(1+k_2)}\\ &= \frac{1+k_1+k_2+k_1k_2}{(1+k_1)(1+k_2)} = 1.\end{aligned}$$

We proceed in an analogous way when the point P is situated in the exterior of triangle ABC.

If the point P is situated on the support line of a side of triangle ABC (i.e., the line determined by two vertices), then

$$z_P = \frac{1}{1+k}b + \frac{k}{1+k}c = 0 \cdot a + \frac{1}{1+k}b + \frac{k}{1+k}c,$$

where $k = \dfrac{PB}{PC}$. $\square$

The real numbers $\mu_a,\ \mu_b,\ \mu_c$ are called the *absolute barycentric coordinates* of P with respect to triangle ABC.

The signs of the numbers $\mu_a,\ \mu_b,\ \mu_c$ depend on the regions of the plane in which the point P is situated. Triangle ABC determines seven such regions (Fig. 4.9).

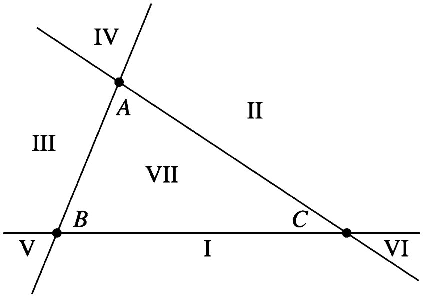

Figure 4.9.

In the following table, we give the signs of $\mu_a,\ \mu_b,\ \mu_c$:

	I	II	III	IV	V	VI	VII
μ_a	−	+	+	+	−	−	+
μ_b	+	−	+	−	+	−	+
μ_c	+	+	−	−	−	+	+

4.7.2 Distance Between Two Points in Barycentric Coordinates

In what follows, in order to simplify the formulas, we will use again the cyclic sum symbol defined above, $\sum_{\text{cyc}} f(x_1,\ x_2,\ \ldots,\ x_n)$. The most important example for our purposes is

$$\sum_{\text{cyc}} f(x_1,\ x_2,\ x_3) = f(x_1,\ x_2,\ x_3) + f(x_2,\ x_3,\ x_1) + f(x_3,\ x_1,\ x_2).$$

Theorem 1. *In the plane of triangle ABC, consider the points P_1 and P_2 with coordinates z_{P_1} and z_{P_2}, respectively. If $z_{P_k} = \alpha_k a + \beta_k b + \gamma_k c$, where $\alpha_k,\ \beta_k,\ \gamma_k$ are real numbers such that $\alpha_k + \beta_k + \gamma_k = 1,\ k = 1, 2$, then*

$$P_1P_2^2 = -\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2.$$

Proof. Choose the origin of the complex plane to be located at the circumcenter O of the triangle ABC. Using properties of the real product, we have

$$P_1P_2^2 = |z_{P_2} - z_{P_1}|^2 = |(\alpha_2 - \alpha_1)a + (\beta_2 - \beta_1)b + (\gamma_2 - \gamma_1)c|^2$$

$$= \sum_{\text{cyc}} (\alpha_2 - \alpha_1)^2 a \cdot \ a + 2\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)a \cdot \ b$$

$$= \sum_{\text{cyc}} (\alpha_2 - \alpha_1)^2 R^2 + 2\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\left(R^2 - \frac{\gamma^2}{2}\right)$$

$$= R^2(\alpha_2 + \beta_2 + \gamma_2 - \alpha_1 - \beta_1 - \gamma_1)^2 - \sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2$$

$$= -\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2,$$

since $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 1$. □

Theorem 2. *The points $A_1,\ A_2,\ B_1,\ B_2,\ C_1,\ C_2$ are situated on the sides $BC,\ CA,\ AB$ of triangle ABC such that lines $AA_1,\ BB_1,\ CC_1$ meet at point P_1, and lines $AA_2,\ BB_2,\ CC_2$ meet at point P_2. If*

$$\frac{BA_k}{A_kC} = \frac{p_k}{n_k},\ \frac{CB_k}{B_kA} = \frac{m_k}{p_k},\ \frac{AC_k}{C_kB} = \frac{n_k}{m_k},\ k = 1, 2,$$

where $m_k,\ n_k,\ p_k$ are nonzero real numbers, $k = 1, 2$, and $S_k = m_k + n_k + p_k$, $k = 1, 2$, then

$$P_1P_2^2 = \frac{1}{S_1^2S_2^2}\left[S_1S_2\sum_{\text{cyc}}(n_{1P2} + p_1n_2)\alpha^2 - S_1^2\sum_{\text{cyc}} n_2p_2\alpha^2 - S_2^2\sum_{\text{cyc}} n_1p_1\alpha^2\right].$$

Proof. The coordinates of points P_1 and P_2 are

$$z_{P_k} = \frac{m_ka + n_kb + p_kc}{m_k + n_k + p_k},\ k = 1, 2.$$

It follows that in this case, the absolute barycentric coordinates of points P_1 and P_2 are given by

$$\alpha_k = \frac{m_k}{m_k + n_k + p_k} = \frac{m_k}{S_k},\ \beta_k = \frac{n_k}{m_k + n_k + p_k} = \frac{n_k}{S_k},$$

$$\gamma_k = \frac{p_k}{m_k + n_k + p_k} = \frac{p_k}{S_k},\ k = 1, 2.$$

Substituting in the formula in Theorem 1 in Sect. 4.7.2, we obtain

$$\begin{aligned}
P_1P_2^2 &= -\sum_{\text{cyc}} \left(\frac{n_2}{S_2} - \frac{n_1}{S_1}\right)\left(\frac{p_2}{S_2} - \frac{p_1}{S_1}\right)\alpha^2 \\
&= -\frac{1}{S_1^2S_2^2}\sum_{\text{cyc}}(S_1n_2 - S_2n_1)(S_1p_2 - S_2p_1)\alpha^2 \\
&= -\frac{1}{S_1^2S_2^2}\sum_{\text{cyc}}\left[S_1^2n_2p_2 + S_2^2n_1p_1 - S_1S_2(n_1p_2 + n_2p_1)\right]\alpha^2 \\
&= \frac{1}{S_1^2S_2^2}\left[S_1S_2\sum_{\text{cyc}}(n_1p_2 + n_2p_1)\alpha^2 - S_1^2\sum_{\text{cyc}} n_2p_2\alpha^2 - S_2^2\sum_{\text{cyc}} n_1p_1\alpha^2\right],
\end{aligned}$$

and the desired formula follows. □

Corollary 1. *For real numbers $\alpha_k, \beta_k, \gamma_k$ with $\alpha_k + \beta_k + \gamma_k = 1$, $k = 1, 2$, the following inequality holds:*

$$\sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2 \le 0,$$

with equality if and only if $\alpha_1 = \alpha_2,\ \beta_1 = \beta_2,\ \gamma_1 = \gamma_2$.

Corollary 2. *For nonzero real numbers $m_k,\ n_k,\ p_k,\ k = 1, 2$, with $S_k = m_k + n_k + p_k,\ k = 1, 2$, the lengths of sides $\alpha,\ \beta,\ \gamma$ of triangle ABC satisfy the inequality*

$$\sum_{\text{cyc}}(n_1p_2 + p_1n_2)^2 \ge \frac{S_1}{S_2}\sum_{\text{cyc}} n_2p_2\alpha^2 + \frac{S_2}{S_1}\sum_{\text{cyc}} n_1p_1\alpha^2,$$

with equality if and only if $\dfrac{p_1}{n_1} = \dfrac{p_2}{n_2},\ \dfrac{m_1}{p_1} = \dfrac{m_2}{p_2},\ \dfrac{n_1}{m_1} = \dfrac{n_2}{m_2}$.

Applications

(1) Let us use the formula in Theorem 2 in Sect. 4.7.2 to compute the distance GI, used in Sect. 4.6.5, where G is the centroid and I is the incenter of the triangle.

We have $m_1 = n_1 = p_1 = 1$ and $m_2 = \alpha,\ n_2 = \beta,\ p_2 = \gamma$; hence

$$S_1 = \sum_{\text{cyc}} m_1 = 3;\ S_2 = \sum_{\text{cyc}} m_2 = \alpha + \beta + \gamma = 2s;$$

$$\sum_{\text{cyc}}(n_1p_2 + n_2p_1)\alpha^2 = (\beta + \gamma)\alpha^2 + (\gamma + \alpha)\beta^2 + (\alpha + \beta)\gamma^2$$

$$= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma = 2s(s^2 + r^2 + 4rR) - 12sRr$$

$$= 2s^3 + 2sr^2 - 4sRr.$$

On the other hand,

$$\sum_{\text{cyc}} n_2 p_2 \alpha^2 = \alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta = \alpha\beta\gamma(\alpha+\beta+\gamma) = 8s^2 Rr$$

and

$$\sum_{\text{cyc}} n_1 p_1 \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = 2s^2 - 2r^2 - 8Rr.$$

Then

$$GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr).$$

(2) Let us prove that in every triangle ABC *with* sides $\alpha,\ \beta,\ \gamma$, the following inequality holds:

$$\sum_{\text{cyc}} (2\alpha - \beta - \gamma)(2\beta - \alpha - \gamma)\gamma^2 \le 0.$$

In the inequality in Corollary 1 in Sect. 4.7.2, we consider the points $P_1 = G$ and $P_2 = I$. Then $\alpha_1 = \beta_1 = \gamma_1 = \dfrac{1}{3}$ and $\alpha_2 = \dfrac{\alpha}{2s},\ \beta_2 = \dfrac{\beta}{2s}$, $\gamma_2 = \dfrac{\gamma}{2s}$, and the above inequality follows. We have equality if and only if $P_1 = P_2$, that is, $G = I$, so the triangle is equilateral.

4.8 The Area of a Triangle in Barycentric Coordinates

Consider the triangle ABC with $a,\ b,\ c$ the respective coordinates of its vertices. Let $\alpha,\ \beta,\ \gamma$ be the lengths of sides $BC,\ CA$, and AB.

Theorem. *Let $P_j(z_{p_j}),\ j = 1,2,3$, be three points in the plane of triangle ABC with $z_{P_j} = \alpha_j a + \beta_j b + \gamma_j c$, where $\alpha_j,\ \beta_j,\ \gamma_j$ are the barycentric coordinates of P_j. If the triangles ABC and $P_1P_2P_3$ have the same orientation, then*

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Proof. Suppose that the triangles ABC and $P_1P_2P_3$ are positively oriented. If O denotes the origin of the complex plane, then using the complex product, we can write

$$2i\ \text{area}[P_1OP_2] = z_{P_1} \times z_{P_2} = (\alpha_1 a + \beta_1 b + \gamma_1 c) \times (\alpha_2 a + \beta_2 b + \gamma_2 c)$$

$$= (\alpha_1\beta_2 - \alpha_2\beta_1) a \times b + (\beta_1\gamma_2 - \beta_2\gamma_1) b \times c + (\gamma_1\alpha_1 - \gamma_2\alpha_1) c \times a$$

$$= \begin{vmatrix} a\times b & b\times c & c\times a \\ \gamma_1 & \gamma_1 & \beta_1 \\ \gamma_2 & \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a\times b & b\times c & 2i \text{ area } [ABC] \\ \gamma_1 & \alpha_1 & 1 \\ \gamma_2 & \alpha_2 & 1 \end{vmatrix}.$$

Analogously, we obtain

$$2i \text{ area } [P_2OP_3] = \begin{vmatrix} a\times b & b\times c & 2i \text{ area } [ABC] \\ \gamma_2 & \alpha_2 & 1 \\ \gamma_3 & \alpha_3 & 1 \end{vmatrix},$$

$$2i \text{ area } [P_3OP_1] = \begin{vmatrix} a\times b & b\times c & 2i \text{ area } [ABC] \\ \gamma_3 & \alpha_3 & 1 \\ \gamma_1 & \alpha_1 & 1 \end{vmatrix}.$$

Assuming that the origin O is situated in the interior of triangle $P_1P_2P_3$, it follows that

$$\text{area}[P_1P_2P_3] = \text{area } [P_1OP_2] + \text{area } [P_2OP_3] + \text{area } [P_3OP_1]$$

$$= \frac{1}{2i}(\alpha_1-\alpha_2+\alpha_2-\alpha_3+\alpha_3-\alpha_1)a\times b - \frac{1}{2i}(\gamma_1-\gamma_2+\gamma_2-\gamma_3+\gamma_3-\gamma_1)b\times c$$

$$+(\gamma_1\alpha_2-\gamma_2\alpha_1+\gamma_2\alpha_3-\gamma_3\alpha_2+\gamma_3\alpha_1-\gamma_1\alpha_3)\text{area } [ABC]$$

$$= (\gamma_1\alpha_2-\gamma_2\alpha_1+\gamma_2\alpha_3-\gamma_3\alpha_2+\gamma_3\alpha_1-\gamma_1\alpha_3) \text{ area } [ABC]$$

$$= \text{area } [ABC] \begin{vmatrix} 1 & \gamma_1 & \alpha_1 \\ 1 & \gamma_2 & \alpha_2 \\ 1 & \gamma_3 & \alpha_3 \end{vmatrix} = \text{area}[ABC] \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

and the desired formula is obtained. □

Corollary 1. *Consider the triangle ABC and the points A_1, B_1, C_1 situated on the respective lines BC, CA, AB (Fig. 4.10) such that*

$$\frac{A_1B}{A_1C} = k_1,\ \frac{B_1C}{B_1A} = k_2,\ \frac{C_1A}{C_1B} = k_3.$$

If $AA_1 \cap BB_1 = \{P_1\}$, $BB_1 \cap CC_1 = \{P_2\}$, and $CC_1 \cap AA_1 = \{P_3\}$, then

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{(1-k_1k_2k_3)^2}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)}.$$

Proof. Applying the well-known Menelaus's theorem to triangle AA_1B, we find that

$$\frac{C_1A}{C_1B}\cdot\frac{CB}{CA_1}\cdot\frac{P_3A_1}{P_3A} = 1.$$

Hence

$$\frac{P_3A}{P_3A_1} = \frac{C_1A}{C_1B}\cdot\frac{CB}{CA_1} = k_3(1+k_1).$$

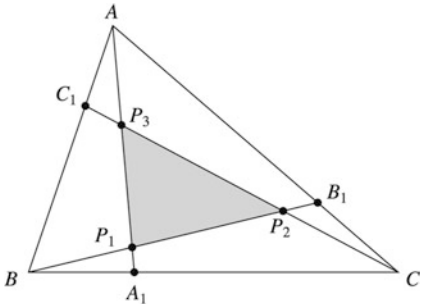

Figure 4.10.

