

Chapter 2

FTS Analysis of Continuous-Time Linear Systems

2.1 Introduction

This chapter deals with the FTS analysis of CT-LTV systems. The first result of the chapter is a necessary and sufficient condition for FTS. Such a condition requires the computation of the state transition matrix of the system; for this reason, it is of little usefulness in practice, both for the intrinsic difficulty of computing the state transition matrix of a linear time-varying system and, more importantly, because it cannot be used in the design context.

Then, by using an approach based on time-varying quadratic Lyapunov functions, two further necessary and sufficient conditions for FTS are stated. The former requires the existence of a feasible solution to a certain DLMI; the latter involves a DLE.

It is shown that the DLE-based condition is more efficient from the computational point of view; however, DLMIs are useful in the context of the design problem. Indeed, in Chap. 3, by the latter approach we shall solve both the state and output feedback finite-time stabilization problems.

At the end of the chapter, the proposed approaches will be used to study the FTS properties of the car suspension system; such an engineering example will be continued in the following chapters to illustrate the effectiveness of the proposed FTS design techniques.

2.2 Problem Statement

In the following, we restrict Definition 1.1 of FTS to the case of CT-LTV systems, and we assume that both the initial set and the trajectory set are ellipsoids. Moreover, we assume that the trajectory set is a *time-varying* ellipsoid. All time-varying matrices, unless otherwise specified, are assumed to be bounded and piecewise continuous functions of time.

Definition 2.1 (FTS of CT-LTV Systems with Time-Varying Ellipsoidal Domains) Given an initial time t_0 , a positive scalar T , a positive definite matrix R , and a positive definite matrix-valued function $\Gamma(\cdot)$, defined over $[t_0, t_0 + T]$, such that $\Gamma(t_0) < R$, the time-varying linear system

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad (2.1)$$

where $A(t) \in \mathbb{R}^{n \times n}$, is said to be finite-time stable with respect to $(t_0, T, R, \Gamma(\cdot))$ if

$$x_0^T R x_0 \leq 1 \Rightarrow x(t)^T \Gamma(t) x(t) < 1, \quad t \in [t_0, t_0 + T]. \quad (2.2)$$

◇

Remark 2.1 The assumption $\Gamma(t_0) < R$ guarantees that the closed ellipsoid

$$\{x_0 : x_0^T R x_0 \leq 1\}$$

is a subset of the open ellipsoid $\{x_0 : x_0^T \Gamma(t_0) x_0 < 1\}$; this in turn guarantees the well-posedness of Definition 2.1. ◇

2.3 Main Results

The following theorem gives a number of necessary and sufficient conditions for the FTS of system (2.1).

Theorem 2.1 *The following statements are equivalent:*

- (i) System (2.1) is finite-time stable with respect to $(t_0, T, R, \Gamma(\cdot))$.
- (ii) For all $t \in [t_0, t_0 + T]$,

$$\Phi(t, t_0)^T \Gamma(t) \Phi(t, t_0) < R,$$

where $\Phi(t, t_0)$ is the state transition matrix of system (2.1).

- (iii) The matrix-valued function $W(\cdot) : [t_0, t_0 + T] \mapsto \mathbb{R}^{n \times n}$, solution of the DLE

$$-\dot{W}(t) + A(t)W(t) + W(t)A^T(t) = 0, \quad t \in [t_0, t_0 + T], \quad (2.3a)$$

$$W(t_0) = R^{-1}, \quad (2.3b)$$

is positive definite and satisfies

$$C(t)W(t)C^T(t) < I, \quad t \in [t_0, t_0 + T], \quad (2.4)$$

where $C(\cdot)$ is any nonsingular matrix-valued function such that $\Gamma(t) = C^T(t)C(t)$ for $t \in [t_0, t_0 + T]$.

