

## Chapter 2

# Preliminary Mathematical Concepts

In this chapter we present several fundamental mathematical concepts that will be used to describe rigid body motion, which is the main topic of this book. A rigid body motion is a motion that preserves the distance between points in the rigid body, i.e., the body is not deformed and it can move freely in space or subject to a set of constraints on the admissible velocities and attainable positions. The main tool that we use to quantify the motion of a rigid body is that of *coordinate systems*. A coordinate system provides us with a measurement system which allows us to measure quantities such as position and orientation, angular and linear velocity, and angular and linear acceleration. We can also describe the forces and torques that act on the rigid body using coordinate systems.

To describe the motion of a rigid body with respect to some reference or observer, we use *reference frames*. A reference frame is a collection of points for which the distance between any two points is constant. It is thus natural to identify a reference frame with each rigid body in the system. In addition we choose a reference frame which we consider fixed in space. This inertial reference frame is often referred to as the world or Newtonian frame and is used as a reference for all other reference frames and thus also the rigid bodies in the system. The inertial reference frame is characterized by the requirement of zero acceleration and we will in most cases also assume zero velocity. On the other hand, a reference frame that accelerates with respect to the world frame is non-inertial, or non-Newtonian. Kinematics is the study of how these reference frames move with respect to the inertial frame and relative to one another. We will attach a non-inertial reference frame to every rigid body in the system. The framework of reference frames thus allows us to describe the positions, orientations, velocities, and accelerations of rigid bodies.

It is important to note that reference frames and coordinate systems are separate entities. While the reference frame determines *what* we want to measure, the coordinate systems determines *how* we want to measure the different quantities. Reference frames and coordinate systems are discussed in more detail in Sects. 2.1 and 2.2, respectively.

The kinematics of a rigid body is the study of the geometry of motion without regard to what causes the motion. Using the framework described above, rigid bodies

are naturally described using reference frames and we will study in great detail how one reference frame relates to another. In the setting of vehicle-manipulator systems, there is a fundamental difference between how the reference frame of a rigid body relates to the inertial frame on one hand, and on the other hand how the reference frames attached to each link in a robotic manipulator relate to each other. Both these transformations describe the location of one frame with respect to another frame, but the difference lies in what motions the transformations allow: For a general rigid object in the 3-dimensional Euclidean space, the transformation from the inertial frame is given by a free motion in six degrees of freedom, while for the links in a robotic manipulator the possible motions are restricted by kinematic constraints and the transformation from one link to another is given by a subgroup of  $SE(3)$ . For conventional robotic manipulators, the transformations allowed by the joints are the 1-parameter subgroups representing pure rotational or pure translational motion.

To get a clear understanding of this difference, we will distinguish between two types of transformations: Euclidean and non-Euclidean transformations. Whether a transformation is Euclidean or not depends on what type of coordinates that can be used to describe the transformation. More specifically, if the velocity variables can be written simply as the time derivative of the position variables, we will say that the transformation is Euclidean. For this type of transformations we can write the position as a vector  $x \in \mathbb{R}^n$  and the velocity as a vector  $v \in \mathbb{R}^n$  where  $v = \dot{x} = \frac{dx}{dt}$ . If the coordinates can be written in this way, we will see that this greatly simplifies the kinematic relations between two reference frames and the kinematics takes a very simple form. The configuration space of a standard robotic manipulator with 1-DoF joints can always be written in this way.

Unfortunately, there exists a rather large group of transformations that cannot be written in this form. This includes transformations such as pure rotational motion, representing for example the attitude of a rigid body, and free transformations of a rigid body in  $\mathbb{R}^3$ , which is the combination of three degrees of freedom translational motion and three degrees of freedom rotational motion. Take for example the attitude of a rigid body. In this case we cannot find a vector  $\Theta \in \mathbb{R}^3$  describing the orientation of the body and at the same time find a vector  $\omega \in \mathbb{R}^3$  describing the angular velocities for which  $\omega = \dot{\Theta}$ . In particular, this is not possible using a minimal representation in three coordinates. In this case we need to find a velocity transformation matrix  $J(\Theta)$  so that the angular velocity relates to the orientation through the linear relation  $\dot{\Theta} = J(\Theta)\omega$ .

Finding a suitable mathematical representation of the velocity transformation matrix  $J(\Theta)$  is challenging and as a result this transformation is often singularity prone. This means that at isolated points in the configuration space some elements of  $J(\Theta)$  go to infinity. This singularity is a *mathematical singularity*. This means that it arises due to a choice of coordinates and not as a result of the physical properties of the system. This kind of singularities should always be avoided because it leads to unstable behavior when implemented in a simulation software or as a control law.

The velocity transformation matrix gives the relation between the velocity of the rigid body expressed in the body or spatial frame and the time derivative of the position coordinates. We will show how to find this transformation in several

different ways. First we will find the relation using the Euler angles, and then we use a more geometric approach for which it is possible to derive this expression without the presence of singularities. Finding these relations is fundamental when deriving the rigid body kinematics, which is treated in detail in Chap. 3.

The remainder of this chapter is organized as follows: Reference frames and coordinate systems are discussed in Sects. 2.1 and 2.2. Section 2.3 describes in detail the difference between Euclidean and non-Euclidean transformations and Sect. 2.4 gives a short introduction to quasi-coordinates which can be used to represent transformations such as the one given by the velocity transformation matrix. We will also use this concept throughout the book to represent this relation in a mathematically sound way. To do this, however, we need some background on manifolds and Lie groups. Lie groups present us with a powerful mathematical tool for representing rigid body transformations in a geometrically meaningful way. Using this formalism we can write the position and velocity variables in a form that allows us to obtain a singularity free formulation of the kinematics and dynamics of these systems. A short introduction to these topics is presented in Sects. 2.5–2.9.

## 2.1 Reference Frames

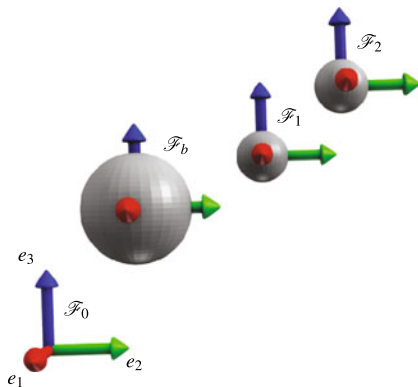
A reference frame is a collection of points for which the distance between any two points is constant at all times (Rao 2006). Even though, as we will see from the definition, any choice of three or more non-collinear points can be used as a reference frame, the easiest way to think of a reference frame is to think of a rigid body. It is also in this context that we will use reference frames: first to identify a reference frame with a rigid body and then to study the motion of this reference frame with respect to an inertial reference frame. Reference frames and rigid bodies are thus used interchangeably throughout the book.

### 2.1.1 Inertial Reference Frames

The first reference frame that we need to identify is the inertial reference frame, also referred to as the world frame. This is chosen such that its points can be used as a reference for all other reference frames. An inertial reference frame is one whose points do not accelerate. There are two types of inertial reference frames depending on whether the points are fixed in space or move with constant velocity.

Choosing a set of points that are absolutely fixed in space and time can of course be done in several different ways. Assume that we choose a reference frame that is fixed at the surface of the Earth. This can be considered fixed in space and time for all objects on the surface of the Earth and will thus serve as a reference for observing the motion of these objects. On the other hand, if we want to observe the motion of the planets in the solar system relative to the sun, the Earth fixed reference frame is not inertial. In this case the inertial reference frame should be chosen as a

**Fig. 2.1** The inertial reference frame  $\mathcal{F}_0$ , a principal non-inertial reference frame  $\mathcal{F}_b$ , and two other frames  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . The reference frame  $\mathcal{F}_0$  has basis vectors  $e_1$ ,  $e_2$ , and  $e_3$



set of points on the sun, or alternatively at some distant star. How we choose the inertial reference frame thus depends on what we want to observe. For most of the applications that we will encounter, a reference frame fixed to the Earth will serve as an inertial frame.

We can also choose a reference frame attached to a system that moves relative to the Earth. Assume for example that we choose a set of points that orbit the Earth slowly at a constant velocity. This can be points on a space station or “imaginary points” that orbit the Earth with constant velocity which allow us to observe the motion of objects relative to a spacecraft or a satellite. From the perspective of the satellite and its nearby objects, such a reference frame will be fixed in space and time. This frame will thus also serve as an inertial reference frame.

### 2.1.2 Non-inertial Reference Frames

Once an inertial reference frame is chosen, we can attach a reference frame to each rigid body in the system. These reference frames will then accelerate with respect to the inertial frame and are thus non-inertial. We will use these reference frames to observe the motion of each rigid body *relative to the inertial frame*. In addition, by attaching a reference frame to each rigid body in the system it is also possible to describe the motion of a rigid body relative to the other rigid bodies in the system. Robot kinematics is the study of how these reference frames move with respect to the inertial frame and each other. This will be discussed in more detail in the subsequent chapters.

We will use  $\mathcal{F}_0$  to denote an inertial reference frame. To observe the motion of a single rigid body, or a vehicle, we attach a reference frame  $\mathcal{F}_b$  to this body. If more rigid bodies are present, for example the links of a robotic arm, we will attach one reference frame to each link, denoted  $\mathcal{F}_i$  for  $i = 1, \dots, n$  where  $n$  is the number of links in the robot. This is illustrated in Fig. 2.1.

## 2.2 Coordinate Systems

Reference frames provide us with a mean to observe motions of rigid bodies. However, we also need a way to describe this motion mathematically. One way to quantify the motion of a reference frame in mathematical terms is to choose a coordinate system that we attach to the reference frame. A coordinate system is determined by two choices:

1. We need to choose a point  $\mathcal{O}$ , called the *origin*, that is fixed in the reference frame.
2. We need to choose a basis in which we can represent vectors in  $\mathbb{R}^3$ . The basis must be a set of three linearly independent “directions” that are also fixed in the reference frame.

Choosing the origin is quite straight forward. However, if we have any information about the motion of the rigid body, we can greatly simplify the expressions if we make a well considered choice. For example, if there are points in the rigid body that do not move for the given motion, one of these points might be a smart choice. This is for example the case for a robotic link that rotates around a fixed axis and thus all the points on this axis will not move for a motion that satisfies the joint constraints. On the other hand, if the origin is chosen at the center of gravity and the basis vectors are chosen as the principal axes of inertia, this will simplify the dynamics because the inertia matrix can be written in a very simple form.

There are many ways to choose a basis  $\{e_1, e_2, e_3\}$  for the coordinate system. In fact, any set of three linearly independent vectors can be chosen, but the most common representation is a *right-handed orthonormal basis*, i.e., a set of unit vectors that are mutually orthogonal and form a right-handed system. We will denote the scalar product (dot product) of two vectors  $x$  and  $y$  as  $x \cdot y$  and the vector product as  $x \times y$ . Formally, a right-handed orthonormal basis is defined as a basis that satisfies the following three properties:

*Property 2.1* (Unit vectors) The basis vectors  $e_1$ ,  $e_2$ , and  $e_3$  are unit vectors if

$$\begin{aligned} e_1 \cdot e_1 &= 1, \\ e_2 \cdot e_2 &= 1, \\ e_3 \cdot e_3 &= 1. \end{aligned} \tag{2.1}$$

*Property 2.2* (Orthogonal vectors) The basis vectors  $e_1$ ,  $e_2$ , and  $e_3$  are mutually orthogonal if

$$\begin{aligned} e_1 \cdot e_2 &= e_2 \cdot e_1 = 0, \\ e_1 \cdot e_3 &= e_3 \cdot e_1 = 0, \\ e_2 \cdot e_3 &= e_3 \cdot e_2 = 0. \end{aligned} \tag{2.2}$$

**Property 2.3 (Right-handed system)** The basis vectors  $e_1$ ,  $e_2$ , and  $e_3$  form a right-handed coordinate system if

$$\begin{aligned} e_1 \times e_2 &= e_3, \\ e_2 \times e_3 &= e_1, \\ e_3 \times e_1 &= e_2. \end{aligned} \tag{2.3}$$

A right-handed orthogonal basis with basis vectors  $e_1$ ,  $e_2$ , and  $e_3$  is illustrated in Fig. 2.1.

### 2.2.1 Cartesian Basis

A *Cartesian basis*  $\{e_x, e_y, e_z\}$  is a basis where the distance from a point  $\mathcal{O}_0$ , fixed in reference frame  $\mathcal{F}_0$ , to a point  $\mathcal{O}_b$ , fixed in reference frame  $\mathcal{F}_b$ , can be represented by a vector

$$p_{0b} = x_{0b}e_x + y_{0b}e_y + z_{0b}e_z \tag{2.4}$$

where  $x_{0b}$ ,  $y_{0b}$ , and  $z_{0b}$  are the components of the position in the directions of the basis vectors  $e_x$ ,  $e_y$ , and  $e_z$ , respectively, and the basis  $\{e_x, e_y, e_z\}$  is fixed in  $\mathcal{F}_0$ . A Cartesian basis is defined such that each component that describes the position of the point  $\mathcal{O}_b$  is the distance from the plane defined by the two other basis vectors in  $\mathcal{F}_0$  to  $\mathcal{O}_b$ . For example, the distance from the point  $\mathcal{O}_b$  to the plane that is spanned by  $e_x$  and  $e_y$  (the  $xy$ -plane), is given by the  $z$ -component, in our case  $z_{0b}$ . As a result, all points that have one common component lie in the same plane, and all points that have two equal components lie on the same line. The reference frame  $\mathcal{F}_0$  in Fig. 2.1 has a Cartesian basis if we let  $e_1 = e_x$ ,  $e_2 = e_y$ , and  $e_3 = e_z$ . The Cartesian basis is the most common choice of basis due to the convenient interpretation that each point can be projected into one of the three planes.

Let the position of the point  $\mathcal{O}_b$  be given by (2.4) and thus described in reference frame  $\mathcal{F}_0$ . Then the velocity of  $\mathcal{O}_b$  can be written using Cartesian coordinates as

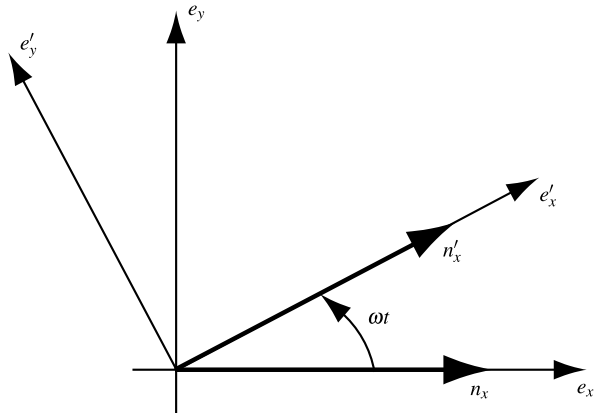
$$v_{0b}^0 = \frac{dp_{0b}}{dt} = \dot{x}_{0b}e_x + \dot{y}_{0b}e_y + \dot{z}_{0b}e_z, \tag{2.5}$$

and the acceleration as

$$\dot{v}_{0b}^0 = \frac{dv_{0b}^0}{dt} = \ddot{x}_{0b}e_x + \ddot{y}_{0b}e_y + \ddot{z}_{0b}e_z, \tag{2.6}$$

where both the velocity and acceleration are observed from reference frame  $\mathcal{F}_0$ . It is important to notice that while a vector is independent of the reference frame in which it is observed, this is not the case with the rate of change of a vector. When writing the velocity and acceleration vectors we must thus always specify with respect to what reference frame the quantities are given. This is denoted by a

**Fig. 2.2** The rotating reference frame in the plane as described in Example 2.1. The body frame  $\{e'_x, e'_y\}$  rotates with constant velocity  $\omega$  with respect to the inertial frame  $\{e_x, e_y\}$



subscript so that  $v_0$  is the velocity with respect to the inertial frame  $\mathcal{F}_0$  and  $v_b$  is the velocity with respect to the body frame  $\mathcal{F}_b$ . This is most easily illustrated with an example:

*Example 2.1* (Rao 2006) Consider the planar rotational motion illustrated in Fig. 2.2 where the body axes  $\{e'_x, e'_y\}$  rotate with respect to a fixed set of inertial axes  $\{e_x, e_y\}$  with a constant angular velocity  $\omega$ .

We will now study how a unit vector  $n'_x$ , fixed in the set of axes  $\{e'_x, e'_y\}$  and pointing in the direction of the  $e'_x$ , changes with respect to time. Of course, as this vector is fixed in the body frame  $\{e'_x, e'_y\}$  the time derivative of  $n'_x$  is zero when written with respect to the body-fixed frame, i.e.,  $v_{bn'_x} = \frac{dn'_x}{dt} = 0$ . However, if we represent  $n'_x$  in terms of the inertial basis  $\{e_x, e_y\}$ , then we have  $n'_x = \cos(\omega t)e_x + \sin(\omega t)e_y$  which gives us  $v_{0n'_x} = \frac{dn'_x}{dt} = (-\sin(\omega t)e_x + \cos(\omega t)e_y)\omega$ . If we let  $n'_y$  be a vector of the same length as  $n'_x$ , but pointing in the direction  $e'_y$ , this can be written as  $v_{0n'_x} = \frac{dn'_x}{dt} = n'_y\omega$  which is not identically equal to zero. In general, we thus obtain different rates of change of the same vector when it is written in different reference frames.

We see that the time derivative of the vector  $n'_x$  with respect to  $\{e'_x, e'_y\}$  is zero. This is a general result and always true: The body velocity relative to the body frame is always zero. However, the body velocity relative to an inertial frame but *observed* from the body frame, is not identically equal to zero. On the contrary, the rate of change of a point moving with constant velocity on a circle is constant:

*Example 2.2* The rate of change of a point  $n'_x$  parameterized by the  $n'_x = \cos(\omega t)e_x + \sin(\omega t)e_y$  in the inertial frame  $\{e_x, e_y\}$  is constant and given by

$$v_{bn'_x} = \begin{bmatrix} 0 \\ n_y \end{bmatrix} \omega \quad (2.7)$$

which represents a constant change in the direction of the  $y$ -axis of the body frame, i.e.,  $v_{bn'_x} = n'_y \omega$ .

In this book we will normally assume that a Cartesian coordinate system is used to quantify the motion. Recall that reference frames and coordinate systems are separate entities, and we thus need to identify both a reference frame and a coordinate system with each rigid body in the system. We will often use the expression *coordinate frame* as short for a reference frame with a Cartesian coordinate system. When we say coordinate frame we thus refer to a reference frame and it is implicitly understood that a Cartesian coordinate system is used to quantify the motion.

## 2.3 Euclidean and Non-Euclidean Transformations

The framework presented in this book is especially suited for modeling robotic systems with joints or transformations that cannot be described by simple one degree of freedom motions. A vehicle-manipulator system is one such system due to the non-Euclidean configuration space of the vehicle. In this setting the difference between Euclidean and non-Euclidean transformations is very important so we start with the formal definitions. We first need to define the terms generalized coordinates and generalized velocities.<sup>1</sup>

**Definition 2.1** (Generalized coordinates) A set of coordinates which uniquely describes the configuration of a body, or system of bodies, is called the generalized coordinates of the system.

The generalized coordinates thus uniquely determine the configuration of the system with respect to some initial configuration. This set is not unique in the sense that there are many different ways to choose these coordinates, but there is normally a set that will allow for an easier representation of the configuration space and a deeper physical insight. For robotic manipulators with 1-DoF joints the joint positions are normally chosen. The minimum number of independent generalized coordinates needed to describe the configuration of a system is known as the *degree of freedom* or *mobility* of the system.

The difference between Euclidean and non-Euclidean transformations lies in the way the velocity variables can be represented. We will start with generalized velocities:

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<sup>1</sup>We note that there are many ways to define generalized velocity and generalized coordinates, depending on the application, and many different definitions can be found in literature. We will use the definitions presented in this section throughout the book. Note also that even though the correct usage of the term “configuration” refers to both position and orientation of the rigid body, we will sometimes use the term position to describe the configuration state of the system when there is no ambiguity.



**Definition 2.2** (Generalized velocities) A generalized velocity is a velocity variable  $\dot{x}_i$  associated with a generalized coordinate  $x_i$  defined as  $\dot{x}_i = \frac{dx_i}{dt}$ .

For each generalized coordinate  $x_i$ , there is a corresponding generalized velocity  $\dot{x}_i$  that is simply the time derivative of the generalized coordinate. The generalized coordinates  $x$  together with the generalized velocities  $\dot{x}$  represent a very convenient way to write the state space of a system. We note, however, that it is not always possible to find a set of generalized velocities to describe the velocity state of the system in this way, i.e., by the time derivative of the position variables. In this case it is thus necessary to describe the velocity state more generally without writing them as the time derivative of the generalized coordinates of the system. This is given in the following:

**Definition 2.3** (Quasi-velocities) A quasi-velocity is a velocity variable  $\dot{\gamma}$ , which uniquely describes the velocity state of the system.

We note that we do not require  $\dot{\gamma} = \frac{dx}{dt}$  to hold for a set of coordinates  $x$  describing the position of the system. Hence, the corresponding quasi-coordinates  $\gamma$  cannot be interpreted as the position variables of the system. We will study quasi-velocities in more detail in the next section.

Generalized velocities are easier to work with than quasi-velocities because they are in the form of Definition 2.2. Whenever we can find a set of coordinates for which Definition 2.2 is true, we chose to work with these coordinates. The main reason for this is that we in this case can apply Lagrange's equations directly to find the dynamics of the system. For a robotic manipulator the joint velocities are generalized velocities, and the state space is then given by  $(q, \dot{q})$ .

On the other hand, when we cannot find a set of coordinates that allows us to find generalized velocities in this way, we need to write the velocity state of the system in terms of quasi-velocities. In this case the velocities and positions do not simply relate by the time derivative, and we need to find a different relation between the two. This is the case for most vehicles that we will encounter, for which the state space is given by  $(x, \dot{\gamma})$ . This is discussed in detail later.

We now turn to the definition of Euclidean and non-Euclidean transformations. In short, a transformation gives a mapping between two different frames or the time evolution of a frame with respect its initial configuration. A Euclidean transformation is a transformation that allows us to write the velocity state in terms of generalized velocities, while a non-Euclidean transformation requires the velocity state to be written in terms of quasi-velocities. A Euclidean transformation is defined as follows:

**Definition 2.4** (Euclidean transformation) A transformation is Euclidean if it can be parameterized in terms of generalized coordinates and a corresponding set of generalized velocities, i.e. the position variables are written as  $x \in \mathbb{R}^n$  and the velocity variables as  $v = \dot{x} \in \mathbb{R}^n$  where  $\dot{x} = \frac{dx}{dt}$  (as defined in Definition 2.2).

Note that we only require that there exists (at least one) parameterization which allows for generalized velocities in the form of Definition 2.2. We will say that a joint is Euclidean if the transformation from one joint position to another is Euclidean. All 1-DoF joints are Euclidean and thus also the most commonly found robotic joints. Also joints with only translational motion are Euclidean. Similarly we will say that the state space is Euclidean if its elements can be written in vector form in this way, i.e., a vector  $x \in \mathbb{R}^n$  representing the position variables and a corresponding vector  $v = \dot{x} = \frac{dx}{dt}$  representing the velocity variables.

**Definition 2.5** (Non-Euclidean transformation) A transformation is denoted non-Euclidean if it cannot be parameterized in terms of generalized coordinates and a corresponding set of generalized velocities, i.e. the position variables are written as  $x \in \mathbb{R}^n$  and the velocity variables must be written as  $\dot{y} = S(x)\dot{x}$  for some transformation matrix  $S(x) \neq I$  so that  $\dot{y} \neq \frac{dx}{dt}$ .