The coordinate of P_3 is given by

$$z_{P_3} = \frac{a + k_3(1+k_1)z_{A_1}}{1 + k_3(1+k_1)} = \frac{a + k_3(1+k_1)\dfrac{b + k_1 c}{1+k_1}}{1 + k_3 + k_3 k_1} = \frac{a + k_3 b + k_3 k_1 c}{1 + k_3 + k_3 k_1}.$$

In an analogous way, we find that

$$z_{P_1} = \frac{k_1 k_2 a + b + k_1 c}{1 + k_1 + k_1 k_2} \text{ and } z_{P_2} = \frac{k_2 a + k_2 k_3 b + c}{1 + k_2 + k_2 k_3}.$$

The triangles ABC and $P_1P_2P_3$ have the same orientation; hence by applying the formula in the above theorem, we find that

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{1}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)} \begin{vmatrix} k_1k_2 & 1 & k_1 \\ k_2 & k_2k_3 & 1 \\ 1 & k_3 & k_3k_1 \end{vmatrix}$$

$$= \frac{(1-k_1k_2k_3)^2}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)}. \qquad \square$$

Remark. When $k_1 = k_2 = k_3 = k$, from Corollary 1 in Sect. 4.8, we obtain Problem 3 in Sect. 4.9.2 from the 23rd Putnam Mathematical Competition.

Let $A_j,\ B_j,\ C_j$ be points on the lines $BC,\ CA,\ AB$, respectively, such that

$$\frac{BA_j}{A_jC} = \frac{p_j}{n_j},\ \frac{CB_j}{B_jA} = \frac{m_j}{p_j},\ \frac{AC_j}{C_jB} = \frac{n_j}{m_j},\ j = 1, 2, 3.$$

Corollary 2. *If P_j is the intersection point of lines AA_j, BB_j, CC_j, $j = 1, 2, 3$, and the triangles ABC, $P_1P_2P_3$ have the same orientation, then*

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{1}{S_1S_2S_3} \begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix},$$

where $S_j = m_j + n_j + p_j$, $j = 1, 2, 3$.

Proof. In terms of the coordinates of the triangle, the coordinates of the points P_j are

$$z_{P_j} = \frac{m_j a + n_j b + p_j c}{m_j + n_j + p_j} = \frac{1}{S_j}(m_j a + n_j b + p_j c),\ j = 1, 2, 3.$$

The formula above follows directly from the above theorem. □

Corollary 3. *In triangle ABC, let us consider the cevians AA', BB', and CC' such that*

$$\frac{A'B}{AC} = m,\ \frac{B'C}{B'A} = n,\ \frac{C'A}{C'B} = p.$$

Then the following formula holds:

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1 + mnp}{(1+m)(1+n)(1+p)}.$$

Proof. Observe that the coordinates of A', B', C' are given by

$$z_{A'} = \frac{1}{1+m}b + \frac{m}{1+m}c,\ z_{B'} = \frac{1}{1+n}c + \frac{n}{1+n}a,\ z_{C'} = \frac{1}{1+p}a + \frac{p}{1+p}b.$$

Applying the formula in Corollary 2 in Sect. 4.8, we obtain

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1}{(1+m)(1+n)(1+p)} \begin{vmatrix} 0 & 1 & m \\ n & 0 & 1 \\ 1 & p & 0 \end{vmatrix}$$

$$= \frac{1 + mnp}{(1+m)(1+n)(1+p)}.$$ □

Applications

(1) (Steinhaus)[6] Let A_j, B_j, C_j be points on lines BC, CA, AB, respectively, $j = 1, 2, 3$. Assume that

$$\frac{BA_1}{A_1C} = \frac{2}{4},\ \frac{CB_1}{B_1A} = \frac{1}{2},\ \frac{AC_1}{C_1B} = \frac{4}{1};$$
$$\frac{BA_2}{A_2C} = \frac{4}{1},\ \frac{CB_2}{B_2A} = \frac{2}{4},\ \frac{AC_2}{C_2B} = \frac{1}{2};$$
$$\frac{BA_3}{A_3C} = \frac{1}{2},\ \frac{CB_3}{B_3A} = \frac{4}{1},\ \frac{AC_3}{C_3B} = \frac{2}{4}.$$

If P_j is the intersection point of lines AA_j, BB_j, CC_j, $j = 1, 2, 3$, and triangles ABC, $P_1P_2P_3$ are of the same orientation, then from Corollary 3 above, we obtain

[6] Hugo Dyonizy Steinhaus (1887–1972), Polish mathematician, made important contributions to functional analysis and other branches of modern mathematics.

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{1}{7\cdot 7\cdot 7}\begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & 4 \\ 4 & 2 & 1 \end{vmatrix} = \frac{49}{7^3} = \frac{1}{7}.$$

(2) If the cevians AA', BB', CC' are concurrent at point P, let us denote by K_P the area of triangle $A'B'C'$. We can use the formula in Corollary 3 above to compute the areas of some triangles determined by the feet of the cevians of some notable points in a triangle.

(i) If I is the incenter of triangle ABC, we have

$$K_I = \frac{1+\frac{\gamma}{\beta}\cdot\frac{\beta}{\alpha}\cdot\frac{\alpha}{\gamma}}{\left(1+\frac{\gamma}{\beta}\right)\left(1+\frac{\beta}{\alpha}\right)\left(1+\frac{\alpha}{\gamma}\right)}\text{area}[ABC]$$

$$= \frac{2\alpha\beta\gamma}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}\text{area}[ABC] = \frac{2\alpha\beta\gamma sr}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}.$$

(ii) For the orthocenter H of the acute triangle ABC, we obtain

$$K_H = \frac{1+\frac{\tan C}{\tan B}\cdot\frac{\tan B}{\tan A}\cdot\frac{\tan A}{\tan C}}{\left(1+\frac{\tan C}{\tan B}\right)\left(1+\frac{\tan B}{\tan A}\right)\left(1+\frac{\tan A}{\tan C}\right)}\text{area}[ABC]$$

$$= (2\cos A\cos B\cos C)\text{area}[ABC] = (2\cos A\cos B\cos C)sr.$$

(iii) For the Nagel point of triangle ABC, we can write

$$K_N = \frac{1+\frac{s-\gamma}{s-\beta}\cdot\frac{s-\alpha}{s-\gamma}\cdot\frac{s-\beta}{s-\alpha}}{\left(1+\frac{s-\gamma}{s-\beta}\right)\left(1+\frac{s-\alpha}{s-\gamma}\right)\left(1+\frac{s-\beta}{s-\alpha}\right)}\text{area}[ABC]$$

$$= \frac{2(s-\alpha)(s-\beta)(s-\gamma)}{\alpha\beta\gamma}\text{area}[ABC] = \frac{4\text{area}^2[ABC]}{2s\alpha\beta\gamma}\text{area}[ABC]$$

$$= \frac{r}{2R}\text{area}[ABC] = \frac{sr^2}{2R}.$$

If we proceed in the same way for the Gergonne point J, we obtain the relation

$$K_J = \frac{r}{2R}\text{area}[ABC] = \frac{sr^2}{2R}.$$

Remark. Two cevians AA' and AA'' are *isotomic* if the points A' and A'' are symmetric with respect to the midpoint of the segment BC. Assuming that

$$\frac{A'B}{A'C} = m, \ \frac{B'C}{B'A} = n, \ \frac{C'A}{C'B} = p,$$

then for the corresponding isotomic cevians, we have

$$\frac{A''B}{A''C} = \frac{1}{m}, \ \frac{B''C}{B''A} = \frac{1}{n}, \ \frac{C''A}{C''B} = \frac{1}{p}.$$

Applying the formula in Corollary 3 above yields that

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1+mnp}{(1+m)(1+n)(1+p)}$$

$$= \frac{1+\dfrac{1}{mnp}}{\left(1+\dfrac{1}{m}\right)\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{1}{p}\right)} = \frac{\text{area}[A''B''C'']}{\text{area}[ABC]}.$$

Therefore, area $[A'B'C'] = \text{area}[A''B''C'']$. A special case of this relation is $K_N = K_J$, since the points N and J are isotomic (i.e., these points are intersections of isotomic cevians).

(3) Consider the excenters $I_\alpha, \ I_\beta, \ I_\gamma$ of triangle ABC. It is not difficult to see that the coordinates of these points are

$$z_{I_\alpha} = -\frac{\alpha}{2(s-\alpha)}a + \frac{\beta}{2(s-\beta)}b + \frac{\gamma}{2(s-\gamma)}c,$$
$$z_{I_\beta} = \frac{\alpha}{2(s-\alpha)}a - \frac{\beta}{2(s-\beta)}b + \frac{\gamma}{2(s-\gamma)}c,$$
$$z_{I_\gamma} = \frac{\alpha}{2(s-\alpha)}a + \frac{\beta}{2(s-\beta)}b - \frac{\gamma}{2(s-\gamma)}c.$$

From the formula in the theorem above, it follows that

$$\text{area}[I_\alpha I_\beta I_\gamma] = \begin{vmatrix} -\frac{\alpha}{2(s-\alpha)} & \frac{\beta}{2(s-\beta)} & \frac{\gamma}{2(s-\gamma)} \\ \frac{\alpha}{2(s-\alpha)} & -\frac{\beta}{2(s-\beta)} & \frac{\gamma}{2(s-\gamma)} \\ \frac{\alpha}{2(s-\alpha)} & \frac{\beta}{2(s-\beta)} & -\frac{\gamma}{2(s-\gamma)} \end{vmatrix} \text{area}[ABC]$$

$$= \frac{\alpha\beta\gamma}{8(s-\alpha)(s-\beta)(s-\gamma)} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \text{area}[ABC]$$

$$= \frac{s\alpha\beta\gamma\text{area}[ABC]}{2s(s-\alpha)(s-\beta)(s-\gamma)} = \frac{s\alpha\beta\gamma\text{area}[ABC]}{2\text{area}^2[ABC]} = \frac{2s\alpha\beta\gamma}{4\text{area}[ABC]} = 2sR.$$

(4) (Nagel line) Using the formula in the theorem above, we give a different proof for the so-called Nagel line: the points I, G, N are collinear. We have seen that the coordinates of these points are

$$z_I = \frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c,$$

$$z_G = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c,$$

$$z_N = \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Then

$$\text{area}[IGN] = \begin{vmatrix} \frac{\alpha}{2s} & \frac{\beta}{2s} & \frac{\gamma}{2s} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 - \frac{\alpha}{s} & 1 - \frac{\beta}{s} & 1 - \frac{\gamma}{s} \end{vmatrix} \cdot \text{area}[ABC] = 0,$$

and hence the points $I,\ G,\ N$ are collinear.

4.9 Orthopolar Triangles

4.9.1 The Simson–Wallace Line and the Pedal Triangle

Consider the triangle ABC, and let M be a point situated in the plane of the triangle. Let $P,\ Q,\ R$ be the projections of M onto lines $BC,\ CA,\ AB$, respectively.

Theorem 1 (The Simson line[7]). *The points $P,\ Q,\ R$ are collinear if and only if M is on the circumcircle of triangle ABC.*

Proof. We will give a standard geometric argument.

Suppose that M lies on the circumcircle of triangle ABC. Without loss of generality, we may assume that M is on the arc $\overset{\frown}{BC}$. In order to prove the collinearity of $R,\ P,\ Q$, it suffices to show that the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent. The quadrilaterals $PRBM$ and $PCQM$ are cyclic (since $\widehat{BRM} \equiv \widehat{BPM}$ and $\widehat{MPC} + \widehat{MQC} = 180°$); hence we have $\widehat{BPR} \equiv \widehat{BMR}$ and $\widehat{CPQ} \equiv \widehat{CMQ}$. But $\widehat{BMR} = 90° - \widehat{ABM} = 90° - \widehat{MCQ}$, since the quadrilateral $ABMC$ is cyclic, too. Finally, we obtain $\widehat{BMR} = 90° - \widehat{MCQ} = \widehat{CMQ}$, so the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent (Fig. 4.11).

To prove the converse, we note that if the points $P,\ Q,\ R$ are collinear, then the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent; hence $\widehat{ABM} + \widehat{ACM} = 180°$, i.e., the quadrilateral $ABMC$ is cyclic. Therefore, the point M is situated on the circumcircle of triangles ABC. □

[7] Robert Simson (1687–1768), Scottish mathematician. This line was attributed to Simson by Poncelet, but it is now generally known as the Simson–Wallace line, since it does not actually appear in any work of Simson. William Wallace (1768–1843) was also a Scottish mathematician, who possibly published the theorem above concerning the Simson line in 1799.

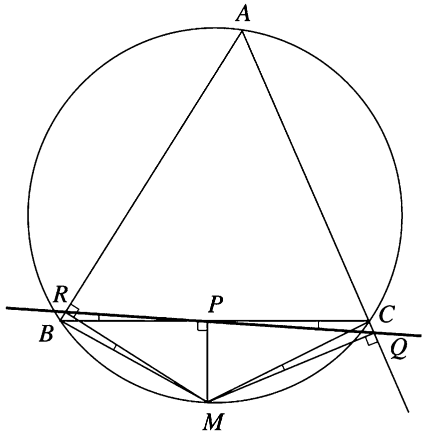

Figure 4.11.