(iv) *Either one of the following inequalities holds:*

$$\lambda_{\max}[C(t)W(t)C^T(t)] < 1, \quad (2.5a)$$

$$\lambda_{\min}[C^{-T}(t)M(t)C^{-1}(t)] > 1, \quad (2.5b)$$

where $W(\cdot)$ is the positive definite solution of (2.3a)–(2.3b), and $M(\cdot)$ is the positive definite solution of

$$\dot{M}(t) + A^T(t)M(t) + M(t)A(t) = 0, \quad t \in [t_0, t_0 + T], \quad (2.6a)$$

$$M(t_0) = R, \quad (2.6b)$$

where $C(\cdot)$ is a nonsingular matrix-valued function such that $\Gamma(t) = C^T(t)C(t)$ for $t \in [t_0, t_0 + T]$.

(v) *The DLMI with terminal and initial conditions*

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) < 0, \quad t \in [t_0, t_0 + T], \quad (2.7a)$$

$$P(t) > \Gamma(t), \quad t \in [t_0, t_0 + T], \quad (2.7b)$$

$$P(t_0) < R, \quad (2.7c)$$

admits a piecewise continuously differentiable symmetric solution $P(\cdot)$ defined over $[t_0, t_0 + T]$. $\square$

Before proving the theorem, some comments are in order.

Remark 2.2 The equivalence of (i) and (ii) has been proved in [10]. The fact that condition (v) implies FTS of system (2.1) dates back to the same work; however, the equivalence of the two conditions and condition (iii) was only proven in the paper [27]. Finally, the proof of the equivalence of (iv) and FTS can be found in [49]. The proof of the theorem presented here follows a slight different machinery, in order to compact all results together. $\diamond$

Remark 2.3 Condition (ii) is hard to apply in the time-varying case, as it requires the computation of the state transition matrix. There is also another important drawback with Condition (ii); indeed, it is simple to recognize that it is not useful for design purposes. $\diamond$

Remark 2.4 In Sect. 2.4, it will be shown that conditions (iii) and (iv), which require the solution of a DLE, are more efficient from the computational point of view in comparison with condition (v) (a similar discussion can be found in [26]). However, (2.7a)–(2.7c) will be the starting point for the solution of the design problem; indeed, the inequality allows us to put the controller design in the convex optimization framework. $\diamond$

Proof (i) $\Leftrightarrow$ (ii) Let x_0 be such that $x_0^T R x_0 \leq 1$. Then

$$\begin{aligned} x^T(t) \Gamma(t) x(t) &= x_0^T \Phi^T(t, t_0) \Gamma(t) \Phi(t, t_0) x_0 \\ &< x_0^T R x_0 \leq 1. \end{aligned} \quad (2.8)$$

Therefore, system (2.25) is finite-time stable.

The necessity is shown by contradiction. Let us assume that for some $\bar{t}$ and $\bar{x}$,

$$\bar{x}^T \Phi^T(t, t_0) \Gamma(t) \Phi(t, t_0) \bar{x} \geq \bar{x}^T R \bar{x}. \quad (2.9)$$

Now let $x_0 = \lambda \bar{x}$, where λ is such that $x_0^T R x_0 = 1$. Then (2.9) implies that

$$x_0^T \Phi^T(t, t_0) \Gamma(t) \Phi(t, t_0) x_0 \geq 1. \quad (2.10)$$

Therefore,

$$x(\bar{t})^T \Gamma(\bar{t}) x(\bar{t}) = x_0^T \Phi^T(t, t_0) \Gamma(t) \Phi(t, t_0) x_0 \geq 1, \quad (2.11)$$

which contradicts the initial assumption that system (2.25) is finite-time stable.

In the sequel, we will first prove that (i) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv); then we show that (iii) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i), which completes the proof.

(i) $\Leftrightarrow$ (iii) First, note that, given a real symmetric and positive definite matrix-valued function $\Gamma(\cdot) \in \mathbb{R}^n$, it is always possible to find a nonsingular matrix-valued function $C(\cdot) : [t_0, t_0 + T] \mapsto \mathbb{R}^{n \times n}$ such that

$$\Gamma(t) = C^T(t) C(t), \quad t \in [t_0, t_0 + T].$$

Since (i) $\Leftrightarrow$ (ii), it follows that system (2.1) is finite-time stable if and only if