A spherical joint which represents the attitude of a rigid body is thus a non-Euclidean joint. We see this if we represent the position variables as the Euler angles. In this case the state variables cannot be written in vector form where the velocity is the time derivative of the position. Because it is not possible to find a set of variables that describes the orientation of a rigid body and whose rate of change describes the instantaneous velocities, the transformation is non-Euclidean.

There is another way to distinguish Euclidean from non-Euclidean transformations. A Euclidean transformation is a transformation which can be written in terms of position variables  $x$  and velocity variables  $\dot{x}$  for which the relation

$$x(t + \Delta t) = x(t) + \dot{x}(t)\Delta t \quad (2.8)$$

makes sense, i.e.,  $x(t + \Delta t)$  can be interpreted as a position variable in the same sense as  $x(t)$ .

For non-Euclidean transformations, on the other hand, this relation does not represent a new position of our system. One example is the Euler angles where

$$\begin{bmatrix} \phi(t + \Delta t) \\ \theta(t + \Delta t) \\ \psi(t + \Delta t) \end{bmatrix} = \begin{bmatrix} \phi(t) \\ \theta(t) \\ \psi(t) \end{bmatrix} + \begin{bmatrix} \dot{\phi}(t) \\ \dot{\theta}(t) \\ \dot{\psi}(t) \end{bmatrix} \Delta t \quad (2.9)$$

is meaningless unless  $\Delta t$  approaches zero. An element of  $SO(3)$  therefore describes a non-Euclidean transformation.

We note that it is of course always possible to write the velocity variables as the time derivative of the position variables, but whenever this set of velocity variables does not represent a geometrically meaningful quantity, as in (2.9) we will say that the system is non-Euclidean and use quasi-velocities to describe the velocity of the system.

## 2.4 Quasi-coordinates and Quasi-velocities

As we have learned from the previous section, we sometimes need to use quasi-velocities to describe the velocity state of the system. One type of systems for which this is true is the class of rigid body motions in the 3-dimensional Euclidean space. From Definition 2.5 a transformation of a rigid body in this space is non-Euclidean because we need to include rotational motion in three degrees of freedom. We will encounter motions of this kind when dealing with single rigid bodies and the vehicle in VM systems. In the setting of this book, this kind of configuration space is thus very important and in this section we address in more detail how we can use quasi-velocities to describe the velocity state of these systems.

For systems with non-Euclidean configuration spaces, it is often easier to formulate the dynamic equations in terms of velocity variables that cannot be written simply as the time derivative of the position variables. For example, when dealing with angular motion, the angular velocity is not the rate of change at which a rotation angle changes, except for the planar case. There is thus no finite change in orientation that corresponds to the angular velocity. One way to deal with the fact that the integral of the velocity variable has no physical interpretation is to introduce quasi-velocities, often represented by  $\dot{\gamma}$ . When using quasi-velocities the corresponding quasi-coordinate  $\gamma$  does not have any useful physical interpretation itself whereas  $\dot{\gamma}$  has the physical interpretation of  $d\gamma = \dot{\gamma}dt$ , i.e. it can be defined in terms of differential increments (Gingsberg 2007).

*Remark 2.1* This observation presents us with an alternative definition of a Euclidean transformation, namely a transformation for which the integral of the velocity variables have the physical meaningful interpretation as the position variables of the system.

Because quasi-velocities are not restricted to be the time derivative of the position coordinates, we have more freedom in choosing the velocity variables. We will often denote the quasi-velocity by  $v$  for a general velocity vector,  $\omega$  for an angular velocity and by  $V$  for a twist. We will also use  $v$  to describe pure translational velocity. We note that for  $\omega$  and  $V$  the integral of the velocity variable does not have a geometric interpretation. To describe the configuration of the system we thus use generalized coordinates, and not quasi-coordinates. It is therefore necessary to find the relation

between the quasi-velocities  $v = \dot{\gamma}$  and the derivative of the configuration variables  $x$  (generalized coordinates). For all the configuration spaces that we will encounter in this book,  $v$  is linear in  $\dot{x}$  and thus of the form

$$v = S(x)\dot{x}. \quad (2.10)$$

In modeling of rigid bodies such as ships and satellites, relations of this kind will always arise.

*Example 2.3* In robotics and modeling of the attitude of vehicles we often use the relation

$$\dot{\Theta} = J(\Theta)\omega \quad (2.11)$$

between the body angular velocity variables  $\omega$  and the Euler angles  $\Theta$ . This is a nice example of how quasi-coordinates are used in robotic applications. We see that if we write  $x = \Theta$  and  $v = \omega$  we get the relation

$$S(x) = J^{-1}(x) \quad (2.12)$$

and Eq. (2.11) is therefore in the form of Eq. (2.10). The matrix that defines the relation between the rate of change of the generalized coordinates and quasi-velocities is thus the inverse of the velocity transformation matrix that we find when we derive the system kinematics.

From this example we see that the quasi-coordinates present us with a well suited framework for describing non-Euclidean transformations. We note that this relation also exists for Euclidean transformations. In this case the mapping between the position and the velocity variables is found simply by taking the time derivative of the position variables. The velocity transformation matrix is then the identity matrix, i.e.,  $S(x) = I$ .

We can also describe the dynamics in terms of local coordinates  $\varphi$ . It is often more convenient to derive the dynamics of systems with a non-Euclidean configuration space in terms of local coordinates because, as we will see in the next section, even though the configuration space is non-Euclidean globally, it is always Euclidean when looked at locally. The drawback of this approach is that the dynamic equations are only valid locally, i.e., in a neighborhood around a given configuration. We will often write the quasi-velocities in terms of the coordinates  $\varphi$  as

$$v = S(\varphi)\dot{\varphi}. \quad (2.13)$$

One advantage of writing the velocity transformation in terms of local coordinates is that we avoid singularities in the representation. As we will see, the relation in Eq. (2.11) is singular and as a result the dynamic equations are not valid globally. This can be avoided by writing the velocity transformation in the form of (2.13). To derive this relation we need some tools from topological spaces and manifolds.

## 2.5 Topological Spaces and Manifolds

We have already seen that when the state space is not Euclidean, the kinematics can pose challenges when it comes to finding a suitable mathematical representation. Thus, the study of evolutionary behavior of a system for which the state space is a curved surface, as opposed to a flat (Euclidean) surface, is important. These surfaces are called manifolds and allow us to describe motion on curved surfaces such as rigid body transformations in the 3-dimensional Euclidean space.

A manifold is a smooth and in general curved surface embedded in the Euclidean space. Locally, however, these surfaces look like the Euclidean space which means that, even though the global structure is curved, when looked at locally they are homeomorphic to the Euclidean space and can be described using vector algebra.

It is important to notice the difference between the Euclidean space and transformations in the Euclidean space. For the 3-dimensional Euclidean space any point (a point has no orientation) can be written as a vector in  $\mathbb{R}^3$  with a corresponding velocity, also in  $\mathbb{R}^3$ , and this space is thus Euclidean. However, the transformation of a rigid body in the 3-dimensional Euclidean space is not Euclidean. As we have already seen, the transformation of a rigid body, for which both position and orientation need to be specified, is not Euclidean. The space that we use to describe rigid body motion is thus not Euclidean itself, but *embedded in the Euclidean space*. This is an important difference that we will discuss in more detail later.

The concepts of Lie groups and Lie algebras are important to describe the motion of rigid bodies where the configuration space is a manifold. Group theory dates back to the work of Cayley who introduced the abstract idea of groups in Cayley (1854). In the context of geometry, early contributions were made by Sophus Lie, see Lie (1888, 1890, 1893), which also gave the name to Lie groups. Other important contributions were made by, among others, Wilhelm Killing (1888), Eduard Study (1903), and Élie Cartan (Cartan and Adam 2000). Over the last decades, Lie theory has also become a very important tool in understanding the kinematics of rigid bodies and multibody systems, and this framework has also been adopted by many researchers in robotics. The reason why this framework has gained such popularity is to a large extent the way it allows us to describe rigid body motion in a mathematically rigorous manner. In the following we will show how to formulate the kinematics of a rigid body in terms of matrix Lie groups, which makes the equations valid globally and the singularities that appear for example in Eq. (2.11) in Sect. 2.4 are avoided. This will in turn be applied to multibody systems with a special focus on vehicle-manipulator systems.

The main objective of this section is to define the state space of rigid bodies, i.e., the set of possible transformations and admissible velocities that a rigid body can take. We first need to define topological spaces. All spaces that we will encounter in this book are topological spaces—in fact the definition of a topological space is very general (Bullo and Lewis 2000; Absil et al. 2008):

**Definition 2.6** (Topological space) A set  $\mathcal{M}$  is called a *topological space* if there exists a collection of open subsets of  $\mathcal{M}$  for which the following axioms hold:

1. The union of a countable number of open sets is an open set.
2. The intersection of a finite number of open sets is an open set.
3. Both  $\mathcal{M}$  and  $\emptyset$  are open sets.

We see that the notion of open sets is very important in topology. The definition of an open set is a generalization of an open interval on  $\mathbb{R}^n$ . The open interval  $(0, 1)$  on  $\mathbb{R}$  includes all real numbers *between* 0 and 1, but not 0 and 1 themselves. On the other hand, the closed interval  $[0, 1]$  includes all real numbers between 0 and 1, including 0 and 1. To choose an element of the open interval  $(0, 1)$  we can thus choose a point arbitrarily close to the limits, but not the limits themselves.

*Example 2.4* (The topology of  $\mathbb{R}^n$ ) Any set  $\mathcal{M} \in \mathbb{R}^n$  is a topological space. We can easily define open balls defined as  $B(x, \rho) = \{x' \in \mathbb{R}^n \mid \|x' - x\| < \rho\}$  which define our open sets. We note that all open sets can be expressed as a countable union of open balls. This representation of  $\mathbb{R}^n$  is not very practical and we often tend to other representations, but it is important to keep in mind that we can express open sets in  $\mathbb{R}^n$  as open balls. We note that sets expressed in the form  $\bar{h} = \{x \in \mathbb{R}^n \mid f(x) = 0\}$  or  $\bar{H} = \{x \in \mathbb{R}^n \mid f(x) \leq 0\}$  define closed sets while sets in the form  $H = \{x \in \mathbb{R}^n \mid f(x) < 0\}$  define open sets.

Another important set consists of all matrices for which  $\det A \neq 0$ . As this is the complement of the closed set defined by  $\det A = 0$ , this set is open. All the matrices that we will use to define the configuration spaces are subgroups of the general linear group that consists of all non-singular matrices i.e., all matrices for which  $\det A \neq 0$ . This set defines all matrices for which the inverse always exists which is the defining property of matrix Lie groups.

*Example 2.5* (The Topology of  $\mathbb{S}^1$ ) Another interesting topology is that of a circle. There are many ways to define a circle, but in topology we can use the notion of *identification*, i.e., to define two points as equivalent. For example, using polar coordinates, 0 and  $2\pi$  represent the same point on the circle. Once 0 and  $2\pi$  are identified as equivalent we can use the notion of open sets on  $\mathbb{R}$  to describe the topology of a circle. For rotational joints, for example, we will describe the joint position as  $q \in \mathbb{R}$ , together with the equivalence that gives us the topology of a circle. We will see that for the special case of rotations in the plane (1-DoF rotational joints) this observation allows us to treat the transformation generated by the rotational joint as Euclidean, even though strictly speaking  $\mathbb{S}^1$  is different from  $\mathbb{R}$  in a topological sense. For the higher dimensional case, however,  $\mathbb{S}^n$  is not a Euclidean transformation.

It is clear that the definition of topological spaces is general enough to include any configuration space that we will encounter in robotics. The definition of topology is in fact very general. It is thus useful to define some other properties that can be used to characterize the most important topological spaces in robotics. We will first define Hausdorff spaces.

**Definition 2.7** (Hausdorff spaces) If, for any distinct  $x_1, x_2 \in \mathcal{M}$ , there exist open sets  $U_1$  and  $U_2$  such that  $x_1 \in U_1$ ,  $x_2 \in U_2$  and  $U_1 \cap U_2 = \emptyset$ , then the topological space  $\mathcal{M}$  is a Hausdorff space.

Thus, for any two points in the set, it is possible to separate the points into two nonoverlapping open sets. The spaces that we are most interested in are manifolds, which are indeed Hausdorff spaces. For example, in  $\mathbb{R}^n$  we can separate any two points simply by choosing open balls around each point that are sufficiently small.

Intuitively, a function is continuous if a small change in the input leads to a small change in the output. In topology, however, we adopt a rather different definition:

**Definition 2.8** (Continuous function—topological definition) Let  $f : \mathcal{M} \rightarrow \mathcal{N}$  denote a function from  $\mathcal{M}$  to  $\mathcal{N}$  where  $\mathcal{M}$  and  $\mathcal{N}$  are topological spaces. Then for any set  $B \subseteq \mathcal{N}$ , the preimage of  $B$  is defined by

$$f^{-1}(B) = \{x \in \mathcal{M} \mid f(x) \in B\}. \quad (2.14)$$

Then the function  $f$  is continuous if for every open set  $U \subseteq \mathcal{N}$ ,  $f^{-1}(U)$  is an open set.

We will make much use of mappings between topological spaces of this kind where it is important to know that the inverse mapping exists. Particularly, when we are to define mappings between the configuration manifold and the Euclidean space in the next section, it is important that these mappings—and their inverses—are well defined.

In topology, we will often need to check if two objects are equivalent. This equivalence relation is called a homeomorphism and allows us to recognize important properties that are true for a group of objects, but where the objects may appear different in most other aspects. The standard example is a donut and a coffee cup, which topologically speaking are the same (because they both have a single hole). On the other hand, a straight line and a circle are not the same in a topological sense.

**Definition 2.9** (Homeomorphism) Given a one-to-one and onto (bijective) function  $f : \mathcal{M} \rightarrow \mathcal{N}$  where  $\mathcal{M}$  and  $\mathcal{N}$  are topological spaces. Then if both  $f$  and  $f^{-1}$  are continuous, then  $f$  is a homeomorphism. If such a homeomorphism exists, the two spaces are homeomorphic.

*Example 2.6* As we have seen, a circle and a straight line are not topologically the same. The reason is that we cannot find a continuous mapping from a circle to the real line. However, if we take a circle and cut away a small part, then the remaining of the circle is topologically the same as the real line  $\mathbb{R}$ , and not the circle  $\mathbb{S}^1$ . We can find a continuous mapping from this open circle to the real line by simply stretching out the circle into a line.

In group theory the corresponding equivalence relation is denoted an isomorphism. An isomorphism defines a one-to-one relation between the elements in two isomorphic groups. Isomorphic groups have the same properties and there is no need to distinguish between them: two isomorphic groups may differ in notation but they are identical for all practical purposes.

An important property of homeomorphisms is that, in addition to map points in a one-to-one manner, they also map open sets in a one-to-one manner. Thus, if  $f : \mathcal{M} \rightarrow \mathcal{N}$  is a homeomorphism, then  $\mathcal{M}$  and  $\mathcal{N}$  are topologically the same. This is an important result because we will define such maps that allow us to describe the neighborhood of a point  $p$  on a manifold  $\mathcal{M}$  using local Euclidean coordinates  $\mathbb{R}^n$ . Such a homeomorphism exists because a differentiable manifold of dimension  $n$  is topologically the same as  $\mathbb{R}^n$  when looked at locally. The mapping  $f$  together with an open set  $U$  is called a chart and provides us with a set of local coordinates. We will discuss charts in more detail below.

The configuration spaces that we will encounter can all be defined on smooth manifolds. A manifold is an abstract mathematical space in which every point has a neighborhood which resembles the Euclidean space. In other words, around every point there is a neighborhood that is topologically the same as an open ball in  $\mathbb{R}^n$ . Globally, however, manifolds allow more complicated structures.

We will use the global properties of manifolds to correctly describe spaces with more complicated topology than those that can be described using vectors in  $\mathbb{R}^n$ , and we will use the local properties to perform operations on these spaces using calculus. Locally we can write both the position and velocity variables as vectors in  $\mathbb{R}^n$  and consequently these are vector spaces.

We will see that in the context of robotics, the configuration space of the most important manifolds can be written as a matrix with certain constraints. If these matrices are smooth manifolds and also satisfy the group property, we will denote them matrix Lie groups. Matrix Lie groups allow us to represent the configuration of a rigid body so that the mathematical representation is consistent with the actual configuration of the rigid body. We will say that for such a representation the configuration does not leave the manifold. Recall that for certain transformations the integral of the velocity variable does not have a physically meaningful interpretation. Thus, if we integrate the velocity variables, the corresponding position variables do not describe a configuration that is attainable for the rigid body. In this case we have “left the manifold” and the actual displacement of the rigid body does not correspond to our mathematical representation. This illustrates the importance of finding a mathematically sound representation of the configuration space. Our main tool in obtaining this is to restrict the configuration space to a manifold. Formally, a manifold is defined in the following way:



**Definition 2.10** (Manifold) A topological space is a *manifold* if for every  $x \in \mathcal{M}$ , there exists an open set  $U \subset \mathcal{M}$  such that

1.  $x \in U$ ,
2.  $U$  is homeomorphic to  $\mathbb{R}^n$ ,
3.  $n$  is fixed for all  $x \in \mathcal{M}$ .

The most important property of a manifold is that locally it is homeomorphic to  $\mathbb{R}^n$ , i.e., it is a topological space with the nice property that locally it behaves like the Euclidean space. This allows us to do calculus on elements of this space, at least locally. This is a very important property that we will use frequently: we first map a point on the manifold to the tangent space on which we can perform an operation on the vector space, before we map back to manifold.

Every manifold has an underlying topology. For example, both a 1-DoF prismatic and a 1-DoF revolute robotic joint can be described by one parameter  $q \in \mathbb{R}$ , but the topology of the prismatic joint is a line while the topology of a revolute joint is a circle. The topology of a space thus gives us information about what this space looks like globally. The second condition in Definition 2.10 tells us that even though revolute and prismatic joints as spaces are different in a global sense, they both look like a line when looked at locally.

### 2.5.1 Coordinate Charts

A coordinate chart on a topological manifold is an invertible map between a subset of the manifold and the Euclidean space. More specifically, a chart, denoted  $(\Psi, U)$ , is a bijection (one-to-one mapping)  $\Psi : \mathcal{M} \rightarrow \mathbb{R}^n$  of a subset  $U \subset \mathcal{M}$  onto an open subset of the Euclidean space  $\mathbb{R}^n$ . The inverse map maps points on  $\mathbb{R}^n$  back to the manifold.

Recall that a homeomorphism not only maps isolated points, but also open sets in a one-to-one manner so that we can define an open set  $U \subset \mathcal{M}$  for which  $U$  and  $\Psi(U)$  are homeomorphic. A *chart*  $(\Psi, U)$  is thus defined by a mapping  $\Psi$  and an open set  $U$  for which  $\Psi$  maps  $U$  homeomorphically to  $\Psi(U)$ .

An important property of manifolds is that we can represent any manifold using a finite number of coordinate charts. The fact that each chart is homeomorphic to an open set in  $\mathbb{R}^n$  means that we can always find a set of local coordinates,  $(\varphi_1, \varphi_2, \dots, \varphi_n)$  that denotes a point in  $\Psi(U)$ . We will see that by introducing these local objects we are able to perform differentiation in the neighborhood of each point on the manifold. A chart that provides us with a set of local coordinates in this way is called a local coordinate system.

When two charts overlap, i.e., when a single point on the manifold can be described using two or more charts, we can define an *overlap map* which maps an

open ball in  $\mathbb{R}^n$  to the manifold and then back to another or the same open ball in  $\mathbb{R}^n$ . Overlap maps allow us to describe a single point using different coordinates, and are therefore often referred to as change of coordinates or coordinate transformations. For two mappings  $\Psi_1 : U_1 \rightarrow \mathbb{R}^n$  and  $\Psi_2 : U_2 \rightarrow \mathbb{R}^n$ , the overlap map is defined as

$$\Psi_2 \circ \Psi_1^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n. \quad (2.15)$$

We will now look at some of the most important coordinate charts and the corresponding overlap maps.

### 2.5.1.1 The Real Line

A straight or curved line is described using  $\mathbb{R}$ . This can thus be mapped to the Euclidean space using the trivial map  $a = \Psi(a)$  for any point  $a \in \mathbb{R}$ . This space is itself Euclidean. The same is the case for the Euclidean space of higher dimension  $\mathbb{R}^n$ .

### 2.5.1.2 The Circle

Consider the circle defined by

$$x^2 + y^2 = 1. \quad (2.16)$$

The simplest description of a circle is obtained by identifying a point on  $\mathbb{S}^1$  by the angular coordinate  $\theta$ . However, this mapping does not take into account the topology of a circle, so in addition we need to define two points as identical if they differ with a multiple of  $2\pi$ . This identification is what characterizes the topology of a circle.

We can also use the mappings shown in Fig. 2.3. The top half of the circle can be mapped into  $\mathbb{R}$  by projecting onto the  $x$ -coordinate using the continuous and invertible chart

$$\Psi_{up}(x, y) = x. \quad (2.17)$$

This will map a point on the manifold to a point in  $(-1, 1)$ . A similar chart can be found for the lower half. We can also define similar charts mapping points to the right and the left, as shown in Fig. 2.3(a). The inverse mapping  $\Psi_{up}^{-1}(x) = (x, y)$  takes us from a point on  $\mathbb{R}$  back to the manifold. This mapping is simply given by  $x = x$  and  $y = \sqrt{1 - x^2}$  for  $\Psi_{up}$ . We can also define a similar map  $\Psi_{down}$  which maps the lower part of the circle ( $y < 0$ ) to the real line. For  $\Psi_{down}$  the inverse map is given by  $x = x$  and  $y = -\sqrt{1 - x^2}$ .