When M lies on the circumcircle of triangle ABC, the line in the above theorem is called the *Simson–Wallace line of* M with respect to triangle ABC.

We continue with a nice generalization of the property contained in Theorem 1 above. For an arbitrary point X in the plane of triangle ABC, consider its projections P, Q, and R on the lines BC, CA and AB, respectively.

The triangle PQR is called the *pedal triangle* of point X with respect to the triangle ABC. Let us choose the circumcenter O of triangle ABC as the origin of the complex plane.

Theorem 2. *The area of the pedal triangle of X with respect to the triangle ABC is given by*

$$\text{area}[PQR] = \frac{\text{area}[ABC]}{4R^2}\left||x|^2 - R^2\right|, \tag{1}$$

where R is the circumradius of triangle ABC.

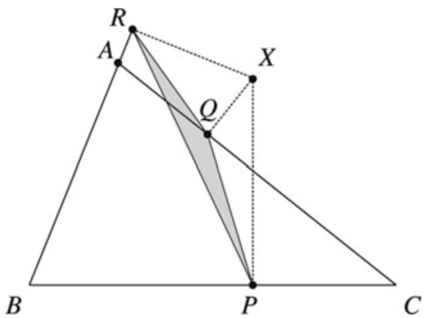

Figure 4.12.

Proof. Applying the formula in the proposition of Sect. 4.5, we obtain the coordinates $p,\ q,\ r$ of the points $P,\ Q,\ R$, respectively (Fig. 4.12):

$$p=\frac{1}{2}\left(x-\frac{bc}{R^2}\overline{x}+b+c\right),$$
$$q=\frac{1}{2}\left(x-\frac{ca}{R^2}\overline{x}+c+a\right),$$
$$r=\frac{1}{2}\left(x-\frac{ab}{R^2}\overline{x}+a+b\right).$$

Taking into account the formula in Sect. 3.5.3, we have

$$\text{area}[PQR]=|\frac{i}{4}\begin{vmatrix} p & \overline{p} & 1\\ q & \overline{q} & 1\\ r & \overline{r} & 1\end{vmatrix}|=|\frac{i}{4}\begin{vmatrix} q-p & \overline{q}-\overline{p}\\ r-p & \overline{r}-\overline{p}\end{vmatrix}|.$$

For the coordinates $p,\ q,\ r$, we obtain

$$\overline{p}=\frac{1}{2}\left(\overline{x}-\frac{\overline{b}\overline{c}}{R^2}x+\overline{b}+\overline{c}\right),$$
$$\overline{q}=\frac{1}{2}\left(\overline{x}-\frac{\overline{c}\,\overline{a}}{R^2}x+\overline{c}+\overline{a}\right),$$
$$\overline{r}=\frac{1}{2}\left(\overline{x}-\frac{\overline{a}\overline{b}}{R^2}x+\overline{a}+\overline{b}\right).$$

It follows that

$$q-p=\frac{1}{2}(a-b)\left(1-\frac{c\overline{x}}{R^2}\right)\ \text{and}\ r-p=\frac{1}{2}(a-c)\left(1-\frac{b\overline{x}}{R^2}\right), \tag{2}$$
$$\overline{q}-\overline{p}=\frac{1}{2abc}(a-b)(x-c)R^2\ \text{and}\ \overline{r}-\overline{p}=\frac{1}{2abc}(a-c)(x-b)R^2.$$

Therefore,

$$\text{area}[PQR]=|\frac{i}{4}\begin{vmatrix} q-p & \overline{q}-\overline{p}\\ r-p & \overline{r}-\overline{p}\end{vmatrix}|=|\frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} 1-\frac{c\overline{x}}{R^2} & (x-c)R^2\\ 1-\frac{b\overline{x}}{R^2} & (x-b)R^2\end{vmatrix}|$$
$$=|\frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} R^2-c\overline{x} & x-c\\ R^2-b\overline{x} & x-b\end{vmatrix}|=|\frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} (b-c)\overline{x} & b-c\\ R^2-b\overline{x} & x-b\end{vmatrix}|$$
$$=|\frac{i(a-b)(b-c)(a-c)}{16abc}\begin{vmatrix} \overline{x} & 1\\ R^2-b\overline{x} & x-b\end{vmatrix}|=|\frac{i(a-b)(b-c)(a-c)}{16abc}(x\overline{x}-R^2)|.$$

We find that

$$\text{area}[PQR] = \frac{|a-b||b-c||c-a|}{16|a||b||c|}\left||x|^2 - R^2\right| = \frac{\alpha\beta\gamma}{16R^3}\left||x|^2 - R^2\right|$$

$$= \frac{\text{area}[ABC]}{4R^2}\left||x|^2 - R^2\right|,$$

where $\alpha,\ \beta,\ \gamma$ are the side lengths of triangle ABC. □

Remarks.

(1) The formula in Theorem 2 above contains the Simson–Wallace line property. Indeed, points $P,\ Q,\ R$ are collinear if and only if area $[PQR] = 0$, that is, $|x\overline{x} - R^2| = 0$, i.e., $x\overline{x} = R^2$. It follows that $|x| = R$, so X lies on the circumcircle of triangle ABC.

(2) If X lies on a circle of radius R_1 and center O (the circumcenter of triangle ABC), then $x\overline{x} = R_1^2$, and from Theorem 2 above, we obtain

$$\text{area}[PQR] = \frac{\text{area}[ABC]}{4R^2}|R_1^2 - R^2|.$$

It follows that the area of triangle PQR does not depend on the point X.

The converse is also true. The locus of all points X in the plane of triangle ABC such that area $[PQR] = k$ (constant) is defined by

$$\left||x|^2 - R^2\right| = \frac{4R^2k}{\text{area}[ABC]}.$$

This is equivalent to

$$|x|^2 = R^2 \pm \frac{4R^2k}{\text{area}[ABC]} = R^2\left(1 \pm \frac{4k}{\text{area}[ABC]}\right).$$

If $k > \frac{1}{4}\text{area}[ABC]$, then the locus is a circle with center O and radius

$$R_1 = R\sqrt{1 + \frac{4k}{\text{area}[ABC]}}.$$

If $k \leq \frac{1}{4}\text{area}[ABC]$, then the locus consists with two circles of center O and radii $R\sqrt{1 \pm \frac{4k}{\text{area}[ABC]}}$, one of which degenerates to O when $k = \frac{1}{4}\text{area}[ABC]$.

Theorem 3. *For every point X in the plane of triangle ABC, we can construct a triangle with sides $AX \cdot BC$, $BX \cdot CA$, $CX \cdot AB$. This triangle is then similar to the pedal triangle of point X with respect to the triangle ABC.*

Proof. Let PQR be the pedal triangle of X with respect to triangle ABC. From formula (2), we obtain

$$q - p = \frac{1}{2}(a-b)(x-c)\frac{R^2 - c\overline{x}}{R^2(x-c)}. \tag{3}$$

Taking moduli in (3), we obtain

$$|q - p| = \frac{1}{2R^2}|a-b||x-c|\left|\frac{R^2 - c\overline{x}}{x-c}\right|. \tag{4}$$

On the other hand,

$$\left|\frac{R^2 - c\overline{x}}{x-c}\right|^2 = \frac{R^2 - c\overline{x}}{x-c}\cdot\frac{R^2 - \overline{c}x}{\overline{x}-\overline{c}} = \frac{R^2 - c\overline{x}}{x-c}\cdot\frac{R^2 - \overline{c}x}{\overline{x}-\dfrac{R^2}{c}}$$
$$= \frac{R^2 - c\overline{x}}{x-c}\cdot\frac{R^2(c-x)}{c\overline{x}-R^2} = R^2,$$

whence from (4), we derive the relation

$$|q - p| = \frac{1}{2R}|a-b||x-c|. \tag{5}$$

Therefore,

$$\frac{PQ}{CX\cdot AB} = \frac{QR}{AX\cdot BC} = \frac{RP}{BX\cdot CA} = \frac{1}{2R}, \tag{6}$$

and the conclusion follows. □

Corollary 1. *In the plane of triangle ABC, consider the point X and denote by $A'B'C'$ the triangle with sides $AX\cdot BC$, $BX\cdot CA$, $CX.\ AB$. Then*

$$\text{area}[A'B'C'] = \text{area}[ABC]\big||x|^2 - R^2\big|. \tag{7}$$

Proof. From formula (6), it follows that area $[A'B'C'] = 4R^2$ area $[PQR]$, where PQR is the pedal triangle of X with respect to triangle ABC. Replacing this result in (1), we obtain the desired formula. □

Corollary 2 (Ptolemy's inequality). *The following inequality holds for every quadrilateral $ABCD$:*

$$AC\cdot BD \le AB\cdot CD + BC\cdot AD. \tag{8}$$

Corollary 3 (Ptolemy's theorem). *The convex quadrilateral $ABCD$ is cyclic if and only if*

$$AC\cdot BD = AB\cdot CD + BC\cdot AD. \tag{9}$$

Proof. If the relation (9) holds, then triangle $A'B'C'$ in Corollary 1 above is degenerate; i.e., area $[A'B'C] = 0$. From formula (7), it follows that $d \cdot \overline{d} = R^2$, where d is the coordinate of D and R is the circumradius of triangle ABC. Hence the point D lies on the circumcircle of triangle ABC.

If quadrilateral $ABCD$ is cyclic, then the pedal triangle of point D with respect to triangle ABC is degenerate. From (6), we obtain the relation (9). □

Corollary 4 (Pompeiu's theorem[8]). *For every point X in the plane of the equilateral triangle ABC, three segments with lengths XA, XB, XC can be taken as the sides of a triangle.*

Proof. In Theorem 3 above, we have $BC = CA = AB$, and the desired conclusion follows. □

The triangle in Corollary 4 above is called the *Pompeiu triangle* of X with respect to the equilateral triangle ABC. This triangle is degenerate if and only if X lies on the circumcircle of ABC. Using the second part of Theorem 3, we find that Pompeiu's triangle of the point X is similar to the pedal triangle of X with respect to triangle ABC and

$$\frac{CX}{PQ} = \frac{AX}{QR} = \frac{BX}{RP} = \frac{2R}{\alpha} = \frac{2\sqrt{3}}{3}. \tag{10}$$

Problem 1. *Let A, B, and C be equidistant points on the circumference of a circle of unit radius centered at O, and let X be any point in the circle's interior. Let d_A, d_B, d_C be the distances from X to A, B, C, respectively. Show that there is a triangle with sides d_A, d_B, d_C, and that the area of this triangle depends only on the distance from X to O.*

(2003 Putnam Mathematical Competition)

Solution. The first assertion is just the property contained in Corollary 4 above. Taking into account the relations (10), we see that the area of Pompeiu's triangle of point X is $\frac{4}{3}\text{area}[PQR]$. From Theorem 2 above, we get that area $[PQR]$ depends only on the distance from X to O, as desired.

Problem 2. *Let X be a point in the plane of the equilateral triangle ABC such that X does not lie on the circumcircle of triangleABC, and let $XA = u$, $XB = v$, $XC = w$. Express the side length α of triangle ABC in terms of real numbers u, v, w.*

(1978 GDR Mathematical Olympiad)

[8] Dimitrie Pompeiu (1873–1954), Romanian mathematician, made important contributions in the fields of mathematical analysis, functions of a complex variable, and rational mechanics. He was a Ph.D student of Henri Poincaré.

Solution. The segments $[XA]$, $[XB]$, $[XC]$ are the sides of Pompeiu's triangle of point X with respect to equilateral triangle ABC. Denote this triangle by $A'B'C'$. From relations (10) and from Theorem 2 in Sect. 4.9.1 it follows that

$$\text{area}[A'B'C'] = \left(\frac{2\sqrt{3}}{3}\right)^2 \text{area}[PQR] = \frac{1}{3R^2}\text{area}[ABC]|x \cdot \overline{x} - R^2|$$
$$= \frac{1}{3R^2} \cdot \frac{\alpha^2\sqrt{3}}{4}\left||x|^2 - R^2\right| = \frac{\sqrt{3}}{4}|XO^2 - R^2|. \tag{1}$$

On the other hand, using the well-known formula of Heron, we obtain, after a few simple computations,

$$\text{area}[A'B'C'] = \frac{1}{4}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)}.$$

Substituting in (1), we obtain

$$|XO^2 - R^2| = \frac{1}{\sqrt{3}}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)}. \tag{11}$$

Now we consider the following two cases:

Case 1. If X lies in the interior of the circumcircle of triangle ABC, then $XO^2 < R^2$. Using the relation (see also formula (4) in Sect. 4.11)

$$XO^2 = \frac{1}{3}(u^2+v^2+w^2-3R^2),$$

from (11) we find that

$$2R^2 = \frac{1}{3}(u^2+v^2+w^2) + \frac{1}{\sqrt{3}}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)},$$

and hence

$$\alpha^2 = \frac{1}{2}(u^2+v^2+w^2) + \frac{\sqrt{3}}{2}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)}.$$

Case 2. If X lies in the exterior of the circumcircle of triangle ABC, then $XO^2 > R^2$, and after some similar computations we obtain

$$\alpha^2 = \frac{1}{2}(u^2+v^2+w^2) - \frac{\sqrt{3}}{2}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)}.$$

4.9.2 Necessary and Sufficient Conditions for Orthopolarity

Consider a triangle ABC and points X, Y, Z situated on its circumcircle. Triangles ABC and XYZ are called *orthopolar triangles* (or *S-triangles*)[9] if the Simson–Wallace line of point X with respect to triangle ABC is perpendicular (orthogonal) to line YZ.

Let us choose the circumcenter O of triangle ABC to lie at the origin of the complex plane. Points A, B, C, X, Y, Z have the coordinates a, b, c, x, y, z with

$$|a| = |b| = |c| = |x| = |y| = |z| = R,$$

where R is the circumradius of the triangle ABC.