$$\Phi^T(t, t_0) \Gamma(t) \Phi(t, t_0) - R < 0, \quad t \in [t_0, t_0 + T],$$

which is equivalent to

$$\begin{aligned} &\Phi^T(t, t_0) C^T(t) C(t) \Phi(t, t_0) - R < 0 \\ &\Leftrightarrow R^{-\frac{1}{2}} \Phi^T(t, t_0) C^T(t) C(t) \Phi(t, t_0) R^{-\frac{1}{2}} - I < 0 \\ &\Leftrightarrow C(t) \Phi(t, t_0) R^{-1} \Phi^T(t, t_0) C^T(t) - I < 0 \end{aligned} \quad (2.12)$$

for all $t \in [t_0, t_0 + T]$. Now, if we let

$$W(t) = \Phi(t, t_0) R^{-1} \Phi^T(t, t_0),$$

then (2.12) can be rewritten as (2.4).

Furthermore, taking into account that the transition matrix $\Phi(t, t_0)$ solves the matrix-valued differential equation

$$\frac{\partial}{\partial t} \Phi(t, t_0) = A(t) \Phi(t, t_0), \quad t \in [t_0, t_0 + T], \quad \Phi(t_0, t_0) = I,$$

it readily follows that $W(\cdot)$ is a positive definite solution of (2.3a)–(2.3b).

(iii) $\Leftrightarrow$ (iv) First, note that from the definition of $W(\cdot)$ and $C(\cdot)$ it readily follows that inequality (2.4) is equivalent to (2.5a).

Furthermore, the equivalence of (2.5a) and (2.5b) follows by letting

$$M(\cdot) = W^{-1}(\cdot)$$

over $[t_0, t_0 + T]$.

(iii) $\Rightarrow$ (v) We have shown that if $W(\cdot)$ satisfies (2.3a)–(2.3b) and (2.4), then system (2.1) is finite-time stable. By continuity arguments, if system (2.1) is finite-time stable wrt $(t_0, T, R, \Gamma(\cdot))$, then there exists a real scalar ε such that also the following system

$$\dot{z}(t) = \left(A(t) + \frac{\varepsilon}{2} I \right) z(t), \quad z(t_0) = x_0,$$

is finite-time stable with respect to $(t_0, T, R, \Gamma(\cdot))$.

Taking again into account the equivalence of conditions (i) and (iii), we denote by $W_\varepsilon(\cdot)$ the continuous positive definite matrix-valued solution of

$$-\dot{W}_\varepsilon(t) + A(t)W_\varepsilon(t) + W_\varepsilon(t)A^T(t) + \varepsilon W_\varepsilon(t) = 0, \quad t \in [t_0, t_0 + T], \quad (2.13a)$$

$$W_\varepsilon(t_0) = R^{-1}, \quad (2.13b)$$

which also satisfies

$$C(t)W_\varepsilon(t)C^T(t) < I, \quad t \in [t_0, t_0 + T].$$

Exploiting continuity arguments once more, it turns out that there exists a real scalar $\alpha > 1$ such that

$$\alpha C(t)W_\varepsilon(t)C^T(t) < I, \quad t \in [t_0, t_0 + T]. \quad (2.14)$$

Let $X(t) = \alpha W_\varepsilon(t)$, $t \in [t_0, t_0 + T]$; inequality (2.14) can be rewritten as

$$C(t)X(t)C^T(t) < I, \quad t \in [t_0, t_0 + T]. \quad (2.15)$$

Since $\dot{X}(t) = \alpha \dot{W}_\varepsilon(t)$, from (2.13a) we obtain

$$-\dot{X}(t) + A(t)X(t) + X(t)A^T(t) + \varepsilon X(t) = 0, \quad t \in [t_0, t_0 + T].$$

Taking into account the positive definitiveness of $X(t)$, it follows that

$$-\dot{X}(t) + A(t)X(t) + X(t)A^T(t) < 0 \quad (2.16)$$

for $t \in [t_0, t_0 + T]$. Furthermore, taking into account (2.13b), we obtain

$$X(t_0) > R^{-1}. \quad (2.17)$$

Eventually, letting

$$P(t) = X^{-1}(t), \quad t \in [t_0, t_0 + T],$$

inequalities (2.7a)–(2.7c) can be easily obtained from (2.15)–(2.17).