We see from Fig. 2.3(a) that the maps overlap, i.e., a point can be written in terms of two different charts. Take a point  $(a, b)$  in the upper right part of the circle. This can be mapped to  $(-1, 1)$  through the right map  $\Psi_{right}(a, b) = b$ . Now, we can perform a manipulation on this point on  $\mathbb{R}$  instead of on  $\mathbb{S}^1$ . We denote this by the function  $b' = f_{right}(b)$ . Finally we can map  $b'$  back to the circle using the inverse map  $\Psi_{right}^{-1}(b') = (a', b')$ . This is illustrated in Fig. 2.3(b). We can also perform the



### 2.5.1.3 The $n$ -Dimensional Sphere

It is not necessary to use four charts to describe the circle. In fact, for any  $n$ -dimensional sphere  $\mathbb{S}^n$  it suffices to define two charts  $U_1 = \mathbb{S}^n - \{(0, \dots, 0, 1)\}$  and  $U_2 = \mathbb{S}^n - \{(0, \dots, 0, -1)\}$ , i.e., removing the north and the south pole, respectively. This is the stereographic projection and is defined by the two maps

$$\psi_1 : U_1 \rightarrow \mathbb{R}^n: (x_1, \dots, x_{n+1}) \rightarrow \left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right), \quad (2.21)$$

$$\psi_2 : U_2 \rightarrow \mathbb{R}^n: (x_1, \dots, x_{n+1}) \rightarrow \left( \frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right). \quad (2.22)$$

The overlap map is given by

$$\psi_2 \circ \psi_1^{-1} = \frac{x}{\|x\|^2}, \quad \forall x \in \mathbb{R}^n - \{0\}. \quad (2.23)$$

For  $\mathbb{S}^1$ ,  $\psi_1(x, y)$  maps all the points on the circle, except the north pole, to the  $x$ -axis while  $\psi_2(x, y)$  maps all the points on the circle, except the south pole, to the  $x$ -axis. The stereographic projection for  $\mathbb{S}^1$  is given by

$$\psi_1(x, y) = \frac{x}{1 - y}, \quad (2.24)$$

$$\psi_2(x, y) = \frac{x}{1 + y}. \quad (2.25)$$

$\psi_1$  is illustrated in Fig. 2.4(a). The overlap map is given by

$$\psi_2 \circ \psi_1^{-1} = \frac{x}{x^2} = \frac{1}{x}, \quad \forall x \in \mathbb{R} - \{0\}. \quad (2.26)$$

Similarly for  $\mathbb{S}^2$ ,  $\psi_1(x, y, z)$  maps all the points on the sphere in  $\mathbb{R}^3$ , except the north pole, to the  $xy$ -plane at  $z = 0$  while  $\psi_2(x, y, z)$  maps all the points on the sphere in  $\mathbb{R}^3$ , except the south pole, to the  $xy$ -plane at the origin. The stereographic projection for  $\mathbb{S}^2$  is given by

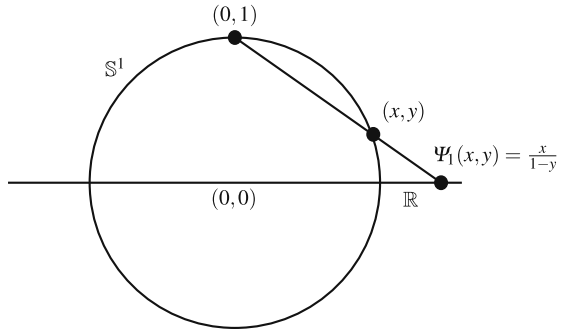
$$\psi_1(x, y, z) = \left( \frac{x}{1 - z}, \frac{y}{1 - z} \right), \quad (2.27)$$

$$\psi_2(x, y, z) = \left( \frac{x}{1 + z}, \frac{y}{1 + z} \right). \quad (2.28)$$

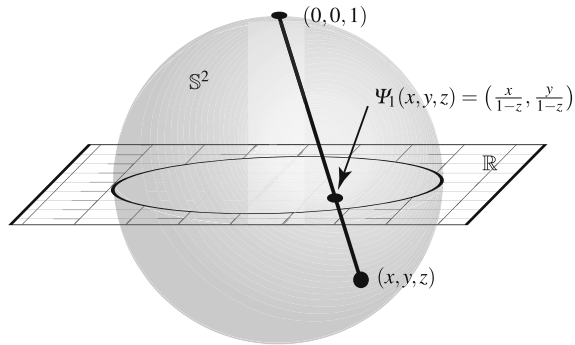
$\psi_1(x, y, z)$  is illustrated in Fig. 2.4(b). The overlap map is given by

$$\psi_2 \circ \psi_1^{-1} = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right), \quad \forall x, y \in \mathbb{R}^2 - \{0\}. \quad (2.29)$$

**Fig. 2.4** The stereographic projection maps a point on the  $\mathbb{S}^n$ -sphere to a point on  $\mathbb{R}^n$



(a) The stereographic projection of  $\mathbb{S}^1$



(b) The stereographic projection of  $\mathbb{S}^2$

### 2.5.2 Manifolds Again

Based on the notion of charts and overlap functions we can give a manifold  $\mathcal{M}$  a topological structure. Recall that the overlap function is in the form

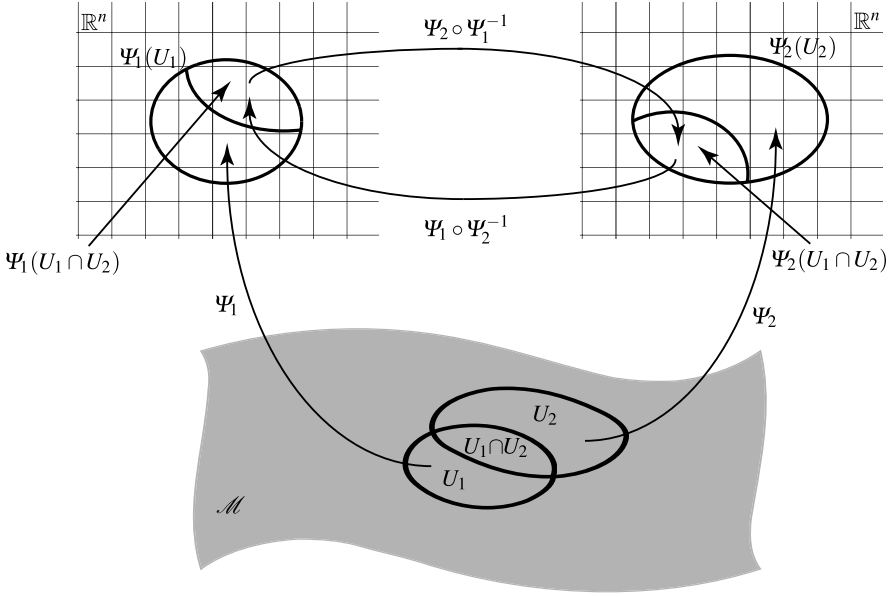
$$f = \Psi_j \circ \Psi_i^{-1}. \quad (2.30)$$

If this mapping is  $C^\infty$ , i.e., derivatives of all orders exist, we denote the underlying manifold a smooth manifold. Note that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a mapping from one open set in  $\mathbb{R}^n$  to another open set in  $\mathbb{R}^n$  through an open set  $U \subset \mathcal{M}$ . Figure 2.5 illustrates the overlap functions on a manifold  $\mathcal{M}$ . We can now define manifolds as in Kwatny and Blankenship (2000):

**Definition 2.11** An  $n$ -dimensional manifold is a set  $\mathcal{M}$  together with a countable collection of subsets  $U_i \subset \mathcal{M}$  and one-to-one mappings  $\Psi_i : U_i \rightarrow V_i$  onto open subsets  $V_i$  of  $\mathbb{R}^n$ , where each pair  $(U_i, \Psi_i)$  is called a coordinate chart, with the following properties:

1. the coordinate charts cover  $\mathcal{M}$ , i.e.,

$$\bigcup_i U_i = \mathcal{M}, \quad (2.31)$$



**Fig. 2.5** The coordinate charts and overlap maps on a manifold  $\mathcal{M}$

2. on the overlap of any pair of charts the composite map

$$f = \Psi_j \circ \Psi_i^{-1} : \Psi_i(U_i \cap U_j) \rightarrow \Psi_j(U_i \cap U_j) \quad (2.32)$$

is a smooth function,

3. (Hausdorff topological spaces) if  $p_i \in U_i$ ,  $p_j \in U_j$  are distinct points of  $\mathcal{M}$ , then there are neighborhoods,  $O_i$  of  $\Psi_i(p_i)$  in  $V_i$  and  $O_j$  of  $\Psi_j(p_j)$  in  $V_j$  such that

$$\Psi_i^{-1}(O_i) \cap \Psi_j^{-1}(O_j) = \emptyset. \quad (2.33)$$

We see that as long as we can define a chart, i.e., a description of the topology of the set, in this way, the set is a manifold. The definition is rather general but again we see that the important aspect here is that there exists a mapping between an open set on the manifold and an open set in  $\mathbb{R}^m$ .

## 2.6 Lie Groups

The kinematics of a mechanical system, such as a rigid body or a robotic manipulator, can be derived globally in terms of Lie group and Lie algebra structures. More specifically, an element of the Lie group corresponds to a configuration of the mechanism while the velocity can be expressed as an element of the Lie algebra. We can thus write the state space in terms of the Lie group and algebra.

A group is a set of elements whose action on a space leaves some aspect of the space invariant. Also, two elements of a group can be combined to produce an element in the same group. Multiplication of two elements of the group is defined by the group operator. It is common to denote this action “multiplication” but this does not necessarily mean that we multiply in the normal sense. The group operator can represent operations such as addition, multiplication, matrix multiplication, complex multiplication, and the quaternion product. For the group of real numbers the group operator is addition while for matrix groups the group operation is matrix multiplication.

A fundamental property of groups is that multiplying two elements of a group will produce another element in the same group. For example, two consecutive rotations that can be represented by two elements of the group of right-handed orthonormal matrices on  $\mathbb{R}^{n \times n}$ , can also be written as a single rotation matrix which is also a right-handed orthonormal matrix on  $\mathbb{R}^{n \times n}$ . Furthermore, the inverse of an element of the group is also an element of the same group. For matrix groups the inverse is given by the matrix inverse and is used to represent the opposite operation. All these properties make the concept of groups a very powerful tool for describing rigid body motion.

Formally, we can define a group  $G$  by identifying four important properties on the elements  $g_1, g_2, g_3 \in G$  given a group operation  $\circ$ :

*Property 2.4 (Closure)* A set  $G$  is closed under the group operation  $\circ$  if for all  $g_1, g_2 \in G$ , then  $g_1 \circ g_2 \in G$ .

*Property 2.5 (Identity)* A set  $G$  has an identity element if it is possible to find an element  $e \in G$  such that  $g \circ e = e \circ g = g$  for every  $g \in G$ .

*Property 2.6 (Inverse)* A set  $G$  is invertible if for each  $g \in G$ , there exists a unique inverse  $g^{-1} \in G$  such that  $g \circ g^{-1} = g^{-1} \circ g = e$ .

*Property 2.7 (Associativity)* A set  $G$  is called associative if for all  $g_1, g_2, g_3 \in G$ , then  $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ .

Based on these properties we can formally define a group in the following way:

**Definition 2.12 (Group)** A set  $G$  with elements  $g_1, g_2, g_3 \in G$  together with a binary operation  $\circ$ , is called a group if it satisfies Properties 2.4–2.7 above.

We see that elements of a group can be multiplied using the group operator and that the resulting element is a member of the same group. This property is very important as it allows us to perform two consecutive operations on a space for which the resulting operation has a meaningful geometrical interpretation. We will see several examples of this in the following. Property 2.6 is also important as it allows us to define “opposite” transformations. Opposite or inverse transformations and the fact that these are unique will also be used frequently in the following. The inverse

of a rotation matrix can for example be interpreted geometrically as the opposite rotation.

Recall that the first reference frame that we need to specify is the inertial frame. This frame is used as a reference for all other frames and is usually chosen as the identity element of the group. All other group elements are thus represented with respect to the identity element and they relate to the identity in the same way that the non-inertial reference frames relate to the inertial world frame. In other words, each group element describes the position of a reference frame with respect to the inertial frame.

In robotics we are mainly interested in sets that are manifolds. Further we want these manifolds to be differentiable, i.e., manifolds that can be represented by several coordinate charts and can be patched together in a smooth manner. This takes us to the important definition of Lie groups:

**Definition 2.13** (Lie Group) A Lie Group is a group  $G$  which is also a smooth manifold and for which the group operation and the inverse are smooth mappings.

The important thing to notice here is that because Lie groups are manifolds, we can always represent an element of a Lie group of dimension  $n$  in terms of local coordinates in  $\mathbb{R}^n$ . This is given by the second condition in Definition 2.10. We will use this property of Lie groups frequently in the subsequent chapters.

In addition to Properties 2.4–2.7 that are always true for Lie groups there are some additional properties that are only true for certain groups. One important property is the one of commutativity. This states whether the sequence in which we perform operations on a space is of importance or not:

*Property 2.8* (Commutativity) A group  $G$  is commutative, or Abelian, if  $g_1 \circ g_2 = g_2 \circ g_1$  for all  $g_1, g_2 \in G$ .

We note that whether a group is commutative or not depends on whether or not the group operator is commutative. Matrix multiplication, for example, is not commutative so in general matrix groups are not commutative. On the other hand, addition of real numbers is commutative, so the group of real numbers with addition as the group operator is commutative.

### 2.6.1 Some Important Lie Groups

There are several examples of Lie groups, many of which are widely used in robotics. The most important Lie group in our setting is the one of free rigid body motion which describes the position and orientation of a rigid body in the



3-dimensional Euclidean space. However, there are several other Lie groups that will be important in the subsequent chapters so we start off by introducing the most important groups briefly.

### 2.6.1.1 The Euclidean Space

The Euclidean space  $\mathbb{R}^n$  with addition as the group operator is a group. Given two elements  $x = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n$  and  $y = [y_1 \ y_2 \ \cdots \ y_n]^T \in \mathbb{R}^n$  the group operation is given by

$$x \circ y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n \quad (2.34)$$

and the inverse of an element is given by

$$x^{-1} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} \in \mathbb{R}^n. \quad (2.35)$$

We see that the identity element is the vector  $e = [0 \ 0 \ \cdots \ 0]^T \in \mathbb{R}^n$  and we note that Properties 2.4–2.7 are satisfied.

*Property 2.9* The Euclidean space  $\mathbb{R}^n$  with addition as group operator is a commutative group.

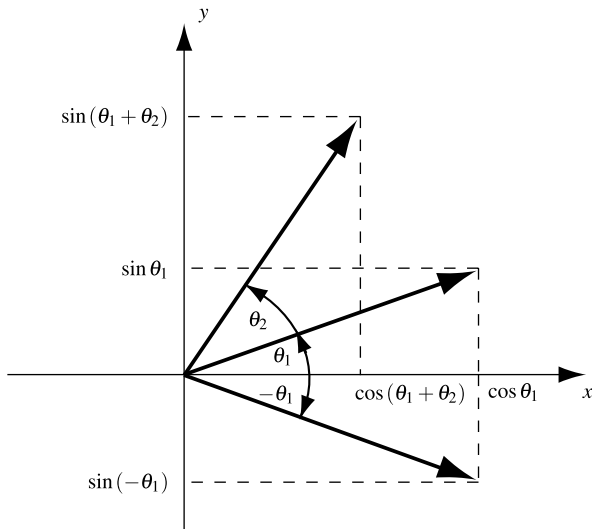
The manifold of this group is the vector space  $\mathbb{R}^n$ . This is the group of linear transformations in  $\mathbb{R}^n$ . If we choose  $n = 1$  we get the motion of a prismatic robotic joint; if we choose  $n = 2$  we get the group of linear transformations in the plane; and if we choose  $n = 3$  we get the group of linear transformations in the 3-dimensional Euclidean space.

### 2.6.1.2 Complex Numbers of Unit Length

If we restrict ourselves to complex numbers written in the form  $z = \cos \theta + i \sin \theta$  and let the group operation be complex multiplication we obtain another important Lie group. We first note that

$$\begin{aligned} z_1 z_2 &= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \end{aligned} \quad (2.36)$$

**Fig. 2.6** The geometric interpretation of the group of complex numbers as rotations in the plane



which shows that the set is closed. We also note that this operation is continuous since addition of real numbers is continuous.

The topology of this group can be interpreted as in Fig. 2.6. We see that we obtain the same group element for  $\theta = 0$  and for  $\theta = 2\pi$ . The manifold thus has the topology of a circle with unit radius, since  $\|z\| = 1$ . The transformation represented by complex numbers in this form is therefore pure rotational motion in the plane, i.e., anticlockwise rotation of the complex plane around the origin of the circle. The inverse element is given by the conjugate

$$\begin{aligned} z^{-1} &= \cos \theta - i \sin \theta \\ &= \cos(-\theta) + i \sin(-\theta). \end{aligned} \quad (2.37)$$

We see from Fig. 2.6 that geometrically this is the opposite rotation, i.e., the rotation of  $-\theta$ .

*Property 2.10* Complex numbers in the form  $z = \cos \theta + i \sin \theta$  is a commutative group.

This is easily seen from Eq. (2.36) if we use that addition of real numbers is commutative.

### 2.6.1.3 The Unit Quaternion

Similarly, the unit quaternion, i.e., one real and three imaginary numbers, is a group. A quaternion is written in terms of the basis  $1, i, j$ , and  $k$  as quadruples in the form

$$\mathcal{H} = \{(a, b, c, d) \mid a, b, c, d \in \mathbb{R}\}. \quad (2.38)$$

A unit quaternion has the additional requirement that  $a^2 + b^2 + c^2 + d^2 = 1$ . The quaternion product of two unit quaternions  $H_1 = a_1 + b_1i + c_1j + d_1k$  and  $H_2 = a_2 + b_2i + c_2j + d_2k$  gives another unit quaternion (the group property) given by

$$\begin{aligned} H_1 \circ H_2 = & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\ & + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k. \end{aligned} \quad (2.39)$$

Because the quaternion product is not commutative, we can thus conclude the following:

*Property 2.11* The unit quaternion is not a commutative group.

The unit quaternion can be used to represent rotations of a rigid body in the 3-dimensional Euclidean space. Because the unit quaternion representation uses four coordinates instead of three, we avoid the singularity that arises when we use the Euler angle representation.

## 2.6.2 Matrix Lie Groups

Of special interest when it comes to representing transformations of rigid bodies in space are Lie groups that can be written in matrix form. The group operation for matrix Lie groups is matrix multiplication. We note that this operation is continuous because it is simply a combination of multiplication and addition of elements in  $\mathbb{R}$ .

Matrix multiplication does not commute, so matrix Lie groups are in general not commutative. This is important to bear in mind when working with these groups. Also, the inverse of a matrix is not always defined. As a first requirement, to make sure that a set of matrices satisfies Property 2.6 we need to make sure that an inverse exists. We do this by restricting ourselves to  $n \times n$  nonsingular matrices. This is our first matrix group.

### 2.6.2.1 The General Linear Group

The general linear group of order  $n$  consists of all  $n \times n$  nonsingular real matrices and is denoted  $GL(n, \mathbb{R})$ . The manifold of  $GL(n, \mathbb{R})$  is thus an open subset of  $\mathbb{R}^{n \times n}$  defined by all matrices in  $\mathbb{R}^{n \times n}$  except the ones that have determinant equal zero. The identity element is given by the  $n \times n$  identity matrix and Property 2.5 is satisfied. As we restrict ourselves to nonsingular matrices the inverse always exists and is given by the matrix inverse. Note that Property 2.6 requires all matrix groups to be subgroups of  $GL(n, \mathbb{R})$ , i.e., that the inverse exists. As a result a matrix group of  $n \times n$  matrices is always a subgroup of  $GL(n, \mathbb{R})$ .

*Property 2.12* The general linear group with matrix multiplication as group operator is not a commutative group.

We can also choose addition as the group operator. In this case the group identity is the zero matrix and the group is commutative. It turns out, however, that choosing matrix multiplication as the group operator allows us to adopt a very nice geometric interpretation of the group elements as points in the configuration space. We will therefore concentrate on matrix groups with multiplication as the group operator.

### 2.6.2.2 The Orthogonal Group

The orthogonal group is a subgroup of the general linear group defined as

$$O(n) = \{R \in GL(n, \mathbb{R}) \mid R^T R = I\}. \quad (2.40)$$

This group consists of all matrices that, in addition to being nonsingular, preserve the scalar product between  $n$ -dimensional vectors. This is an important property that we will use later, for example when we find the corresponding Lie algebra, see Sect. 2.6.4. The fact that these matrices preserve the bilinear form means that the scalar product of two vectors does not change (is invariant) when both vectors are acted on by a matrix in  $O(n)$ . If we write  $x' = Rx$  and  $y' = Ry$  we see that

$$x' \cdot y' = x^T R^T R y = x \cdot y \quad (2.41)$$

because  $R^T R = I$ , which defines this group.

*Property 2.13* The orthogonal group with matrix multiplication as group operator is not a commutative group.

### 2.6.2.3 The Special Orthogonal Group

The special orthogonal group is a subgroup of the orthogonal group defined as

$$SO(n) = \{R \in GL(n, \mathbb{R}) \mid R^T R = I, \det(R) = +1\}. \quad (2.42)$$

Note that for  $O(n)$  the determinant is  $\pm 1$  as  $\det R^T \det R = \det I = 1$ . The special orthogonal group  $SO(n)$  consists of all elements with determinant  $+1$ . This group is thus a subset of the orthogonal group with the additional requirement that the determinant is positive, which is what “special” refers to. An element of the special orthogonal group of dimension 3 is a rotation matrix and can be interpreted as pure rotational motion of a rigid body. As  $R^T R = I$  we can conclude that the inverse of an element  $R$  is the same as the transpose, i.e.,  $R^{-1} = R^T$ . This property is very useful and will simplify the computations in the chapters to come. This property is true for all orthogonal groups, not only the special orthogonal group.

*Property 2.14* The special orthogonal group with matrix multiplication as group operator is not a commutative group.

Note that even though  $SO(n)$  is not commutative in general, the special case of  $SO(2)$  is commutative. This is the group of rotations in the plane which, as we have already seen, is commutative.

### 2.6.2.4 The Special Euclidean Group

The special Euclidean group  $SE(n)$  is the group of *rigid body transformations* on  $\mathbb{R}^n$ . We are especially interested in the special Euclidean group that act on  $\mathbb{R}^3$ , denoted  $SE(3)$ . This is the set of rigid body transformations on  $\mathbb{R}^3$  defined as the set of mappings  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $g(x) = Rx + p$  where  $R \in SO(3)$  and  $p \in \mathbb{R}^3$ . The matrix representation of  $SE(3)$  is typically given as

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}. \quad (2.43)$$

Matrix multiplication of two elements of  $SE(3)$  gives

$$g_2 g_1 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_2 R_1 & R_2 p_1 + p_2 \\ 0 & 1 \end{bmatrix} = g \quad (2.44)$$

where  $R_2 R_1 \in SO(3)$  and  $R_2 p_1 + p_2 \in \mathbb{R}^3$  and thus  $g = g_2 g_1 \in SE(3)$ . The inverse is given by the matrix inverse, i.e.,

$$g^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \quad (2.45)$$

where we have used that  $R^{-1} = R^T$  for orthogonal matrices. We see that because  $R^T \in SO(3)$  and  $-R^T p \in \mathbb{R}^3$  Property 2.6 is satisfied as  $g^{-1} \in SE(3)$ .

$SE(3)$  can be also written as a semidirect product of  $SO(3)$  and  $\mathbb{R}^3$ , i.e. as  $SO(3) \ltimes \mathbb{R}^3$ . This means that  $SE(3)$ , as a manifold, can be looked upon as the product  $SO(3) \times \mathbb{R}^3$ , but its group structure includes the action of  $SO(3)$  on  $\mathbb{R}^3$  which is illustrated by using  $\ltimes$  instead of  $\times$ . Intuitively this is easy to see. Translating a rigid object in space will only change the position of the object, the attitude of the object remains fixed. However, if we rotate an object around an arbitrary line, this will not only change the attitude, but also the position of the rigid body. Hence, the group structure allows for actions of  $SO(3)$  on  $\mathbb{R}^3$ . This can also be seen from the matrix representation of two consecutive transformations in (2.44).

The matrix representation of this group given in (2.43) is an injective homomorphism from  $SE(3)$  to  $GL(4, \mathbb{R})$ , i.e.,

$$(R, p) \rightarrow \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}. \quad (2.46)$$

We thus obtain a convenient way of writing an element of this group at the expense of increasing the number of parameters from 6 to 16 (or 12 assuming the final row as fixed).

*Property 2.15* The special Euclidean group  $SE(3)$  with matrix multiplication as group operator is not a commutative group.

### 2.6.2.5 Planar Motion

We can also restrict the motion of a rigid body to the set of rigid transformations in the plane. This is the group  $SE(2)$  and can be written in matrix form as

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad (2.47)$$

where  $R \in SO(2)$  and  $p \in \mathbb{R}^2$ . This representation is thus an injective homomorphism from  $SE(2)$  to  $GL(3, \mathbb{R})$ . Here  $R \in SO(2)$  is the rotation matrix representing rotations around the axis orthogonal to the plane, and  $p \in \mathbb{R}^2$  represents translational motion in the plane.