Theorem. *Triangles ABC and XYZ are orthopolar triangles if and only if $abc = xyz$.*

Proof. Let P, Q, R be the feet of the orthogonal lines from the point X to the lines BC, CA, AB, respectively.

Points P, Q, R are on the same line, namely the Simson–Wallace line of point X with respect to triangle ABC.

The coordinates of P, Q, R are denoted by p, q, r, respectively. Using the formula in Proposition of Sect. 4.5, we have

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right),$$
$$q = \frac{1}{2}\left(x - \frac{ca}{R^2}\overline{x} + c + a\right),$$
$$r = \frac{1}{2}\left(x - \frac{ab}{R^2}\overline{x} + a + b\right).$$

We study two cases.

Case 1. Point X is not a vertex of triangle ABC.

Then PQ is orthogonal to YZ if and only if $(p - q) \cdot (y - z) = 0$. That is,

$$\left[(b - a)\left(1 - \frac{c\overline{x}}{R^2}\right)\right] \cdot (y - z) = 0,$$

or

$$(\overline{b} - \overline{a})(R^2 - \overline{c}x)(y - z) + (b - a)(R^2 - c\overline{x})(\overline{y} - \overline{z}) = 0.$$

We obtain

$$\left(\frac{R^2}{b} - \frac{R^2}{a}\right)\left(R^2 - \frac{R^2}{c}x\right)(y - z) + (b - a)\left(R^2 - c\frac{R^2}{x}\right)\left(\frac{R^2}{y} - \frac{R^2}{z}\right) = 0;$$

[9] This definition was given in 1915 by the Romanian mathematician Traian Lalescu (1882–1929). He is famous for his book *La géometrie du triangle* [43].

hence

$$\frac{1}{abc}(a-b)(c-x)(y-z) - \frac{1}{xyz}(a-b)(c-x)(y-z) = 0.$$

The last relation is equivalent to

$$(abc - xyz)\ (a-b)(c-x)(y-z) = 0,$$

and finally, we get $abc = xyz$, as desired.

Case 2. Point X is a vertex of triangle ABC. Without loss of generality, assume that $X = B$.

Then the Simson–Wallace line of point $X = B$ is the orthogonal line from B to AC. It follows that BQ is orthogonal to YZ if and only if lines AC and YZ are parallel. This is equivalent to $ac = yz$. Because $b = x$, we obtain $abc = xyz$, as desired.

□

Remark. Due to the symmetry of the relation $abc = xyz$, we observe that the Simson–Wallace line of every vertex of triangle XYZ with respect to ABC is orthogonal to the opposite side of the triangle XYZ. Moreover, the same property holds for the vertices of triangle ABC.

Hence ABC and XYZ are orthopolar triangles if and only if XYZ and ABC are orthopolar triangles. Therefore the orthopolarity relation is symmetric.

Problem 1. *The median and the orthic triangles of a triangle ABC are orthopolar in the nine-point circle.*

Solution. Consider the origin of the complex plane at the circumcenter O of triangle ABC. Let $M,\ N,\ P$ be the midpoints of $AB,\ BC,\ CA$ and let $A',\ B',\ C'$ be the feet of the altitudes of triangles ABC from $A,\ B,\ C$, respectively.

If $m,\ n,\ p,\ a',\ b',\ c'$ are coordinates of $M,\ N,\ P,\ A',\ B',\ C'$, then we have

$$m = \frac{1}{2}(a+b),\ n = \frac{1}{2}(b+c),\ p = \frac{1}{2}(c+a)$$

and

$$a' = \frac{1}{2}\left(a+b+c-\frac{bc}{R^2}\overline{a}\right) = \frac{1}{2}(a+b+c-\frac{bc}{a}),$$
$$b' = \frac{1}{2}\left(a+b+c-\frac{ca}{b}\right),\ c' = \frac{1}{2}\left(a+b+c-\frac{ab}{2}\right).$$

The nine-point center O_9 is the midpoint of the segment OH, where $H(a+b+c)$ is the orthocenter of triangle ABC. The coordinate of O_9 is $\omega = \dfrac{1}{2}(a+b+c)$.

Now observe that

$$(a' - \omega)(b' - \omega)(c' - \omega) = (m - \omega)(n - \omega)(p - \omega) = -\frac{1}{8}abc,$$

and the claim is proved.

Problem 2. *The altitudes of triangle ABC meet its circumcircle at points $A_1,\ B_1,\ C_1$, respectively. If $A_1',\ B_1',\ C_1'$ are the antipodal points of $A_1,\ B_1,\ C_1$ on the circumcircle ABC, then ABC and $A_1'B_1'C_1'$ are orthopolar triangles.*

Solution. The coordinates of A_1, B_1, C_1 are $-\frac{bc}{a},\ -\frac{ca}{b},\ -\frac{ab}{c}$, respectively. Indeed, the equation of line AH in terms of the real product is

$$AH : (z - a) \cdot (b - c) = 0.$$

It suffices to show that the point with coordinate $-\frac{bc}{a}$ lies both on AH and on the circumcircle of triangle ABC. First, let us note that

$$\left|-\frac{bc}{a}\right| = \frac{|b||c|}{|a|} = \frac{R \cdot R}{R} = R;$$

hence this point is situated on the circumcircle of triangle ABC. Now we shall show that the complex number $-\frac{bc}{a}$ satisfies the equation of the line AH. This is equivalent to

$$\left(\frac{bc}{a} + a\right) \cdot (b - c) = 0.$$

Using the definition of the real product, this reduces to

$$\left(\frac{\overline{b}\overline{c}}{\overline{a}} + \overline{a}\right)(b - c) + \left(\frac{bc}{a} + a\right)(\overline{b} - \overline{c}) = 0,$$

or

$$\left(\frac{a\overline{b}\overline{c}}{R^2} + \overline{a}\right)(b - c) + \left(\frac{bc}{a} + a\right)\left(\frac{R^2}{b} - \frac{R^2}{c}\right) = 0.$$

Finally, this comes down to

$$(b - c)\left(\frac{a\overline{b}\overline{c}}{R^2} + \overline{a} - \frac{R^2}{a} - \frac{aR^2}{bc}\right) = 0,$$

a relation that is clearly true.

It follows that $A_1',\ B_1',\ C_1'$ have coordinates $\frac{bc}{a},\ \frac{ca}{b},\ \frac{ab}{c}$, respectively. Because

$$\frac{bc}{a} \cdot \frac{ca}{c} \cdot \frac{ab}{c} = abc,$$

we obtain that the triangles ABC and $A_1'B_1'C_1'$ are orthopolar (Fig. 4.13).

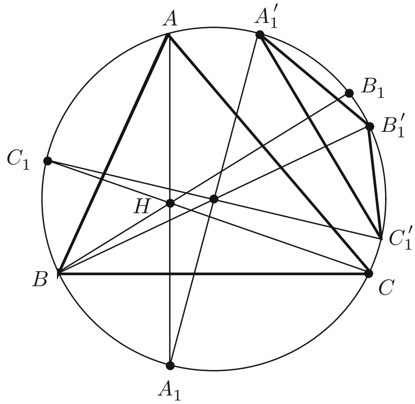

Figure 4.13.

Problem 3. *Let P and P' be distinct points on the circumcircle of triangle ABC such that lines AP and AP' are symmetric with respect to the bisector of angle $\widehat{BAC}$. Then triangles ABC and APP' are orthopolar* (Fig. 4.14).

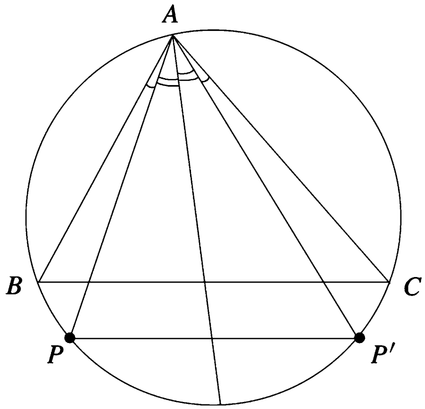

Figure 4.14.

Solution. Let us consider p and p' the coordinates of points P and P', respectively. It is clear that the lines PP' and BC are parallel. Using the complex product, it follows that $(p - p') \times (b - c) = 0$. This relation is equivalent to

$$(p - p')(\overline{b} - \overline{c}) - (\overline{p} - \overline{p'})(b - c) = 0.$$

Considering the origin of the complex plane at the circumcenter O of triangle ABC, we have

$$(p-p')\left(\frac{R^2}{b}-\frac{R^2}{c}\right)-\left(\frac{R^2}{p}-\frac{R^2}{p}\right)(b-c)=0,$$

so

$$R^2(p-p')(b-c)\left(\frac{1}{bc}-\frac{1}{pp'}\right)=0.$$

Therefore, $bc = pp'$, i.e., $abc = app'$. From the theorem at the beginning of this subsection, it follows that ABC and APP' are orthopolar triangles.

4.10 Area of the Antipedal Triangle

Consider a triangle ABC and a point M. The perpendicular lines from A, B, C to MA, MB, MC, respectively, determine a triangle; we call this triangle the *antipedal* triangle of M with respect to ABC (Fig. 4.15).

Recall that M' is the *isogonal point* of M if the pairs of lines AM, AM'; BM, BM'; CM, CM' are isogonal, i.e., the following relations hold:

$$\widehat{MAC} \equiv \widehat{M'AB},\ \widehat{MBC} \equiv \widehat{M'BA},\ \widehat{MCA} \equiv \widehat{M'CB}.$$

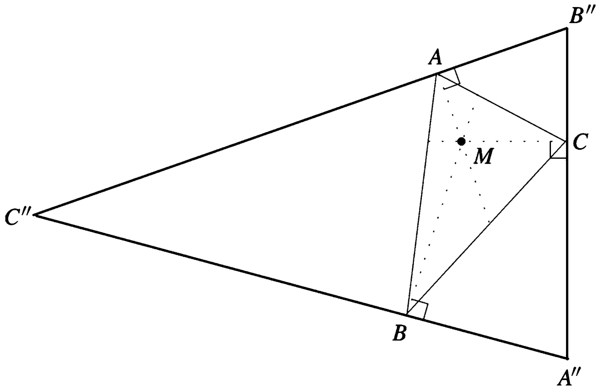

Figure 4.15.

Theorem. *Consider M a point in the plane of triangle ABC, M' the isogonal point of M, and $A''B''C''$ the antipedal triangle of M with respect to ABC. Then*

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']}=\frac{|R^2-OM'^2|}{4R^2}=\frac{|\rho(M')|}{4R^2},$$

where $\rho(M')$ is the power of M' with respect to the circumcircle of triangle ABC.

Proof. Consider point O the origin of the complex plane and let $m,\ a,\ b,\ c$ be the coordinates of $M,\ A,\ B,\ C$. Then

$$R^2 = a\overline{a} = b\overline{b} = c\overline{c} \text{ and } \rho(M) = R^2 - m\overline{m}. \tag{1}$$

Let $O_1,\ O_2,\ O_3$ be the circumcenters of triangles $BMC,\ CMA,\ AMB$, respectively. It is easy to verify that $O_1,\ O_2,\ O_3$ are the midpoints of segments $MA'',\ MB'',\ MC''$, respectively, and so

$$\frac{\text{area}[O_1O_2O_3]}{\text{area}[A''B''C'']} = \frac{1}{4}. \tag{2}$$

The coordinate of the circumcenter of the triangle with vertices with coordinates $z_1,\ z_2,\ z_3$ is given by the following formula (see formula (1) in Sect. 3.6.1):

$$z_O = \frac{z_1\overline{z_1}(z_2 - z_3) + z_2\overline{z_2}(z_3 - z_1) + z_3\overline{z_3}(z_1 - z_2)}{\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}}.$$

The bisector line of the segment $[z_1,\ z_2]$ has the following equation in terms of the real product: $\left[z - \frac{1}{2}(z_1 + z_2)\right] \cdot (z_1 - z_2) = 0$. It is sufficient to check that zo satisfies this equation, since that implies, by symmetry, that z_0 belongs to the perpendicular bisectors of segments $[z_2,\ z_3]$ and $[z_3,\ z_1]$.

The coordinate of O_1 is

$$z_{O_1} = \frac{m\overline{m}(b - c) + b\overline{b}(c - m) + c\overline{c}(m - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}}$$

$$= \frac{(R^2 - m\overline{m})(c - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}} = \frac{\rho(M)(c - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}}.$$

Let

$$\Delta = \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}$$

and consider

$$\alpha = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}, \quad \beta = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ c & \overline{c} & 1 \\ a & \overline{a} & 1 \end{vmatrix},$$

and

$$\gamma = \frac{1}{\Delta} \begin{vmatrix} m & \overline{m} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix}.$$

With this notation we obtain

$$(\alpha a + \beta b + \gamma c) \cdot \Delta = \sum_{\text{cyc}} m(a\overline{b} - a\overline{c}) - \sum_{\text{cyc}} \overline{m}(ab - ac) + \sum_{\text{cyc}} a(b\overline{c} - \overline{b}c)$$

$$= m\Delta - \overline{m} \cdot 0 + \sum_{\text{cyc}} a \left(b\frac{R^2}{c} - \frac{R^2}{c}a \right) = m\Delta + R^2 \sum_{\text{cyc}} \left(\frac{ab}{c} - \frac{ac}{b} \right) = m\Delta,$$

and consequently,

$$\alpha a + \beta b + \gamma c = m,$$

since it is clear that $\Delta \neq 0$.

We note that $\alpha,\ \beta,\ \gamma$ are real numbers and $\alpha + \beta + \gamma = 1$, so $\alpha,\ \beta,\ \gamma$ are the barycentric coordinates of point M.