(v) $\Rightarrow$ (i) Let us consider $V(t, x) = x^T(t)P(t)x(t)$. Given a system trajectory $x(\cdot)$, the time derivative of $V(t, x)$ reads

$$\dot{V}(t, x) = x^T(t)(\dot{P}(t) + A(t)^T P(t) + P(t)A(t))x(t),$$

which is negative definite by virtue of (2.7a). This implies that $V(t, x)$ is strictly decreasing along the trajectories of system (2.1). Now, if we consider an initial state x_0 such that $x_0^T R x_0 \leq 1$, we have, for all $t \in [t_0, t_0 + T]$,

$$\begin{aligned} x(t)^T \Gamma(t)x(t) &< x(t)^T P(t)x(t) \quad \text{by (2.7b)} \\ &< x_0^T P(t_0)x_0 \\ &< x_0^T R x_0 \leq 1 \quad \text{by (2.7c),} \end{aligned}$$

which implies that system (2.1) is finite-time stable with respect to $(t_0, T, R, \Gamma(\cdot))$. $\diamond$

2.4 Computational Issues

The numerical example considered in this section is introduced to discuss the effectiveness and some computational issues related to the necessary and sufficient conditions for the FTS of CT-LTV systems proposed in Theorem 2.1.

Let us first consider the second-order LTV system

$$\dot{x}(t) = \begin{pmatrix} 0.4 \cdot t & 1 \\ -1 & -1 + 0.4 \cdot t \end{pmatrix} x(t) \quad (2.18)$$

and the time interval $[0, 1]$. We set

$$R = \begin{pmatrix} 2.5 & 0 \\ 0 & 2.5 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \quad (2.19)$$

Table 2.1 Values of $\gamma_{\max}$ satisfying Theorem 2.1 for the LTV system (2.18)

Condition	Sample time (T_s) [s]	$\gamma_{\max}$	Average computation time for a single iteration [s]
DLMI (2.7a)–(2.7c)	0.1	1.9226	2.0
	0.05	1.9321	12.3
	0.025	1.9367	139.6
Solution of (2.3a)–(2.3b) and check of inequality (2.4)	$2 \cdot 10^{-4}$	1.9402	1.1

and we exploit the necessary and sufficient conditions (iii) and (v) of Theorem 2.1 in a linear search, in order to estimate the maximum value of the parameter γ , say $\gamma_{\max}$, such that system (2.18) is finite-time stable wrt $(0, 1, R, \Gamma)$.

Note that, in order to recast the DLMI condition (2.7a)–(2.7c) in terms of LMIs, the matrix-valued function $P(\cdot)$ has been assumed piecewise linear by dividing the time interval $[0, T]$ in $n = T/T_s$ subintervals and assuming the time derivative of $P(\cdot)$ constant in each subinterval:

$$P(t) = \begin{cases} P_0 + \Theta_1(t - t_0), & t \in [t_0, t_0 + T_s], \\ P_0 + \sum_{h=1}^j \Theta_h T_s + \Theta_{j+1}(t - jT_s - t_0), & t \in (t_0 + jT_s, t_0 + (j+1)T_s], \\ j = 1, \dots, J, \end{cases}$$

where $J = \max\{j \in \mathbb{N} : j < T/T_s\}$, $T_s \ll T$, and $P_0, \Theta_l, l = 1, \dots, J+1$, are the optimization variables. It is straightforward to recognize that such a piecewise function can approximate a generic continuous $P(\cdot)$ with adequate accuracy, provided that the length of T_s is sufficiently small.

The obtained estimates of $\gamma_{\max}$, the corresponding values T_s , and the average computation times for a single iteration of the linear search are shown in Table 2.1. These results have been obtained by running the Matlab LMI Toolbox [48] on a PC equipped with an Intel i5-460M processor and 6 GB of RAM.