*Property 2.16* The special Euclidean group  $SE(2)$  with matrix multiplication as group operator is not a commutative group.

It is fairly straight forward to see that  $SE(2)$  is not commutative also from a geometric point of view. For example, a linear motion in the direction of one of the axes followed by a rotation will give a different transformation than a rotation followed by a linear motion because the axis of the linear motion changes when we chose to perform the rotation before the translation.

### 2.6.2.6 The Schönflies Group

The Schönflies group describes three degrees of freedom linear motion and one degree of freedom rotational motion. We normally assume that the rotational axis is the  $z$ -axis. This is a normal configuration space for pick-and-place tasks in robotic manufacturing. An element of the Schönflies group  $\mathcal{S}$  is normally written in matrix form as

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad (2.48)$$

where  $p \in \mathbb{R}^3$  and  $R$  is the  $3 \times 3$  matrix representation of  $SO(2)$ . We will find these rotation matrices for the three coordinate axes in Sect. 3.2.1.

**Table 2.1** The Lie subgroups of  $SE(3)$ . Only the normal forms are shown. The dimension of each group is given in the parenthesis

Group	Name	Basis	Description
$SE(3)$	Special Euclidean Group	$\{e_1, e_2, e_3, e_4, e_5, e_6\}$	Rigid body motions in $\mathbb{R}^3$ (6)
$\mathcal{Z}(z)$	Schönflies Group	$\{e_1, e_2, e_3, e_6\}$	Translations in $\mathbb{R}^3$ and rotations around the $z$ -axis (4)
$\mathbb{R}^3$	Translational Group	$\{e_1, e_2, e_3\}$	Translations in $\mathbb{R}^3$ (3)
$SE(2)$	Planar Group	$\{e_1, e_2, e_6\}$	Planar motion (3)
$SO(3)$	Special Orthogonal Group	$\{e_4, e_5, e_6\}$	Rigid body rotations around a fixed point (3)
$\mathcal{Y}_\rho$		$\{e_1, e_2, e_6 + \rho e_3\}$	Translational motion in the plane and screw motion around the $z$ -axis (3)
$\mathbb{R}^2$	Translational Group	$\{e_1, e_2\}$	Translational motion in the plane (2)
$\mathcal{C}$		$\{e_3, e_6\}$	Translations along and rotations around the $z$ -axis (2)
$\mathbb{R}$	Translational Group	$\{e_1\}$	Linear motion (1)
$SO(2)$	Special orthogonal Group	$\{e_3\}$	Rotational motion in the plane (1)
$\mathcal{H}_\rho$		$\{e_6 + \rho e_3\}$	Screw motion (1)

### 2.6.2.7 The Lie Subgroups

All the Lie subgroups that we are interested in, i.e., the ones that describe rigid body motion, can be written in terms of the basis of  $SE(3)$  and are thus subgroups of  $SE(3)$ . If we write the basis of the special Euclidean group as  $SE(3) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$  we can write the normal form of each subgroup in terms of this basis. Here,  $e_1, e_2, e_3$  represent linear motion in the directions of the  $x$ -,  $y$ -, and  $z$ -axes, respectively, and  $e_4, e_5, e_6$  represent angular motion around the same axes. There are a total of 10 subgroups of  $SE(3)$ . A short description is given in Table 2.1. We see that there are three 1-dimensional subgroups, these are all screw motions of zero ( $SO(2)$ ), infinite ( $\mathbb{R}$ ), and non-zero ( $\mathcal{H}_\rho$ ) pitch. These are the motions of the most important robotic joints. The higher-dimensional groups are frequently used to describe the configuration space of single rigid bodies or the robot end effector. We see that  $SE(3)$  is of dimension 6 and that there are no subgroups of dimension 5. Furthermore there is only one 4-dimensional subgroup and four 3-dimensional subgroups, which includes the three important groups of pure rotational motion, planar motion, and pure translational motion. More details on the normal forms of the Lie subgroups can be found in Meng et al. (2005, 2007).

### 2.6.3 Local Coordinates of Matrix Lie Groups

We saw in Sect. 2.5.1 that we can find a mapping from a subset of a manifold to the Euclidean space and we studied these maps for  $\mathbb{S}^1$  and  $\mathbb{S}^2$ . We will now look at the mapping from  $SO(3)$  to the Euclidean space  $\mathbb{R}^3$ . As  $SO(3)$  is a 3-dimensional space this mapping cannot be illustrated by simple drawings in the same intuitive way as we could with  $\mathbb{S}^1$  and  $\mathbb{S}^2$ , but we need to think of  $SO(3)$  as the orientation of a rigid body in space. A point on the manifold  $SO(3)$  can be represented by a rotation matrix  $R \in \mathbb{R}^{3 \times 3}$ . It is possible to find an invertible one-to-one mapping

$$\psi_1 : U_1 \rightarrow \mathbb{R}^3 \quad (2.49)$$

from a subset  $U_1 \subset SO(3)$  to the Euclidean space  $\mathbb{R}^3$ . As for  $\mathbb{S}^1$  and  $\mathbb{S}^2$  the subset  $U_1$  consists of all points on the manifold, except one.  $\Theta \in \mathbb{R}^3$  is the Euler angles and is one way to find a minimum representation of  $SO(3)$ . As there is one point for which this mapping is not well defined we will define sets of the form  $\mathbb{R}^3 - \{0, \pm \frac{\pi}{2}, 0\}$  for which we can find a one-to-one mapping to  $U_1$ . We see that we have removed one point, similarly to what we did for  $\mathbb{S}^1$  and  $\mathbb{S}^2$ . Depending on how we choose to represent the elements of  $SO(3)$ , we need to remove the single point on the manifold where the mapping does not exist for the chosen representation. We will see how we arrive at this later in this chapter. The three variables in  $\Theta$  are normally interpreted as rotations around the coordinate axes. The problem is that no matter what sequence of axes we choose, there always exists one point in one of the variables for which one of the other two variables can be chosen arbitrarily without changing the configuration. If we for example use consecutive rotations around the  $x$ ,  $y$ , and  $z$ -axes, the coordinates  $(\alpha - \psi, \pm \frac{\pi}{2}, \psi)$  describe the same rotation independently of how  $\psi$  is chosen. As a result a continuous one-to-one mapping between  $SO(3)$  and  $\Theta$  cannot be found for a rotation  $\pm \frac{\pi}{2}$  around the  $y$ -axis. This point represents the well known Euler angles singularity.

It is interesting to note the similarities with the stereographic mappings of  $\mathbb{S}^1$  and  $\mathbb{S}^2$ . Recall that we found two mappings  $\psi_1$  and  $\psi_2$  which had different domains  $U_1 = \mathbb{S}^n - \{(0, \dots, 0, 1)\}$  and  $U_2 = \mathbb{S}^n - \{(0, \dots, 0, -1)\}$ . Also for  $SO(3)$  we get different domains depending on what mappings we choose. The different mappings are obtained by choosing different sequences of rotation matrices. A different order of rotations leads to singularities at different points and thus also different domains.

### 2.6.4 Lie Algebra

Lie algebras can be thought of as tangent vectors to the manifold, or more specifically the tangent space of the identity element  $e$ . Imagine a path through the identity in a group  $G$  defined by  $\gamma : \mathbb{R} \rightarrow G$  such that  $\gamma(0) = e$ . The Lie algebra describes the derivatives in some local coordinate system around  $e$ . If the parameter is time, i.e.,  $\gamma(t)$ , then the tangent space can be identified with the space of all admissible velocities at the identity.



Taking this approach we can also define the tangent space as an equivalence relation. If we say that two paths are equivalent if their first derivatives are identical at the identity, we see that a tangent vector is the equivalence class that allows us to recognize all such paths. Such a space of equivalence classes is actually a vector space. This is a very important property of the tangent space that we will use frequently in the subsequent sections.

If this vector space  $\mathcal{V}$  allows a bilinear operation  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ , called the Lie bracket, which satisfies both skew-symmetry and the Jacobi identities, then this vector space is a Lie algebra. For two elements  $v_1, v_2 \in \mathcal{V}$  the Lie bracket is given by

$$[v_1, v_2] = v_1 v_2 - v_2 v_1, \quad v_1, v_2 \in \mathcal{V}. \quad (2.50)$$

An important property of the Lie bracket is that the resulting product is also an element of  $\mathcal{V}$ , i.e.,  $[v_1, v_2] \in \mathcal{V}$ . Formally a Lie algebra can be defined as a vector space which allows a Lie bracket:

**Definition 2.14** (Lie algebra) A vector space  $\mathcal{V}$  is a *Lie algebra* if there exists a bilinear operation  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ , denoted  $[\cdot, \cdot]$ , satisfying the two properties:

- skew-symmetry, i.e.,

$$[v_1, v_2] = -[v_2, v_1], \quad \forall v_1, v_2 \in \mathcal{V}, \quad (2.51)$$

- the Jacobi identity, i.e.,

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0, \quad \forall v_1, v_2, v_3 \in \mathcal{V}. \quad (2.52)$$

The Lie bracket  $[v_1, v_2]$  plays a very important role in Lie theory as it is a mean to differentiate a vector field  $v_2$  with respect to another vector field  $v_1$ . The Lie algebras associated with the groups described in the previous section are described in the following.

### 2.6.4.1 The General Linear Group

The Lie algebra of  $GL(n, \mathbb{R})$  is the set of all  $n \times n$  matrices, denoted  $gl(n, \mathbb{R})$ . It can be shown (Tu 2008) that  $[A, B]_{ij} = (AB - BA)_{ij}$ , and the Lie bracket is thus given by

$$[A, B] = AB - BA, \quad A, B \in gl(n, \mathbb{R}). \quad (2.53)$$

### 2.6.4.2 The Euclidean Space

The Lie algebra associated with the Lie group  $\mathbb{R}^n$  is  $\mathbb{R}^n$  itself. We will write an element of  $\mathbb{R}^n$  as  $p$  and an element of the corresponding Lie algebra as  $v$ . As we

have already seen, this is a very convenient way of writing the velocity variables. Because the position and velocity variables live on the same space we can write the velocity variables as  $v = \dot{p}$ . We then see that the Lie bracket is given by

$$[v_1, v_2] = 0, \quad \forall v_1, v_2 \in \mathbb{R}^n. \quad (2.54)$$

Thus, for a commutative Lie algebra the Lie bracket is zero. We will make much use of the fact that the velocity variables can be written as the time derivative of the position variables. Recall from Sect. 2.3 that this was in fact the definition of a Euclidean transformation:

*Property 2.17* The transformation associated with an element of the Euclidean space  $\mathbb{R}^n$  is Euclidean.

### 2.6.4.3 The Special Orthogonal Group

The Lie algebra of  $SO(3)$  is denoted  $so(3)$  and can be identified with a skew-symmetric traceless matrix in  $\mathbb{R}^{3 \times 3}$ . We will start by defining the skew-symmetric operator and the opposite operation, the vee operator. For now we will concentrate on  $SO(3)$  so we only need the operations on  $\mathbb{R}^3$ . However, the hat and vee operators are not restricted to  $\mathbb{R}^3$  but can be defined for a general vector  $\mathbb{R}^n$ . We will find these matrices in the next sections.

**Definition 2.15** The hat operator  $\wedge$  on  $\mathbb{R}^3$  maps an element  $\omega = [\omega_x \ \omega_y \ \omega_z]^\top \in \mathbb{R}^3$  into a traceless symmetric matrix

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \quad (2.55)$$

**Definition 2.16** The vee operator  $\vee$  is the inverse of the hat operator in Definition 2.15. For  $so(3)$  it maps a skew-symmetric matrix into a vector in  $\mathbb{R}^3$ .

We will also use the  $\vee$  map to write the vector representation of a matrix in general. As an example,  $g^\vee$  is the vector representation of the homogeneous transformation matrix  $g$ .

If we write an element of  $SO(3)$  as  $R(t)$  and recall that  $R(t)^\top R(t) = I$ , we can differentiate and get

$$\frac{d}{dt} R(t)^\top R(t) + R(t)^\top \frac{d}{dt} R(t) = 0. \quad (2.56)$$

We see that the constraints that make  $R(t)$  an element of  $SO(3)$  can be translated into the constraint that  $R^\top \dot{R}$  and  $\dot{R}^\top R$  are skew-symmetric. We see this if we rewrite (2.56) as  $\dot{R}^\top R + (\dot{R}^\top R)^\top = 0$ . This shows that  $\dot{R}^\top R$  and  $R^\top \dot{R}$  must be in the form of (2.55).

Note also that  $R^\top$  takes  $\dot{R}$  back to the identity. Thus, if  $\dot{R}$  is the tangent space at  $R$  then  $\widehat{\omega} = R^\top \dot{R}$  is the tangent space at the identity. The Lie algebra is defined as the tangent space at the identity which is the set of skew-symmetric  $3 \times 3$  matrices. An element of  $so(3)$  can thus be identified by the vector  $\omega = [\omega_x \ \omega_y \ \omega_z]^\top$ .

*Property 2.18* The transformation associated with an element of the special orthogonal group  $SO(3)$  is not Euclidean.

This follows directly from the observation that there is no way to write the orientation as a vector in  $\mathbb{R}^3$  for which Definition 2.2 is satisfied. We will study this kind of transformations in more detail in Chap. 3.

#### 2.6.4.4 The Special Euclidean Group

The Lie algebra of  $SE(3)$  is denoted  $se(3)$  and can be identified with the matrix

$$se(3) = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}, \quad (2.57)$$

where  $\omega, v \in \mathbb{R}^3$  and  $\widehat{\omega} \in so(3)$ . Elements of  $se(3)$  are also referred to in the robotics literature as twists. Physically, twists represent the quasi-velocities of a rigid body and can be written in vector form in terms of twist coordinates as

$$V = \begin{bmatrix} v \\ \omega \end{bmatrix} \quad (2.58)$$

where  $v = [v_x \ v_y \ v_z]^\top$  represents the linear velocity and  $\omega = [\omega_x \ \omega_y \ \omega_z]^\top$  represents the angular velocity.

As for  $SO(3)$ , the constraints on  $g \in SE(3)$  are translated into the constraints that  $\widehat{V} \in se(3)$  is in the form of (2.57). As we will see in the next chapter when we study rigid body motion in more detail, this is the same as requiring that  $\widehat{V}$  equals  $\dot{g}$  after it has been transformed back to the identity.

*Property 2.19* The transformation associated with an element of the special Euclidean group  $SE(3)$  is not Euclidean.

#### 2.6.4.5 The Planar Group

The Lie algebra of  $SE(2)$  is denoted  $se(2)$  and can be identified with the matrix

$$se(2) = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}, \quad (2.59)$$

where  $\omega \in \mathbb{R}$ ,  $v \in \mathbb{R}^2$  and  $\widehat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \in so(2)$ . Elements of  $se(2)$  can be written in vector form in terms of twist coordinates as

$$V = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^3 \quad (2.60)$$

where  $v = [v_x \ v_y]^\top$  represents the linear velocity and  $\omega = \omega_z$  represents the angular velocity. This is the group of linear motion in the plane and rotation around an axis orthogonal to this plane.

*Property 2.20* The transformation associated with an element of the special Euclidean group  $SE(2)$  is Euclidean.

Note that this transformation is Euclidean because there exists at least one parameterization for which the variables can be written in the form of Definition 2.2. One such parameterization is when the position variables are written in the normal way as  $(x, y, \psi)$  and the velocity variables are written as  $(v_x, v_y, \omega_z) = (\dot{x}, \dot{y}, \dot{\psi})$ .

#### 2.6.4.6 The Schönflies Group

The Lie algebra of  $\mathcal{X}(z)$  is denoted  $\mathfrak{x}$  and can be identified with the matrix

$$\mathfrak{x} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}, \quad \widehat{\omega} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.61)$$

and thus only angular velocity around the  $z$ -axis is allowed. If the rotational motion is in another direction,  $\widehat{\omega}$  is modified correspondingly, or we choose the coordinate system so that the  $z$ -axis is aligned with the revolute axis. We allow linear velocity in all three directions so  $v = [v_x \ v_y \ v_z]^\top$ .

*Property 2.21* The transformation associated with an element of the Schönflies group  $\mathcal{X}(z)$  is Euclidean.

### 2.6.5 Geometric Interpretation of Lie Group Representations

We have now developed several tools that we can use to represent spaces and transformations on these spaces. In this section we will look into how the different matrix Lie groups are interpreted geometrically. One of the most important tasks in Lie theory is to find suitable matrix representations for the different spaces, i.e., we want to find a set of matrices that satisfy certain properties so that they can be interpreted as geometrically meaningful objects or transformations. We will discuss how these group elements are interpreted geometrically and what they represent.

In general we can say that there are two main interpretations of the elements of a Lie group that are widely used in robotics: either as the configuration of a rigid body or as a transformation of a rigid body. The configuration of a rigid body can be specified with respect to an inertial frame or with respect to another rigid body. The configuration can thus be interpreted as the difference in position and orientation between two different frames.

On the other hand, a transformation can describe the position and orientation of the same rigid body before and after some action. In this case the transformation is to be interpreted as an active transformation of a rigid body, often expressed in the observer frame.

We will also look at how we can use an element of the Lie algebra to describe the velocity of a rigid body to obtain a complete representation of the state space. In the previous section we used twists to represent the velocity of the rigid body. However, as we have already seen, we need to specify the frame in which the velocity is represented. There are two main ways to represent twists that are of importance in robotics, using either body or spatial coordinates.

### 2.6.5.1 Representations of Matrix Lie Groups

We will now study different ways to represent the location of a rigid body in space. This can be written in terms of three variables  $(x_{12}, y_{12}, z_{12})$  describing the position and three variables  $(\phi_{12}, \theta_{12}, \psi_{12})$  describing the orientation, for example by

$$g_{12}^{\vee} = \begin{bmatrix} x_{12} \\ y_{12} \\ z_{12} \\ \phi_{12} \\ \theta_{12} \\ \psi_{12} \end{bmatrix}. \quad (2.62)$$

Here,  $g^{\vee}$  denotes the vector representation of the homogeneous transformation matrix  $g$ .

If we also know what convention is used for the Euler angles, this vector uniquely describes the location of a frame  $\mathcal{F}_2$  in another frame  $\mathcal{F}_1$ . Suppose now that after we have performed this transformation, we want to perform another transformation from frame  $\mathcal{F}_2$  to frame  $\mathcal{F}_3$ . We thus want to perform two consecutive transformations, the first represented by  $g_{12}^{\vee}$  and the second by  $g_{23}^{\vee}$ . Unfortunately, the final location of the rigid body, represented by  $g_{13}^{\vee}$ , cannot be found by

$$g_{13}^{\vee} = g_{12}^{\vee} + g_{23}^{\vee} \quad (2.63)$$

because geometrically the transformation represented by  $g_{13}^{\vee}$  is not the same as simply adding the transformation corresponding to  $g_{12}^{\vee}$  and  $g_{23}^{\vee}$  using normal addition. One way to see that this is not correct is to note that by simply adding these, the

rotational part of the first transformation will not affect the translational part of the second transformation, as it should (because it is a semi-direct product).

It turns out, however, that if we write each transformation in matrix form as in (2.43), the transformation of  $g_{12}$  followed by the transformation  $g_{23}$  can be written as

$$g_{13} = g_{12}g_{23} \quad (2.64)$$

where  $g \in SE(3)$  is the homogeneous transformation matrix introduced in Sect. 2.6.2. This is the main motivation for writing the transformations in this form. We see that at the cost of increasing the number of variables needed to describe the configuration of a rigid body from 6 to 16 (a  $4 \times 4$  matrix) we obtain the benefits of this convenient representation.

Another transformation that can be performed in an elegant manner using this matrix representation is the transformation of a point in space. We now write the point in homogeneous coordinates  $\bar{q}^b = [(q^b)^T \ 1]^T$ . Hence, the homogeneous coordinates of a point are obtained by augmenting the coordinates with an identity element. We will now see how this representation allows us to use homogeneous transformation matrices to perform operations on this point. Let  $\bar{q}^b$  represent a point that is fixed in frame  $\mathcal{F}_b$ . Given another frame  $\mathcal{F}_a$  that relates to  $\mathcal{F}_b$  through  $g_{ab}(t)$  this point can be expressed in coordinate frame  $\mathcal{F}_a$  by

$$\bar{q}^a(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} \bar{q}^b. \quad (2.65)$$

This result is obtained by rewriting the well known equations of rotating and translating a point, which is given by  $q^a = R_{ab}q^b + p_{ab}$ . The same applies to the velocity  $v$  of the same point which can be described either in coordinate frame  $\mathcal{F}_a$  or in  $\mathcal{F}_b$ :

$$\bar{v}^a = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \bar{v}^b \quad (2.66)$$

where  $\bar{v}^b = [(v^b)^T \ 0]^T$ . Also Eq. (2.66) is only a reformulation of the well known equation  $v^a = R_{ab}v^b$ . We note that the homogeneous coordinates of a vector are obtained by adding a zero to the end of the vector. Here the velocity  $v^b$  expressed in  $\mathcal{F}_b$  is transformed so that it is written as observed from frame  $\mathcal{F}_a$ , denoted  $v^a$ . Using this interpretation of the transformation (2.66) the same vector is represented in two different coordinate frames, namely  $\mathcal{F}_a$  and  $\mathcal{F}_b$ , and the transformation thus describes what the same vector looks like for two observers located in two different frames.

*Remark 2.2* Note that the homogeneous coordinates of a point  $q$  are obtained by adding 1 at the end, i.e., a point  $q$  in homogeneous coordinates is written as  $\bar{q} = [q^T \ 1]^T$ . For a vector, however, the homogeneous coordinates are obtained by adding 0 to the vector, i.e., the vector  $v$  is written in homogeneous coordinates as  $\bar{v} = [v^T \ 0]^T$ . We note that in this way, adding a point and a vector gives another point, adding two vectors gives a vector, and adding two points is meaningless.

The expression in the form of (2.65) can also be interpreted as a transformation of the point  $\bar{q}_a$  to another point  $\bar{q}_b$  by first a translation along  $p_{ab}$  and then a rotation by  $R_{ab}$ . In this case both  $\bar{q}_a$  and  $\bar{q}_b$  are given with respect to the same reference frame and  $\bar{q}_a$  is transformed by  $g_{ab}$  to  $\bar{q}_b$ . The transformation is then written as

$$\bar{q}_b = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \bar{q}_a. \quad (2.67)$$

*Remark 2.3* Note that a superscript is used to denote in what reference frame a point or vector is represented while a subscript is used to distinguish two different points or vectors:  $\bar{q}_a$  and  $\bar{q}_b$  are two different points, while  $\bar{q}^a$  and  $\bar{q}^b$  are two different representations of the same point.

Adapting the interpretation above we can look at an element  $g \in SE(3)$  as a transformation from some initial coordinates  $\bar{q}(0)$  to the coordinates after a rigid body motion is applied. In this case, both the initial and final coordinates are viewed *from the same coordinate system*. Take a point  $\bar{q}(0)$  represented in the inertial frame  $\mathcal{F}_0$  and let  $g(t, X)$  represent a transformation of  $t$  units (or  $t$  units of time with constant unitary velocity) along the direction of the twist  $X$ . Then, if  $\bar{q}(0)$  is transformed into another point by  $g(t, X)$  we get

$$\bar{q}(t) = g(t, X)\bar{q}(0) \quad (2.68)$$

where  $\bar{q}(0)$  is the point before and  $\bar{q}(t)$  is the point after the transformation by  $g(t, X)$ . Note that both  $\bar{q}(0)$  and  $\bar{q}(t)$  are represented in  $\mathcal{F}_0$ . In this case the transformation is not interpreted as a change of observer but rather as an active action on the point  $\bar{q}(t)$ , in other words a motion. If we want to be specific about what reference frame the point is expressed in, for example the inertial frame  $\mathcal{F}_0$ , we can denote the point before the transformation as  $q^0(0)$  and after the transformation as  $q^0(t)$ .