Since

$$z_{O_1} = \frac{(c-b) \cdot \rho(M)}{\alpha \cdot \Delta},\ z_{O_2} = \frac{(c-a) \cdot \rho(M)}{\beta\Delta},\ z_{O_3} = \frac{(a-b) \cdot \rho(M)}{\gamma \cdot \Delta},$$

we have

$$\frac{\text{area}[O_1O_2O_3]}{\text{area}[ABC]} = \left| \frac{\frac{i}{4} \begin{vmatrix} z_{O_1} & \overline{z_{O_1}} & 1 \\ z_{O_2} & \overline{z_{O_2}} & 1 \\ z_{O_3} & \overline{z_{O_3}} & 1 \end{vmatrix}}{\frac{i}{4}\Delta} \right|$$

$$= \left| \frac{1}{\Delta} \cdot \frac{\rho^2(M)}{\Delta^2} \cdot \frac{1}{\alpha\beta\gamma} \begin{vmatrix} b-c & \overline{b}-\overline{c} & \alpha \\ c-a & \overline{c}-\overline{a} & \beta \\ a-b & \overline{a}-\overline{b} & \gamma \end{vmatrix} \right|$$

$$= \left| \frac{\rho^2(M)}{\Delta^3} \cdot \frac{1}{\alpha\beta\gamma} \cdot \begin{vmatrix} c-a & \overline{c}-\overline{a} \\ a-b & \overline{a}-\overline{b} \end{vmatrix} \right|$$

$$= \left| \frac{\rho^2(M)}{\Delta^3} \cdot \frac{1}{\alpha\beta\gamma} \cdot \Delta \right| = \left| \frac{\rho^2(M)}{\Delta^2} \cdot \frac{1}{\alpha\beta\gamma} \right|. \tag{3}$$

Relations (2) and (3) imply that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|\Delta^2\alpha\beta\gamma|}{4\rho^2(M)}. \tag{4}$$

Because $\alpha,\ \beta,\ \gamma$ are the barycentric coordinates of M, it follows that

$$z_M = \alpha z_A + \beta z_B + \gamma z_C.$$

Using the real product, we find that

$$\begin{aligned} OM^2 = z_M \cdot z_M &= (\alpha z_A + \beta z_B + \gamma z_C) \cdot (\alpha z_A + \beta z_B + \gamma z_C) \\ &= (\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\sum_{\text{cyc}} \alpha\beta z_A \cdot z_B \\ &= (\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\sum_{\text{cyc}} \alpha\beta \left(R^2 - \frac{AB^2}{2}\right) \\ &= (\alpha + \beta + \gamma)^2 R^2 - \sum_{\text{cyc}} \alpha\beta AB^2 = R^2 - \sum_{\text{cyc}} \alpha\beta AB^2. \end{aligned}$$

Therefore, the power of M' with respect to the circumcircle of triangle ABC can be expressed in the form

$$\rho(M) = R^2 - OM^2 = \sum_{\text{cyc}} \alpha\beta AB^2.$$

On the other hand, if $\alpha,\ \beta,\ \gamma$ are the barycentric coordinates of the point M, then its isogonal point M' has barycentric coordinates given by

$$\alpha' = \frac{\beta\gamma BC^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2},\ \beta' = \frac{\gamma\alpha CA^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2},$$

$$\gamma' = \frac{\alpha\beta AB^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2}.$$

Therefore,

$$\begin{aligned} \rho(M') &= \sum_{\text{cyc}} \alpha'\beta' AB^2 \\ &= \frac{\alpha\beta\gamma AB^2 \cdot BC^2 \cdot CA^2}{(\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2)^2} = \frac{\alpha\beta\gamma AB^2 \cdot BC^2 \cdot CA^2}{\rho^2(M)}. \end{aligned} \tag{5}$$

On the other hand, we have

$$\Delta^2 = \left| \left(\frac{4}{i} \cdot \frac{i}{4}\Delta \right)^2 \right| = \left| \frac{4}{i} \cdot \text{area}[ABC] \right|^2 = \frac{AB^2 \cdot BC^2 \cdot CA^2}{R^2}. \tag{6}$$

The desired conclusion follows from the relations (4), (5), and (6). □

Applications

(1) If M is the orthocenter H, then M' is the circumcenter O, and

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{R^2}{4R^2} = \frac{1}{4}.$$

(2) If M is the circumcenter O, then M' is the orthocenter H, and we obtain

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OH^2|}{4R^2}.$$

Using the formula in the theorem of Sect. 4.6.4, it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|(2R+r)^2 - s^2|}{2R^2}.$$

(3) If M is the Lemoine point K, then M' is the centroid G, and

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OG^2|}{4R^2}.$$

Applying the formula in Corollary 1 in Sect. 4.6.4, then the first formula in Corollary of Sect. 4.6.1, it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{2(s^2 - r^2 - 4Rr)}{36R^2} = \frac{\alpha^2 + \beta^2 + \gamma^2}{36R^2},$$

where $\alpha,\ \beta,\ \gamma$ are the sides of triangle ABC.
From the inequality $\alpha^2 + \beta^2 + \gamma^2 \leq 9R^2$ (Corollary 2 in Sect. 4.6.4), we obtain

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} \leq \frac{1}{4}.$$

(4) If M is the incenter I of triangle ABC, then $M' = I$, and using Euler's formula $OI^2 = R^2 - 2Rr$ (see the theorem of Sect. 4.6.2), we find that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OI^2|}{4R^2} = \frac{2Rr}{4R^2} = \frac{r}{4R}.$$

Applying Euler's inequality $R \geq 2r$ (corollary of Sect. 4.6.2), it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} \leq \frac{1}{4}.$$

4.11 Lagrange's Theorem and Applications

Consider the distinct points $A_1(z_1),\ \ldots,\ A_n(z_n)$ in the complex plane. Let $m_1,\ \ldots,\ m_n$ be nonzero real numbers such that $m_1 + \cdots + m_n \neq 0$. Let $m = m_1 + \cdots + m_n$.

The point G with coordinate

$$z_G = \frac{1}{m}(m_1 z_1 + \cdots + m_n z_n)$$

is called the *barycenter of the set* $\{A_1, \ldots, A_n\}$ with respect to the weights $m_1, \ldots, m_n$.

In the case $m_1 = \cdots = m_n = 1$, the point G is the *centroid* of the set $\{A_1, \ldots, A_n\}$.

When $n = 3$ and the points A_1, A_2, A_3 are not collinear, we obtain the absolute barycentric coordinates of G with respect to the triangle $A_1A_2A_3$ (see Sect. 4.7.1):

$$\mu_{z_1} = \frac{m_1}{m},\ \mu_{z_2} = \frac{m_2}{m},\ \mu_{z_3} = \frac{m_3}{m}.$$

Theorem 1 (Lagrange[10]). *Consider the points $A_1, \ldots, A_n$ and the nonzero real numbers $m_1, \ldots, m_n$ such that $m = m_1 + \cdots + m_n \neq 0$. If G denotes the barycenter of the set $\{A_1, \ldots, A_n\}$ with respect to the weights $m1, \ldots, m_n$, then for every point M in the plane, the following relation holds:*

$$\sum_{j=1}^{n} m_j MA_j^2 = mMG^2 + \sum_{j=1}^{n} m_j GA_j^2. \tag{1}$$

Proof. Without loss of generality, we can assume that the barycenter G is the origin of the complex plane; that is, $z_G = 0$.

Using properties of the real product, we obtain for all $j = 1, \ldots, n$, the relations

$$\begin{aligned} MA_j^2 &= |z_M - z_j|^2 = (z_M - z_j) \cdot (z_M - z_j) \\ &= |z_M|^2 - 2z_M \cdot z_j + |z|^2, \end{aligned}$$

i.e.,

$$MA_j^2 = |z_M|^2 - 2z_M \cdot z_j + |z_j|^2.$$

Multiplying by m_j and adding the relations obtained for $j = 1, \ldots, n$ yields

$$\begin{aligned} \sum_{j=1}^{n} m_j MA_j^2 &= \sum_{j=1}^{n} m_j(|z_M|^2 - 2z_M \cdot z_j + |z_j|^2) \\ &= m|z_M|^2 - 2z_M \cdot \left(\sum_{j=1}^{n} m_j z_j\right) + \sum_{j=1}^{n} m_j |z_j|^2 \end{aligned}$$

[10] Joseph Louis Lagrange (1736–1813), French mathematician, one of the greatest mathematicians of the eighteenth century. He made important contributions in all branches of mathematics, and his results have greatly influenced modern science.

$$= m|z_M|^2 - 2z_M \cdot (mz_G) + \sum_{j=1}^{n} m_j|z_j|^2$$

$$= m|z_M|^2 + \sum_{j=1}^{n} m_j|z_j|^2 = m|z_M - z_G|^2 + \sum_{j=1}^{n} m_j|z_j - z_G|^2$$

$$= mMG^2 + \sum_{j=1}^{n} m_j GA_j^2. \qquad \square$$

Corollary 1. *Consider the distinct points $A_1, \ldots, A_n$ and the nonzero real numbers $m_1, \ldots, m_n$ such that $m_1 + \cdots + m_n \neq 0$. The following inequality holds for every point M in the plane:*

$$\sum_{j=1}^{n} m_j MA_j^2 \geq \sum_{j=1}^{n} m_j GA_j^2, \tag{2}$$

with equality if and only if $M = G$, the barycenter of set $\{A_1, \ldots, A_n\}$ with respect to the weights $m_1, \ldots, m_n$.

Proof. The inequality (2) follows directly from Lagrange's relation (1). $\square$

If $m_1 = \cdots = m_n = 1$, then from Theorem 1 above, one obtains the following corollary.

Corollary 2 (Leibniz[11]). *Consider the distinct points $A_1, \ldots, A_n$ and the centroid G of the se t $\{A_1, \ldots, A_n\}$. The following relation holds for every point M in the plane:*

$$\sum_{j=1}^{n} MA_j^2 = nMG^2 + \sum_{j=1}^{n} GA_j^2. \tag{3}$$

Remark. The relation (3) is equivalent to the following identity: For all complex numbers $z, z_1, \ldots, z_n$, we have

$$\sum_{j=1}^{n} |z - z_j|^2 = n\left|z - \frac{z_1 + \cdots + z_n}{n}\right|^2 + \sum_{j=1}^{n}\left|z_j - \frac{z_1 + \cdots + z_n}{n}\right|^2.$$

Applications. We will use formula (3) in determining some important distances in a triangle. Let us consider the triangle ABC and let us take $n = 3$ in the formula (3). We find that the following formula holds for every point M in the plane of triangle ABC:

$$MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2, \tag{4}$$

[11] Gottfried Wilhelm Leibniz (1646–1716) was a German philosopher, mathematician, and logician who is probably best known for having invented the differential and integral calculus independently of Sir Isaac Newton.

where G is the centroid of triangle ABC. Assume that the circumcenter O of the triangle ABC is the origin of the complex plane.

(1) In the relation (4) we choose $M = O$, and we get

$$3R^2 = 3OG^2 + GA^2 + GB^2 + GC^2.$$

Applying the well-known median formula yields

$$\begin{aligned} GA^2 + GB^2 + GC^2 &= \frac{4}{9}(m_\alpha^2 + m_\beta^2 + m_\gamma^2) \\ &= \frac{4}{9}\sum_{\text{cyc}} \frac{1}{4}[2(\beta^2 + \gamma^2) - \alpha^2] = \frac{1}{3}(\alpha^2 + \beta^2 + \gamma^2), \end{aligned}$$

where $\alpha,\ \beta,\ \gamma$ are the sides of triangle ABC. We obtain

$$OG^2 = R^2 - \frac{1}{9}(\alpha^2 + \beta^2 + \gamma^2). \tag{5}$$

An equivalent form of the distance OG is given in terms of the basic invariants of a triangle in Corollary 1, Sect. 4.6.4.

(2) Using the collinearity of points $O,\ G,\ H$ and the relation $OH = 3OG$ (see Theorem 1 in Sect. 3.1), it follows that

$$OH^2 = 9OG^2 = 9R^2 - (\alpha^2 + \beta^2 + \gamma^2). \tag{6}$$

An equivalent form for the distance OH was obtained in terms of the fundamental invariants of the triangle in the theorem of Sect. 4.6.4.

(3) In (4), consider $M = I$, the incenter of triangle ABC (Fig. 4.16). We obtain

$$IA^2 + IB^2 + IC^2 = 3IG^2 + \frac{1}{3}(\alpha^2 + \beta^2 + \gamma^2).$$

On the other hand, we have the following relations:

$$IA = \frac{r}{\sin\dfrac{A}{2}},\ IB = \frac{r}{\sin\dfrac{B}{2}},\ IC = \frac{r}{\sin\dfrac{C}{2}},$$

where r is the inradius of triangle ABC. It follows that

$$IG^2 = \frac{1}{3}\left[r^2\left(\frac{1}{\sin^2\dfrac{A}{2}} + \frac{1}{\sin^2\dfrac{B}{2}} + \frac{1}{\sin^2\dfrac{C}{2}}\right) - \frac{1}{3}(\alpha^2 + \beta^2 + \gamma^2)\right].$$

Taking into account the well-known formula

$$\sin^2\frac{A}{2} = \frac{(s-\beta)(s-\gamma)}{\beta\gamma},$$

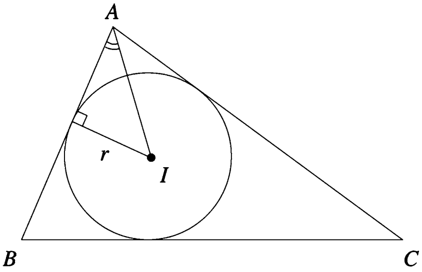

Figure 4.16.

we obtain

$$\sum_{\text{cyc}} \frac{1}{\sin^2 \frac{A}{2}} = \sum_{\text{cyc}} \frac{\beta\gamma}{(s-\beta)(s-\gamma)} = \sum_{\text{cyc}} \frac{\beta\gamma(s-\alpha)}{(s-\alpha)(s-\beta)(s-\gamma)}$$

$$= \frac{s}{K^2} \sum_{\text{cyc}} \beta\gamma(s-\alpha) = \frac{s}{K^2}\left[s\sum \beta\gamma - 3\alpha\beta\gamma\right]$$

$$= \frac{s}{K^2}[s(s^2+r^2+4Rr) - 12sRr] = \frac{1}{r^2}(s^2+r^2-8Rr),$$

where we have used the formulas in Sect. 4.6.1. Therefore,

$$IG^2 = \frac{1}{3}\left[s^2+r^2-8Rr-\frac{1}{3}(\alpha^2+\beta^2+\gamma^2)\right]$$

$$= \frac{1}{3}\left[s^2+r^2-8Rr-\frac{2}{3}(s^2-r^2-4Rr)\right] = \frac{1}{9}(s^2+5r^2-16Rr),$$

where the first formula in Corollary 1 in this section was used. That is,

$$IG^2 = \frac{1}{9}(s^2+5r^2-16Rr), \tag{7}$$

and hence we obtain again the formula in Application 1 of Sect. 4.7.2.

