

# Chapter 2

## Pseudo-Gradients

In this chapter, we define, construct and study pseudo-gradient fields, whose trajectories connect the critical points of a Morse function. These vector fields allow us to define the stable and unstable manifolds of the critical points, which will play an important role. We call attention to the “Smale property” because of which, for example, there are only finitely many trajectories connecting two critical points with consecutive indices and we prove the existence of pseudo-gradient fields satisfying this property.

### 2.1 Gradients, Pseudo-Gradients and Morse Charts

#### 2.1.a Gradients and Pseudo-Gradients

If  $f$  is a function defined on  $\mathbf{R}^n$ , then we are familiar with its *gradient*, the vector field  $\text{grad } f$ , whose coordinates in the canonical basis of  $\mathbf{R}^n$  are

$$\text{grad}_x f = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

More succinctly, it is (also) the vector field defined by

$$\langle \text{grad}_x f, Y \rangle = (df)_x(Y)$$

for every vector  $Y \in \mathbf{R}^n$  (where, of course, the angle brackets  $\langle \cdot, \cdot \rangle$  denote the usual Euclidean inner product in  $\mathbf{R}^n$ ). The most important properties of this vector field are due to the fact that this inner product is a *positive definite* symmetric bilinear form:

- (1) It vanishes exactly at the critical points of the function  $f$ .

(2) The function  $f$  is decreasing along the flow lines of the field  $-\text{grad } f$ :

$$\frac{d}{ds}(f(\varphi^s(x))) = -\|\text{grad}_{\varphi^s(x)} f\|^2 < 0.$$

**Remark 2.1.1.** More generally, if the manifold  $V$  is endowed with a Riemannian metric (see Section A.5), then a function on  $V$  has a gradient defined by the same formula,

$$\langle \text{grad}_x f, Y \rangle = (df)_x(Y)$$

for every  $Y \in T_x V$ .

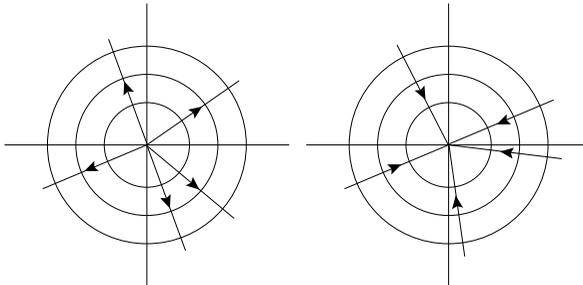
**Remark 2.1.2.** There is no doubt that the conjunction of the terms “flow” and “height” is at the source (if we dare say so) of the (recent) tradition to use the negative gradient: the flow moves downward. We have conformed to this convention, as do [66] and [49], but not [46].

Let  $f : V \rightarrow \mathbf{R}$  be a Morse function on a manifold  $V$ . A *pseudo-gradient field* or *pseudo-gradient* adapted to  $f$  is a vector field  $X$  on  $V$  such that:

- (1) We have  $(df)_x(X_x) \leq 0$ , where equality holds if and only if  $x$  is a critical point.
- (2) In a Morse chart in the neighborhood of a critical point,  $X$  coincides with the negative gradient for the canonical metric on  $\mathbf{R}^n$ .

**Remark 2.1.3.** A pseudo-gradient is a rather particular vector field. See Exercise 10 on p. 51 for vector fields that are not pseudo-gradients.

### 2.1.b Morse Charts



**Fig. 2.1** Maximum and minimum

This notion allows us to make the Morse charts more precise by specifying the trajectories of a pseudo-gradient field  $X$ . For example, Figure 2.1 finally

shows the difference between a maximum (on the left) and a minimum (on the right).

Figure 2.2 shows a Morse chart for a critical point of index  $i$ . The chart or, strictly speaking, the model in  $\mathbf{R}^n$ , appears on the left. On the right we can see its image in the manifold (the function is the height). Let us fix some notation. In  $\mathbf{R}^n$ , the quadratic form  $Q$  is negative definite on  $V_-$ , a subspace of dimension  $i$ , and positive definite on  $V_+$ . We set

$$U(\varepsilon, \eta) = \{x \in \mathbf{R}^n \mid -\varepsilon < Q(x) < \varepsilon \text{ and } \|x_-\|^2 \|x_+\|^2 \leq \eta(\varepsilon + \eta)\}.$$

Since we are in  $\mathbf{R}^n$ , the function  $Q$  has a gradient, namely

$$-\text{grad}_{(x_-, x_+)} Q = 2(x_-, -x_+).$$

The boundary of  $U(\varepsilon, \eta)$  is made up of three parts:

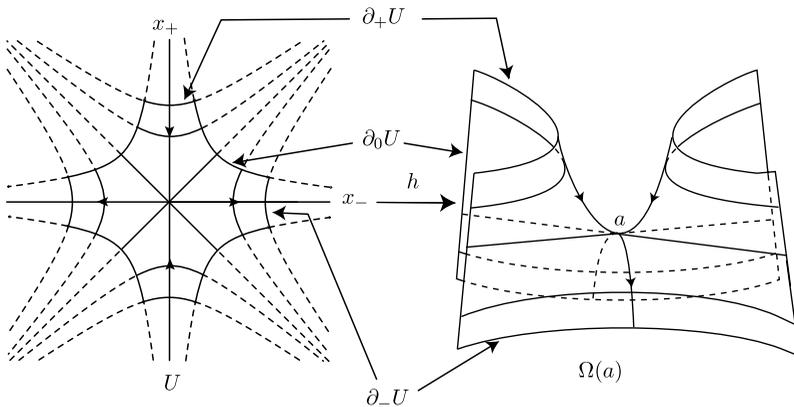
- Two subsets of the sublevel sets of  $Q$ :

$$\partial_{\pm} U = \{x \in U \mid Q(x) = \pm\varepsilon \text{ and } \|x_{\mp}\|^2 \leq \eta\}$$

- A set of pieces of trajectories of the gradient  $\text{grad} Q$ :

$$\partial_0 U = \{x \in \partial U \mid \|x_-\|^2 \|x_+\|^2 = \eta(\varepsilon + \eta)\}.$$

See Figure 2.2. The notation we use here is that of [46].



**Fig. 2.2** A Morse chart

**Remark 2.1.4.** By the definition of the gradient in  $\mathbf{R}^n$ , the trajectories of the gradient are orthogonal to the level sets of the function (the tangent space

to a level set is the kernel of the differential of the function). It is therefore not a coincidence that on the model shown in Figure 2.2, they show up as equilateral hyperbolas from two orthogonal families.

Taking a closer look at the two parts of the figure proves to be useful. In both parts, we have indicated the critical level set (let us say that it is level 0), two positive regular level sets and two negative regular level sets and trajectories of the gradient that delimit the chart, as well as trajectories of the negative gradient that end or begin at the critical point.

If  $a$  is a critical point of the function  $f$ , then a neighborhood of  $a$  is described by the Morse lemma as the image of some  $U(\varepsilon, \eta)$  under a diffeomorphism  $h$ . We will denote such a neighborhood by  $\Omega(a)$ . We will also write

$$\partial_{\pm}\Omega(a) = h(\partial_{\pm}U), \quad \partial_0\Omega = h(\partial_0U),$$

etc. We will try to use the following notation consistently, as in Figure 2.2:

- $\Omega$  for the images of charts ( $\Omega$  is a subset of the manifold)
- $U$  for the chart domains, the models ( $U$  is a subset of  $\mathbf{R}^n$ ).

### 2.1.c Existence of Pseudo-Gradient Fields

Pseudo-gradient fields exist for all Morse functions on all manifolds. This is, for example, a consequence of the existence of Riemannian metrics and, more exactly, of the existence of Riemannian metrics with a prescribed form on a given subset of the manifold (a neighborhood of the critical points). In any case, it is a simple consequence of the existence of partitions of unity, as we will show now.

*Proof of the Existence of Pseudo-Gradients.* Let  $c_1, \dots, c_r$  be the critical points of  $f$  on the manifold  $V$  (there are finitely many because  $V$  is compact and the critical points of a Morse function are isolated) and let  $(U_1, h_1), \dots, (U_r, h_r)$  be Morse charts in the neighborhoods of these points. The open images  $\Omega_j$  are, of course, assumed to be disjoint. We add more sets and obtain a finite open cover  $(\Omega_j)_{1 \leq j \leq N}$  of  $V$  by images of open sets of charts  $(U_j, h_j)$ . We may, and do, assume that in this cover, the critical point  $c_i$  is contained only in the open set  $\Omega_i$ .