Up until now we have only looked at transformations of vectors and points. Similarly, the configuration of a rigid body can be observed from different frames and transformed in space. The configuration of a rigid body is given by the matrix  $g_{0b}$ , i.e., the position and orientation of the body frame  $\mathcal{F}_b$  with respect to the inertial frame  $\mathcal{F}_0$ . Using the concept of change of observer's frame we can instead write the position and orientation of the rigid body with respect to an arbitrary frame  $\mathcal{F}_i$  as

$$g_{ib} = g_{i0}g_{0b} \quad (2.69)$$

where  $g_{0i}$  is the transformation from the inertial frame to frame  $i$  and  $g_{i0} = g_{0i}^{-1}$ . The interpretation of (2.69) is the same as that of (2.65), i.e., a change of observer's frame.

Furthermore, if  $g_{0b}(0)$  is the configuration of a rigid body with a reference frame  $\mathcal{F}_b$  attached to it with respect to the inertial frame  $\mathcal{F}_0$ , then the configuration of the same rigid body after a transformation by  $g(t, X)$  is given by

$$g_{0b}(t) = g(t, X)g_{0b}(0) \quad (2.70)$$

also with respect to  $\mathcal{F}_0$ . This is the motion of a rigid body parameterized by  $t$  and denoted  $g(t, X)$ . We will normally parameterize the motion in terms of the time variable  $t$ . We then get a time variant homogeneous transformation matrix  $g(t)$  that describes the position and orientation of a rigid body as a function of time. We will use both interpretations of the homogeneous transformation matrix frequently in the subsequent chapters.

Summing up, there are two different interpretations of a matrix Lie group. Firstly, a transformation  $g$  can be interpreted as a change of the frame in which a set of points is represented. We have already seen that for a point  $\bar{q}^b$  represented in coordinate frame  $\mathcal{F}_b$  and a homogeneous transformation matrix  $g_{ab}$ , this point can be represented in  $\mathcal{F}_a$  by the transformation in (2.65). This transformation is referred to as a change of observer's frame. An element  $g$  of the Lie group can also be interpreted as an active transformation, i.e., we consider the observer's frame as fixed and let  $g$  represent a transformation of a point, a vector, or a rigid body from an initial to a final configuration.

### 2.6.5.2 Representations of Lie Algebras

Also the velocity of a rigid body can be represented either in vector or matrix form. We will see that as for the position variables, there are several advantages of representing the velocity in matrix form. Consider a point  $q^b$  fixed to a rigid body with coordinate frame  $\mathcal{F}_b$  that moves corresponding to the 1-parameter subgroup of  $SE(3)$ . Given a rigid body motion  $g_{ab}(t)$  relating  $\mathcal{F}_a$  and  $\mathcal{F}_b$ , this point can be represented with respect to  $\mathcal{F}_a$  by

$$\bar{q}^a(t) = g_{ab}(t)\bar{q}^b. \quad (2.71)$$

The velocity of this point is given by

$$\bar{v}_{q^a}(t) = \frac{d}{dt}\bar{q}^a(t) = \dot{g}_{ab}(t)\bar{q}^b. \quad (2.72)$$

There is, however a more convenient and compact way to write this relation. First write

$$\bar{v}_{q^a}(t) = \dot{g}_{ab}(t)g_{ab}^{-1}(t)g_{ab}(t)\bar{q}^b. \quad (2.73)$$

Using (2.71) we can rewrite this as

$$\bar{v}_{q^a}(t) = \dot{g}_{ab}(t)g_{ab}^{-1}(t)\bar{q}^a. \quad (2.74)$$

It can be shown that  $\dot{g}_{ab}(t)g_{ab}^{-1}(t) \in se(3)$  has a very useful geometric interpretation. If we let  $\mathcal{F}_a$  be the inertial frame denoted  $\mathcal{F}_0$ , and let  $\mathcal{F}_b$  be the body frame, we get the spatial velocity variables  $\hat{V}_{0b}^S$  defined by

$$\hat{V}_{0b}^S = \dot{g}_{0b}g_{0b}^{-1} = \begin{bmatrix} \hat{\omega}_{0b}^S & v_{0b}^S \\ 0 & 0 \end{bmatrix}. \quad (2.75)$$



The spatial velocity variables  $\widehat{V}_{0b}^S$  map the point  $\bar{q}^0$  to the velocity of the same point also represented in the inertial frame  $\mathcal{F}_0$ , i.e.  $\bar{v}_{q^0}(t) = \widehat{V}_{0b}^S \bar{q}^0$ . Using the terminology of Lie groups,  $\widehat{V}_{0b}^S$  is the right translate of  $\dot{g}_{0b} \in T_g SE(3)$  which maps  $\dot{g}_{0b}$  back to the identity  $T_I SE(3)$ .  $\dot{g}_{0b}$  is thus in the tangent space to  $SE(3)$  at configuration  $g_{0b}$ , denoted  $T_g SE(3)$ . If we recall that the Lie algebra was defined as the tangent space at the identity, we can conclude that  $\widehat{V}_{0b}^S$  is an element of the Lie algebra  $se(3)$ . Furthermore, we can write  $\widehat{V}_{0b}^S$  in vector form and get

$$V_{0b}^S = \begin{bmatrix} v_{0b}^S \\ \omega_{0b}^S \end{bmatrix} = \begin{bmatrix} -\dot{R}_{0b} R_{0b}^\top p_{0b} + \dot{p}_{0b} \\ (\dot{R}_{0b} R_{0b}^\top)^\vee \end{bmatrix}. \quad (2.76)$$

$V_{0b}^S$  is thus the vector representation of the velocity of a reference frame  $\mathcal{F}_b$  with respect to  $\mathcal{F}_0$  in spatial coordinates, which is what the superscript refers to.

We can also write the velocity in the body frame. Using  $\bar{q}^0(t) = g_{0b}(t) \bar{q}^b$  and Eq. (2.74) we get

$$\begin{aligned} \bar{v}_{q^b}(t) &= g_{0b}^{-1}(t) \bar{v}_{q^0}(t) \\ &= g_{0b}^{-1}(t) \dot{g}_{0b}(t) g_{0b}^{-1}(t) \bar{q}^0 \\ &= g_{0b}^{-1}(t) \dot{g}_{0b}(t) \bar{q}^b. \end{aligned} \quad (2.77)$$

Also  $g_{0b}^{-1}(t) \dot{g}_{0b}(t) \in se(3)$  has a geometrically meaningful interpretation and represents the same velocity as above, but now as observed from the reference frame attached to the body  $\mathcal{F}_b$ , i.e., in body coordinates. The body velocity variable  $\widehat{V}_{0b}^B$  is defined as

$$\widehat{V}_{0b}^B = g_{0b}^{-1} \dot{g}_{0b} = \begin{bmatrix} \widehat{\omega}_{0b}^B & v_{0b}^B \\ 0 & 0 \end{bmatrix}. \quad (2.78)$$

In this case  $\widehat{V}_{0b}^B$  is obtained by left translating  $\dot{g}_{0b} \in T_g SE(3)$  back to the identity, and consequently  $\widehat{V}_{0b}^B$  is also an element of the Lie algebra. If we write the body velocity variables in vector form we get

$$V_{0b}^B = \begin{bmatrix} v_{0b}^B \\ \omega_{0b}^B \end{bmatrix} = \begin{bmatrix} R_{0b}^\top \dot{p}_{0b} \\ (R_{0b}^\top \dot{R}_{0b})^\vee \end{bmatrix}. \quad (2.79)$$

The map  $\widehat{V}_{0b}^B$  thus takes a point represented in the frame attached to the rigid body to the velocity of this point by  $\bar{v}_{q^b}(t) = \widehat{V}_{0b}^B \bar{q}^b$ , also represented in the body frame. This can also be written as

$$v_{q^b}(t) = \omega_{0b}^B \times q^b + v_{0b}^B \quad (2.80)$$

or using homogeneous coordinates as

$$\bar{v}_{q^b}(t) = \widehat{V}_{0b}^B \bar{q}^b. \quad (2.81)$$

We will study body and spatial velocities in more detail in Chap. 3 where we derive the kinematics of a single rigid body.

### 2.6.5.3 The Velocity of a Point

Recall from (2.74) that the velocity of a point  $q^0$  (fixed in reference frame  $\mathcal{F}_b$ ), whose motion is given by  $\bar{q}^0(t) = g_{0b}(t)\bar{q}^b$  can be written as

$$\bar{v}_{q^0}(t) = \frac{d}{dt}\bar{q}^0 = \widehat{V}_{0b}^S \bar{q}^0. \quad (2.82)$$

This gives us the velocity of a point  $q^0$  fixed in the body frame, but expressed in the inertial frame  $\mathcal{F}_0$ , when the body frame moves with a constant velocity  $V_{0b}^S$ . In fact, this is a general result. Given a fixed point  $q^0$  and a matrix representation of the Lie algebra given by  $\widehat{X}$ , then the velocity of the body-fixed point  $q^0$  represented in the inertial frame  $\mathcal{F}_0$  is given by

$$v_{q^0}(t) = \widehat{X}_{0b}^0 q^0(t). \quad (2.83)$$

*Example 2.7* For  $SE(3)$  this becomes

$$\bar{v}_{q^0}(t) = \widehat{V}_{0b}^S \bar{q}^0(t) \quad (2.84)$$

written in spatial velocity variables. If we write the twist coordinates as  $V_{0b}^S = [u^S \ v^S \ w^S \ p^S \ q^S \ r^S]^T$ , the twist is defined in matrix form as

$$\widehat{V}_{0b}^S = \begin{bmatrix} 0 & -r^S & q^S & u^S \\ r^S & 0 & -p^S & v^S \\ -q^S & p^S & 0 & w^S \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.85)$$

In body velocity variables the expression becomes

$$\bar{v}_{q^b}(t) = \widehat{V}_{0b}^B \bar{q}^b(t). \quad (2.86)$$

*Example 2.8* Similarly, for pure rotational motion the corresponding expression for the spatial velocity variables is given as

$$\bar{v}_{q^0}(t) = \widehat{\omega}_{0b}^S \bar{q}^0(t) \quad (2.87)$$

where  $\widehat{\omega}$  is the standard skew-symmetric matrix representation of the Lie algebra  $so(3)$ , so (2.87) can also be written as  $v_{q^0}(t) = \omega_{0b}^S \times q^0$ . Rotational motion in this form can also be written in terms of body velocity variables as

$$v_{q^b}(t) = \widehat{\omega}_{0b}^B q^b(t) \quad (2.88)$$

which is the same as  $v_{q^b}(t) = \omega_{0b}^B \times q^b$ .

*Example 2.9* Also for planar motion we can obtain the same relation by writing

$$v_{q^0}(t) = \begin{bmatrix} \dot{q}_x^0 \\ \dot{q}_y^0 \\ 0 \end{bmatrix}, \quad \bar{q}^0(t) = \begin{bmatrix} q_x^0 \\ q_y^0 \\ 1 \end{bmatrix}, \quad \widehat{V}_{0b}^S = \begin{bmatrix} 0 & -r^S & u^S \\ r^S & 0 & v^S \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.89)$$

$\bar{v}_{q^0}(t) = \widehat{V}_{0b}^S \bar{q}^0(t)$  can be written as

$$\begin{aligned} \dot{q}_x^0 &= -r^S q_y^0 + u^S, \\ \dot{q}_y^0 &= r^S q_x^0 + v^S. \end{aligned} \quad (2.90)$$

We note that the formulation of the twist in (2.89) is just a compact way of writing the more general expression that we found for  $se(3)$ . That is, we could also write this as an element of  $se(3)$  as

$$\widehat{V}_{0b}^S = \begin{bmatrix} 0 & -r^S & 0 & u^S \\ r^S & 0 & 0 & v^S \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.91)$$

## 2.6.6 Actions on Lie Groups

An element of a Lie group can act on different mathematical spaces, and depending on what space they act on we get different geometrical interpretations. First of all, the elements of a group can act on a vector space: we saw an example of this when a point or a vector was transformed by a homogeneous transformation matrix or a rotation matrix. In this case the group element, for example the rotation matrix, acts on a vector, i.e., it transforms the vector in some way. A group can also act on itself: examples of this are the rigid body motions in Eq. (2.70) and a change of observer's frame. Finally a group element can act on its tangent space. This is useful as it allows us to transform the tangent vectors from the identity to any group element and to represent velocities in different frames. We will discuss such actions in more detail in this section.

### 2.6.6.1 Conjugations

If we let  $G$  be a Lie group and let  $g$  and  $x$  be elements of  $G$ , then we define a *conjugation* as a homomorphism given by

$$h_g(x) = gxg^{-1}. \quad (2.92)$$

This is thus a mapping from the manifold of  $G$  to itself, i.e.,  $h_g : G \rightarrow G$  which follows directly from Properties 2.4 and 2.6. If we let  $X$  be an element of a Lie

algebra, we get a similar transformation from the tangent space at the identity to itself, i.e.,  $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$  given by

$$\text{Ad}_g(X) = g \widehat{X} g^{-1}. \quad (2.93)$$

We see that a conjugation represents the same linear transformation, but as observed from a different reference frame and using a different basis. Let  $A \in \mathbb{R}^{n \times n}$  denote a matrix. Then left multiplication  $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defines a correspondence between the matrix  $A$  and linear functions from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  given by

$$L_A(X) = AX, \quad X \in \mathbb{R}^n. \quad (2.94)$$

Using this notation we can write the conjugation

$$\text{Ad}_g(X) = g \widehat{X} g^{-1} \quad (2.95)$$

as

$$L_{gXg^{-1}} = L_g \circ L_X \circ L_{g^{-1}}. \quad (2.96)$$

We recall that an element of the Lie algebra  $X$  can be interpreted as a linear transformation, for example as in (2.82), of a vector or a point represented in the body frame  $\mathcal{F}_b$ . If we want to perform the same operation on a point  $q^a$  represented in the reference frame  $\mathcal{F}_a$  which relates to the body frame through the transformation  $g_{ab}$ , we need to perform the following operations, which also gives us an intuitive interpretation of conjugations: First we need to transform the point  $q^a$  to frame  $\mathcal{F}_b$ . This is obtained by a transformation by  $g_{ab}^{-1}$  which gives us  $q^b = L_{g_{ab}^{-1}} q^a$ . Now that the point is represented in the body frame we can perform the desired operation, represented by  $X$ , which is applied to  $q^b$  by  $L_X q^b$ . Finally we need to transform the vector back into the frame  $\mathcal{F}_a$  by applying  $L_{g_{ab}}$ . Applying all these transformations gives us  $L_{gXg^{-1}} q^a = L_g \circ L_X \circ L_{g^{-1}} q^a$ . This is the same transformation as  $L_X q^b = \widehat{X} q^b$  but represented in reference frame  $\mathcal{F}_a$ . A conjugation thus represents a change of basis or change of observer and allows us to represent the linear transformation  $X$  but in a different basis or reference frame. In the following we discuss this kind of transformations in more detail.

We can define right multiplication in a similar way to left multiplication. Right multiplications  $R_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  define linear functions from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  given by

$$R_A(X) = XA, \quad X \in \mathbb{R}^n. \quad (2.97)$$

### 2.6.6.2 The Adjoint Map $\text{Ad}_g$

The Adjoint map  $\text{Ad}_g$  represents an action of an element of the Lie group on its Lie algebra. The Adjoint map is important as it allows us to represent an element of the Lie algebra in different coordinate frames. The Adjoint map is given by the conjugation by an element  $g$ . Take for example the conjugation by  $g$  on the identity

element, i.e.,  $geg^{-1} = e$ . This operation maps the identity element of the Lie group to itself. The differential of this map is the Jacobian, and maps the tangent space at the identity to itself. More specifically we can write the Adjoint map  $\text{Ad} : G \times \mathfrak{g} \rightarrow \mathfrak{g}$  as

$$\text{Ad}_g X = g \widehat{X} g^{-1}, \quad \forall g \in G, \widehat{X} \in \mathfrak{g}. \quad (2.98)$$

$\text{Ad}$  is thus a function of  $g$  for which (2.98) is true. The Adjoint map satisfies the following properties:

*Property 2.22* The inverse of the Adjoint map is given as

$$\text{Ad}_g^{-1} = \text{Ad}_{g^{-1}}. \quad (2.99)$$

*Property 2.23* The Adjoint map given by the conjugation by  $gh$  is found as

$$\text{Ad}_g \text{Ad}_h = \text{Ad}_{gh}. \quad (2.100)$$

This can be seen from Eq. (2.98). We will now look at the adjoint maps of the most important Lie groups used in robotics.

### 2.6.6.3 The Special Orthogonal Group

Let  $R$  be an element of  $SO(3)$  and  $\omega$  an element of  $so(3)$ . Then the Adjoint map  $\text{Ad}_R : so(3) \rightarrow so(3)$  is given by

$$\text{Ad}_R \omega = R \widehat{\omega} R^T. \quad (2.101)$$

The Adjoint map on  $so(3)$  also admits a  $3 \times 3$  matrix representation, i.e.,

$$\text{Ad}_R = R \quad (2.102)$$

and we get an alternative action on the vector  $\omega$ :

$$\text{Ad}_R \omega = R \omega. \quad (2.103)$$

The fact that  $\text{Ad}_R = R$  is an “accidental property” of  $SO(3)$  and can not be taken as a general result.

*Remark 2.4* When the Adjoint map acts on a matrix, it is implicitly understood that it is of the form of (2.101), while when the Adjoint map acts on a vector it is in the form (2.103). Equations (2.101) and (2.103) represent two different formulations of the same transformation.

### 2.6.6.4 The Special Euclidean Group

Also for  $SE(3)$  we can find similar relations. Let  $g = (R, p)$  be an element of  $SE(3)$  and  $V = [v^\top \omega^\top]^\top$  an element of  $se(3)$ , the Adjoint map  $\text{Ad}_g : se(3) \rightarrow se(3)$  is then defined as

$$\text{Ad}_g V = g \widehat{V} g^{-1}. \quad (2.104)$$

We can also find a  $6 \times 6$  matrix representation of the Adjoint map that acts on the vector  $V \in \mathbb{R}^6$ . Let  $V_{ab}^B$  be the velocity of a rigid body expressed in the body frame with respect to a fixed frame  $\mathcal{F}_a$  and  $g_{ab}$  the transformation from  $\mathcal{F}_a$  to  $\mathcal{F}_b$ . Then the linear and angular velocities given in spatial coordinates can be written as

$$\begin{aligned} v_{ab}^S &= p_{ab} \times (R_{ab} \omega_{ab}^B) + R_{ab} v_{ab}^B, \\ \omega_{ab}^S &= R_{ab} \omega_{ab}^B. \end{aligned} \quad (2.105)$$

This can be written in vector form as

$$V_{ab}^S = \text{Ad}_{g_{ab}} V_{ab}^B = \begin{bmatrix} R_{ab} & \widehat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} v_{ab}^B \\ \omega_{ab}^B \end{bmatrix} \quad (2.106)$$

and we thus get

$$\text{Ad}_{g_{ab}} = \begin{bmatrix} R_{ab} & \widehat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}. \quad (2.107)$$

We see that if we let  $\mathcal{F}_a$  be the inertial frame, i.e.,  $\mathcal{F}_0$ , we get the relation between body and spatial velocity variables that we have seen earlier, i.e.,

$$V_{0b}^S = \text{Ad}_{g_{0b}} V_{0b}^B \quad (2.108)$$

or alternatively

$$\begin{bmatrix} v_{0b}^S \\ \omega_{0b}^S \end{bmatrix} = \begin{bmatrix} R_{0b} & \widehat{p}_{0b} R_{0b} \\ 0 & R_{0b} \end{bmatrix} \begin{bmatrix} v_{0b}^B \\ \omega_{0b}^B \end{bmatrix}. \quad (2.109)$$

The Adjoint map can also be used to find the spatial velocities with respect to different frames in the same way.

We can check that the above corresponds to the expressions that we found in (2.76) and (2.79). The right hand side is given by

$$\begin{aligned} \begin{bmatrix} R_{0b} & \widehat{p}_{0b} R_{0b} \\ 0 & R_{0b} \end{bmatrix} \begin{bmatrix} R_{0b}^\top \dot{p}_{0b} \\ (R_{0b}^\top \dot{R}_{0b})^\vee \end{bmatrix} &= \begin{bmatrix} R_{0b} R_{0b}^\top \dot{p}_{0b} + \widehat{p}_{0b} R_{0b} (R_{0b}^\top \dot{R}_{0b})^\vee \\ R_{0b} (R_{0b}^\top \dot{R}_{0b})^\vee \end{bmatrix} \\ &= \begin{bmatrix} \dot{p}_{0b} + \widehat{p}_{0b} R_{0b} (R_{0b}^\top \dot{R}_{0b})^\vee \\ R_{0b} (R_{0b}^\top \dot{R}_{0b})^\vee \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \dot{p}_{0b} + \widehat{p}_{0b} R_{0b} R_{0b}^\top (\dot{R}_{0b} R_{0b}^\top)^\vee \\ R_{0b} R_{0b}^\top (\dot{R}_{0b} R_{0b}^\top)^\vee \end{bmatrix} \\
&= \begin{bmatrix} \dot{p}_{0b} + \widehat{p}_{0b} (\dot{R}_{0b} R_{0b}^\top)^\vee \\ (\dot{R}_{0b} R_{0b}^\top)^\vee \end{bmatrix} \\
&= \begin{bmatrix} \dot{p}_{0b} - \dot{R}_{0b} R_{0b}^\top p_{0b} \\ (\dot{R}_{0b} R_{0b}^\top)^\vee \end{bmatrix} \tag{2.110}
\end{aligned}$$

where we have used that

$$(R_{0b}^\top \dot{R}_{0b})^\vee = \omega_{0b}^B = R_{0b}^\top \omega_{0b}^S = R_{0b}^\top (\dot{R}_{0b} R_{0b}^\top)^\vee \tag{2.111}$$

and we get (2.76) as required.

For  $SE(3)$  we can show Property 2.22 by straight forward calculation:

$$\text{Ad}_g^{-1} = \begin{bmatrix} R^\top & -\widehat{R^\top p} R^\top \\ 0 & R^\top \end{bmatrix} = \begin{bmatrix} R^\top & -R^\top \hat{p} \\ 0 & R^\top \end{bmatrix} = \text{Ad}_{g^{-1}}. \tag{2.112}$$

The adjoint map on  $SE(3)$  is discussed in detail in Chap. 3.