Problem. *Let $z_1,\ z_2,\ z_3$ be distinct complex numbers having modulus R. Prove that*

$$\frac{9R^2 - |z_1+z_2+z_3|^2}{|z_1-z_2|\cdot|z_2-z_3|\cdot|z_3-z_1|} \geq \frac{\sqrt{3}}{R}.$$

Solution. Let $A,\ B,\ C$ be the geometric images of the complex numbers $z_1,\ z_2,\ z_3$ and let G be the centroid of the triangle ABC.

The coordinate of G is equal to $\dfrac{z_1+z_2+z_3}{3}$, and $|z_1-z_2| = \gamma$, $|z_2-z_3| = \alpha$, $|z_3-z_1| = \beta$.

The inequality becomes

$$\frac{9R^2 - 9OG^2}{\alpha\beta\gamma} \geq \frac{\sqrt{3}}{R}. \tag{1}$$

Using the formula

$$OG^2 = R^2 - \frac{1}{9}(\alpha^2 + \beta^2 + \gamma^2),$$

we see that (1) is equivalent to

$$\alpha^2 + \beta^2 + \gamma^2 \geq \frac{\alpha\beta\gamma\sqrt{3}}{R} = \frac{4RK}{R}\sqrt{3} = 4K\sqrt{3}.$$

Here is a proof of this famous inequality using Heron's formula and the arithmetic–geometric mean (AM-GM) inequality:

$$K = \sqrt{s(s-\alpha)(s-\beta)(s-\gamma)} \leq \sqrt{s\frac{(s-\alpha+s-\beta+s-\gamma)^3}{27}} = \sqrt{s\frac{s^3}{27}}$$
$$= \frac{s^2}{3\sqrt{3}} = \frac{(\alpha+\beta+\gamma)^2}{12\sqrt{3}} \leq \frac{3(\alpha^2+\beta^2+\gamma^2)}{12\sqrt{3}} = \frac{\alpha^2+\beta^2+\gamma^2}{4\sqrt{3}}.$$

We now extend Leibniz's relation in Corollary 2 above. First, we need the following result.

Theorem 2. *Let $n \geq 2$ be a positive integer. Consider the distinct points $A_1, \ldots, A_n$, and let G be the centroid of the set $\{A_1, \ldots, A_n\}$. Then the following formula holds for every point in the plane:*

$$n^2 MG^2 = n\sum_{j=1}^{n} MA_j^2 - \sum_{1\leq i<k\leq n} A_iA_k^2. \tag{8}$$

Proof. We assume that the barycenter G is the origin of the complex plane. Using properties of the real product, we have

$$MA_j^2 = |z_M - z_j|^2 = (z_M - z_j)\cdot(z_M - z_j) = |z_M|^2 - 2z_M \cdot z_j + |z_j|^2$$

and

$$A_iA_k^2 = |z_i - z_k|^2 = |z_i|^2 - 2z_i \cdot z_k + |z_k|^2,$$

where the complex number z_j is the coordinate of the point A_j, $j = 1, 2, \ldots, n$.

The relation (8) is equivalent to

$$n^2|z_M|^2 = n\sum_{j=1}^{n}(|z_M|^2 - 2z_M\cdot z_j + |z_j|^2) - \sum_{1\leq i<k\leq n} |(|z_i|^2 - 2z_i\cdot z_k + |z_k|^2).$$

That is,

$$n\sum_{j=1}^{n}|z_j|^2 = 2n\sum_{j=1}^{n} z_M \cdot z_j + \sum_{1\le i<k\le n}(|z_i|^2 - 2z_i z_k + |z_k|^2).$$

Taking into account the hypothesis that G is the origin of the complex plane, we have

$$\sum_{j=1}^{n} z_M \cdot z_j = z_M \cdot \left(\sum_{j=1}^{n} z_j\right) = n(z_M \cdot z_G) = n(z_M \cdot 0) = 0.$$

Hence, the relation (8) is equivalent to

$$\sum_{j=1}^{n}|z_j|^2 = -2\sum_{1\le i<k\le n} z_i \cdot z_k.$$

The last relation can be obtained as follows:

$$\begin{aligned} 0 = |z_G|^2 = z_G \cdot z_G &= \frac{1}{n^2}\left(\sum_{i=1}^{n} z_i\right)\cdot\left(\sum_{k=1}^{n} z_k\right) \\ &= \frac{1}{n^2}\cdot\left(\sum_{j=1}^{n}|z_j|^2 + 2\sum_{1\le i<k\le n} z_i \cdot z_k\right). \end{aligned}$$

Therefore the relation (8) is proved. □

Remark. The formula (8) is equivalent to the following identity: for all complex numbers $z,\ z_1,\ \ldots,\ z_n$, we have

$$\frac{1}{n}\sum_{j=1}^{n}|z-z_j|^2 - \left|z - \frac{z_1+\cdots+z_n}{n}\right|^2 = \frac{1}{n}\sum_{1\le i<k\le n}|z_i - z_k|^2.$$

Applications

(1) If $A_1,\ \ldots,\ A_n$ are points on the circle with center O and radius R, then if we take $M = O$ in (8), it follows that

$$\sum_{1\le i<k\le n} A_iA_k^2 = n^2(R^2 - OG^2).$$

If $n = 3$, we obtain the formula (5).

(2) The following inequality holds for every point M in the plane:

$$\sum_{j=1}^{n} MA_j^2 \ge \frac{1}{n}\sum_{1\le i<k\le n} A_iA_k^2,$$

with equality if and only if $M = G$, the centroid of the set $\{A_1,\ \ldots,\ A_n\}$.

Let $n \geq 2$ be a positive integer, and let k be an integer such that $2 \leq k \leq n$. Consider the distinct points $A_1, \ldots, A_n$ and let G be the centroid of the set $\{A_1, \ldots, A_n\}$. For indices $i_1 < \cdots < i_k$, let us denote by $G_{i_1,\ldots,i_k}$ the centroid of the set $\{A_{i_1}, \ldots, A_{i_k}\}$. We have the following result:

Theorem 3. *For every point M in the plane,*

$$(n-k)\binom{n}{k}\sum_{j=1}^{n} MA_j^2 + n^2(k-1)\binom{n}{k} MG^2$$
$$= kn(n-1)\sum_{1\leq i_1<\cdots<i_k\leq n} MG_{i_1\cdots i_k}^2. \tag{9}$$

Proof. It is not difficult to see that the barycenter of the set $\{G_{i_1\cdots i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ is G. Applying Leibniz's relation, one obtains

$$\sum_{j=1}^{n} MA_j^2 = nMG^2 + \sum_{j=1}^{n} GA_j^2, \tag{10}$$

$$\sum_{1\leq i_1<\cdots<i_k\leq n} MG_{i_1\cdots i_k}^2 = \binom{n}{k} MG^2 + \sum_{1\leq i_1<\cdots<i_k\leq n} GG_{i_1\cdots i_k}^2, \tag{11}$$

$$\sum_{s=1}^{k} MA_{i_s}^2 = kMG_{i_1\cdots i_k}^2 + \sum_{s=1}^{k} G_{i_1\cdots i_k}A_{i_s}^2. \tag{12}$$

Considering in (12) $M = G$ and adding all these relations yields

$$\sum_{1\leq i_1<\cdots<i_k\leq n}\sum_{s=1}^{k} GA_{i_s}^2 = k\sum_{1\leq i_1<\cdots<i_k\leq n} GG_{i_1\cdots i_k}^2$$
$$+ \sum_{1\leq i_1<\cdots<i_k\leq n}\sum_{s=1}^{k} G_{i_1\cdots i_k}A_{i_s}^2. \tag{13}$$

Applying formula (8) in Theorem 3 above to the sets $\{A_1, \ldots, A_n\}$ and $\{A_{i_1}, \ldots, A_{i_k}\}$, respectively, we get

$$n^2MG^2 = n\sum_{j=1}^{n} MA_j^2 - \sum_{1\leq i<k\leq n} A_iA_k^2, \tag{14}$$

$$k^2MG_{i_1\cdots i_k}^2 = k\sum_{s=1}^{k} MA_{i_s}^2 - \sum_{1\leq p<q\leq k} A_{i_p}A_{i_q}^2. \tag{15}$$

Taking $M = G_{i_1\cdots i_k}$ in (15) yields

$$\sum_{s=1}^{k} G_{i_1\cdots i_k}A_{i_s}^2 = \frac{1}{k}\sum_{1\leq p<q\leq k} A_{i_p}A_{i_q}^2. \tag{16}$$

From (16) and (13), we obtain

$$\sum_{1\le i_1<\cdots<i_k\le n}\sum_{s=1}^{k} GA_{i_s}^2 = k\sum_{1\le i_1<\cdots<i_k\le n} GG_{i_1\cdots i_k}^2 + \frac{1}{k}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{1\le p<q\le n} A_{i_p}A_{i_q}^2. \tag{17}$$

If we rearrange the terms in formula (17), we get

$$\frac{\binom{k}{1}\binom{n}{k}}{\binom{n}{1}}\sum_{j=1}^{n} GA_j^2 = k\sum_{1\le i_1<\cdots<i_k\le n} GG_{i_1\cdots i_k}^2 + \frac{1}{k}\frac{\binom{k}{2}\binom{n}{k}}{\binom{n}{2}}\sum_{1\le i<k\le n} A_iA_j^2. \tag{18}$$

From relations (10), (11), (14), and (18), we readily derive formula (9). $\square$

Remark. The relation (9) is equivalent to the following identity: for all complex numbers z, $z_1, \ldots, z_n$, we have

$$(n-k)\binom{n}{k}\sum_{j=1}^{n}|z-z_j|^2 + n^2(k-1)\binom{n}{k}\left|z-\frac{z_1+\cdots+z_n}{n}\right|^2 = kn(n-1)\sum_{1\le i_1<\cdots<i_k\le n}\left|z-\frac{z_{i_1}+\cdots+z_{i_k}}{k}\right|^2.$$

Applications

(1) In the case $k=2$, from (9) we obtain that the following relation holds for every point M in the plane:

$$(n-2)\sum_{j=1}^{n} MA_j^2 + n^2MG^2 = 4\sum_{1\le i_1<i_2\le n} MG_{i_1i_2}^2.$$

In this case, $G_{i_1i_2}$ is the midpoint of the segment $[A_{i_1}A_{i_2}]$.

(2) If $k=3$, from (9) we get that the relation

$$(n-3)(n-2)\sum_{j=1}^{n} MA_j^2 + 2n^2(n-2)MG^2 = 18\sum_{1\le i_1<i_2<i_3\le n} MG_{i_1i_2i_3}^2$$

holds for every point M in the plane. Here the point $G_{i_1i_2i_3}$ is the centroid of triangle $A_{i_1}A_{i_2}A_{i_3}$.

4.12 Euler's Center of an Inscribed Polygon

Consider a polygon $A_1A_2\cdots A_n$ inscribed in a circle centered at the origin of the complex plane and let $a_1,\ a_2,\ \ldots,\ a_n$ be the coordinates of its vertices.

By definition, the point E with coordinate

$$z_E = \frac{a_1 + a_2 + \cdots + a_n}{2}$$

is called *Euler's center* of the polygon $A_1A_2\cdots A_n$. In the case $n = 3$, it is clear that E is equal to O_9, the center of Euler's nine-point circle.

Remarks.

(a) Let $G(z_G)$ and $H(z_H)$ be the centroid and orthocenter of the inscribed polygon $A_1A_2\cdots A_n$. Then

$$z_E = \frac{nz_G}{2} = \frac{z_H}{2} \text{ and } OE = \frac{nOG}{2} = \frac{OH}{2}.$$

Recall that the orthocenter of the polygon $A_1A_2\cdots A_n$ is the point H with coordinate $z_H = a_1 + a_2 + \cdots + a_n$.

(b) For $n = 4$, point E is also called *Mathot's point* of the inscribed quadrilateral $A_1A_2A_3A_4$.