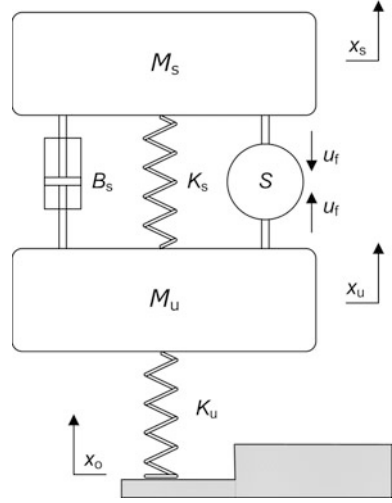
The example shows that the condition based on the solution of the DLE is more efficient than the DLMI-based approach, both in terms of computing time and in terms of estimation of $\gamma_{\max}$.

It is clear from Table 2.1 that, when the DLMIs are involved, increasing the sampling time causes only a marginal improvement in terms of the $\gamma_{\max}$ estimation, while the time efficiency decreases dramatically.

2.5 Application to the Car Suspension System

In this section, in order to illustrate the proposed technique, we present a typical engineering case study, namely a vehicle active suspension system.

Fig. 2.1 Schematic representation of the active suspension system



2.5.1 The Model

The model considered to this aim is taken from [40], where the authors focus on the design of an $\mathcal{H}_\infty$ control scheme satisfying some output and control constraints.

The scheme of a two-degrees-of-freedom (2-DOF) quarter-car model is reported in Fig. 2.1: the system comprises the sprung mass M_s , the unsprung mass M_u , the suspension damper with damping coefficient B_s , the suspension spring with elastic coefficient K_s , the elastic effect caused by the tire deflection, modeled by means of a spring with elastic coefficient K_u , and the hydraulic actuator S generating a scalar active force u_f . As state variables, we choose the suspension stroke $x_s - x_u$, the tire deflection $x_u - x_o$, and the derivatives with respect to time of x_s and x_u , that is,

$$x_1 = x_s - x_u,$$

$$x_2 = \dot{x}_s,$$

$$x_3 = x_u - x_o,$$

$$x_4 = \dot{x}_u,$$

where x_s and x_u are the vertical displacements of the sprung and unsprung masses, respectively, and x_o is the vertical ground displacement caused by the road unevenness. The resulting open-loop dynamical model reads

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{K_s}{M_s} & -\frac{B_s}{M_s} & 0 & \frac{B_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \frac{K_s}{M_u} & \frac{B_s}{M_u} & -\frac{K_u}{M_u} & -\frac{B_s}{M_u} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{u_{\max}}{M_s} \\ 0 \\ -\frac{u_{\max}}{M_u} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} w(t), \quad (2.20)$$

where the normalized active force $u = u_f / u_{\max}$ is the control input, and $w = \dot{x}_o$ is an exogenous disturbance generated by the road roughness. The values of the model parameters used in this example are

$$\begin{aligned} M_s &= 320 \text{ kg}, & K_s &= 50 \frac{\text{kN}}{\text{m}}, & B_s &= 2000 \frac{\text{Ns}}{\text{m}}, \\ K_u &= 200 \frac{\text{kN}}{\text{m}}, & M_u &= 40 \text{ kg}, & u_{\max} &= 100 \text{ kN}. \end{aligned}$$

2.5.2 FTS Analysis

For the system illustrated above, it is important to prevent excessive suspension bottoming, which can lead to a rapid deterioration of ride comfort and possible structural damage. This goal can be translated into a constraint on the suspension stroke limitation in the form

$$|x_1| = |x_s(t) - x_u(t)| \leq SS_{\max}, \quad t \geq 0, \quad (2.21)$$

where $SS_{\max}$ is the maximum allowable suspension stroke.

For simplicity, here we will limit our scope to the case of road bumps (or holes): in this case, the vertical ground displacement can be approximated to a step function, and the disturbance input $w(t)$ is impulsive. Therefore, the bump effect can be reduced to an instantaneous change of the initial value of x_3 , and the disturbance term is identically zero for all $t > 0$ (we set $t = 0$ as the instant when the tyre hits the bump).