For a function  $g$  defined on an open subset of  $\mathbf{R}^n$ , we let  $\text{grad } g$  be its gradient for the standard Euclidean metric on  $\mathbf{R}^n$ . For every index  $j$ , we define a vector field  $X_j$  on the open set  $\Omega_j$  by pulling back the gradient of  $f \circ h_j$  to  $V$ , that is, by the formula

$$X_j(x) = -(T_{h_j^{-1}(x)}h_j)(\text{grad}_{h_j^{-1}(x)}(f \circ h_j))$$

(the formula may seem complicated, but the object itself is very simple: we have taken the gradient of the function  $f \circ h_j$ , a vector field on  $U_j$ , and have transformed it in a natural manner into a vector field on  $\Omega_j$ ). By the very definition,  $X_j \cdot f \leq 0$  on  $\Omega_j$  (this is one of the properties of the negative gradient). Moreover,  $X_j$  vanishes only at the critical point of  $f$  on  $U_j$  (for  $j \leq r$ ).

Next, we use a partition of unity  $(\varphi_j)_j$  associated with the cover  $(\Omega_j)_j$  to extend the local vector fields  $X_j$  to vector fields  $\tilde{X}_j$  defined on all of  $V$ , setting

$$\tilde{X}_j(x) = \begin{cases} \varphi_j(x)X_j(x) & \text{if } x \in \Omega_j \\ 0 & \text{otherwise.} \end{cases}$$

The last step is to set

$$X = \sum_{j=1}^N \tilde{X}_j.$$

As expected, the resulting vector field  $X$  is a pseudo-gradient adapted to  $f$ , since we indeed have

$$(df)_x(X_x) = \sum_{j=1}^N (df)_x((\tilde{X}_j)_x) \leq 0.$$

If this inequality is an equality, then  $\varphi_j(x)X_j(x) = 0$  for every  $j$ , so that either  $x$  is a critical point, or  $\varphi_j(x) = 0$  for every  $j$  (which is absurd).

Let  $c_i$  be one of the critical points of  $f$ . By construction,  $X$  coincides with the image of the Euclidean gradient on the complement in  $U_i$  of the union of the other open sets  $U_j$ , a complement that is a neighborhood of  $c_i$  that contains a small Morse chart in the neighborhood of  $c_i$ .  $\square$

### 2.1.d Stable and Unstable Submanifolds

Let  $a$  be a critical point of  $f$ . Denote by  $\varphi^s$  the flow of a pseudo-gradient. We define its stable manifold to be

$$W^s(a) = \left\{ x \in V \mid \lim_{s \rightarrow +\infty} \varphi^s(x) = a \right\}$$

and its unstable manifold to be

$$W^u(a) = \left\{ x \in V \mid \lim_{s \rightarrow -\infty} \varphi^s(x) = a \right\}.$$

In an open subset  $U = U(\varepsilon, \eta)$  of  $\mathbf{R}^n$  as in Section 2.1.b, we have

$$W^s(0) = U \cap V_+ \quad \text{and} \quad W^u(0) = U \cap V_-.$$

In the notation of Subsection 2.1.b, the stable manifold of  $a$  is obtained from the union of  $h(U \cap V_+) = h(W^s(0))$  and

$$h(\partial_+U \cap V_+) \times \mathbf{R}$$

by identifying  $(x, s)$ , for  $x$  on the boundary and  $s \geq 0$ , with  $\varphi^s(x)$ . See Figure 2.3. If  $k$  is the index of the critical point  $a$ , then  $h(\partial_+U \cap V_+)$  is a sphere of dimension  $n - k - 1$ ; it is the image under the diffeomorphism  $h$  of the sphere  $\|x_+\|^2 = \varepsilon$  in  $V_+$  (vector space of dimension  $n - k$ ).

Hence the stable manifolds and, likewise, the unstable manifolds, are submanifolds: outside of the critical point, the stable manifold is the image of the embedding  $(x, s) \mapsto \varphi^s(x)$  and in the neighborhood of the critical point, it is the image of  $V_+$ . This argument also shows that  $W^s(a)$  is diffeomorphic to the disk of dimension  $n - k$  (and likewise that  $W^u(a)$  is diffeomorphic to the disk of dimension  $k$ ):  $W^s(a)$  can be obtained by compactifying  $S^{n-k-1} \times \mathbf{R}$  by adding the unique “point at infinity”  $a$ . It is also the quotient of  $S^{n-k-1} \times ]-\infty, +\infty]$  by the equivalence relation that identifies the sphere  $S^{n-k-1} \times \{+\infty\}$  with a unique point.

We have proved the following result.

**Proposition 2.1.5.** *The stable and unstable manifolds of the critical point  $a$  are submanifolds of  $V$  that are diffeomorphic to open disks. Moreover, we have*

$$\dim W^u(a) = \operatorname{codim} W^s(a) = \operatorname{Ind}(a). \quad \square$$

Here,  $\operatorname{Ind}(a)$  denotes the index of the point  $a$  as a critical point of  $f$ .

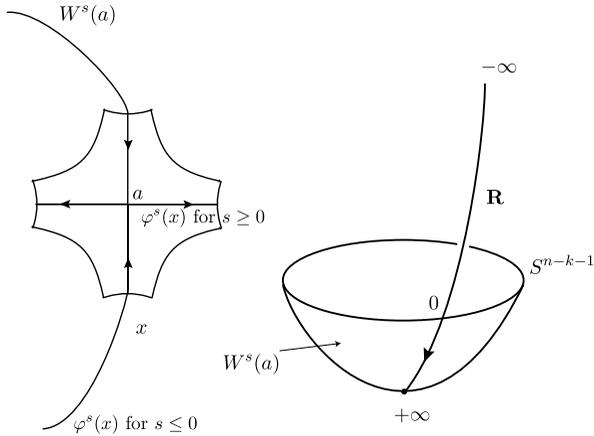
### *Trajectories of the Pseudo-Gradient Field.*

Let  $\varphi_X^s$  or  $\varphi^s$  denote the flow of the pseudo-gradient  $X$ . The most important property of the flow lines or trajectories of the vector field  $X$  is that they all connect critical points of the function  $f$ : all trajectories come from a critical point and go toward another critical point.

**Proposition 2.1.6.** *We suppose that the manifold  $V$  is compact. Let  $\gamma : \mathbf{R} \rightarrow V$  be a trajectory of the pseudo-gradient field  $X$ . Then there exist critical points  $c$  and  $d$  of  $f$  such that*

$$\lim_{s \rightarrow -\infty} \gamma(s) = c \quad \text{and} \quad \lim_{s \rightarrow +\infty} \gamma(s) = d.$$

*Proof.* Let us show that  $\gamma(t)$  has a limit when  $t$  tends to  $+\infty$  (for example) and that this limit is a critical point. We must prove that  $\gamma(t)$  reaches  $S_+(d) = \partial_+\Omega(d) \cap W^s(d)$  for some critical point  $d$  of  $f$ . We suppose that this is not



**Fig. 2.3** Stable manifolds

true. Then every time that the trajectory  $\gamma$  enters a Morse neighborhood, it must also leave it without ever being able to return to it since  $f$  is decreasing along  $\gamma$ . Let  $s_0$  be the time at which  $\gamma$  leaves the (finite) union of the Morse charts of the critical points, that is,

$$\Omega = \bigcup_{c \in \text{Crit}(f)} \Omega(c)$$

(of course,  $\text{Crit}(f)$  denotes the set of critical points of the function  $f$ ) for the last time. There exists an  $\varepsilon_0 > 0$  such that

$$\forall x \in V - \Omega, \quad (df)_x(X_x) \leq -\varepsilon_0.$$

Consequently, for every  $s \geq s_0$ , we have

$$\begin{aligned} f(\gamma(s)) - f(\gamma(s_0)) &= \int_{s_0}^s \frac{d(f \circ \gamma)}{du} du \\ &= \int_{s_0}^s (df)_{\gamma(u)}(X_{\gamma(u)}) du \\ &\leq -\varepsilon_0(s - s_0), \end{aligned}$$

so that

$$\lim_{s \rightarrow +\infty} f(\gamma(s)) = -\infty,$$

which is absurd. □

### 2.1.e Topology of the Sublevel Sets: When We Do Not Cross a Critical Value

The topology of the level sets does not change as long as we do not cross a critical value. The same holds for that of the sublevel sets. Let

$$V^a = f^{-1}(]-\infty, a])$$

denote the sublevel set of  $f$  for  $a$ . In Subsection A.2.c, we said that if  $a$  is a regular value, then  $V^a$  is a manifold with boundary.