Proposition. *In the above notation, the following relation holds:*

$$\sum_{i=1}^{n} EA_i^2 = nR^2 + (n-4)EO^2. \tag{1}$$

Proof. Using the identity (8) in Theorem 4, Sect. 2.17 for $M = E$ and $M = O$, namely

$$n^2 \cdot MG^2 = n\sum_{i=1}^{n} MA_i^2 - \sum_{1\le i<j\le n} A_iA_j^2,$$

we obtain

$$n^2 \cdot EG^2 = n\sum_{i=1}^{n} EA_i^2 - \sum_{1\le i<j\le n} A_iA_j^2 \tag{2}$$

and

$$n^2 \cdot OG^2 = nR^2 - \sum_{1\le i<j\le n} A_iA_j^2. \tag{3}$$

Setting $s = \sum\limits_{i=1}^{n} a_i$, we have

$$EG = |z_E - z_G| = \left|\frac{s}{2} - \frac{s}{n}\right| = \left|\frac{s}{2}\right| \cdot \frac{n-2}{n} = \frac{n-2}{n} \cdot OE. \tag{4}$$

From the relations (2), (3), and (4), we derive that

$$n\sum_{i=1}^{n} EA_i^2 = n^2 \cdot EG^2 - n^2 \cdot OG^2 + n^2R^2$$

$$= (n-2)^2OE^2 - 4OE^2 + n^2R^2 = n(n-4)\cdot EO^2 + n^2R^2,$$

or equivalently,

$$\sum_{i=1}^{n} EA_i^2 = nR^2 + (n-4)EO^2,$$

as desired. □

Applications

(1) For $n = 3$, from relation (1), we obtain

$$O_9A_1^2 + O_9A_2^2 + O_9A_3^2 = 3R^2 - OO_9^2. \tag{5}$$

Using the formula in Corollary 1 in Sect. 4.6.4, we can express the right-hand side in (5) in terms of the fundamental invariants of triangle $A_1A_2A_3$:

$$O_9A_1^2 + O_9A_2^2 + O_9A_3^2 = \frac{3}{4}R^2 - \frac{1}{2}r^2 - 2Rr + \frac{1}{2}s^2. \tag{6}$$

From formula (5), it follows that the following inequality holds for every triangle $A_1A_2A_3$:

$$O_9A_1^2 + O_9A_2^2 + O_9A_3^2 \le 3R^2, \tag{7}$$

with equality if and only if the triangle is equilateral.

(2) For $n = 4$, we obtain the interesting relation

$$\sum_{i=1}^{4} EA_i^2 = 4R^2. \tag{8}$$

The point E is the unique point in the plane of the quadrilateral $A_1A_2A_3A_4$ satisfying relation (8).

(3) For $n > 4$, from relation (1), the inequality

$$\sum_{i=1}^{n} EA_i^2 \ge nR^2 \tag{9}$$

follows. Equality holds only in the polygon $A_1A_2\cdots A_n$ with the property $E = O$.

(4) The Cauchy–Schwarz inequality and inequality (7) give

$$\left(\sum_{i=1}^{3} R \cdot O_9A_i\right)^2 \leq (3R^2) \sum_{i=1}^{3} O_9A_i^2 \leq 9R^2.$$

This is equivalent to

$$O_9A_1 + O_9A_2 + O_9A_3 \leq 3R. \tag{10}$$

(5) Using the same inequality and the relation (8), we have

$$\left(R \sum_{i=1}^{4} EA_i\right)^2 \leq 4R^2 \cdot \sum_{i=1}^{4} EA_i = 16R^4,$$

or equivalently,

$$\sum_{i=1}^{4} EA_i \leq 4R. \tag{11}$$

(6) Using the relation

$$2EA_i = 2|e - a_i| = 2\left|\frac{s}{2} - a_i\right| = |s - 2a_i|,$$

the inequalities (4), (5) become respectively

$$\sum_{\text{cyc}} |-a_1 + a_2 + a_3| \leq 6R$$

and

$$\sum_{\text{cyc}} |-a_1 + a_2 + a_3 + a_4| \leq 8R.$$

The above inequalities hold for all complex numbers of the same modulus R.

4.13 Some Geometric Transformations of the Complex Plane

4.13.1 Translation

Let z_0 be a fixed complex number and let t_{z_0} be the mapping defined by

$$t_{z_0} : \mathbb{C} \to \mathbb{C},\ t_{z_0}(z) = z + z_0.$$

The mapping t_{z_0} is called the *translation* of the complex plane by complex number z_0.

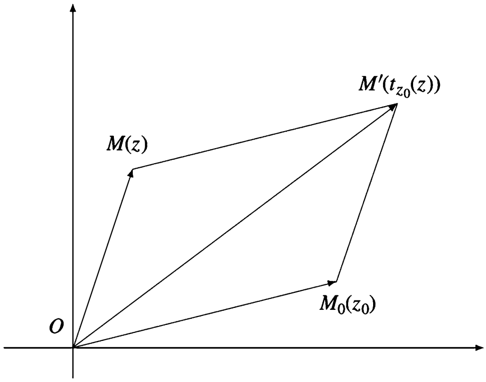

Figure 4.17.

Taking into account the geometric interpretation of the addition of two complex numbers (see Sect. 1.2.3), we have Fig. 4.17, giving the geometric image of $t_{z_0}(z)$.

In Fig. 4.17, $OM_0M'M$ is a parallelogram and OM' is one of its diagonals. Therefore, the mapping t_{z_0} corresponds in the complex plane $\mathbb{C}$ to the translation $t_{\overrightarrow{OM_0}}$ by the vector $\overrightarrow{OM_0}$ in the case of the Euclidean plane.

It is clear that the composition of two translations t_{z_1} and t_{z_2} satisfies the relation

$$t_{z_1} \circ t_{z_2} = t_{z_1+z_2}.$$

It is also clear that the set $\mathcal{T}$ of all translations of the complex plane is a group with respect to the composition of mappings. The group $(\mathcal{T}, \circ)$ is abelian, and its unit is $t_O = 1_{\mathbb{C}}$, translation by the complex number 0.

4.13.2 Reflection in the Real Axis

Consider the mapping $s : \mathbb{C} \to \mathbb{C},\ s(z) = \overline{z}$. If M is the point with coordinate z, then the point $M'(s(z))$ is obtained by reflecting M across the real axis (see Fig. 4.18). The mapping s is called the *reflection in the real axis*. It is clear that $s \circ s = 1_{\mathbb{C}}$.

4.13.3 Reflection in a Point

Consider the mapping $s_0 : \mathbb{C} \to \mathbb{C},\ s_0(z) = -z$. Since $s_0(z)+z = 0$, the origin O is the midpoint of the segment $[M(z)M'(z)]$; hence M' is the reflection of point M across O (Fig. 4.19).

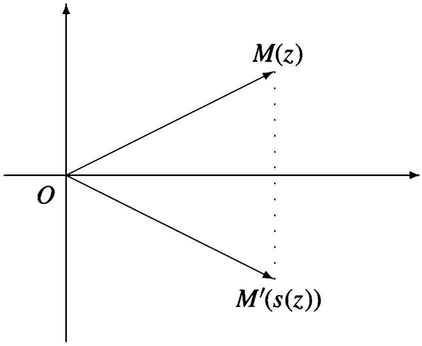

Figure 4.18.

The mapping s_0 is called the *reflection in the origin*.

Consider a fixed complex number z_0 and the mapping

$$s_{z_0} : \mathbb{C} \to \mathbb{C},\ s_{z_0}(z) = 2z_0 - z.$$

If $z_0,\ z,\ s_{z_0}(z)$ are the coordinates of points $M_0,\ M,\ M'$, then M_0 is the midpoint of the segment $[MM']$. Hence M' is the reflection of M in M_0 (Fig. 4.20).

The mapping s_{z_0} is called the *reflection in the point* $M_0(z_0)$. It is clear that the following relation holds: $s_{z_0} \circ s_{z_0} = 1_{\mathbb{C}}$.

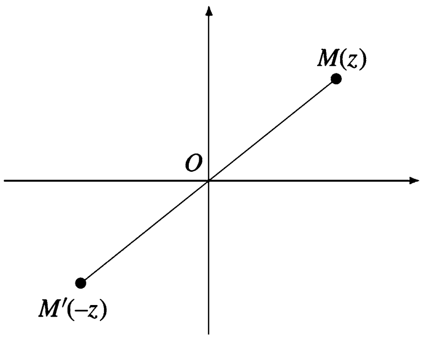

Figure 4.19.

4.13.4 Rotation

Let $a = \cos t_0 + i \sin t_0$ be a complex number having modulus 1 and let r_a be the mapping given by $r_a : \mathbb{C} \to \mathbb{C},\ r_a(z) = az$. If $z = \rho(\cos t\ + i \sin t)$, then

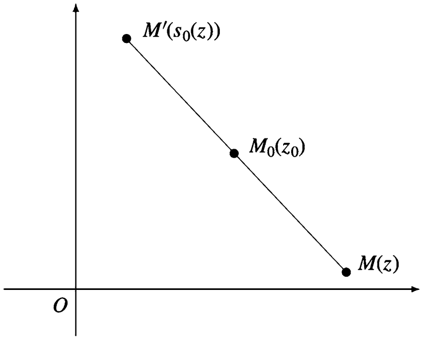

Figure 4.20.

$$r_a(z) = az = \rho[\cos(t + t_0) + i \sin(t + t_0)],$$

and hence $M'(r_a(z))$ is obtained by rotating point $M(z)$ about the origin through the angle t_0 (Fig. 4.21).

The mapping r_a is called the *rotation* with center O and angle $t_0 = \arg a$.

4.13.5 Isometric Transformation of the Complex Plane

A mapping $f : \mathbb{C} \to \mathbb{C}$ is called an *isometry* if it preserves distance, i.e., for all $z_1,\ z_2 \in \mathbb{C},\ |f(z_1) - f(z_2)| = |z_1 - z_2|$.

Theorem 1. *Translations, reflections (in the real axis or in a point), and rotations about center O are isometries of the complex plane.*

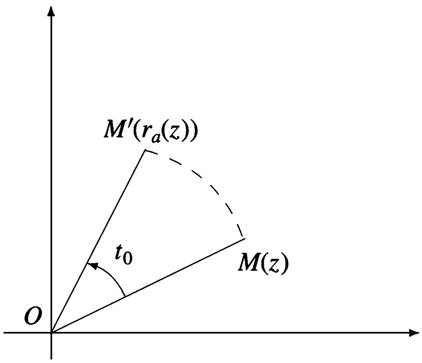

Figure 4.21.

Proof. For the translation t_{z_0}, we have

$$|t_{z_0}(z_1) - t_{z_0}(z_2)| = |(z_1 + z_0) - (z_2 + z_0)| = |z_1 - z_2|.$$

For the reflection s across the real axis, we obtain

$$|s(z_1) - s(z_2)| = |\overline{z_1} - \overline{z_2}| = |\overline{z_1 - z_2}| = |z_1 - z_2|,$$

and the same goes for the reflection in a point. Finally, if r_a is a rotation, then

$$|r_a(z_1) - r_a(z_2)| = |az_1 - az_2| = |a||z_1 - z_2| = |z_1 - z_2|, \text{ since } |a| = 1. \quad \square$$

We can easily check that the composition of two isometries is also an isometry. The set Iso($\mathbb{C}$) of all isometries of the complex plane is a group with respect to the composition of mappings, and $(\mathcal{T}, \circ)$ is a subgroup of that group.

Problem. *Let $A_1A_2A_3A_4$ be a cyclic quadrilateral inscribed in a circle with center O, and let H_1, H_2, H_3, H_4 be the orthocenters of triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$, respectively.*

Prove that quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent.

(Balkan Mathematical Olympiad, 1984)

Solution. Consider the complex plane with origin at the circumcenter, and denote by the corresponding lowercase letter the coordinates of a point denoted by an uppercase letter.

If $s = a_1 + a_2 + a_3 + a_4$, then $h_1 = a_2 + a_3 + a_4 = s - a_1$, $h_2 = s - a_2$, $h_3 = s - a_3$, $h_4 = s - a_4$. Hence the quadrilateral $H_1H_2H_3H_4$ is the reflection of quadrilateral $A_1A_2A_3A_4$ across the point with coordinate $\dfrac{s}{2}$.

The following result describes all isometries of the complex plane.

Theorem 2. *Every isometry of the complex plane is a mapping $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = az + b$ or $f(z) = a\overline{z} + b$, where a, $b \in \mathbb{C}$ and $|a| = 1$.*

Proof. Let $b = f(0)$, $c = f(1)$, and $a = c - b$. Then

$$|a| = |c - b| = |f(1) - f(0)| = |1 - 0| = 1.$$

Consider the mapping $g : \mathbb{C} \to \mathbb{C}$, given by $g(z) = az+b$. It is not difficult to prove that g is an isometry, with $g(0) = b = f(0)$ and $g(1) = a+b = c = f(1)$. Hence $h = g^{-1}$ is an isometry, with 0 and 1 as fixed points. By definition, it follows that every real number is a fixed point of h, and hence $h = 1_{\mathbb{C}}$ or $h = s$, the reflection in the real axis. Hence $g = f$ or $g = f \circ s$, and the proof is complete. $\square$

The above result shows that every isometry of the complex plane is the composition of a rotation and a translation or the composition of a rotation with a reflection in the origin O and a translation.

4.13.6 Morley's Theorem

In 1899, Frank Morley, then professor of mathematics at Haverford College, came across a result so surprising that it entered mathematical folklore under the name "Morley's Miracle." Morley's marvelous theorem states that *the three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle.*

The theorem was mistakenly attributed to Napoleon Bonaparte, who made some contributions to geometry.

There are various proofs of this nice result, such as those by J. Conway, D.J. Newman, L. Bankoff, and N. Dergiades.

Here we present a new proof published in 1998, by Alain Connes. His proof is derived from the following result:

Theorem 1 (Alain Connes). *Consider the transformations $f_i : \mathbb{C} \to \mathbb{C}$, $f_i(z) = a_i z + b_i$, $i = 1, 2, 3$, of the complex plane, where all coefficients a_i are different from zero. Assume that the mappings $f_1 \circ f_2$, $f_2 \circ f_3$, $f_3 \circ f_1$, and $f_1 \circ f_2 \circ f_3$ are not translations, equivalently, that a_1a_2, a_2a_3, a_3a_1, $a_1a_2a_3 \in \mathbb{C}\backslash\{1\}$. Then the following statements are equivalent:*

(1) $f_1^3 \circ f_2^3 \circ f_3^3 = 1_{\mathbb{C}}$.
(2) $j^3 = 1$ *and* $\alpha + j\beta + j^2\gamma = 0$, *where* $j = a_1a_2a_3 \neq 1$ *and* α, β, γ *are the respective unique fixed points of the mappings* $f_1 \circ f_2$, $f_2 \circ f_3$, $f_3 \circ f_1$.