This problem can be readily cast in the framework of FTS: first, let us characterize the family of bumps for which we want to perform the FTS analysis by defining an ellipsoidal constraint on the admissible initial conditions; assuming that the system is at steady state at $t = 0^-$, that is, all the state variables but x_3 are zero, the weighting matrix would ideally read

$$R = \text{diag} \left(\infty \quad \infty \quad \frac{1}{x_{3\max}^2} \quad \infty \right). \quad (2.22)$$

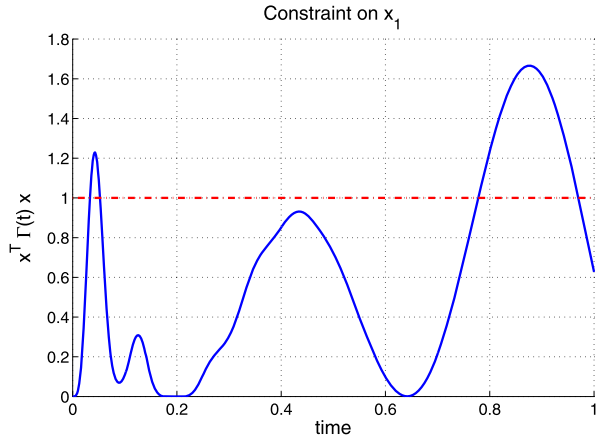
This choice of R considers all the possible disturbances yielding an instantaneous change of $|x_3| \leq x_{3\max}$.

The suspension stroke constraint can be taken into account through the choice of a suitable weighting matrix function $\Gamma(\cdot)$. The matrix

$$\Gamma(t) = \text{diag} \left(\frac{1}{(x_{1\max} e^{-\frac{t}{\tau}})^2} \quad 0 \quad 0 \quad 0 \right) \quad (2.23)$$

defines an exponentially decaying threshold on the suspension stroke; for the initial maximum value, we set $x_{1\max} = 0.02 \text{ m}$ and $\tau = 0.4 \text{ s}$.

Fig. 2.2 Time history of the squared weighted norm of the state of system (2.20); free evolution from initial condition (0 0 0.08 0)



With the given choices of R and $\Gamma(\cdot)$, we can now tackle the problem of analyzing the suspension stroke constraint by solving the FTS analysis problem for system (2.20) (with $w \equiv 0$) with respect to $(0, 1, R, \Gamma(\cdot))$.

In order to avoid numerical singularity issues, the above ideal values of the weighting matrices cannot be used. Instead, we relax the constraints by setting the ∞ entries of R to very large values and the zero entries of $\Gamma(\cdot)$ to very small values.

Figure 2.2 reports the time evolution of $x^T(t)\Gamma(t)x(t)$ for $x_3(0) = 0.08$ m, showing the violation of the constraint on x_1 . The corresponding state response is reported in Fig. 2.3.

Applying condition (iii) of Theorem 2.1, it can be easily checked that the system under consideration is not finite-time stable wrt $(0, 1, R, \Gamma(\cdot))$ when $x_{3\max} = 0.08$, which is consistent with the simulation results shown above.

Now, we can exploit the analysis conditions to find the maximum value of $x_3(0)$ that does not yield a violation of the assigned suspension stroke constraint. To this aim, it is sufficient to iteratively apply Theorem 2.1 changing the values of $x_{3\max}$ in (2.22). By means of a dichotomous search it is readily found that such a value is $\bar{x}_3 = 0.049$ m. This result is confirmed by the state response depicted in Fig. 2.4.

2.6 Summary

In this chapter we have discussed the FTS problem for the class of CT-LTV systems. To this end, a methodology based on the use of time-varying quadratic Lyapunov functions has been developed.

The main result of the section consists of some necessary and sufficient conditions for FTS. The first condition, which involves the state transition matrix, is hard from the computational point of view (unless simple cases are considered); also, it is not useful for design purposes. Therefore, condition (ii) in Theorem 2.1 is not practical.