**Theorem 2.1.7.** *Let  $a$  and  $b$  be two real numbers such that  $f$  does not have any critical value in the interval  $[a, b]$ . We suppose that  $f^{-1}([a, b])$  is compact. Then  $V^b$  is diffeomorphic to  $V^a$ .*

*Proof.* We use the flow of a pseudo-gradient  $X$  to retract  $V^b$  onto  $V^a$ . We fix a function  $\rho : V \rightarrow \mathbf{R}$  with values

$$\begin{cases} -\frac{1}{(df)_x(X)} & \text{on } f^{-1}([a, b]) \\ 0 & \text{outside of a compact neighborhood of this subset.} \end{cases}$$

The vector field  $Y = \rho X$  is zero outside of a compact set, so that its flow  $\psi^s$  is defined for every  $s \in \mathbf{R}$ . For a fixed point  $x \in V$ , we consider the function  $s \mapsto f \circ \psi^s(x)$  in one real variable. If  $\psi^s(x) \in f^{-1}([a, b])$ , then we have

$$\begin{aligned} \frac{d}{ds} f \circ \psi^s(x) &= (df)_{\psi^s(x)} \left( \frac{d}{ds} \psi^s(x) \right) \\ &= (df)_{\psi^s(x)} (Y_{\psi^s(x)}) \\ &= -1. \end{aligned}$$

Hence, for  $\psi^s(x) \in f^{-1}([a, b])$ , we have

$$f \circ \psi^s(x) = -s + f(x).$$

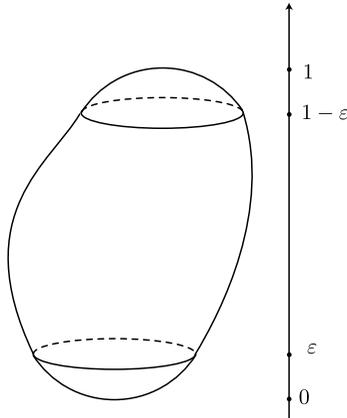
It follows that the diffeomorphism  $\psi^{b-a}$  of  $V$  sends  $V^b$  onto  $V^a$ . □

**Remark 2.1.8.** The map

$$\begin{aligned} r : V^b \times [0, 1] &\longrightarrow V^b \\ (x, s) &\longmapsto \begin{cases} x & \text{if } f(x) \leq a \\ \psi^{s(f(x)-a)}(x) & \text{if } a \leq f(x) \leq b \end{cases} \end{aligned}$$

is a deformation retraction of  $V^b$  onto  $V^a$ , that is,  $r_0 = \text{Id}$ ,  $r_t$  equals the inclusion on  $V^a$  for all  $t$  and  $r_1$  has image  $V^a$ . Consequently  $V^a$  is a deformation retract of  $V^b$ .

**Corollary 2.1.9 (Reeb’s theorem).** *Let  $V$  be a compact manifold. Suppose that there exists a Morse function on  $V$  that has only two critical points. Then  $V$  is homeomorphic to a sphere.*



**Fig. 2.4** Reeb’s theorem

*Proof.* The two critical points must be the minimum and the maximum. We may, and do, assume that  $f(V) = [0, 1]$ . Then for  $\varepsilon > 0$  sufficiently small, the Morse lemma asserts that  $f^{-1}([0, \varepsilon])$  and  $f^{-1}([1 - \varepsilon, 1])$  are disks  $D^n$ . By Theorem 2.1.7, the sublevel sets  $V^\varepsilon$  and  $V^{1-\varepsilon}$  are diffeomorphic. Hence  $V^{1-\varepsilon}$  is also a disk  $D^n$  and  $V$  is the union of two disks glued along their boundaries. It is a classic result that  $V$  is then homeomorphic to a sphere: we can construct an explicit map

$$h : D_1^n \cup_{\text{Id}} D_2^n \longrightarrow D_1^n \cup_\varphi D_2^n$$

by setting

$$h(x) = \begin{cases} x & \text{if } x \in D_1^n \\ \|x\| \varphi(x/\|x\|) & \text{if } x \in D_2^n - \{0\} \\ 0 & \text{if } x = 0 \in D_2^n. \end{cases}$$

This is a homeomorphism from the standard sphere (where the two disks are glued via the identity map on  $S^{n-1}$ ) onto our manifold (where the two disks are glued via the diffeomorphism  $\varphi$ ). □

**Remarks 2.1.10.** First, the theorem remains true if the two critical points are not assumed to be nondegenerate. Second, it is not true that the manifold  $V$  is *diffeomorphic* to a sphere; this is even an important argument in the construction of manifolds that are homeomorphic to the sphere without being diffeomorphic to it. For these two results, see the references in the book [54, p. 25].

### 2.1.f Topology of the Sublevel Sets: When We Cross a Critical Value

Both the topology of the level set and that of the sublevel set will change. The following theorem expresses how.

**Theorem 2.1.11.** *Let  $f : V \rightarrow \mathbf{R}$  be a function. Let  $a$  be a nondegenerate critical point of index  $k$  of  $f$  and let  $\alpha = f(a)$ . We suppose that for some sufficiently small  $\varepsilon > 0$ , the set  $f^{-1}([\alpha - \varepsilon, \alpha + \varepsilon])$  is compact and does not contain any critical point of  $f$  other than  $a$ . Then for every sufficiently small  $\varepsilon > 0$ , the homotopy type of the space  $V^{\alpha+\varepsilon}$  is that of  $V^{\alpha-\varepsilon}$  with a cell of dimension  $k$  attached (the unstable manifold of  $a$ ).*

**Remark 2.1.12.** This theorem allows us to justify the appearance, or rather the parachuting in, of the Morse function on  $\mathbf{P}^n(\mathbf{C})$  that was discussed in Exercise 5 on p. 18.

We begin with a very natural description of the (say, complex) projective space. It is, as all geometers know, the union of an affine space and a hyperplane “at infinity”. By considering things the other way around, we can say that the complex projective space  $\mathbf{P}^n(\mathbf{C})$  is obtained from  $\mathbf{P}^{n-1}(\mathbf{C})$  by adding a  $\mathbf{C}^n$ , or disk  $D^{2n}$ . Step by step, this defines a “cellular decomposition” of  $\mathbf{P}^n(\mathbf{C})$ : we begin with a point (this is  $\mathbf{P}^0$ , the point  $[1, 0, \dots, 0]$ ), we attach a disk of (real) dimension 2 (we now have  $\mathbf{P}^1(\mathbf{C})$ , the cell of points  $[a, b, 0, \dots, 0]$ ), and so on.

The Morse function in Exercise 5 enables this reconstruction, as is explained in Theorem 2.1.11. We begin with the minimum (this is precisely our point  $[1, 0, \dots, 0]$ ), where the function has value 0. The first critical value is then 1. This corresponds to the critical point  $[0, 1, 0, \dots, 0]$ , which has index 2... and enables us to attach a cell of dimension 2, and so on. In this sense, this function is “perfect”.<sup>1</sup>

*Proof of Theorem 2.1.11.* Let us begin by presenting the ideas of the proof while contemplating Figure 2.5. The cell  $D^k$  is the piece of the unstable

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<sup>1</sup> There is also a precise mathematical definition of the expression “perfect Morse function”, of which the one considered here is the prototype.

manifold of  $a$  shown in this figure. We proceed as follows:

- (1) By modifying  $f$ , we construct a function  $F$  that coincides with  $f$  outside of a neighborhood of  $a$  where  $F < f$ , so that  $F^{-1}(]-\infty, \alpha - \varepsilon])$  will be the union of  $V^{\alpha - \varepsilon}$  and a small neighborhood of  $a$  (the part with horizontal hatching in Figure 2.5).
- (2) Now, Theorem 2.1.7 and Remark 2.1.8 applied to the function  $F$  give the hatched part  $F^{-1}(]-\infty, \alpha + \varepsilon])$  as a retract of  $V^{\alpha + \varepsilon}$  (which is also the sublevel set of the modified function  $F$  for  $\alpha + \varepsilon$ ).
- (3) We can then position ourselves in a Morse chart to show that the subset of  $V$  consisting of the piece of the unstable manifold together with  $V^{\alpha - \varepsilon}$  is a deformation retract of  $F^{-1}(]-\infty, \alpha + \varepsilon])$ .

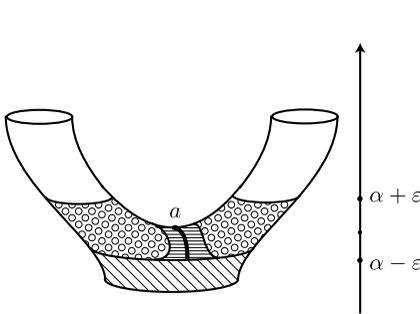


Fig. 2.5

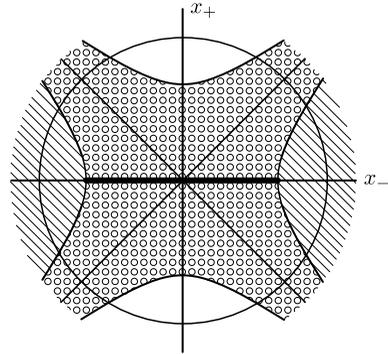


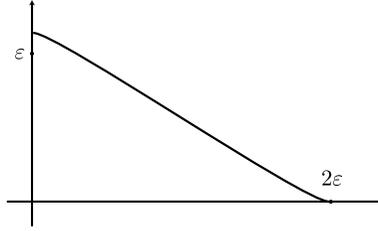
Fig. 2.6

*Construction of  $F$ .*

We choose a Morse chart  $(U, h)$  in the neighborhood of  $a$  and an  $\varepsilon > 0$  that is sufficiently small that  $f^{-1}(] \alpha - \varepsilon, \alpha + \varepsilon])$  is compact and that  $U$  contains the ball of radius  $\sqrt{2\varepsilon}$  with center 0. The disk  $D^k$  is the subset of  $U$  consisting of the  $(x_-, x_+)$  such that  $\|x_-\|^2 < \varepsilon$  and  $x_+ = 0$ . In Figure 2.5, as in Figure 2.6, the sublevel set  $V^{\alpha - \varepsilon}$  is indicated with oblique hatching while  $f^{-1}(] \alpha - \varepsilon, \alpha + \varepsilon])$  is dotted. The cell, the disk  $D^k$ , is a thick line segment.