### 2.6.6.5 Planar Motion

For planar motion we write

$$\widehat{\omega}_{ab}^B = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad R_{ab} = \begin{bmatrix} \cos \theta_{ab} & -\sin \theta_{ab} \\ \sin \theta_{ab} & \cos \theta_{ab} \end{bmatrix}. \tag{2.113}$$

First we note that

$$\widehat{\omega}_{ab}^S = R_{ab} \widehat{\omega}_{ab}^B R_{ab}^\top = \widehat{\omega}_{ab}^B \tag{2.114}$$

and thus  $\widehat{V}_{ab}^S = g_{ab} \widehat{V}_{ab}^B g_{0b}^{-1}$  can be written as

$$\begin{aligned}
v_{ab}^S &= \widehat{p}_{ab} \omega_{ab}^B + R_{ab} v_{ab}^B, \\
\omega_{ab}^S &= \omega_{ab}^B
\end{aligned} \tag{2.115}$$

where we have defined  $\widehat{p}_{ab} = [y_{ab} \ -x_{ab}]^\top$ . The Adjoint map  $\text{Ad}_g : se(2) \rightarrow se(2)$  can then be written in vector form as

$$V_{ab}^S = \text{Ad}_{g_{ab}} V_{ab}^B = \begin{bmatrix} R_{ab} & \widehat{p}_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{ab}^B \\ \omega_{ab}^B \end{bmatrix} \tag{2.116}$$

and we get

$$\text{Ad}_{gab} = \begin{bmatrix} R_{ab} & \hat{p}_{ab} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta_{ab} & -\sin \theta_{ab} & y_{0b} \\ \sin \theta_{ab} & \cos \theta_{ab} & -x_{0b} \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.117)$$

### 2.6.6.6 The Adjoint Map $\text{ad}_X$

As we have seen, there exists a commutator, or a Lie bracket, on every Lie algebra which maps two elements on the Lie algebra to itself. In this section we will look into how this transformation, denoted  $\text{ad}_X Y = [\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X}$  can be interpreted geometrically. This will lead us to the definition of another action on a Lie group, the *adjoint map*  $\text{ad}$ .

Close to the identity, a group element  $g \in G$  can be approximated by  $g = I + t\hat{X} + o(t^2)$  and its inverse by  $g^{-1} = I - t\hat{X} + o(t^2)$  where  $o(t^2)$  guarantees that  $g$  is still an element in  $G$ . Recall that conjugation of an element of the Lie algebra is given by  $g\hat{Y}g^{-1}$ . Substituting the expression for  $g$  gives

$$g\hat{Y}g^{-1} = (I + t\hat{X} + o(t^2))\hat{Y}(I - t\hat{X} + o(t^2)) = \hat{Y} + t(\hat{X}\hat{Y} - \hat{Y}\hat{X}) + o(t^2). \quad (2.118)$$

Differentiating and setting  $t = 0$  allows us to write this locally in terms of the Lie algebra element  $\hat{X}$  as (recall that  $\hat{Y}$  is fixed)

$$\frac{d}{dt}g\hat{Y}g^{-1} = \hat{X}\hat{Y} - \hat{Y}\hat{X} = [\hat{X}, \hat{Y}]. \quad (2.119)$$

Note that this is the derivative of the Adjoint map  $\text{Ad}_g$ , i.e., we define

$$\text{ad}_X Y = \frac{d}{dt}\text{Ad}_g Y. \quad (2.120)$$

The adjoint map  $\text{ad}$  can also be written in terms of the important Lie bracket:

**Definition 2.17** The adjoint map  $\text{ad}_X$  is defined as

$$\text{ad}_X Y = [\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X} \quad (2.121)$$

where  $[\cdot, \cdot]$  is the Lie bracket.

Again we note the slight abuse of notation because  $Y$  is a matrix on the left hand side, but a vector on the right. We will assume that it is implicitly understood that the adjoint map  $\text{ad}_X$  is given by an  $n \times n$ -matrix when acting on a vector and by the expression in Definition 2.17 when acting on a matrix. The two formulations will of course give us the same result.



The Lie bracket is thus an action of the Lie algebra on itself. We note that the Lie bracket is linear, i.e.,

$$[a\widehat{X}_1 + b\widehat{X}_2, \widehat{Y}] = a[\widehat{X}_1, \widehat{Y}] + b[\widehat{X}_2, \widehat{Y}]. \quad (2.122)$$

This means that all the elements of the Lie algebra can be found from the pairs of the basis elements of the same Lie algebra.

Equation (2.119) provides us with an intuition of the action imposed by the adjoint map  $\text{ad}_g$ : Given an element  $\widehat{X} \in \mathfrak{g}$ . This is an element of the Lie algebra and gives us a 1-parameter subgroup of  $G$  around the identity parameterized by  $t$ . Locally this is written as  $g(t) = I + t\widehat{X} + o(t^2)$ . The action of  $g$  on another element of the Lie group  $Y$  is given by  $\text{Ad}_g Y$  and transforms a (fixed) element of the Lie algebra  $Y$  by  $g$ . The time derivative of this transformation, denoted  $\text{ad}_X Y$ , thus gives us how  $Y$  changes with time as a result of the action imposed by  $X$ .

To get a deeper understanding of the geometrical interpretation of the Lie bracket, we first write the following corollary:

**Corollary 2.1** *For  $\widehat{X}, \widehat{Y} \in \mathfrak{g}$  and small  $t$  the following relation is true:*

$$e^{-t\widehat{Y}} e^{-t\widehat{X}} e^{t\widehat{Y}} e^{t\widehat{X}} = e^{t^2[\widehat{X}, \widehat{Y}] + o(t^3)}. \quad (2.123)$$

Equation (2.123) provides us with a geometric interpretation of the Lie bracket. Recall that  $\text{Ad}_g Y$  tells us how  $Y$  is transformed by an element of the Lie group  $g$ . The differential of this, namely  $\text{ad}_X Y$ , thus tells us how  $Y$  changes with  $X$  near the identity. Assume a curve given by  $e^{-t\widehat{Y}} e^{-t\widehat{X}} e^{t\widehat{Y}} e^{t\widehat{X}}$ , then the Lie bracket  $[\widehat{X}, \widehat{Y}]$  represents the tangent vector to this curve at the identity element as illustrated in Fig. 2.7. If the initial point on the manifold is  $x$ , the Lie bracket is given by

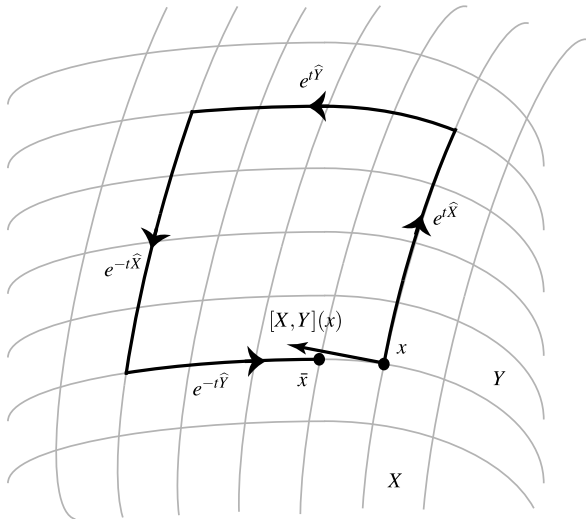
$$[X, Y](x) = \lim_{t \rightarrow 0^+} \frac{e^{-t\widehat{Y}} e^{-t\widehat{X}} e^{t\widehat{Y}} e^{t\widehat{X}} x - x}{t} \quad (2.124)$$

which is illustrated in Fig. 2.7. Exponential maps of this kind are discussed in more detail in Sect. 2.8.

We can also look at the Lie bracket as a measure of how  $X$  fails to commute with  $Y$ . More specifically, the Lie bracket tells us how elements of  $G$  near the identity in the direction of  $X$  fail to commute with elements in the direction of  $Y$ . For instance, if  $X$  and  $Y$  commute, then  $[X, Y] = 0$ . We also see that if  $[X, Y] \neq 0$  then this tells us that if we first follow the trajectories of  $X$  followed by  $Y$  we do not get the same result as if we first follow the trajectories of  $Y$  followed by  $X$ . To gain some more insight into the structure of the Lie bracket we include an example taken from Bullo and Lewis (2000).

**Example 2.10** Consider the vector fields  $X = \frac{\partial}{\partial y}$  and  $Y = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$  on a manifold  $\mathcal{M} = \mathbb{R}^3$ . The Lie bracket is then given by  $[X, Y] = \frac{\partial}{\partial z}$  which means that if we

**Fig. 2.7** A geometric interpretation of the Lie bracket  $[X, Y]$ . The path that takes us from an initial point  $x$  to a final point  $\bar{x} = e^{-t\hat{Y}}e^{-t\hat{X}}e^{t\hat{Y}}e^{t\hat{X}}x$ . The Lie bracket tells us how  $Y$  changes in the direction of  $X$



follow a flow first along  $X$ , then along  $Y$ , then along  $-X$  and finally along  $-Y$ , all for the same amount of time, then this motion will move us upwards in the direction of the  $z$ -axis.

We will now look at how we find the adjoint map  $\text{ad}_X$  for the most important groups.

### 2.6.6.7 Euclidean Space

The Lie algebra associated with the Euclidean space  $\mathbb{R}^n$  is  $\mathbb{R}^n$  itself. As this is commutative for all  $n$  the Lie bracket is trivially zero. The adjoint map  $\text{ad}_v$  corresponding to the Euclidean space of dimension  $n$  is thus an  $n \times n$  zero matrix.

### 2.6.6.8 The Special Orthogonal Group

The Lie algebra of the special orthogonal group can be written in terms of the basis elements

$$E_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.125)$$

We can find an adjoint representation of  $E_x$ ,  $E_y$ , and  $E_z$  in a very simple form. Take for example the adjoint action of  $E_x$  on a vector  $Y = [a \ b \ c]^T \in \mathbb{R}^3$ , then

$$\begin{aligned} E_x \widehat{Y} - \widehat{Y} E_x &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -b & -c \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix}. \end{aligned} \quad (2.126)$$

If we write the skew-symmetric matrix  $\widehat{Y}$  as a vector  $Y$  we can find the adjoint map  $\text{ad}_{E_x}$  through the relation

$$\text{ad}_{E_x} Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -c \\ b \end{bmatrix} \quad (2.127)$$

whose skew-symmetric form is the same as (2.126). The same can be shown for  $E_y$  and  $E_z$ , i.e.,

$$\text{ad}_{E_x} = E_x, \quad \text{ad}_{E_y} = E_y, \quad \text{ad}_{E_z} = E_z. \quad (2.128)$$

To find the adjoint map of an element  $\omega$  in  $so(3)$  we write  $\omega = [p \ q \ r]^T$ , and write the adjoint map as

$$\text{ad}_\omega = p \text{ad}_{E_x} + q \text{ad}_{E_y} + r \text{ad}_{E_z} = p E_x + q E_y + r E_z = \widehat{\omega} \quad (2.129)$$

where

$$\widehat{\omega} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}. \quad (2.130)$$

For  $SO(3)$  we thus have the rather special case that the adjoint representation  $\text{ad}_\omega$  equals the matrix representation of the Lie algebra  $\widehat{\omega}$  of the angular velocities. This property is only true for  $SO(3)$ , however, and cannot be taken as a general rule. This will be clear when we find the adjoint representation of an element of  $se(3)$ .

### 2.6.6.9 The Special Euclidean Group

The Lie algebra of the Special Euclidean group can be written in terms of the basis elements

$$\begin{aligned}
 \widehat{v}_x &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \widehat{v}_y &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 \widehat{v}_z &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \widehat{\omega}_x &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 \widehat{\omega}_y &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \widehat{\omega}_z &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
 \end{aligned} \tag{2.131}$$

For two twists  $X = [v_1^\top \ \omega_1^\top]^\top$  and  $Y = [v_2^\top \ \omega_2^\top]^\top$  we find

$$\widehat{X}\widehat{Y} - \widehat{Y}\widehat{X} = \begin{bmatrix} \widehat{\omega_1 \times \omega_2} & \omega_1 \times v_2 - \omega_2 \times v_1 \\ 0 & 0 \end{bmatrix}. \tag{2.132}$$

If we compare this to the vector representation of the adjoint map we will find the basis elements of the corresponding adjoint maps

$$\begin{aligned}
 \text{ad}_{v_x} &= \begin{bmatrix} 0 & E_x \\ 0 & 0 \end{bmatrix}, & \text{ad}_{v_y} &= \begin{bmatrix} 0 & E_y \\ 0 & 0 \end{bmatrix}, & \text{ad}_{v_z} &= \begin{bmatrix} 0 & E_z \\ 0 & 0 \end{bmatrix}, \\
 \text{ad}_{\omega_x} &= \begin{bmatrix} E_x & 0 \\ 0 & E_x \end{bmatrix}, & \text{ad}_{\omega_y} &= \begin{bmatrix} E_y & 0 \\ 0 & E_y \end{bmatrix}, & \text{ad}_{\omega_z} &= \begin{bmatrix} E_z & 0 \\ 0 & E_z \end{bmatrix}.
 \end{aligned}$$

Let  $V = [u \ v \ w \ p \ q \ r]^\top$  and write

$$\text{ad}_V = u \text{ad}_{v_x} + v \text{ad}_{v_y} + w \text{ad}_{v_z} + p \text{ad}_{\omega_x} + q \text{ad}_{\omega_y} + r \text{ad}_{\omega_z} \tag{2.133}$$

which gives us the adjoint map as

$$\text{ad}_V = \begin{bmatrix} 0 & -r & q & 0 & -w & v \\ r & 0 & -p & w & 0 & -u \\ -q & p & 0 & -v & u & 0 \\ 0 & 0 & 0 & 0 & -r & q \\ 0 & 0 & 0 & r & 0 & -p \\ 0 & 0 & 0 & -q & p & 0 \end{bmatrix}. \quad (2.134)$$

For a twist  $V_2$  the adjoint map  $\text{ad}_{V_1} V_2$  tells us how the twist  $V_2$  changes with  $V_1$  in the vicinity of the identity element.

### 2.6.6.10 Planar Motion

The basis elements of the planar group are given by

$$\widehat{v}_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widehat{v}_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widehat{\omega}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.135)$$

We can find an adjoint representation for the planar group similar to the ones that we found for  $SO(3)$  and  $SE(3)$ . The adjoint maps corresponding to each degree of freedom is given by

$$\text{ad}_{v_x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{ad}_{v_y} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{ad}_{\omega_z} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.136)$$

and we see that

$$\text{ad}_{v_x} = -\widehat{v}_y, \quad \text{ad}_{v_y} = \widehat{v}_x, \quad \text{ad}_{\omega_z} = \widehat{\omega}_z. \quad (2.137)$$

The adjoint representation is thus given by

$$\text{ad}_V = \begin{bmatrix} 0 & -r & v \\ r & 0 & -u \\ 0 & 0 & 0 \end{bmatrix} \quad (2.138)$$

where  $V = [u \ v \ r]^\top$ .

### 2.6.6.11 The Schönflies Group

The Schönflies group allows translation in three degrees of freedom and rotation around one of the coordinate axes, normally the  $z$ -axis. The Lie algebra is therefore

given by

$$\hat{X} = \begin{bmatrix} 0 & -\omega & 0 & v_x \\ \omega & 0 & 0 & v_y \\ 0 & 0 & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.139)$$

We can now use that  $\text{ad}_X Y = \hat{X}\hat{Y} - \hat{Y}\hat{X}$  to find the adjoint map. This is not normally shown in standard text books on robotics, so we show the derivation in some more detail. If we write  $X = [X_x \ X_y \ X_z \ X_\omega]^\top$  and  $Y = [Y_x \ Y_y \ Y_z \ Y_\omega]^\top$  we find the adjoint map as

$$\begin{aligned} \text{ad}_X Y &= \begin{bmatrix} 0 & -X_\omega & 0 & X_x \\ X_\omega & 0 & 0 & X_y \\ 0 & 0 & 0 & X_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -Y_\omega & 0 & Y_x \\ Y_\omega & 0 & 0 & Y_y \\ 0 & 0 & 0 & Y_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & -Y_\omega & 0 & Y_x \\ Y_\omega & 0 & 0 & Y_y \\ 0 & 0 & 0 & Y_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -X_\omega & 0 & X_x \\ X_\omega & 0 & 0 & X_y \\ 0 & 0 & 0 & X_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -X_\omega Y_\omega & 0 & 0 & -X_\omega Y_y \\ 0 & -X_\omega Y_\omega & 0 & X_\omega Y_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} -Y_\omega X_\omega & 0 & 0 & -Y_\omega X_y \\ 0 & -Y_\omega X_\omega & 0 & Y_\omega X_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & -X_\omega Y_y + Y_\omega X_y \\ 0 & 0 & 0 & X_\omega Y_x - Y_\omega X_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (2.140)$$

We can now extract the matrix  $\text{ad}_X$  as

$$\text{ad}_X = \begin{bmatrix} 0 & -X_\omega & 0 & X_y \\ X_\omega & 0 & 0 & -X_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.141)$$

because

$$\begin{bmatrix} -X_\omega Y_y + Y_\omega X_y \\ X_\omega Y_x - Y_\omega X_x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -X_\omega & 0 & X_y \\ X_\omega & 0 & 0 & -X_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_x \\ Y_y \\ Y_z \\ Y_\omega \end{bmatrix}. \quad (2.142)$$

The basis of the adjoint map  $\text{ad}$  for the Schönflies group is thus given by

$$\begin{aligned} \text{ad}_{v_x} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \text{ad}_{v_y} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \text{ad}_{v_z} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \text{ad}_\omega &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (2.143)$$

The adjoint map  $\text{ad}_X$  of the Schönflies group is normally written as

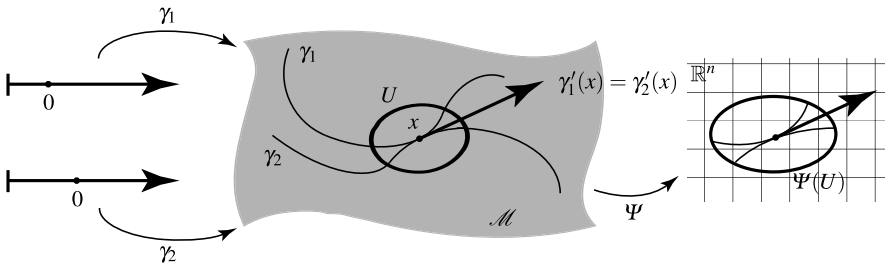
$$\text{ad}_X = \begin{bmatrix} 0 & -r & 0 & v \\ r & 0 & 0 & -u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.144)$$

## 2.7 Tangent Spaces, Vector Fields, and Integral Curves

When we are to derive the dynamic equations of a mechanical system we want to know what the state space of the system looks like. In this section we will look at the tangent bundle which associates a set of admissible velocities with each position that the mechanical system can take. We will see that it is not meaningful to talk about velocity without regard to the configuration: we need to define the velocity at a certain configuration of the mechanical system. Thus, at each point on the manifold we can assign a tangent vector. If we do this in a smooth way we obtain a vector field which gives us information about the flow, or motion, of the system. Finally, a specific motion, i.e., a specified initial condition and flow, allows us to find a curve for which the tangent vector corresponds to the vector field for every point on the curve. These curves are called integral curves.

### 2.7.1 Tangent Spaces

At any point  $x$  on a manifold  $\mathcal{M}$  we can identify a tangent vector as an equivalence relation between two paths that pass through  $x$ . This equivalence relation tells us



**Fig. 2.8** The definition of a tangent vector at a point  $x$  on a manifold  $\mathcal{M}$

that the first order derivative of the two paths are equal at  $x$ . Intuitively we can imagine a particle that moves along a path on a surface. Then, at every point in the path the velocity vector will be a tangent vector for that path (and possibly other paths that pass through the same point).

We will use the definition presented in Bullo and Lewis (2000) and illustrated in Fig. 2.8. Define a curve  $\gamma : \mathbb{R} \rightarrow \mathcal{M}$  as a mapping from an interval on  $\mathbb{R}$  which contains 0 to the manifold  $\mathcal{M}$  where  $\gamma(0) = x$ . Then two curves  $\gamma_1$  and  $\gamma_2$  are equivalent at  $x$  if, for some local coordinate system,  $\gamma_1$  and  $\gamma_2$  have the same derivative at 0. Intuitively this means that the two curves move in the same direction when they pass through the point  $x$ . The collection of all possible directions at  $x$  is called the *tangent space* at  $x$  and is denoted  $T\mathcal{M}_x$ .

We are mainly concerned with manifolds that are submanifolds of the Euclidean space: a sphere  $\mathbb{S}^2$  can for example be thought of as a submanifold of  $\mathbb{R}^3$ . In this case we can visualize the tangent space at each point as a plane tangent to the sphere at that point.

We can also define a tangent vector in a somewhat more intuitive way:

**Definition 2.18** Define  $\gamma : \mathbb{R} \rightarrow \mathcal{M}$  so that  $\gamma(t)$  is a curve in a manifold  $\mathcal{M}$ . The tangent vector  $\dot{\gamma}$  to the curve  $\gamma(t)$  at a point  $\gamma(t_0)$  is defined by

$$v = \dot{\gamma}(t_0) = \lim_{t \rightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \quad (2.145)$$

We see the definition of the tangent vector corresponds well with the classical definition of the velocity of a particle when  $\gamma$  is a function of time  $t$ .

At every point in  $\mathcal{M}$  we can define such a tangent space. The tangent bundle is the collection of all the tangent spaces at every point in the manifold (Kwatny and Blankenship 2000):



**Definition 2.19** (Tangent bundle) The union of all the tangent spaces to  $\mathcal{M}$  is called the *tangent bundle*  $T\mathcal{M}$  and is defined as

$$T\mathcal{M} = \bigcup_{x \in \mathcal{M}} T\mathcal{M}_x. \quad (2.146)$$

The tangent bundle plays a very important role in mechanics. A point in the tangent bundle  $T\mathcal{M}$  is a pair  $(x, v)$  where  $x \in \mathcal{M}$  and  $v \in T\mathcal{M}_x$ . We see that the velocity vector therefore is position dependent, i.e., it is taken from the tangent space at position  $x$ , i.e.,  $T\mathcal{M}_x$ . This is a manifold of dimension  $\dim T\mathcal{M} = 2 \dim \mathcal{M}$ . *The tangent bundle is normally referred to as the state space in mechanics.* We can write the local coordinates on  $\mathcal{M}$  as  $(x_1, x_2, \dots, x_n)$  and the components of the tangent vectors on  $T\mathcal{M}_x$  as  $(v_1, v_2, \dots, v_n)$ . The coordinates of the tangent bundle completely describe the state space of the system in coordinates  $(x_1, x_2, \dots, x_n, v_1, v_2, \dots, v_n)$ .

Given the local coordinates as a chart  $(U, \Psi)$  we get the following important result (Kwatny and Blankenship 2000):

**Definition 2.20** The components of the tangent vector  $v$  to the curve  $\gamma(t)$  on  $\mathcal{M}$  in the local coordinate chart  $(U, \Psi)$  are the numbers  $v_1, v_2, \dots, v_m$  where

$$v_i = \frac{d\Psi_i}{dt}. \quad (2.147)$$

We will study in great detail what these velocity variables look like for different mechanical systems.

### 2.7.2 Vector Fields, Flows, and Integral Curves

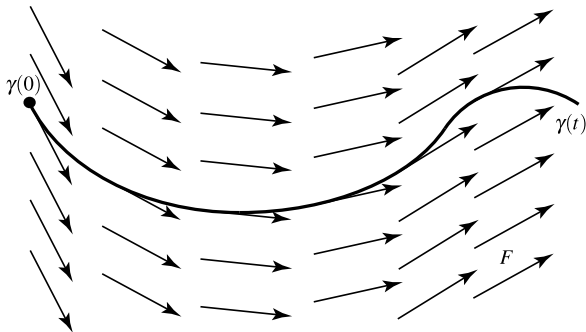
A vector field tells us in what direction our system moves at each position. Using the concept of tangent spaces, a vector field smoothly assigns a tangent vector to each point on the manifold. The vector field thus presents us with a solution to an ordinary differential equation for a given initial condition. Figure 2.9 shows a vector field  $F$  and the solution to the corresponding differential equation with initial condition  $\gamma(0)$ .