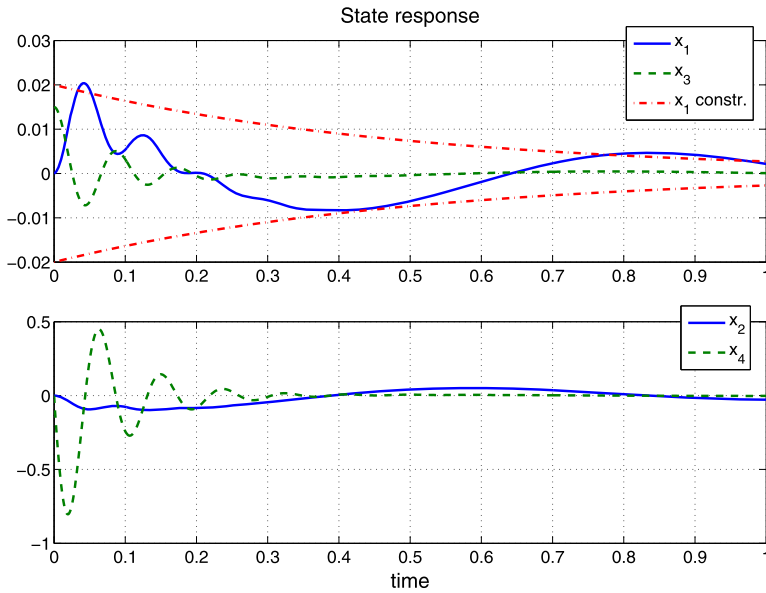


Fig. 2.3 State response of system (2.20): free evolution from initial condition $(0\ 0\ 0.08\ 0)$

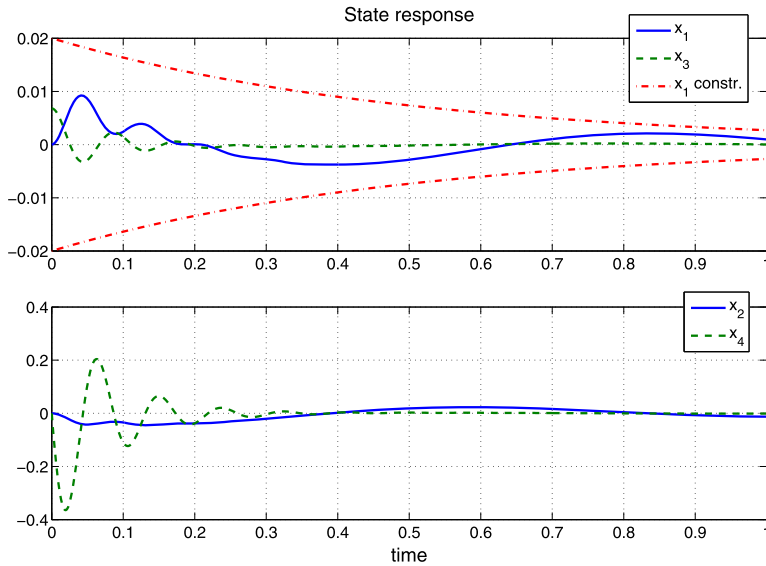


Fig. 2.4 State response of system (2.20): free evolution from initial condition $(0\ 0\ 0.049\ 0)$

Condition (v) in Theorem 2.1 allows us to check FTS of system 2.1 via a DLMI coupled with a pair of (time-varying) LMIs. In the former papers [10, 19], the sufficiency of the condition was only proved. In a later work (see [27]), the necessity was demonstrated, together with the equivalence with condition (iii) in Theorem 2.1,

which, differently from (v), requires the solution of a DLE. The latter condition is more efficient from the computational point of view, as it is shown in the example section. Finally, it is equivalent to Condition (iv), which was independently derived by Garcia et al. [49].

As it will be shown in the following chapter, the DLMI-based condition plays an important role in the derivation of the design conditions. Indeed, the optimization of the controller matrices requires the availability of a convex condition to be numerically implemented.

The methodology has been applied to design a dynamical controller for the car suspension system derived in Sect. 3.4. This engineering example gave also the opportunity of illustrating some approaches for the numerical solution of DLEs and DLMIs and for showing that DLEs are more efficient from the computational point of view than DLMIs.

By restricting Definition 1.1 to the case of continuous-time linear time-invariant (CT-LTI) systems and assuming that both the initial set and the trajectory set are time-invariant ellipsoids having the same shape, a different approach for the FTS study is proposed in [6]. To this end, consider a system of the form

$$\dot{x}(t) = Ax(t), \quad (2.24)$$

where $A \in \mathbb{R}^{n \times n}$. For CT-LTI systems, the FTS definition (2.1) particularizes as follows.