We construct the function  $F$  by using a  $C^\infty$  function  $\mu : [0, +\infty[ \rightarrow [0, +\infty[$  with the following properties:

- $\mu(0) > \varepsilon$
- $\mu(s) = 0$  for  $s \geq 2\varepsilon$
- $-1 < \mu'(s) \leq 0$  for every  $s$



**Fig. 2.7** The function  $\mu$

We define  $F$  by setting

$$F(x) = \begin{cases} f(x) & \text{if } x \notin \Omega(a) \\ \alpha - \|x_-\|^2 + \|x_+\|^2 \mu(\|x_-\|^2 + 2\|x_+\|^2) & \text{if } x = h(x_-, x_+). \end{cases}$$

Note that the sublevel set of  $F$  for  $\alpha + \varepsilon$  is exactly the sublevel set  $V^{\alpha+\varepsilon}$  of  $f$ . Indeed:

- Outside of  $\|x_-\|^2 + 2\|x_+\|^2 \leq 2\varepsilon$ , we have  $F = f$ .
- In the interior of the ellipsoid in question, we have

$$F(x) \leq f(x) = \alpha - \|x_-\|^2 + \|x_+\|^2 \leq \alpha + \frac{1}{2}\|x_-\|^2 + \|x_+\|^2 \leq \alpha + \varepsilon.$$

Moreover, the critical points of  $F$  are the same as those of  $f$ , since

$$dF = \underbrace{(-1 - \mu'(\|x_-\|^2 + 2\|x_+\|^2))}_{<0} 2x_- \cdot dx_- + \underbrace{(1 - 2\mu'(\|x_-\|^2 + 2\|x_+\|^2))}_{\geq 1} 2x_+ \cdot dx_+$$

vanishes only for  $x_- = x_+ = 0$ , that is, at  $a$ .

We now know that

$$F^{-1}([\alpha - \varepsilon, \alpha + \varepsilon]) \subset f^{-1}([\alpha - \varepsilon, \alpha + \varepsilon]);$$

in particular, this region is compact. Moreover, it does not contain any critical points of  $F$ : the only possible candidate would be  $a$ , but

$$F(a) = \alpha - \mu(0) < \alpha - \varepsilon.$$

It follows that  $F^{-1}(]-\infty, \alpha + \varepsilon])$  is a deformation retract of  $V^{\alpha+\varepsilon}$ . Let  $H$  be the horizontally hatched part in Figure 2.5 (it is clear in Figure 2.8), that is,

the closure of  $F^{-1}(]-\infty, \alpha + \varepsilon]) - V^{\alpha-\varepsilon}$ . We have, in particular,

$$F^{-1}(]-\infty, \alpha + \varepsilon]) = V^{\alpha-\varepsilon} \cup H.$$

*The Retraction.*

We define the retraction by following the arrows indicated in Figure 2.9. Explicitly,  $r_t$  is the identity outside of  $\Omega(a)$  and we define  $r_t$  on  $U$  (rather

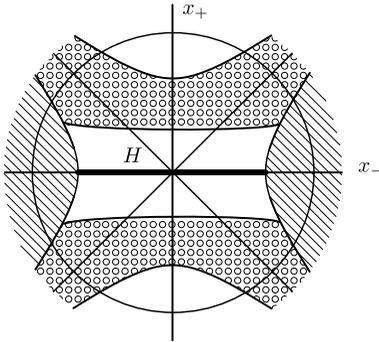


Fig. 2.8

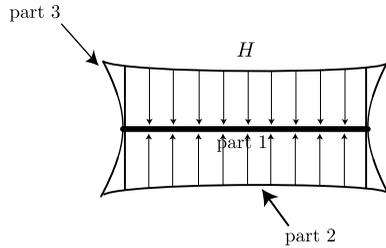


Fig. 2.9

than on  $\Omega(a)$ , to simplify the notation) as follows:

- On region 1 (Figure 2.9), that is, on  $\|x_-\|^2 \leq \varepsilon$ ,

$$r_t(x_-, x_+) = (x_-, tx_+).$$

- On region 2, defined by  $\varepsilon \leq \|x_-\|^2 \leq \varepsilon\|x_+\|^2$ , we set

$$r_t(x_-, x_+) = (x_-, s_t x_+),$$

where

$$s_t = t + (1 - t) \frac{\sqrt{\|x_-\|^2 - \varepsilon}}{\|x_+\|}$$

is the appropriate number to make the formulas continuous.

- On region 3, which corresponds to  $V^{\alpha-\varepsilon}$  and where  $\|x_+\|^2 + \varepsilon \leq \|x_-\|^2$ , we simply take  $r_t = \text{Id}$ .  $\square$

## 2.2 The Smale Condition

Let us return to the stable and unstable manifolds of the critical points.

### 2.2.a Examples of Stable and Unstable Manifolds

Here are the stable and unstable manifolds of the critical points of the examples considered earlier.

*The Height on the Round Sphere.*

Let  $a$  be the minimum and let  $b$  be the maximum. We have

$$W^s(a) = S^2 - \{b\}, \quad W^u(a) = \{a\}$$

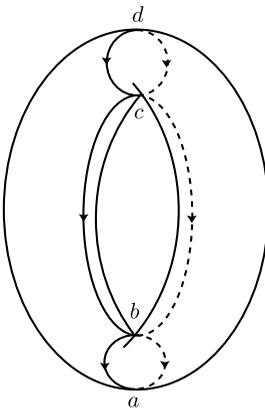
and likewise

$$W^s(b) = \{b\}, \quad W^u(b) = S^2 - \{a\}$$

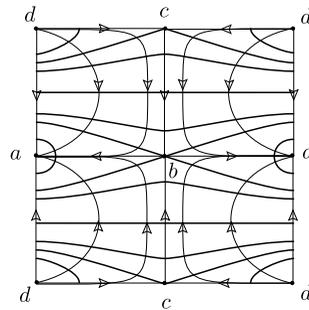
(for every pseudo-gradient field).

*The Torus.*

We begin with the height function on the inner tube torus. Let  $a, b, c, d$  be the critical points ordered according to the values that the function takes in them (Figure 2.10). The pseudo-gradient field used here is simply the gradient for



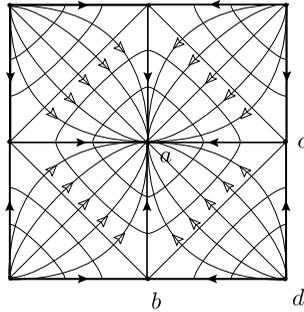
**Fig. 2.10** The height on the torus



**Fig. 2.11** Another point of view on the same thing

the metric induced by that on  $\mathbf{R}^3$ . The stable manifold of  $a$  consists of all points that descend to  $a$ , that is, the complement of the trajectories in the

figure that end at  $b$  or at  $c$ . Hence  $W^s(a)$  is homeomorphic to an open disk. The stable manifold of  $b$  consists of the two trajectories starting at  $c$  and ending at  $b$ . Hence  $W^s(b)$  is diffeomorphic to an open interval. The same holds for the stable manifold of  $c$ , which consists of the two trajectories starting at  $d$  that we see in the figure. We should note that in this example (which was chosen for this reason), the unstable manifold of  $c$  and the stable manifold of  $b$  have two open intervals in common. Figure 2.11 shows a few level sets of the same function in a square, which solves an exercise suggested in the previous chapter. It also shows a few trajectories of the gradient.



**Fig. 2.12** The torus, again

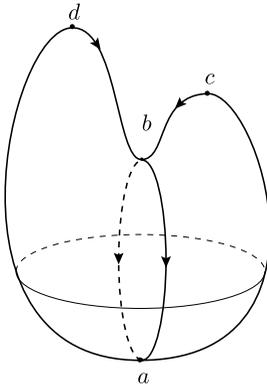
The situation is somewhat different in the case of the other Morse function that we encountered on the torus  $T^2$ , namely

$$f(x, y) = \cos(2\pi x) + \cos(2\pi y).$$

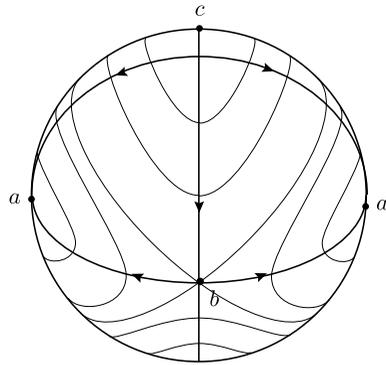
Let us again denote the extrema by  $a$  and  $d$  and the critical points of index 1 by  $b$  and  $c$ . Figure 2.12 shows the gradient lines that connect the two critical points (here we use the gradient for the “flat” metric on the torus, that is, for the usual metric on  $\mathbf{R}^2$ ; as we can see, the lines in question form true right angles with the level sets). We can clearly see that  $W^s(a)$  is the open square (that is, an open disk), that  $W^s(b)$  is an open interval (the horizontal side of the square in the figure), as are  $W^u(b)$  (vertical segment),  $W^s(c)$  (vertical side) and  $W^u(c)$  (horizontal segment), while  $W^s(d)$  is reduced to  $d$ . Note that  $W^u(b)$  and  $W^s(c)$  do not meet.