The solution  $\gamma(t)$  to the differential equation is called an integral curve:

**Definition 2.21** A differentiable curve  $\gamma(t)$  is called an integral curve at  $x \in \mathcal{M}$  for a vector field  $F$  if

$$\dot{\gamma}(t) = F(\gamma(t)), \quad \gamma(0) = x. \quad (2.148)$$

**Fig. 2.9** A vector field  $F$  and the solution to a differential equation with initial condition  $\gamma(0)$



We see that the vector field tells us what the tangent vector looks like for the curve  $\gamma(t)$  at each point on the curve (Fig. 2.10).

A vector field is a mapping  $F : \mathcal{M} \rightarrow T\mathcal{M}$  that for every point  $x \in \mathcal{M}$  assigns a tangent vector  $F(x) \in T\mathcal{M}$ . A Lie algebra element is a tangent vector at the identity and we can therefore find a one-to-one correspondence between the Lie algebra and a vector field. In order to do this we need to define a tangent vector at every point on the manifold. This can be obtained by right translating the tangent vector at the identity to every element of the group. We write the group elements as  $g$  and a corresponding tangent vector at  $g$  as  $\hat{X}g$ . This follows directly from Eq. (2.78). In this case we have  $F(g) = R_g \hat{X} = \hat{X}g$  and the integral curve is thus found by solving the differential equation

$$\frac{dg(t)}{dt} = \hat{X}g(t). \quad (2.149)$$

The solution to this differential equation is very important and will be treated in the next section when we look at the exponential map.

Recall that the tangent space at a point  $g(t)$  in a Lie group  $G$ , denoted  $\dot{g}(t)$  is transformed back to the identity, i.e., to the Lie algebra  $\hat{X}$ , by

$$\hat{X} = \dot{g}(t)g(t)^{-1}. \quad (2.150)$$

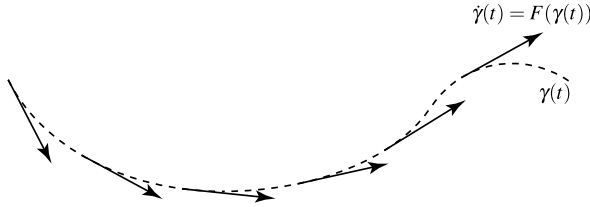
The Lie algebras are in this case interpreted as spatial velocity variables. If we instead use left translation the Lie algebra is in the form  $\hat{X} = g^{-1}\dot{g}$  and interpreted as body frame velocities.

*Example 2.11* An element of the Lie algebra  $so(2)$  can be represented as

$$\hat{\omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in so(2). \quad (2.151)$$

A differentiable path  $R(t)$  can be written in the normal way as

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \in SO(2). \quad (2.152)$$



**Fig. 2.10** The derivative of the integral curve is given by the vector field as  $\dot{\gamma}(t) = F(\gamma(t))$

For a point  $y_0$  we can study the velocity of this point at time  $t = 0$  due to the vector field by looking at the Lie algebra. The instantaneous velocity of this point is given by  $L_X y_0 = \hat{X} y_0$ . A point on the positive  $y$ -axis represented by  $y_0 = [0 \ 1]^T$  will for example move to the left at time  $t = 0$  because

$$L_X y_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \quad (2.153)$$

However, because  $y(t) = L_{R(t)} y_0$  is an integral curve of the vector field  $L_X, \hat{X}$  tells us how  $y$  moves for all time, not just at  $t = 0$ . This vector field is given by  $L_X R(t)$ :

$$F(R(t)) = L_X R(t) y_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}. \quad (2.154)$$

We see that if we associate  $y_0$  with the identity  $R = I$  at time  $t = 0$  we can find the tangent vector to the curve  $R(t)$  at any time  $t$  by Eq. (2.154). In Fig. 2.11 this is illustrated for  $t = \frac{\pi}{2}$  which corresponds to a clockwise rotation of  $\frac{\pi}{2}$  radians around the origin.

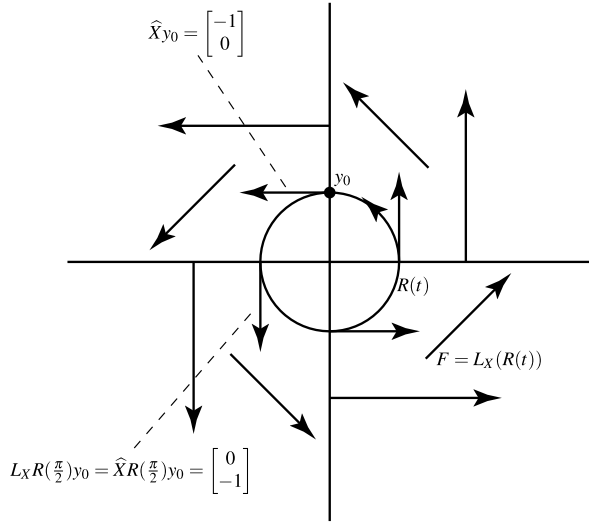
## 2.8 The Exponential Map

One of the most important tools that we will use when deriving the kinematics is the fact that there exists an onto map from the Lie algebra to the Lie group. In the context of robotics the exponential map was introduced by Brockett (1984) who derived the forward kinematics of a robotic manipulator with 1-DoF joints using the exponential map. In Chap. 4 we will show how to use the exponential map to write the forward kinematics of a robotic manipulator in terms of the joint twists.

We have seen that an integral curve of a vector field  $F(g)$ , where  $g$  is a point on the manifold  $\mathcal{M}$ , is a curve  $\gamma(t)$  for which  $\gamma(0) = I$  and  $\dot{\gamma}(0) = X$ . In other words, the derivative at  $t = 0$  is the Lie algebra  $X$ . In this section we will look at the solution to differential equations in the form

$$\dot{g}(t) = F(g(t)) \quad (2.155)$$

**Fig. 2.11** The vector field  $F = L_X(R(t))$  and an integral curve  $R(t)$ . The Lie algebra  $\hat{X} \in \mathfrak{so}(2)$  tells us what the tangent vector to the integral curve looks like for all  $t$



where  $g(t)$  is a path on a manifold  $\mathcal{M}$  and  $F(g(t)) : \mathcal{M} \rightarrow T\mathcal{M}$  is a vector field in the form  $F(g(t)) = L_X g(t)$ . We will study the exponential map which presents us with a solution to the initial value problem on left invariant vector fields of the Lie algebras.

We will use the exponential map extensively when deriving the dynamics of multibody systems. There are two main interpretations of the exponential map. First of all the exponential map allows us to map an element of the Lie algebra to the corresponding Lie group. Thus, for a given twist, there is a neighborhood around 0 in the Lie algebra for which the exponential map maps this twist homeomorphically to a neighborhood of the identity in the Lie group. The Lie algebra therefore gives us much information about how the Lie group behaves locally.

We can also think of the exponential mapping as a solution to a left invariant vector field. The solution to differential equations of the form  $\dot{g}(t) = L_X g(t)$  form 1-parameter subgroups that can be described in terms of the Lie algebra. These paths are often called optimal paths and have the nice property that  $g(t)$  is the integral curve of the vector field  $F(g(t)) = L_X g(t)$ .

### 2.8.1 The Exponential of a Matrix

For the group of  $n \times n$  nonsingular real matrices  $GL(n)$  the exponential map is given by the power series of the Lie algebra elements (Murray et al. 1994). Let  $A \in \mathfrak{gl}(n)$ , where  $\mathfrak{gl}(n)$  is the Lie algebra associated with  $GL(n)$ . Then the exponential map

$\exp(A)$  is given by

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad (2.156)$$

where  $I$  is the identity matrix. This expression is valid for all subgroups of  $SE(3)$  and  $SE(3)$  itself by replacing  $A$  with the matrix representation of the Lie algebra associated with the Lie group. We denote the matrix representation of the corresponding Lie algebra by  $\hat{X}$  and thus get

$$e^{\hat{X}} = I + \hat{X} + \frac{\hat{X}^2}{2!} + \frac{\hat{X}^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{\hat{X}^n}{n!}. \quad (2.157)$$

We found  $\hat{X}$  for the different Lie groups in Sect. 2.6.4. Later in this section we will look at what the exponential map looks like for the most important Lie groups in robotics. We will first look at how we can use the exponential map to solve differential equations and how these present us with a solution to the natural paths on the Lie groups.

### 2.8.2 Differential Equations

Assume a standard homogeneous differential equation in the form

$$\frac{dy}{dt} = ky \quad (2.158)$$

where  $k$  is a constant and  $y$  is the variable. We know that the solution to this differential equation is given by

$$y(t) = y(0)e^{kt} \quad (2.159)$$

where  $y(0)$  is the initial condition. We can see this by substituting this expression back into (2.158):

$$\frac{dy}{dt} = \frac{d}{dt}(y(0)e^{kt}) = y(0)\frac{de^{kt}}{dt} = y(0)ke^{kt} = k(y(0)e^{kt}) = ky. \quad (2.160)$$

Similarly, if  $y$  is a vector we have

$$\frac{dy}{dt} = Ky \quad (2.161)$$

where  $K$  is a matrix and  $y$  the vector of variables, we get the solution

$$y(t) = e^{Kt}y(0) \quad (2.162)$$

where  $e^{Kt} = Ae^{At}A^{-1}$  for a matrix  $A$  with the Eigenvalues and a matrix  $A$  with the Eigenvectors of  $K$  (Jordan and Smith 2004).

We will see that we can find a similar solution when the underlying space is a general manifold.

### 2.8.3 The Natural Path in a Matrix Group

The following proposition is valid for any Lie group  $\mathfrak{g}$  and will play an important role in the remaining of the book (Tapp 2005):

**Proposition 2.1** *Let  $X \in \mathfrak{g}$  and  $\gamma(t) = e^{t\hat{X}}$ .*

1. *For all  $y_0 \in \mathbb{R}^n$ ,  $y(t) = L_{\gamma(t)}y_0$  is an integral curve of  $L_X$ .*
2.  *$\gamma(t)$  is itself an integral curve of the vector field on  $G$ . The value of the vector field at  $g$  is  $g\hat{X}$ .*

Part 1 is illustrated in Fig. 2.11. We need to choose  $\gamma(t)$  so that  $y(t) = L_{\gamma(t)}y_0$  is an integral curve of the vector field  $L_X$ . In other words, we need  $y(0) = y_0$  to satisfy the initial condition and  $\dot{y}(t) = L_X y(t)$  for  $y(t)$  to be an integral curve of  $L_X$ . Following the approach in Tapp (2005) we can show this by writing the path  $y(t)$  by the power series

$$y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots \quad (2.163)$$

and its derivative as

$$\dot{y}(t) = c_1 + 2c_2 t + 3c_3 t^2 + \dots \quad (2.164)$$

Because  $\dot{y}(t) = L_X y(t)$  we get

$$\begin{aligned} (c_1 + 2c_2 t + 3c_3 t^2 + \dots) &= \hat{X}(c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots) \\ &= (\hat{X}c_0 + \hat{X}c_1 t + \hat{X}c_2 t^2 + \hat{X}c_3 t^3 + \dots) \end{aligned} \quad (2.165)$$

We first note that  $y(0) = L_{\gamma(0)}y_0 = y_0$  and (2.163) then gives us  $c_0 = y(0) = y_0$ . From (2.165) we then find  $c_1 = \hat{X}y_0$  so that the terms with no time variables are correct. Next, the terms with  $t$  give us  $c_2 = \frac{1}{2}\hat{X}^2 y_0$ . Continuing in this way gives us the recursive formula  $c_i = \frac{1}{i!}\hat{X}^i c_{i-1}$ . If we now write the integral curve in terms of these  $c_i$ 's we get

$$y(t) = y_0 + \hat{X}t y_0 + \frac{1}{2!}(\hat{X}t)^2 y_0 + \frac{1}{3!}(\hat{X}t)^3 y_0 + \dots \quad (2.166)$$

and we have found an expression for the integral curve in terms of the Lie algebra  $X$ . The final step is to recognize that this power series can be written as an exponential map of the form

$$\begin{aligned}
 y(t) &= y_0 + \widehat{X}t y_0 + \frac{1}{2!}(\widehat{X}t)^2 y_0 + \frac{1}{3!}(\widehat{X}t)^3 y_0 + \cdots \\
 &= \left( I + \widehat{X}t + \frac{1}{2!}(\widehat{X}t)^2 + \frac{1}{3!}(\widehat{X}t)^3 + \cdots \right) y_0 \\
 &= e^{\widehat{X}t} y_0 \\
 &= L_{e^{\widehat{X}t}} y_0.
 \end{aligned} \tag{2.167}$$

We have thus shown that for  $\gamma(t) = e^{t\widehat{X}}$ , then  $y(t) = L_{\gamma(t)} y_0$  is an integral curve of  $L_X$ .

The second part of Proposition 2.1 says that also  $\gamma(t)$  is an integral curve of a vector field with values  $g\widehat{X}$  at  $g$ . For this to be true we need  $\gamma(0) = I$  and  $\dot{\gamma}(0) = \widehat{X}$ , and also that  $\dot{\gamma}(t) = L_g \widehat{X}$ . First write

$$\gamma(t) = e^{\widehat{X}t} = I + \widehat{X}t + \frac{1}{2!}(\widehat{X}t)^2 + \frac{1}{3!}(\widehat{X}t)^3 + \cdots \tag{2.168}$$

and

$$\dot{\gamma}(t) = \frac{d}{dt} e^{\widehat{X}t} = \widehat{X} + \widehat{X}^2 t + \frac{1}{2!} \widehat{X}^3 t^2 + \cdots \tag{2.169}$$

which gives  $\gamma(0) = I$  and  $\dot{\gamma}(0) = \widehat{X}$  as required. We also see that if we factor  $\widehat{X}$  out on the right we get

$$\begin{aligned}
 \dot{\gamma}(t) &= \left( I + \widehat{X}t + \frac{1}{2!}(\widehat{X}t)^2 + \cdots \right) \widehat{X} \\
 &= \gamma(t) \widehat{X}
 \end{aligned} \tag{2.170}$$

which shows that  $\gamma(t)$  is an integral curve to the vector field generated by the Lie algebra  $X$ .

*Remark 2.5* We note that we could also have factored  $\widehat{X}$  out on the left hand side in Eq. (2.170). This is a solution to the differential equation  $\dot{\gamma} = R_g \widehat{X}$ . Even though the vector fields  $L_X$  and  $R_X$  are in general different, they agree along the trajectory of  $\gamma(t)$ .

What we have shown here is that a curve  $\gamma(t) = e^{\widehat{X}t}$  is actually a 1-parameter subgroup of  $G$ , i.e., the exponential mapping maps  $\widehat{X}t \in \mathfrak{g}$  to a 1-parameter subgroup  $\gamma(t) = e^{\widehat{X}t} \in G$ . More specifically it maps it to the 1-parameter subgroups that are tangent to  $X$  at  $I$  (because  $\dot{\gamma}(0) = X$ ). These 1-parameter subgroups play

a very important role in the derivation of the dynamic equations of mechanical systems such as robotic manipulators.

We are now ready to present the following important theorem:

**Theorem 2.1** *If  $X \in \mathfrak{g}$ , then  $e^{\hat{X}t} \in G$ .*

The exponential map is thus a mapping from the Lie algebra  $\hat{X}$  to the corresponding Lie group. We will not prove this for a general Lie group, but refer to Tapp (2005) for the proof. In the following sections we will, however, show that this is true for the most important Lie groups and find the explicit expressions whenever possible.

### 2.8.4 Rigid Body Motion in Terms of Exponential Coordinates

Recall that a point  $\bar{q}^a(0)$  can be transformed into another point by a homogeneous transformation  $g(t, X)$  by

$$\bar{q}^a(t) = g(t, X)\bar{q}^a(0) \quad (2.171)$$

where  $\bar{q}^a(0)$  is the point before and  $\bar{q}^a(t)$  is the point after the transformation, both expressed in the same coordinate frame  $\mathcal{F}_a$ . For a constant twist, this transformation can be written in terms of exponential coordinates as

$$\bar{q}^a(t) = e^{\hat{X}t}\bar{q}^a(0). \quad (2.172)$$

We have seen that if  $\hat{X} \in se(3)$ , then  $e^{\hat{X}t} \in SE(3)$ , so this expression is in the form of (2.171). Equation (2.172) represents a 1-parameter transformation of  $\bar{q}^a(0)$  and can for example give us information about how a one degree of freedom revolute or translational joint acts on a point or a vector.

Similarly, we can look at the transformation of a rigid body with initial position  $g_{0b}(0)$  transformed by  $g(t, X)$  as in (2.70):

$$g_{0b}(t) = g(t, X)g_{0b}(0). \quad (2.173)$$

Also this transformation can be written in terms of exponential coordinates as

$$g_{0b}(t) = e^{\hat{X}t}g_{0b}(0). \quad (2.174)$$

If for example  $g_{0b}(0)$  denotes the position of a rigid body, then  $g_{0b}(t)$  denotes the position after a 1-parameter transformation  $e^{\hat{X}t}$ . We note that both  $g_{0b}(0)$  and  $g_{0b}(t)$  are represented in the same inertial reference frame  $\mathcal{F}_0$  so Eq. (2.174) can be interpreted as a rigid body motion. We will see that we can derive the kinematics of robots using exponential coordinates in this way.



### 2.8.5 The Exponential Map of the Most Important Lie Groups

The exponential map is used to represent motion in a fixed direction, i.e., for a given Lie algebra. In this section we will look at what this motion looks like for the most important Lie groups. We will see that these motions are 1-parameter subgroups of  $SE(3)$ .

#### 2.8.5.1 The Euclidean Group $\mathbb{R}^n$

We have seen that the Lie algebra of  $\mathbb{R}^n$  is  $\mathbb{R}^n$  itself. The integral curve of the constant vector field  $F(x) = v$  is thus given by  $\gamma(t) = vt$ . For the Euclidean group  $\mathbb{R}^n$  we can find the solution to the exponential map by simple reasoning. From Eq. (2.174) we see that the exponential map gives us the transformation by an element  $\hat{X}$  of the Lie algebra by  $t$  units of time given an initial condition  $g_{ob}(0)$ . Correspondingly, for the Euclidean group we write the elements of the Lie group as  $x(t)$  with initial condition  $x(0)$  and the elements of the Lie algebra as  $v$ . We then know that this transformation is given by

$$x(t) = x(0) + vt \quad (2.175)$$

for the Euclidean group with addition as the group operator and a constant vector field  $F(x) = v$ .

#### 2.8.5.2 The Special Orthogonal Group $SO(2)$

For rotational motion in the plane the exponential map can be written as in (2.175). This is because, even though  $\mathbb{S}^1$  and  $\mathbb{R}$  strictly speaking are not topologically the same, we can treat 1-DoF rotations as Euclidean. We saw this in Sect. 2.5. The exponential map of  $SO(2)$  is thus the same as for  $\mathbb{R}^n$ .

#### 2.8.5.3 The Special Orthogonal Group $SO(3)$

For rotational motion in the three dimensional Euclidean space we have the following important result:

**Theorem 2.2** *Given an element  $\hat{\omega} \in so(3)$ . Then the exponential map is given by*

$$e^{\hat{\omega}t} = I + \hat{\omega} \sin t + \hat{\omega}^2 (1 - \cos t) \quad (2.176)$$

for  $\|\omega\| = 1$  and

$$e^{\hat{\omega}t} = I + \frac{\hat{\omega}}{\|\omega\|} \sin \|\omega\|t + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos \|\omega\|t) \quad (2.177)$$

for  $\|\omega\| \neq 1$ . Equation (2.176) is normally referred to as Rodrigues' formula.

Following the proof found in Murray et al. (1994) we will show how we arrive at Rodrigues' formula when  $\|\omega\| = 1$ . First write

$$e^{\widehat{\omega}t} = I + t\widehat{\omega} + \frac{t^2}{2!}\widehat{\omega}^2 + \frac{t^3}{3!}\widehat{\omega}^3 + \dots \quad (2.178)$$

It can be shown (see Murray et al. 1994) that for  $\widehat{\omega} \in so(3)$ , this can be written as

$$e^{\widehat{\omega}t} = I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} \dots\right)\widehat{\omega} + \left(\frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} \dots\right)\widehat{\omega}^2. \quad (2.179)$$

If we use the relations  $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} \dots$  and  $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \dots$  we get an expression for the exponential map on  $so(3)$  given by

$$e^{\widehat{\omega}t} = I + \widehat{\omega} \sin t + \widehat{\omega}^2 (1 - \cos t). \quad (2.180)$$

The exponential map of an element of  $so(3)$  produces an element of  $SO(3)$ :

**Theorem 2.3** For  $\widehat{\omega} \in so(3)$  and  $t \in \mathbb{R}$  we have

$$e^{\widehat{\omega}t} \in SO(3). \quad (2.181)$$

In other words, the exponential map of  $\widehat{\omega}$  gives us a rotation matrix that corresponds to  $t$  units of rotation around the axis  $\omega$ . To prove Theorem 2.3 we follow the approach in Murray et al. (1994) and show that  $e^{\widehat{\omega}t}$  satisfies the group properties  $R^T R = I$  and  $\det R = 1$ . First note that

$$(e^{\widehat{\omega}t})^{-1} = e^{-\widehat{\omega}t} = e^{\widehat{\omega}^T t} = (e^{\widehat{\omega}t})^T. \quad (2.182)$$

This shows that  $R^{-1} = R^T$  and we have shown that  $R^T R = I$ . Further, we know that the determinant of a matrix exponential is given simply by the exponential of the trace of the matrix, i.e.,  $\det e^{\widehat{\omega}} = e^{\text{Tr}(\widehat{\omega})}$ . As the trace of  $\widehat{\omega} \in so(3)$  is zero, the determinant of  $e^{\widehat{\omega}t}$  equals 1.  $e^{\widehat{\omega}t}$  is therefore an element of  $SO(3)$ . Finally we note that the exponential map on  $so(3)$  is surjective.

Using the expression in (2.172) we can use the exponential map to transform a point corresponding to a 1-parameter motion represented by a rotation of  $t$  radians around the axis  $\omega$  which is then in the form

$$q^a(t) = e^{\widehat{\omega}t} q^a(0) = R(\omega, t) q^a(0) \quad (2.183)$$

where  $R(\omega, t)$  is the rotation matrix that represents a rotation of  $t$  radians around  $\omega$ .

#### 2.8.5.4 The Special Euclidean Group $SE(3)$

We can find a theorem for  $SE(3)$  similar to Theorem 2.2 for  $SO(3)$ :

**Theorem 2.4** *Given an element  $V = [v^\top \ \omega^\top]^\top \in se(3)$ . Then the exponential map is given by*

$$e^{\widehat{V}t} = \begin{bmatrix} I & vt \\ 0 & 1 \end{bmatrix}, \quad \omega = 0 \quad (2.184)$$

$$e^{\widehat{V}t} = \begin{bmatrix} e^{\widehat{\omega}t} & (I - e^{\widehat{\omega}t})(\omega \times v) + \omega \omega^\top vt \\ 0 & 1 \end{bmatrix}, \quad \omega \neq 0 \quad (2.185)$$

where  $e^{\widehat{\omega}t}$  is given as in (2.176) and we have assumed  $\|\omega\| = 1$ .

We refer to Tapp (2005) or Murray et al. (1994) for the proofs. Also for the Special Euclidean group it can be shown that the exponential map of an element of  $se(3)$  produces an element of  $SE(3)$ :

**Theorem 2.5** *For  $\widehat{V} \in se(3)$  and  $t \in \mathbb{R}$  we have*

$$e^{\widehat{V}t} \in SE(3). \quad (2.186)$$

The proof follows directly by observing that (2.184) and (2.185) are elements of  $SE(3)$ . This is a very important result as it allows us to derive the kinematics of multibody systems in terms of exponential coordinates by Eq. (2.174).