Proof. Note that $(f_1 \circ f_2)(z) = a_1a_2z + a_1b_2 + b_1,\ a_1a_2 \neq 1,$

$$
\begin{aligned}
(f_2 \circ f_3)(z) &= a_2a_3z + a_2b_3 + b_2,\ a_2a_3 \neq 1,\\
(f_3 \circ f_1)(z) &= a_3a_1z + a_3b_1 + b_3,\ a_3a_1 \neq 1,\\
\text{Fix } (f_1 \circ f_2) &= \left\{\frac{a_1b_2 + b_1}{1 - a_1a_2}\right\} = \left\{\frac{a_1a_3b_2 + a_3b_1}{a_3 - j} =: \alpha\right\},\\
\text{Fix } (f_2 \circ f_3) &= \left\{\frac{a_2b_3 + b_2}{1 - a_2a_3}\right\} = \left\{\frac{a_1a_2b_3 + a_1b_2}{a_1 - j} =: \beta\right\},\\
\text{Fix } (f_3 \circ f_1) &= \left\{\frac{a_3b_1 + b_3}{1 - a_3a_1}\right\} = \left\{\frac{a_2a_3b_1 + a_2b_3}{a_2 - j} =: \gamma\right\},
\end{aligned}
$$

where $\mathrm{Fix}(f)$ denotes the set of fixed points of the mapping f.

For the cubes of f_1, f_2, f_3, we have the formulas

$$
\begin{aligned}
f_1^3(z) &= a_1^3z + b_1(a_1^2 + a_1 + 1),\\
f_2^3(z) &= a_2^3z + b_2(a_2^2 + a_2 + 1),\\
f_3^3(z) &= a_3^3 + b_3(a_3^2 + a_3 + 1),
\end{aligned}
$$

whence

$$
\begin{aligned}
(f_1^3 \circ f_2^3 \circ f_3^3)(z) = a_1^3a_2^3a_3^3z &+ a_1^3a_2^3b_3(a_3^2 + a_3 + 1)\\
&+ a_1^3b_2(a_2^2 + a_2 + 1) + b_1(a_1^2 + a_1 + 1).
\end{aligned}
$$

Therefore, $f_1^3 \circ f_2^3 \circ f_3^3 = id_C$ if and only if $a_1^3 a_2^3 a_3^3 = 1$ and

$$a_1^3 a_2^3 b_3(a_3^2 + a_3 + 1) + a_1^3 b_2(a_2^2 + a_2 + 1) + b_1(a_1^2 + a_1 + 1) = 0.$$

To prove the equivalence of statements (1) and (2) we have to show that $\alpha + j\beta + j^2\gamma$ is different from the free term of $f_1^3 \circ f_2^3 \circ f_3^3$ by a multiplicative constant. Indeed, using the relation $j^3 = 1$ and implicitly $j^2 + j + 1 = 0$, we have successively

$$\alpha + j\beta + j^2\gamma = \alpha + j\beta + (-1 - j)\gamma = \alpha - \gamma + j(\beta - \gamma)$$

$$= \frac{a_1a_3b_2 + a_3b_1}{a_3 - j} - \frac{a_2a_3b_1 + a_2b_3}{a_2 - j} + j\left(\frac{a_1a_2b_3 + a_1b_2}{a_1 - j} - \frac{a_2a_3b_1 + a_2b_3}{a_2 - j}\right)$$

$$= \frac{a_1a_2a_3b_2 + a_2a_3b_1 - a_1a_3b_2j - a_3b_1j - a_2a_3^2b_1 - a_2a_3b_3 + a_2a_3b_1j + a_2b_3j}{(a_2 - j)(a_3 - j)}$$

$$+ j\frac{a_1a_2^2b_3 + a_1a_2b_2 - a_1a_2b_3j - a_1b_2j - a_1a_2a_3b_1 - a_1a_2b_3 + a_2a_3b_1j + a_2b_3j}{(a_1 - j)(a_2 - j)}$$

$$= \frac{1}{a_2 - j}\left(\frac{b_2j - a_2a_3b_1j^2 - a_1a_3b_2j - a_3b_1j - a_2a_3^2b_1 - a_2a_3b_3 + a_2b_3j}{a_3 - j}\right.$$

$$\left. + \frac{a_1a_2^2b_3j + a_1a_2b_2j + a_1a_2b_3 - a_1b_2j^2 - b_1j^2 + a_2a_3b_1j^2 + a_2b_3j^2}{a_1 - j}\right)$$

$$= \frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}(a_1b_2j - b_1 - a_1^2a_3b_2j - a_1a_3b_1j - a_1a_2a_3^2b_1 - b_3j$$

$$+ a_1a_2b_3j - b_2j^2 + a_2a_3b_1 + a_1a_3b_2j^2 + a_3b_1j^2 + a_2a_3^2b_1j + a_2a_3b_3j - a_2b_3j^2$$

$$+ a_2b_3j^2 + b_2j^2 + b_3j - a_1a_3b_2j^2 - a_3b_1j^2 + a_2a_3b_1j^2 + a_2a_3b_3j^2$$

$$- a_1a_2^2b_3j^2 - a_1a_2b_2j^2 - a_1a_2b_3j + a_1b_2 + b_1 - a_2a_3b_1 - a_2b_3)$$

$$= \frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}(-a_1b_2j^2 - a_1^2a_3b_2j - a_1a_3b_1j - a_3b_1j$$

$$- a_2a_3^2b_1 - a_2a_3b_3 - aa_2^2b_3j^2 - a_1a_2b_2j^2 - a_2b_3)$$

$$= -\frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}(a_1^2a_2^2a_3^2b_2 + a_1^3a_2a_3^2b_2$$

$$+ a_1^2a_2a_3^2b_1 + a_1a_2a_3^2b_1 + a_2a_3^2b_1 + a_2a_3b_3 + a_1^3a_2^4a_3^2b_3 + a_1^3a_2^3a_3^2b_2 + a_2b_3)$$

$$= -\frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}[a_2a_3^2b_1(1 + a_1 + a_1^2) + a_1^3a_2a_3^2b_2(1 + a_2 + a_2^2)$$

$$+ a_2b_3(1 + a_3 + a_1^3 + a_1^3a_2^3a_3^2)]$$

$$= -\frac{a_2a_3^2}{(a_1 - j)(a_2 - j)(a_3 - j)}[a_1^3a_2^3b_3(1 + a_3 + a_3^2)$$

$$+ a_1^3b_2(1 + a_2 + a_2^2) + b_1(1 + a_1 + a_1^2)]. \qquad \square$$

Theorem 2 (Morley). *The three points $A'(\alpha)$, $B'(\beta)$, $C'(\gamma)$ of the adjacent trisectors of the angles of any triangle ABC form an equilateral triangle.*

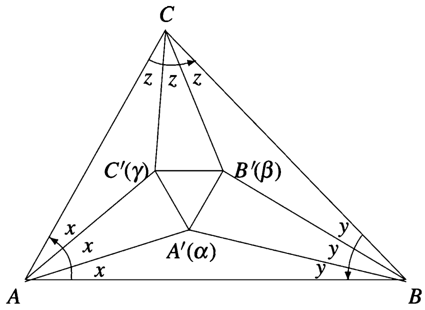

Figure 4.22.

Proof (Alain Connes). Let us consider the rotations $f_1 = r_{A,2x}$, $f_2 = r_{B,2y}$, $f_3 = r_{C,2z}$ with centers A, B, C and angles $x = \frac{1}{3}\hat{A}$, $y = \frac{1}{3}\hat{B}$, $z = \frac{1}{3}\hat{C}$ (Fig. 4.22).

Note that $\mathrm{Fix}\,(f_1 \circ f_2) = \{A'\}$, $\mathrm{Fix}\,(f_2 \circ f_3) = \{B'\}$, $\mathrm{Fix}\,(f_3 \circ f_1) = \{C'\}$ (see Fig. 4.23).

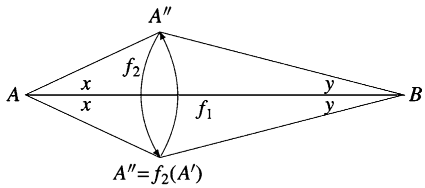

Figure 4.23.

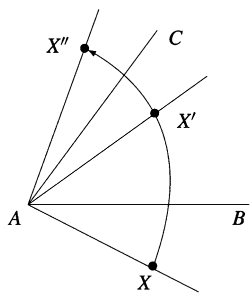

Figure 4.24.

To prove that triangle $A'B'C'$ is equilateral, it is sufficient to show, by Proposition 2, in Sect. 2.4 and above Theorem 1 in Sect. 4.13.6, that $f_1^3 \circ f_2^3 \circ f_3^3 = 1_{\mathbb{C}}$. The composition $s_{AC} \circ s_{AB}$ of reflections s_{AC} and s_{AB} across the lines AC and AB is a rotation about center A through angle $6x$ (Fig. 4.24).

Therefore, $f_1^3 = s_{AC} \circ s_{AB}$, and analogously, $f_2^3 = s_{BA} \circ s_{BC}$ and $f_3^3 = s_{CB} \circ s_{CA}$. It follows that

$$f_1^3 \circ f_2^3 \circ f_3^3 = s_{AC} \circ s_{AB} \circ s_{BA} \circ s_{BC} \circ s_{CB} \circ s_{CA} = 1_{\mathbb{C}}.$$

□

4.13.7 Homothecy

Given a fixed nonzero real number k, the mapping $h_k : \mathbb{C} \to \mathbb{C},\ h_k(z) = kz$, is called the *homothety* of the complex plane with center O and magnitude k.

Figures 4.25 and 4.26 show the position of point $M'(h_k(z))$ in the cases $k > 0$ and $k < 0$.

Points $M(z)$ and $M'(h_k(z))$ are collinear with center O, which lies on the line segment MM' if and only if $k < 0$.

Moreover, the following relation holds:

$$|OM'| = |k||OM|.$$

Point M' is called the *homothetic* point of M with center O and magnitude k.

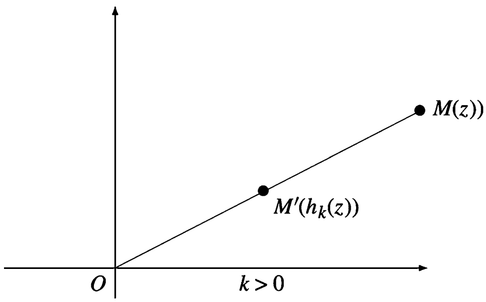

Figure 4.25.

It is clear that the composition of two homotheties h_{k_1} and h_{k_2} is also a homothety, that is,

$$h_{k_1} \circ h_{k_2} = h_{k_1 k_2}.$$

The set $\mathcal{H}$ of all homotheties of the complex plane is an abelian group with respect to the composition of mappings. The identity of the group $(\mathcal{H}, \circ)$ is $h_1 = 1_{\mathbb{C}}$, the homothety of magnitude 1.

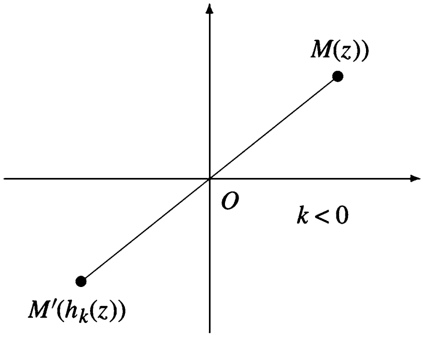

Figure 4.26.

Problem. *Let M be a point inside an equilateral triangle ABC and let M_1, M_2, M_3 be the feet of the perpendiculars from M to the sides BC, CA, AB, respectively. Find the locus of the centroid of the triangle $M_1M_2M_3$.*

Solution. Let 1, ε, ε^2 be the coordinates of points A, B, C, where $\varepsilon = \cos 120° + i \sin 120°$. Recall that

$$\varepsilon^2 + \varepsilon + 1 = 0 \text{ and } \varepsilon^3 = 1.$$

If m, m_1, m_2, m_3 are the coordinates of points M, M_1, M_2, M_3, we have

$$m_1 = \frac{1}{2}(1 + \varepsilon + m - \varepsilon\overline{m}),$$

$$m_2 = \frac{1}{2}(\varepsilon + \varepsilon^2 + m - \overline{m}),$$

$$m_3 = \frac{1}{2}(\varepsilon^2 + 1 + m - \varepsilon^2\overline{m}).$$

Let g be the coordinate of the centroid of the triangle $M_1M_2M_3$. Then

$$g = \frac{1}{3}(m_1 + m_2 + m_3) = \frac{1}{6}(2(1 + \varepsilon + \varepsilon^2) + 3m - \overline{m}(1 + \varepsilon + \varepsilon^2)) = \frac{m}{2},$$

and hence $OG = \frac{1}{2}OM$.

The locus of G is the interior of the triangle obtained from ABC under a homothety of center O and magnitude $\frac{1}{2}$. In other words, the vertices of this triangle have coordinates $\frac{1}{2}$, $\frac{1}{2}\varepsilon$, $\frac{1}{2}\varepsilon^2$.

4.13.8 Problems

1. Prove that the composition of two isometries of the complex plane is an isometry.

2. An isometry of the complex plane has two fixed points A and B. Prove that every point M of line AB is a fixed point of the transformation.

3. Prove that every isometry of the complex plane is a composition of a rotation with a translation and possibly also with a reflection in the real axis.

4. Prove that the mapping $f : \mathbb{C} \to \mathbb{C},\ f(z) = i \cdot \overline{z} + 4 - i$, is an isometry. Analyze f as in the previous problem.

5. Prove that the mapping $g : \mathbb{C} \to \mathbb{C},\ g(z) = -iz + 1 + 2i$, is an isometry. Analyze g as in the previous problem.