### *The Height on the “Other” Sphere.*

From Figure 2.13, the readers will be able to determine the stable and unstable manifolds of the four critical points of the height function (for the gradient of the metric induced by that on  $\mathbf{R}^3$ ).



**Fig. 2.13** The other sphere



**Fig. 2.14** The projective plane

*The Morse Function on  $\mathbf{P}^2(\mathbf{R})$ .*

Figure 2.14, in turn, shows the stable and unstable manifolds of the three critical points  $a$  (minimum),  $b$  (index 1) and  $c$  (maximum) for the function considered before (in Exercise 6 on p. 19) on the real projective plane  $\mathbf{P}^2(\mathbf{R})$ .

**2.2.b The Smale Condition**

We say that a pseudo-gradient field adapted to the Morse function  $f$  satisfies the *Smale condition* if all stable and unstable manifolds of its critical points meet transversally, that is, if

$$\text{for all critical points } a, b \text{ of } f, \quad W^u(a) \pitchfork W^s(b).$$

**Remark 2.2.1.** Certain stable and unstable manifolds always meet transversally. For example, we always have:

- $W^u(a) \pitchfork W^s(a)$  (for the same critical point), which is what we see in a Morse chart around  $a$ .
- $W^u(a) \cap W^s(b) = \emptyset$  if  $a$  and  $b$  are distinct and  $f(a) \leq f(b)$  (in particular, these stable and unstable manifolds are transversal).

If the vector field satisfies the Smale condition, then for all critical points  $a$  and  $b$ , we have

$$\text{codim}(W^u(a) \cap W^s(b)) = \text{codim } W^u(a) + \text{codim } W^s(b),$$

that is,

$$\dim(W^u(a) \cap W^s(b)) = \text{Ind}(a) - \text{Ind}(b).$$

Under our condition, this intersection is a submanifold of  $V$ , which we will denote by  $\mathcal{M}(a, b)$ . It consists of all points on the trajectories connecting  $a$  to  $b$ :

$$\mathcal{M}(a, b) = \left\{ x \in V \mid \lim_{s \rightarrow -\infty} \varphi^s(x) = a \text{ and } \lim_{s \rightarrow +\infty} \varphi^s(x) = b \right\}.$$

**Proposition 2.2.2.** *The group  $\mathbf{R}$  of translations in time acts on  $\mathcal{M}(a, b)$  by  $s \cdot x = \varphi^s(x)$ . This action is free if  $a \neq b$ .*

*Proof.* The fact that this is a group operation is clear. If  $a \neq b$ , then there is no critical point in  $\mathcal{M}(a, b)$ . Let  $x \in \mathcal{M}(a, b)$ . Since  $x$  is not a critical point, we know that  $f(\varphi^s(x))$  is a decreasing function of  $s$ , so that if  $\varphi^s(x) = \varphi^{s'}(x)$ , we necessarily have  $s = s'$ . Hence the action is free.  $\square$

The quotient is therefore a manifold, which we will call  $\mathcal{L}(a, b)$ . Its dimension is

$$\dim \mathcal{L}(a, b) = \text{Ind}(a) - \text{Ind}(b) - 1.$$

**Remark 2.2.3.** It is clear that the quotient is a separated space. In fact, the most convenient way to consider this quotient is the following. If  $\alpha$  is a value of  $f$  lying between  $f(a)$  and  $f(b)$ , then  $\mathcal{M}(a, b)$  is transversal to the level set  $f^{-1}(\alpha)$ : this level set has codimension 1 and the vector field  $X$  is transversal to it (by definition, it is not tangent to the level set, or we would have  $df(X) = 0$  at a noncritical point). All trajectories starting at  $a$  meet this intermediate level set at exactly one point, so that  $\mathcal{L}(a, b)$  can be identified with  $\mathcal{M}(a, b) \cap f^{-1}(\alpha)$ .

Hence, if  $a$  and  $b$  are two (distinct) critical points and if the gradient that is used satisfies the Smale condition, then for  $\mathcal{M}(a, b)$  or  $\mathcal{L}(a, b)$  to be nonempty, we must have

$$\text{Ind}(a) > \text{Ind}(b).$$

In other words, the index decreases along the gradient lines.

We will come back to these spaces at length in the next chapter.

**Examples 2.2.4.** All examples presented above satisfy the Smale condition,<sup>2</sup> except for that of the height function on the torus. Indeed, the manifolds  $W^s(b)$  and  $W^u(c)$  are not transversal, as we have already noted without knowing the example in Subsection 2.2.a and Figure 2.10. Moreover, we have trajectories connecting two critical points of index 1, which, as we just saw, is forbidden. This “bad” vector field is the gradient for the Riemannian metric on the torus induced by the surrounding Euclidean metric.

---

<sup>2</sup> In the cases of the “other sphere” and of the projective plane, this holds, for example, by virtue of Exercise 11 (p. 51).

The Smale condition forbids the existence of flow lines such as those shown in Figure 2.15. Figure 2.16 shows the trajectories of a neighboring field satisfying the Smale condition (obtained by the general method explained in the following subsection).

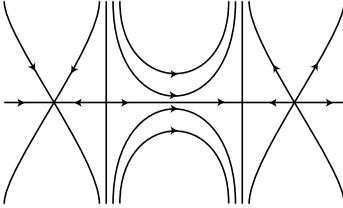


Fig. 2.15

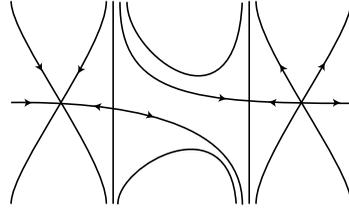


Fig. 2.16

### 2.2.c Existence

Following [72], let us now show the existence and genericness of pseudo-gradient fields satisfying the Smale condition.

We first note that after replacing  $f$  by another function that is arbitrarily close in the  $\mathcal{C}^1$  sense, if necessary, we may, and do, assume that  $f$  takes on distinct values at all of its critical points. Indeed, outside of  $\Omega$ , we have (by compactness)  $df(X) < -\varepsilon_0$  for some  $\varepsilon_0 > 0$ . We then choose a function  $h$  that is constant on each Morse chart  $\Omega_i$ , satisfies  $|dh| < \frac{1}{2}\varepsilon_0$  and for which

$$f(c_i) + h(c_i) \neq f(c_j) + h(c_j) \quad \text{for } i \neq j.$$

The function  $f+h$  is still a Morse function, with the same critical points as  $f$ , and the vector field  $X$  is still an adapted pseudo-gradient, but the critical values are now distinct.

**Theorem 2.2.5 (Smale Theorem [72]).** *Let  $V$  be a manifold with boundary and let  $f$  be a Morse function on  $V$  with distinct critical values. We fix Morse charts in the neighborhood of each critical point of  $f$ . Let  $\Omega$  be the union of these charts and let  $X$  be a pseudo-gradient field on  $V$  that is transversal to the boundary. Then there exists a pseudo-gradient field  $X'$  that is close to  $X$  (in the  $\mathcal{C}^1$  sense), equals  $X$  on  $\Omega$  and for which we have*

$$W_{X'}^s(a) \cap W_{X'}^u(b)$$

for all critical points  $a, b$  of  $f$ .

Let us clarify the notion of  $\mathcal{C}^1$  *proximity* used here: the statement asserts that for every  $\varepsilon > 0$ , for every cover of  $V$  by charts  $\varphi_i(U_i)$  and for every compact subset  $K_i \subset U_i$ , there exists a vector field  $X'$  such that

$$\|T\varphi_i^{-1}(X') - T\varphi_i^{-1}(X)\| < \varepsilon$$

for the  $\mathcal{C}^1$  norm on  $K_i$ , as well as the stated properties.

**Remark 2.2.6.** A vector field  $X'$  sufficiently close to the pseudo-gradient field  $X$  in the  $\mathcal{C}^1$  sense and equal to  $X$  on  $\Omega$  is itself a pseudo-gradient field. For the sake of simplicity, we will call such an  $X'$  a “good approximation” of  $X$ .

**Remark 2.2.7.** This theorem is sometimes called the “Kupka–Smale” theorem because Kupka also gave a proof. The one we imitate here is Smale’s proof.