Again, using the expression in (2.172) we can use the exponential map to transform a point corresponding to a 1-parameter motion represented by the constant twist  $V$  which is then given by

$$\bar{q}^a(t) = e^{\widehat{V}t} \bar{q}^a(0) = g(V, t) \bar{q}^a(0) \quad (2.187)$$

where  $g(V, t)$  is the homogeneous transformation matrix that corresponds to a 1-parameter motion of  $t$  units in the direction of  $V$ .

### 2.8.5.5 The Planar Group $SE(2)$

For consistency we include the following theorem from  $SE(2)$ :

**Theorem 2.6** *For  $\widehat{X} \in se(2)$  and  $t \in \mathbb{R}$  we have*

$$e^{\widehat{X}t} \in SE(2) \quad (2.188)$$

which is the homogeneous transformation matrix in the plane.

This theorem follows from Theorem 2.4 by selecting the basis  $\widehat{v}_x$ ,  $\widehat{v}_y$ , and  $\widehat{\omega}_z$  corresponding to  $SE(2)$  in (2.131).

### 2.8.5.6 The General Linear Group $GL(n)$

For the General Linear Group with a corresponding Lie algebra  $X \in \mathfrak{g}$  the exponential map is given by the series expansion

$$e^{\hat{X}t} = \sum_{i=0}^{\infty} \frac{t^i \hat{X}^i}{i!}. \quad (2.189)$$

We note that the expressions found for  $SO(3)$ ,  $SE(3)$  and  $SE(2)$  are specific formulas of this general expression. It is straight forward to show that  $e^{t\hat{X}}$  is an integral curve of the vector field  $L_X$ :

$$\frac{d}{dt}e^{\hat{X}t} = \sum_{i=1}^{\infty} \frac{t^{i-1} \hat{X}^i}{(i-1)!} = \sum_{i=1}^{\infty} \frac{t^{i-1} \hat{X}^{i-1}}{(i-1)!} \hat{X} = \sum_{i=0}^{\infty} \frac{t^i \hat{X}^i}{i!} \hat{X} = e^{\hat{X}t} \hat{X}. \quad (2.190)$$

We also have the following general result:

**Theorem 2.7** For  $\hat{X} \in \mathfrak{gl}(n)$  and  $t \in \mathbb{R}$  we have

$$e^{\hat{X}t} \in GL(n) \quad (2.191)$$

where  $GL(n)$  is the general linear group and  $\mathfrak{gl}(n)$  the corresponding Lie algebra.

### 2.8.6 Charts and Exponential Maps

As we have seen, the exponential map can be interpreted as a mapping from the Lie algebra to the corresponding Lie group, e.g., the mapping  $\exp : \mathfrak{se}(3) \rightarrow SE(3)$ . There is, however, a slightly different interpretation of the exponential map which leads to a different way of utilizing the results from the previous section. Recall that a chart  $\Psi$  maps an element on the manifold, for example  $SE(3)$ , onto the Euclidean space with the same dimension. In this section we will look at the inverse of this mapping, i.e., the mapping  $\Phi$  which maps an element of  $\mathbb{R}^n$ —which in our case are the exponential coordinates—to an element of the Lie group. As the Lie algebra also lives on the Euclidean space it is thus natural to ask whether we can use the exponential map as a mapping from  $\mathbb{R}^n$  (where  $\mathbb{R}^n$  denotes the Euclidean space in the neighborhood of some point) to the manifold, for example  $SE(3)$ . From the previous section it is quite intuitive that the transformation represented by a quantity  $t$  in the direction of a Lie algebra  $X$  can be *interpreted as a displacement*, and not only as an element of the Lie algebra.

We will denote local coordinates in the Euclidean space as  $\varphi \in \mathbb{R}^n$ . These local coordinates are the exponential coordinates that we found in the previous section, just interpreted a little differently. We interpret these coordinates as tiny displacements in the neighborhood of some point on  $\mathbb{R}^n$ . From (2.168) we see that we can

interpret  $vt$  as position coordinates for small  $t$ . We can therefore replace  $vt$  with  $\varphi$  in the exponential map. The exponential map thus provides us with a transformation between local and global variables. The local variables are represented in  $\mathbb{R}^n$  and the global variables are defined on the manifold itself.

Recall that the time derivative of an element of a Lie group does not have a physical meaningful interpretation. That is, we cannot simply write  $\dot{g}(t)$  to get the velocity that corresponds to a path  $g(t)$ . However, as the local position variables are written as  $\varphi \in \mathbb{R}^n$  the time derivative can be written simply as  $\dot{\varphi} = \frac{d\varphi}{dt}$ . Because  $\varphi \in \mathbb{R}^n$  this is a physically meaningful quantity and gives us the local velocity variables. We can then use this to find a relation between the local and global velocity variables.

A transformation can be uniquely described by a matrix  $Q \in GL(n)$ . We have seen several examples of such matrices: a rotation matrix ( $Q = R$ ) can uniquely describe a ball joint, a homogeneous transformation matrix ( $Q = g$ ) can represent unconstrained rigid body motion in  $\mathbb{R}^3$  and a scalar ( $Q = \mathbb{R}$ ) can represent a 1-dimensional linear or revolute transformation. Let's assume that  $Q$  is constant and denote this by  $\bar{Q}$ . The exponential map allows us to express the dynamics in exponential coordinates  $\varphi$  so that locally every state  $\bar{Q}$  is described by a set of Euclidean coordinates  $\varphi \in \mathbb{R}^n$ . Thus, in the neighborhood of  $\bar{Q}$  there exist a function  $\Phi(\bar{Q}, \varphi)$  that defines a local diffeomorphism between a neighborhood of  $0 \in \mathbb{R}^n$  and a neighborhood of  $\bar{Q}$ .  $\bar{Q}$  is locally described by  $Q = \Phi(\bar{Q}, \varphi)$  with  $\Phi(\bar{Q}, 0) = \bar{Q}$ . Using the results from the previous section, the mapping  $\Phi(\bar{Q}, \varphi)$  can be expressed in terms of the exponential map.

In this way the position of a rigid body in space is given globally by a matrix Lie group  $Q$ , but locally around an element  $\bar{Q}$  it is also described by the local coordinates  $\varphi \in \mathbb{R}^n$ . Also the velocity variables can be defined both locally and globally. Locally the velocity variables are given simply by  $\dot{\varphi} = \frac{d\varphi}{dt}$  and globally by a vector  $v \in \mathbb{R}^n$ . The vector  $v$  uniquely describes the joint twist as  $V_{0b}^B = H(Q)v$  where  $H(Q)$  is called a selection matrix. The selection matrix maps the  $n$ -dimensional velocity vector to a 6-dimensional twist. The term *selection matrix* arises because in most cases  $H(Q) = H$  consists of only ones and zeros, and selects what components of the twist the elements of the vector  $v$  correspond to.

We will now look at some examples of how we can use the coordinate charts to describe the neighborhood of a point  $\bar{Q}$  on a manifold in terms of local coordinates  $\varphi$ .

### 2.8.6.1 The Euclidean Space $\mathbb{R}$

For the Euclidean space the allowed velocity is given as an element of the tangent space of the Lie group and is uniquely described by a vector  $v \in \mathbb{R}^n$ . For Euclidean 1-DoF transformations, such as translational and rotational joints, we have  $Q \in \mathbb{R}$  and  $v = \dot{Q} \in \mathbb{R}$ . The coordinate mapping is given by  $\Phi(\bar{Q}, \varphi) = \bar{Q} + \varphi$  with  $\varphi \in \mathbb{R}$ . We see that if we replace  $tv$  with  $\varphi$ —which we can do because both are

representations of displacements—this corresponds to the exponential map of  $\mathbb{R}^n$  that we found in the previous section.

Also other transformations can be written in this way, for example pure translational motion in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . In this case we get  $Q \in \mathbb{R}^n$  and  $v = \dot{Q} \in \mathbb{R}^n$ . The coordinate mapping is given by

$$\Phi(\bar{Q}, \varphi) = \bar{Q} + \varphi \quad (2.192)$$

with  $\varphi \in \mathbb{R}^n$ .

### 2.8.6.2 Matrix Lie Groups $GL(n)$

An important group of transformations consists of all transformations with a Lie group topology.  $\Phi$  is then given by the exponential map, i.e.

$$\Phi(\bar{Q}, \varphi) = \bar{Q} e^{\sum_{i=1}^n b_i \varphi_i} \quad (2.193)$$

where  $b_i$  represents the basis elements of the Lie algebra found in (2.131).  $n$  is the dimension of the manifold and the basis elements determine in what directions we allow displacements. For example, for pure rotational motion  $b_i$  represents the three axes that we can rotate around and  $\varphi_i$  determines the size of the rotation in the direction of  $b_i$  in the normal way. In this case we have  $b_1 = \hat{\omega}_x$ ,  $b_2 = \hat{\omega}_y$ , and  $b_3 = \hat{\omega}_z$ .

Equation (2.193) can be interpreted as first a transformation  $g_{0\bar{Q}}$  from the inertial frame to the configuration represented by  $\bar{Q}$  followed by a transformation  $g_{\bar{Q}Q}$  from  $\bar{Q}$  to  $Q$ . This transformation is only valid locally, i.e., in the neighborhood of  $\bar{Q}$  and can therefore be written in terms of the local coordinates  $\varphi$ . For  $SE(3)$  Eq. (2.193) can thus be written as

$$g_{0Q} = g_{0\bar{Q}} g_{\bar{Q}Q}. \quad (2.194)$$

$g_{0Q}$  is a homogeneous transformation matrix  $g_{0Q} \in SE(3)$  and we can find the spatial velocities from

$$\hat{v}_{0Q}^S = \dot{g}_{0Q} g_{0Q}^{-1} = \begin{bmatrix} \hat{\omega}_{0Q}^S & v_{0Q}^S \\ 0 & 0 \end{bmatrix}. \quad (2.195)$$

More generally, for a general manifold with a matrix representation  $Q$  we can write

$$\hat{v}_{0b}^B = \Phi^{-1}(\bar{Q}, \varphi) \dot{\Phi}(\bar{Q}, \varphi) \quad (2.196)$$

in body coordinates, or in spatial coordinates as

$$\hat{v}_{0b}^S = \dot{\Phi}(\bar{Q}, \varphi) \Phi^{-1}(\bar{Q}, \varphi). \quad (2.197)$$

## 2.9 Local Coordinates and Velocity Transformation Matrices

We have seen that  $SE(3)$  is topologically different from the Euclidean space and that it is not possible to continuously and globally cover it using six coordinates. As a direct result of this, there is no way to write the configuration of a rigid body as a vector in  $\mathbb{R}^6$  in a way that is globally valid. There are, however, several representation methods that describe  $SE(3)$  either only locally continuously using six numbers, or globally continuously using more than six numbers.

In this section we will describe the location of a rigid body locally using vectors in  $\mathbb{R}^6$  and we will see how we can use the relation in (2.197) to find a velocity transformation matrix in terms of these local coordinates. This will give us the global velocity variables in terms of the local velocity variables in  $\mathbb{R}^6$  and the velocity transformation matrix which depends only on the local position variables. This velocity transformation matrix is important when we are to derive the dynamics of the system in the subsequent chapters.

### 2.9.1 Velocity Transformation Matrices in Dynamics

In general, the topology of a Lie group is not Euclidean. When deriving the dynamic equations for vehicles such as ships (Fossen 2002), AUVs (Antonelli 2006), and spacecraft (Hughes 2002), this is normally dealt with by introducing a transformation matrix that relates the velocity variables represented in the different frames. However, forcing the dynamics into a vector representation in this way, without taking the topology of the configuration space into account, leads to singularities in the representation. To preserve the topology of the configuration space we will use quasi-coordinates, i.e., velocity coordinates that are not given by the time-derivative of position coordinates, but by a linear relation. Thus, there exist differentiable matrices  $S_i$  such that we can write  $v_i = S_i(Q_i, \varphi_i)\dot{\varphi}_i$  for every  $Q_i$ . For Euclidean joints this relation is given simply by the identity map while for joints with a Lie group topology we can use the exponential map to derive this relation.

We will represent the configuration of the multibody system as a set of configuration states  $Q = \{Q_i\}$ . The configuration state  $Q_i$  of joint  $i$  is then the matrix representation of the Lie group corresponding to the topology of the joint. The corresponding block  $S_i(Q_i, \varphi_i)$  relating the velocity variables is well known from the Lie theory and can be found in terms of the Lie bracket or the exponential map (Rossmann 2002). For standard revolute and prismatic joints  $Q_i$  becomes a scalar  $Q_i = q_i$  and  $S_i = 1$  while for joints or transformations with a Lie group topology,  $Q_i$  is the matrix Lie group. We will now look at what form  $S_i(Q_i, \varphi_i)$  will take in this case.

### 2.9.2 The Velocity Transformation Matrix in Terms of Exponential Coordinates

We have seen that the velocity transformation matrix, i.e., the mapping from the velocity variables to the time derivative of the position variables is not always well defined. In this section we will show that we can rewrite the expression in (2.197) to find a matrix in this form. We will start by stating the following important theorem:

**Proposition 2.2** *It is always possible to find a matrix  $S(\bar{Q}, \varphi)$  such that*

$$v = S(\bar{Q}, \varphi)\dot{\varphi}, \quad (2.198)$$

*i.e., there is a linear relation between  $v$  and  $\dot{\varphi}$ .*

*Proof* (Duindam 2006) Note that when  $g(\bar{Q}, \varphi)$  is in the form of (2.193), i.e.,  $\bar{Q}$  is constant, then the time derivative of the transformation  $g(\bar{Q}, \varphi)$  is a function  $\dot{g}(\bar{Q}, \varphi, \dot{\varphi})$  which is linear in  $\dot{\varphi}$ . There is also a linear relation from  $\dot{g}(\bar{Q}, \varphi, \dot{\varphi})$  to the twist, given by the conjugation  $\hat{V}_{ab}^c = g_{ca}\dot{g}_{ab}g_{bc}$ . If we now write the vector  $v$  in terms of the twist as  $V_{ab}^c = H(\bar{Q})v$  we conclude that there exists a linear relation between  $v$  and  $\dot{\varphi}$ .  $\square$

We will see several examples of typical expressions for the relation in (2.198). For the cases we are interested in we can find simplified expressions for this velocity transformation map. In general we will use a series representation of the exponential map. This is because, when we are to derive the dynamics later on, we will not use the exponential coordinates themselves, but rather their differential properties. After differentiating we will also evaluate the functions at  $\varphi = 0$ . We see that if we differentiate the map given in (2.193) and evaluate at  $\varphi = 0$  all higher order terms of  $\varphi$  vanish. From this point of view a series representation is easier to work with than analytical expressions. We will now look at what the expression in (2.193) looks like for the most important Lie groups.

#### 2.9.2.1 The Euclidean Space $\mathbb{R}^n$

We note that for a Euclidean transformation the Jacobian is given simply by setting  $S(Q, \varphi) = I$ . We can see this by simple inspection or from (2.192). Equation (2.198) thus simplifies to

$$v = \dot{\varphi}. \quad (2.199)$$



### 2.9.2.2 The Special Orthogonal Group $SO(3)$

For  $SO(3)$  the exponential map of an element  $\widehat{\varphi} \in so(3)$  is given in the normal way as

$$e^{\widehat{\varphi}} = I + \widehat{\varphi} + \frac{1}{2!}\widehat{\varphi}^2 + \frac{1}{3!}\widehat{\varphi}^3 + o(\varphi^4) \quad (2.200)$$

where  $o(\varphi^n)$  collects the terms with order  $n$  or higher. The time derivative is given by

$$\begin{aligned} \frac{d}{dt}e^{\widehat{\varphi}} &= \frac{d}{dt}I + \frac{d}{dt}\widehat{\varphi} + \frac{1}{2!}\frac{d}{dt}\widehat{\varphi}^2 + \frac{1}{3!}\frac{d}{dt}\widehat{\varphi}^3 + o(\varphi^3) \\ &= \dot{\widehat{\varphi}} + \frac{1}{2}(\widehat{\varphi}\dot{\widehat{\varphi}} + \dot{\widehat{\varphi}}\widehat{\varphi}) + \frac{1}{6}(\dot{\widehat{\varphi}}\widehat{\varphi}^2 + \widehat{\varphi}\dot{\widehat{\varphi}}\widehat{\varphi} + \widehat{\varphi}^2\dot{\widehat{\varphi}}) + o(\varphi^3) \end{aligned} \quad (2.201)$$

and the inverse is given by

$$\begin{aligned} (e^{\widehat{\varphi}})^{-1} &= (e^{-\widehat{\varphi}}) = I + (-\widehat{\varphi}) + \frac{1}{2!}(-\widehat{\varphi})^2 + \frac{1}{3!}(-\widehat{\varphi})^3 + o(\varphi^4) \\ &= I - \widehat{\varphi} + \frac{1}{2!}\widehat{\varphi}^2 - \frac{1}{3!}\widehat{\varphi}^3 + o(\varphi^4). \end{aligned} \quad (2.202)$$

We can now write the *body frame* angular velocities in (2.196) as

$$\begin{aligned} \widehat{\omega}_{0b}^B &= \Phi^{-1}(\bar{Q}, \varphi) \dot{\Phi}(\bar{Q}, \varphi) = (\bar{Q}e^{\widehat{\varphi}})^{-1} \frac{d}{dt}(\bar{Q}e^{\widehat{\varphi}}) = (e^{\widehat{\varphi}})^{-1} \bar{Q}^T \bar{Q} \frac{d}{dt}(e^{\widehat{\varphi}}) \\ &= \left( I - \widehat{\varphi} + \frac{1}{2}\widehat{\varphi}^2 + o(\varphi^3) \right) \\ &\quad \times \left( \dot{\widehat{\varphi}} + \frac{1}{2}(\widehat{\varphi}\dot{\widehat{\varphi}} + \dot{\widehat{\varphi}}\widehat{\varphi}) + \frac{1}{6}(\dot{\widehat{\varphi}}\widehat{\varphi}^2 + \widehat{\varphi}\dot{\widehat{\varphi}}\widehat{\varphi} + \widehat{\varphi}^2\dot{\widehat{\varphi}}) + o(\varphi^3) \right) \\ &= \dot{\widehat{\varphi}} + \frac{1}{2}(\widehat{\varphi}\dot{\widehat{\varphi}} + \dot{\widehat{\varphi}}\widehat{\varphi}) - \widehat{\varphi}\dot{\widehat{\varphi}} - \frac{1}{2}(\widehat{\varphi}^2\dot{\widehat{\varphi}} - \dot{\widehat{\varphi}}\widehat{\varphi}^2) + \frac{1}{2}(\widehat{\varphi}\dot{\widehat{\varphi}} + \dot{\widehat{\varphi}}\widehat{\varphi}) \\ &\quad + \frac{1}{6}(\dot{\widehat{\varphi}}\widehat{\varphi}^2 + \widehat{\varphi}\dot{\widehat{\varphi}}\widehat{\varphi} + \widehat{\varphi}^2\dot{\widehat{\varphi}}) + \frac{1}{2}\widehat{\varphi}\dot{\widehat{\varphi}}\widehat{\varphi} + o(\varphi^3) \\ &= \dot{\widehat{\varphi}} - \frac{1}{2}(\widehat{\varphi}\dot{\widehat{\varphi}} - \dot{\widehat{\varphi}}\widehat{\varphi}) + \frac{1}{6}(\dot{\widehat{\varphi}}\widehat{\varphi}^2 - 2\widehat{\varphi}\dot{\widehat{\varphi}}\widehat{\varphi} + \widehat{\varphi}^2\dot{\widehat{\varphi}}) + o(\varphi^3). \end{aligned} \quad (2.203)$$

We will use that for two vectors  $a, b \in \mathbb{R}^3$  we have the relation  $\widehat{ab} = \widehat{a}\widehat{b} - \widehat{b}\widehat{a}$  so that the second term is simplified by

$$(\widehat{\varphi}\dot{\widehat{\varphi}} - \dot{\widehat{\varphi}}\widehat{\varphi}) = \widehat{\dot{\varphi}\varphi} \quad (2.204)$$

and the body angular velocity can now be written in matrix form as

$$\widehat{\omega}_{0b}^B = \dot{\widehat{\varphi}} - \frac{1}{2}\widehat{\dot{\varphi}\varphi} + o(\varphi^2) \quad (2.205)$$

which can also be written in vector form as

$$\omega_{0b}^B = \left( I - \frac{1}{2}\widehat{\varphi} + o(\varphi^2) \right) \dot{\varphi}. \quad (2.206)$$

We do not calculate the higher order terms because these vanish when we differentiate and evaluate the expressions at  $\varphi = 0$ . The velocity transformation matrix relating the time derivative of the exponential coordinates and the body frame angular velocities can now be written in terms of the exponential coordinates as

$$S(\varphi) = I - \frac{1}{2}\widehat{\varphi} + o(\varphi^2). \quad (2.207)$$

We note that this expression does not depend on the current configuration  $\bar{Q}$ , which is reasonable because the transformation represents a mapping to the *body* velocity twist.

### 2.9.2.3 The Special Euclidean Group $SE(3)$

For the special Euclidean group the exponential coordinates  $\varphi \in \mathbb{R}^6$  are chosen using the standard basis so that the first three coordinates represent translation and the last three rotational motion. We can find a formulation for the special Euclidean group similar to the one that we found for the special orthogonal group by using the adjoint representation  $\text{ad}_\varphi$ . In fact the twist written in the body frame for  $SE(3)$  given by (2.196) becomes

$$V_{0b}^B = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)!} \text{ad}_\varphi^i \dot{\varphi}, \quad (2.208)$$

which, if we write the first three terms explicitly, becomes

$$V_{0b}^B = \left( I - \frac{1}{2}\text{ad}_\varphi + \frac{1}{6}\text{ad}_\varphi^2 + o(\varphi^3) \right) \dot{\varphi}. \quad (2.209)$$

We will not show it here, but these expressions can be found in the same way as we did for the special orthogonal group. We refer to Rossmann (2002) for the proofs.

## 2.10 Geometric Integrators

Numerical integration methods are normally designed to give us the evolution of a differential equation whose configuration space is a vector space  $\mathbb{R}^n$ . These approaches cannot be directly applied to cases where the configuration space is a curved manifold because there is no guarantee that we stay on the manifold. When the domain is a Lie group, for example, it is important that we can guarantee that the

computed solution remains in the same group. Geometric integration techniques are a group of integrators that guarantee that the computed solution stays on a certain manifold. There will always be a numerical error, but this is an error on the manifold itself.

There are two important families of geometric integrators: embedded and intrinsic integrators (Munthe-Kaas 1998). Firstly, embedded integrators require the manifold to be embedded in  $\mathbb{R}^n$  and integrates using a standard integrator such as Euler's method or Runge-Kutta. The problem with embedding the manifold in  $\mathbb{R}^n$  is that these classical integration method do not in general stay on the right manifold, and the computed solution therefore has to be projected onto the manifold after each iteration.

Intrinsic methods, on the other hand, compute the flow and therefore guarantee that we stay on the manifold. This is normally performed by computing the Lie algebra at each iteration step and then find the computed solution by the exponential map (Iserles 1984). We have already seen that the exponential map maps an element of the Lie algebra to the Lie group, and we can therefore conclude that we do not leave the manifold.

An early and comprehensive discussion on geometric integrators is found in Crouch and Grossman (1993) and a more recent treatment in McLachlan and Quispel (2006). An overview of different methods is given in Engø and Marthinsen (1997) and Hairer (2001). Geometric integrators in the context of multibody systems is discussed in Park and Chung (2005) and the special case of attitude dynamics of a rigid body is discussed in Lee et al. (2005, 2011).

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