Definition 2.2 (FTS of CT-LTI Systems with Ellipsoidal Domains) Given a positive scalar T and positive definite matrices R and Γ with $\Gamma > R$, the CT-LTI system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad (2.25)$$

is said to be finite-time stable with respect to (T, R, Γ) if

$$x_0^T R x_0 \leq 1 \Rightarrow x(t)^T \Gamma x(t) < 1, \quad t \in [0, T]. \quad (2.26)$$

◇

The following theorem is derived under the assumption that the initial set and the trajectory set have the same shape, namely $\Gamma = \rho R$, for a given positive scalar $\rho < 1$.

Theorem 2.2 [6] System (2.25) is finite-time stable with respect to $(T, R, \rho R)$ with $0 < \rho < 1$ if letting $\tilde{Q} = R^{-\frac{1}{2}} Q R^{-\frac{1}{2}}$, there exist a nonnegative scalar α and a positive definite matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$A \tilde{Q} + \tilde{Q} A^T - \alpha \tilde{Q} < 0, \quad (2.27a)$$

$$\text{cond}(Q) < \frac{1}{\rho} e^{-\alpha T}, \quad (2.27b)$$

where $\text{cond}(Q) = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}$ denotes the condition number of Q . □

To prove Theorem 2.2, consider the Lyapunov function $V(x) := x^T \tilde{Q}^{-1}x$. Suppose that

$$\dot{V}(x(t)) < \alpha V(x(t)) \quad (2.28)$$

for all $t \in [0, T]$. The first step consists of proving that conditions (2.28) and (2.27b) imply that system (2.25) is finite-time stable with respect to $(T, R, \rho R)$. Indeed, by dividing both sides of (2.28) by $V(x)$, and integrating from 0 to t , with $t \in (0, T]$, we obtain

$$\log \frac{V(x(t))}{V(x(0))} < \alpha t. \quad (2.29)$$

We have, for $t \in (0, T]$,

$$x^T(t) R^{1/2} Q^{-1} R^{1/2} x(t) \geq \lambda_{\min}(Q^{-1}) x^T(t) R x(t) \quad (2.30)$$

and

$$x^T(0) R^{1/2} Q^{-1} R^{1/2} x(0) e^{\alpha t} \leq \lambda_{\max}(Q^{-1}) e^{\alpha T}. \quad (2.31)$$

Putting together (2.29)–(2.31), we have

$$x^T(t) R x(t) < \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)} e^{\alpha T} = \frac{1}{\rho}. \quad (2.32)$$

To conclude the proof, note that condition (2.28) is equivalent to (2.27a).

In Theorem 2.2 an exponential time weighting $e^{-\alpha t/2}$ of the state with a *negative* exponent is used, while in the Lyapunov stability context one uses an exponential time weighting with a positive exponent. This allows us to relax the classical conditions for asymptotic stability and to establish the FTS of the system under consideration even if it is not asymptotically stable.

It is easy to check that condition (2.27b) can be turned into the LMI in the matrix variable Q

$$\rho e^{\alpha T} I < Q < I.$$

Therefore, from a computational point of view, it is important to notice that, once a value for α has been fixed, the feasibility of the conditions stated in Theorem 2.2 can be turned into an LMI-based feasibility problem.

As said, one strong drawback of the approach followed in Theorem 2.2 is that we have to restrict the matrix Γ to be in the form $\Gamma = \rho R$ with $\rho < 1$; in other words, the (time-invariant) trajectory set is constrained to be a scaled version of the initial set. This is a weakness with respect to the approach followed in Theorem 2.1. However, Theorem 2.2 plays an important role in the following developments since it probably represents the first attempt to applying an LMI-based approach to solve FTS problems.

Conditions for finite-time stabilization via state and output feedback, consistent with Definition 2.2, are proposed in [6] and [12], respectively, and will be briefly illustrated in Sect. 3.5.

Finite-Time Stability and Control

Amato, F.; Ambrosino, R.; Ariola, M.; Cosentino, C.; De Tommasi, G.

2014, XII, 146 p. 37 illus., 14 illus. in color., Softcover

ISBN: 978-1-4471-5663-5