*Proof of Theorem 2.2.5.* Since the critical values of  $f$  are distinct, let us arrange them in order:

$$\text{Crit}(f) = \{c_1, \dots, c_q\} \quad \text{with } f(c_1) > f(c_2) > \dots > f(c_q),$$

and let  $\alpha_i = f(c_i)$ . The proof of the theorem will use an induction based on the following lemma.

**Lemma 2.2.8.** *Let  $j \in \{1, \dots, q\}$  and let  $\varepsilon > 0$ . There exists a good approximation  $X'$  (in the  $\mathcal{C}^1$  sense) of  $X$  such that:*

- (1) *The vector field  $X'$  coincides with  $X$  on the complement of  $f^{-1}([\alpha_j + \varepsilon, \alpha_j + 2\varepsilon])$  in  $V$ .*
- (2) *The stable manifold of  $c_j$  (for  $X'$ ) is transversal to the unstable manifolds of all critical points, that is,*

$$W_{X'}^s(c_j) \pitchfork W_{X'}^u(c_i).$$

Let us (for the time being) admit the lemma and prove the theorem. We let  $\mathcal{P}(r)$  denote the following property: there exists a good approximation  $X'_r$  of  $X$  such that for every  $p \leq r$  and every  $i$ , we have

$$W_{X'_r}^s(c_p) \pitchfork W_{X'_r}^u(c_i).$$

Note that:

- Property  $\mathcal{P}(q)$  is exactly the theorem.

- Property  $\mathcal{P}(1)$  is true for a trivial reason: the critical point  $c_1$  is the maximum of  $f$  and its stable manifold is reduced to itself, so that it does not meet any unstable manifold of any critical point lying below  $c_1$ .
- And Property  $\mathcal{P}(2)$  follows from the lemma with  $j = 2$ .

Let us therefore assume that  $\mathcal{P}(r - 1)$  is true and show that  $\mathcal{P}(r)$  is then also true. We have a vector field  $X'_{r-1}$  such that the stable manifold of  $c_{r-1}$  is transversal to all the unstable manifolds. We apply the lemma to the vector field  $X'_{r-1}$  and  $j = r$ . This gives a vector field  $X'_r$  that, in particular (this is the first property given by the lemma), coincides with  $X'_{r-1}$  outside of the narrow strip where  $\alpha_r + \varepsilon \leq f \leq \alpha_r + 2\varepsilon$ . Moreover, since for every  $p \leq r - 1$ , the stable manifold of  $c_p$  for  $X'_{r-1}$  lies above this strip, the stable manifold is the same for  $X'_{r-1}$  as it is for  $X'_r$  (see Figure 2.17). We therefore have

$$W_{X'_{r-1}}^s(c_p) \cap W_{X'_{r-1}}^u(c_i) = W_{X'_r}^s(c_p) \cap W_{X'_r}^u(c_i)$$

for  $p \leq r - 1$  and for every  $i$ , so that

$$W_{X'_r}^s(c_p) \pitchfork W_{X'_r}^u(c_i).$$

For  $p = r$ , the lemma implies (this is the second property) that

$$W_{X'_r}^s(c_r) \pitchfork W_{X'_r}^u(c_i). \quad \square$$

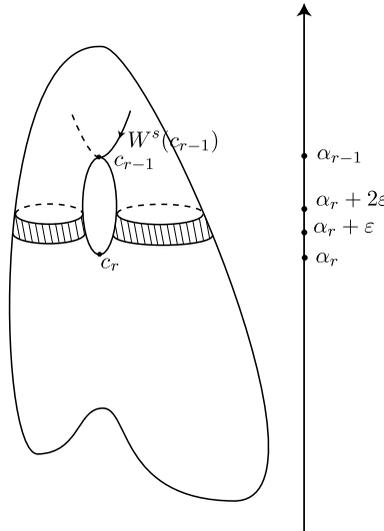
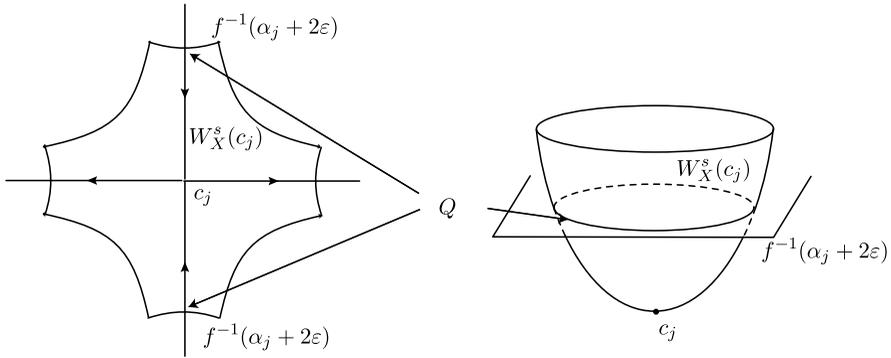


Fig. 2.17

*Proof of Lemma 2.2.8.* It can be useful to consider a Morse chart in the neighborhood of  $c_j$ . Figure 2.18 shows one (twice). We choose an  $\varepsilon$  sufficiently

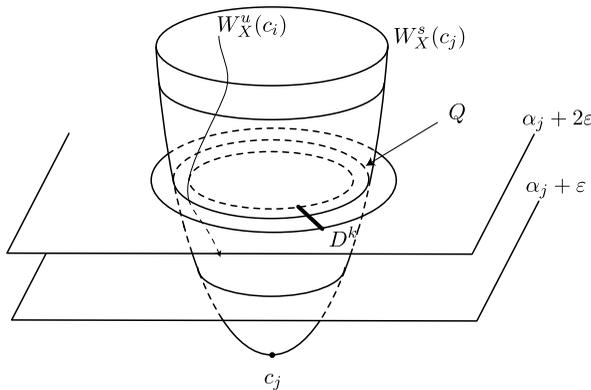


**Fig. 2.18** In a Morse chart

small so that

$$\alpha_j + 2\varepsilon < \alpha_{j-1}.$$

Let  $k$  denote the index of  $c_j$ , and let  $Q = W^s(c_j) \cap f^{-1}(\alpha_j + 2\varepsilon)$ . Then  $Q$  is a sphere of dimension  $n - k - 1$ . We consider a tubular neighborhood of this sphere  $Q$  in  $f^{-1}(\alpha_j + 2\varepsilon)$ , of the form  $Q \times D^k$ , which we see in a Morse chart in Figure 2.18, and in Figure 2.19.



**Fig. 2.19** In the manifold

Then there exists an embedding

$$\Psi : D^k \times Q \times [0, m] \longrightarrow f^{-1}([\alpha_j + \varepsilon, \alpha_j + 2\varepsilon])$$

such that:

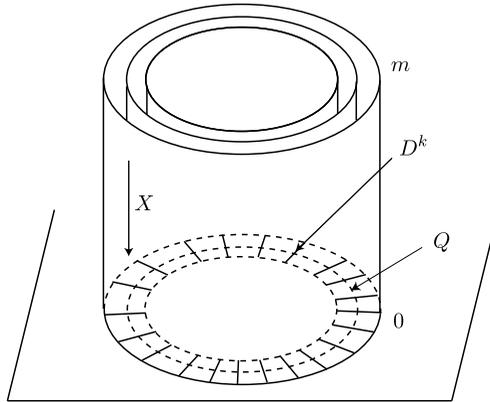
- $\Psi$  restricted to  $\{0\} \times Q \times \{0\}$  is the embedding of  $Q$  in  $f^{-1}(\alpha_j + \varepsilon)$ .
- $\Psi$  restricted to  $\{0\} \times Q \times \{m\}$  is the embedding of  $Q$  in  $f^{-1}(\alpha_j + 2\varepsilon)$ .
- If  $z$  is the coordinate in  $[0, m] \subset \mathbf{R}$ , then

$$\Psi_* \left( -\frac{\partial}{\partial z} \right) = X.$$

The unstable manifolds are transversal to the level sets. In particular, they meet  $D^k \times Q$  along a manifold  $P'$  (that is not connected in general). If  $W_{X'}^s(c_j) \pitchfork P'$ , then there is nothing to prove. The proof will therefore consist in modifying  $X$  into  $X'$  on  $f^{-1}([\alpha_j + \varepsilon, \alpha_j + 2\varepsilon])$  in such a way that

$$W_{X'}^s(c_j) \pitchfork P'.$$

The modification will take place inside the image of  $\Psi$ . Let us therefore position ourselves in  $D^k \times Q \times [0, m]$  with  $X = -\partial/\partial z$ . Figure 2.19 shows what happens in the manifold, while Figure 2.20 shows the model.



**Fig. 2.20** The model

Let  $P$  be the submanifold  $\Psi^{-1}(P') \subset D^k \times Q \times [0, m]$ . As seen in the model, the desired transversality condition  $W_{X'}^s(c_j) \pitchfork P'$  can be written as

$$W_{X'}^s \pitchfork P \quad \text{with } W_{X'}^s = \{ \phi_{X'}^{-s}(0, q, 0) \mid s > 0, q \in Q \}.$$

In the initial situation,  $X' = X = -\partial/\partial z$ , so that

$$P \cap W_X^s = P \cap \{(0, q, s) \mid s > 0, q \in Q\} = P \cap \{(0, q, 0) \mid q \in Q\} = g^{-1}(0),$$

where  $g : P \rightarrow D^k$  is the projection  $(x, q, z) \mapsto x$ .

The proof will consist in making 0 a regular value of  $g$ . By Sard's theorem, there is a vector  $w$  in  $D^k$ , as close to 0 as we want (say  $\|w\| = \delta$ ), such that  $w$  is a regular value of  $g$ . We are going to construct a perturbation  $X'$  of the vector field  $X$  such that

$$W_{X'}^s \cap \{z = m\} = \phi_{X'}^{-m}(0, q, 0) = (w, q, m).$$

We will then have  $W_{X'}^s \cap P = g^{-1}(w)$ , which implies that  $W_{X'}^s \cap P$  is a submanifold of codimension  $k$  in  $P$ . The equality

$$\text{codim}_P W_{X'}^s \cap P = \text{codim}_V W_{X'}^s,$$

implies the transversality of the two submanifolds  $W_{X'}^s$  and  $P$ .

The lemma "in the manifold" now results from the following lemma "in the model".

**Lemma 2.2.9.** *There exists a vector field  $X'$  close to  $-\partial/\partial z$  (in the  $\mathcal{C}^1$  sense) such that:*

- (1)  $X' = -\partial/\partial z$  near  $\partial(D^k \times Q \times [0, m])$ .
- (2)  $\varphi_{X'}^{-m}(0, q, 0) = (w, q, m)$ .

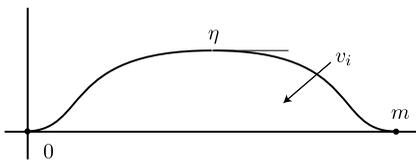
*Proof.* Let  $(v_1, \dots, v_k)$  be the coordinates of  $w \in D^k$ . We set

$$X' = -\frac{\partial}{\partial z} - \sum_{i=1}^k \beta_i(z) \gamma(x) \frac{\partial}{\partial x_i},$$

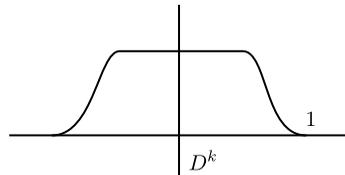
where:

- The function  $\beta_i$  is zero outside of  $[0, m]$  and satisfies  $|\beta_i(s)| < \eta$ ,  $|\beta'_i(s)| < \eta$  and  $\int_0^m \beta_i(t) dt = v_i$  (where  $\eta$  is a small fixed positive number corresponding to the desired precision of the approximation).
- The function  $\gamma$  is, in turn, defined on  $D^k$ , has values in  $[0, 1]$ , is identically zero near  $\partial D^k$  and satisfies  $\gamma \equiv 1$  on  $\|x\| \leq 1/3$  and  $|\partial\gamma/\partial x_i| \leq 2$ .

See Figures 2.21 and 2.22.



**Fig. 2.21** The function  $\beta_i$



**Fig. 2.22** The function  $\gamma$

It is clear that the vector field  $X'$  satisfies the first stated property. To prove that it also satisfies the second one, we first consider the vector field

$$X'' = -\frac{\partial}{\partial z} - \sum_{i=1}^k \beta_i(z) \frac{\partial}{\partial x_i}$$

(we have left out  $\gamma(x)$ ). To determine  $\varphi_{X''}^{-m}(0, q, 0)$ , we must solve the differential system

$$\begin{cases} \frac{\partial x_i}{\partial s} = -\beta_i(z(s)), & x(0) = 0 \\ \frac{\partial q}{\partial s} = 0, & q(0) = q \\ \frac{\partial z}{\partial s} = -1, & z(0) = 0 \end{cases}$$

whose solution is

$$\begin{cases} x_i(s) = \int_0^s -\beta_i(-t) dt = \int_0^{-s} \beta_i(-t) dt \\ q(s) = q \\ z(s) = -s. \end{cases}$$

We therefore have

$$\varphi_{X''}^{-m}(0, q, 0) = (w, q, m)$$

and, since for  $s \in [-m, 0]$ , the norm satisfies  $\|x(s)\| \leq 1/3$  (for sufficiently small  $\beta_i$ , that is, for sufficiently small  $w$ ), we remain in the part of the disk where  $\gamma \equiv 1$ , so that the formula indeed gives the flow of  $X'$ .  $\square$

This completes the proof of Lemma 2.2.8.  $\square$

### 2.2.d An Illustration, the Height Function on the Torus

Let us return to the example of the height function on the torus of dimension 2, with gradient field  $X$  that does not satisfy the Smale property, as noted in Examples 2.2.4. We first copy Figures 2.10 and 2.11, indicating the two level sets for  $\alpha + \varepsilon$  and  $\alpha + 2\varepsilon$  above the critical point  $b$  between which we need to modify the vector field.

A Morse neighborhood of the critical point  $b$  is clearly visible in the middle of Figure 2.24. We extract it from this figure (Figure 2.25) and modify the vector field in the useful part of the model (Figure 2.26). This last figure

shows the same model as Figure 2.20, namely a  $Q \times D^k \times [0, m]$ , but  $Q$  is now a sphere of dimension  $n - k - 1 = 0$  and  $D^k$  is a disk of dimension  $k = 1$ .

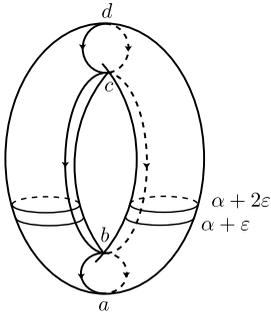


Fig. 2.23

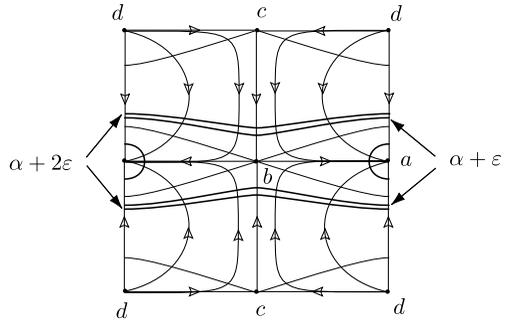


Fig. 2.24

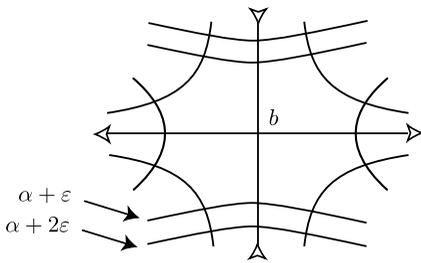


Fig. 2.25 Extract of the model

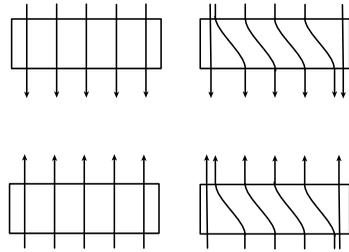


Fig. 2.26 The modification in the model

Put back into the surface, the modification gives a vector field with the expected property (Figure 2.27).

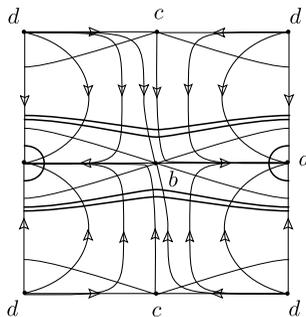


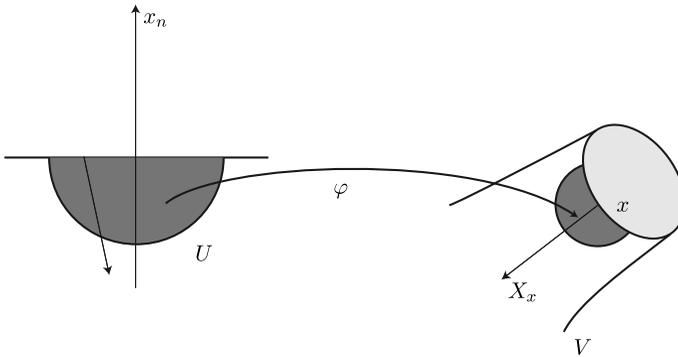
Fig. 2.27 The modified field

## 2.3 Appendix: Classification of the Compact Manifolds of Dimension 1

### 2.3.a Morse Functions and Adapted Vector Fields on a Manifold with Boundary

Let us now consider a manifold with boundary  $V$ . We fix a vector field  $X$  defined in a neighborhood of  $\partial V$  in  $V$ , which we assume to be *incoming*, that is, such that for every chart  $\varphi : U \rightarrow V$  (where  $U$  is an open subset of the half-space  $x_n \leq 0$  in  $\mathbf{R}^n$ ) and every  $x \in \partial V \cap \varphi(U)$ , we have

$$T_{\varphi^{-1}(x)}\varphi^{-1}(X_x) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \quad \text{with } a_n(\varphi^{-1}(x)) < 0.$$



**Fig. 2.28** Incoming vector field

Constructing such a vector field is easy, for example by starting out with the vector field  $-\partial/\partial x_n$  on  $\mathbf{R}^n$  and using a partition of unity.

We can also construct a Morse function  $f$  on  $V$  such that

$$df(X) < 0 \quad \text{in the neighborhood of } \partial V$$

by first defining it in the neighborhood of the boundary, setting

$$f(\varphi_X^{-s}(x)) = s \quad \text{for } s \in [0, \delta[ \text{ if } x \in \partial V,$$

then extending it arbitrarily to  $V$  and finally taking a perturbation, if necessary, in order to have a Morse function.

Next, we extend  $X$ , which up to now has only been defined in the neighborhood of the boundary, to a pseudo-gradient field adapted to  $f$  that we will still call  $X$ .

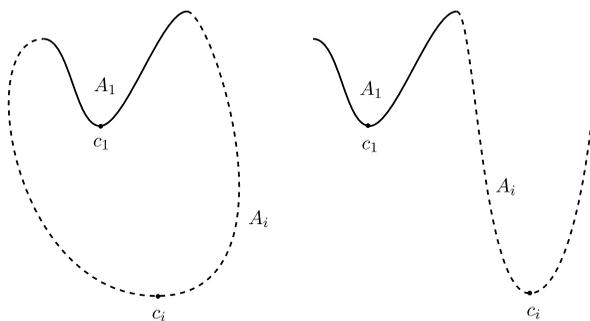
**Remark 2.3.1.** It is not true that every Morse function on a manifold with boundary admits a pseudo-gradient field transversal to the boundary (think of the projection of the unit disk onto a straight line). That is why in this construction we have chosen to start out with the vector field.

### 2.3.b The Classification Theorem

**Theorem 2.3.2.** *Let  $V$  be a compact connected manifold of dimension 1. Then  $V$  is diffeomorphic to  $S^1$  if  $\partial V = \emptyset$  and diffeomorphic to  $[0, 1]$  otherwise.*

*Proof.* Let  $X$  be a vector field that is incoming along the boundary, let  $f$  be a Morse function for which  $X$  is an adapted pseudo-gradient field (we can construct such a vector field and such a function as indicated above). The critical points of  $f$  are local minima and maxima. The proof is based on the fact that all trajectories that are not stationary at a maximum end up at a minimum. Let  $c_1, \dots, c_k$  be the minima of  $f$ . The stable manifold  $W^s(c_i)$  is diffeomorphic to an (open) interval; it consists of the two trajectories ending at  $c_i$  and the point  $c_i$  itself. In the closure  $A_i$  of this stable manifold there are (moreover) the starting points of these two trajectories. These starting points:

- either are both maxima (in which case they can either coincide or not)
- or at least one of them is a boundary point of  $V$  (in which case they are distinct).



**Fig. 2.29**

It is also clear that, as the closure of a connected space,  $A_i$  is connected. Clearly, if  $A_i$  consists of  $W^s(c_i)$  and a unique point, then the latter is a maximum and  $A_i$  is diffeomorphic to a circle. Likewise, if  $A_i$  consists of the stable manifold and two added points, then  $A_i$  is diffeomorphic to a closed interval.

Note that the union of the  $A_i$  is all of  $V$ : if  $x \in V$ , then its trajectory tends to a minimum and lies in one of the stable manifolds, unless  $x$  is a maximum, but then it lies in the closure of a stable manifold.

If  $k = 1$ , then the theorem has now been proved. Otherwise, since  $V$  is connected, there exists an  $i \geq 2$  such that  $A_1 \cap A_i \neq \emptyset$ . This intersection contains only local maxima, since these are the only points from which we can descend to two different minima. In particular,  $\partial V \cap (A_1 \cap A_i) = \emptyset$ . In  $A_1 \cap A_i$ , there are at most two points. We have two possibilities:

- If this intersection contains two points, then the two are maxima,  $A_1 \cup A_i$  is diffeomorphic to  $S^1$  and we are done.
- If, on the contrary, it contains only one point, then  $A_1 \cup A_i$  is diffeomorphic to  $[0, 1]$ . If  $A_1 \cup A_i = V$ , then we are done.

And if this is not the case, then we continue adding  $A_i$ 's until they run out. □

There exist other ways to prove this theorem, which may be simpler (see [55]). This proof using Morse theory has the advantage of preparing other, analogous, proofs, such as that of Proposition 4.5.1.

### 2.3.c An Application, the Brouwer Fixed Point Theorem

We will now use Sard's theorem and our knowledge of manifolds of dimension 1 to prove Brouwer's famous theorem (this proof comes from Milnor's book [55]).

**Theorem 2.3.3.** *Let  $\varphi : D^n \rightarrow D^n$  be a continuous map; then it has a fixed point.*

*Proof.* The first part of the proof consists in reducing to the case where  $\varphi$  is a  $C^\infty$  map. We will not give the details here (see [45]). Next, starting from  $\varphi$ , which is assumed to be  $C^\infty$  and without fixed points, we construct a retraction

$$r : D^n \longrightarrow S^{n-1}$$

by sending  $x \in D^n$  to the intersection point of the sphere  $S^{n-1}$  and the ray starting at  $\varphi(x)$  and going through  $x$ . This way, if  $x$  is a point of the sphere, it stays in place. We thus have a  $C^\infty$  map  $r$  that restricts to the identity on the boundary. Sard's theorem asserts that this map has regular values. Let

$a \in S^{n-1}$  be one of them; then  $r^{-1}(a)$  is a submanifold of dimension 1 of  $D^n$ , with boundary

$$\partial r^{-1}(a) = r^{-1}(a) \cap \partial D^n = \{a\}.$$

But a manifold of dimension 1 with boundary is diffeomorphic to a union of circles and closed intervals, so that its boundary consists of an even number of points. This gives a contradiction, and therefore the existence of a fixed point.  $\square$

## Exercises

**Exercise 10.** Show that the vector fields whose flows are drawn in Figure 2.30 are not pseudo-gradient fields.

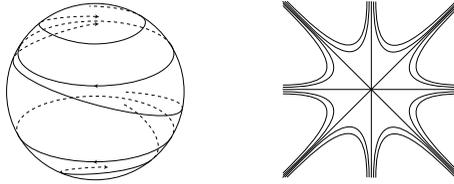


Fig. 2.30

**Exercise 11.** Let  $V$  be a manifold of dimension 2 endowed with a Morse function with a unique critical point of index 1. Show that every pseudo-gradient field adapted to this function satisfies the Smale condition.

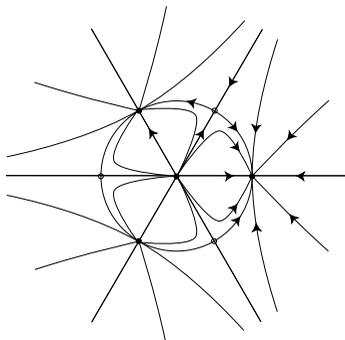
**Exercise 12.** We fix an integer  $m \geq 2$ . Find all critical points of the function  $f : \mathbf{P}^1(\mathbf{C}) \rightarrow \mathbf{R}$  defined by

$$f([z_0, z_1]) = \frac{|z_0^m + z_1^m|^2}{(|z_0|^2 + |z_1|^2)^m} = \frac{|z^m + 1|^2}{(|z|^2 + 1)^m}$$

(in homogeneous coordinates or in the affine chart  $z_1 \neq 0$ ). Verify that for  $m = 2$ , the function  $f$  is not a Morse function.<sup>3</sup>

We suppose that  $m \geq 3$ . Show that  $f$  is a Morse function and has two local maxima: the points 0 and  $\infty$ ;  $m$  local minima: the  $m$ -th roots of  $-1$ ; and  $m$  critical points of index 1: the  $m$ -th roots of 1.

<sup>3</sup> It is a Mores–Bott function (see [14]): its critical points form submanifolds (here  $\mathbf{P}^1(\mathbf{R})$  for the maximum) and the second-order derivative is transversally nondegenerate.



**Fig. 2.31**

Hint: We can determine the critical points using the derivatives with respect to  $z$  and  $\bar{z}$ , and then use a second-order Taylor expansion of  $f(u)$  with respect to  $u$  in the neighborhood of 0 (to study the critical points at 0 and  $\infty$ ) or the analogous expansion of  $f(\zeta(1+u))$  (to study the critical points at  $\zeta$  with  $\zeta^m = \pm 1$ ).

Show that there exists a pseudo-gradient field such as that shown (in an affine chart) in Figure 2.31 (for  $m = 3$ ). More generally, see the article [9] in which an analogous function (defined on  $\mathbf{P}^n(\mathbf{C})$ ) plays an important role.



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