

Chapter 2

PDF Method: A Stochastic Framework

To model the desired macroscopic quantities, the easiest way is to write closed PDEs for specific quantities. In this process, additional unknowns are added to the system and the success of this method depends on the accuracy with which the unclosed terms explicitly can be based on macroscopic laws or using a model which does not depart dramatically from reality. As discussed in Chap. 1, the applicability of this method to complicated multiphase flows is limited because the average of a complicated function of a variable such as $\langle f(\varphi) \rangle$ must be defined based on the available information, which is typically limited to the first- and second-order moments of the variable φ , i.e. $\langle \varphi \rangle$ and $\langle \varphi^2 \rangle$. For complicated reactive or poly-dispersed flows, this information is typically insufficient to accurately define such functions. A full or direct numerical simulation is also not feasible for complicated flows and geometries, and the only reasonable solution would be what is usually referred to as a mesoscopic approach.

In this monograph, we mainly focus on the mesoscopic approach based on PDFs, and hence, a brief introduction to the existing stochastic and statistical methods in a form suitable for modelling poly-dispersed turbulent particulate flows is required and is provided in this chapter. The material in this chapter is presented without formal proof, and the interested reader can consult [1–4] for further information. Several examples are, however, provided to demonstrate the significance of different stochastic processes.

2.1 Definition of a Stochastic Process

A stochastic variable, X , in applied sciences is usually defined directly from its probability density function (PDF). For now, assume that this random (stochastic) variable, X , is a scalar, e.g. one-dimensional velocity in Brownian motion or temperature of a particle. This stochastic variable, X , can take a range of possible values

$x \in S$ where S can be a discrete or continuous set such as \mathbb{R} or \mathbb{R}^d and the probability that X takes the values between x and $x + dx$ is

$$\Pr(x < X < x + dx) = \mathcal{P}(x)dx. \quad (2.1)$$

In the same fashion, we can define a multivariate distribution, also called joint probability distribution of r variables X_1, \dots, X_r . Taking a subset s of r , ($s < r$), marginal distribution of s variables is the probability that X_1, \dots, X_s take the values x_1, \dots, x_s , regardless of values of X_{s+1}, \dots, X_r , i.e.

$$\mathcal{P}(x_1, \dots, x_s) = \int \mathcal{P}(x_1, \dots, x_s, x_{s+1}, \dots, x_r) dx_{s+1}, \dots, dx_r \quad (2.2)$$

It is also possible for one to assign fixed values to X_{s+1}, \dots, X_r and ask for the joint probability of the remaining variables which is called conditional probability of X_1, \dots, X_s and defined by $\mathcal{P}(x_1, \dots, x_s | x_{s+1}, \dots, x_r)$. The joint probability of X_1, \dots, X_r is equal to the marginal probability of X_{s+1}, \dots, X_r , to have the values x_{s+1}, \dots, x_r , multiplied by the conditional probability of X_1, \dots, X_s , given the values of X_{s+1}, \dots, X_r :

$$\mathcal{P}(x_1, \dots, x_r) = \mathcal{P}(x_{s+1}, \dots, x_r) \mathcal{P}(x_1, \dots, x_s | x_{s+1}, \dots, x_r) \quad (2.3)$$

Once the stochastic variable X is defined, other stochastic variables, namely Y , can be derived from it by a mapping, f . These new variables can also be functions of time, t , and we can write

$$Y_X(t) = f(X, t) \quad (2.4)$$

By replacing X by x , one possible value, we get $Y_x(t) = f(x, t)$, an ordinary function called a realization of the process. Therefore, the stochastic process in physical sense is an ensemble of these realizations. By measuring the values, x_0, \dots, x_n at times t_0, \dots, t_n where $t_0 \leq t_n$, we can completely describe the process by the joint probability density function $\mathcal{P}(x_n, t_n; \dots; x_0, t_0)$. In case of complete independence, we can write

$$\mathcal{P}(x_n, t_n; \dots; x_0, t_0) = \prod_i \mathcal{P}(x_i, t_i) \quad (2.5)$$

This means that the values of X at time t are independent of its values in the past or future. Also note that the x_i values could each be a vector, e.g. velocity components or the whole phase space vector, i.e. all the velocity and position components in addition to other scalars. We use bold symbols such as \mathbf{X} or \mathbf{Z} , and their corresponding values \mathbf{x} and \mathbf{z} , to indicate that these stochastic variables are in fact vectors, which are also functions of time, defining the whole phase space.

2.2 Markov Process

The general process defined by¹:

$$\mathcal{P}(\mathbf{x}_n, t_n; \dots; \mathbf{x}_0, t_0) \quad (2.6)$$

is very difficult to handle since one needs the knowledge of all the previous points in time to describe such a process. Accordingly, the problem is usually restricted to a family of processes known as Markov processes where the knowledge of present state of the system completely describes the whole process. In other words, the past history of process has no effect on the future evolution of the process or mathematically

$$\mathcal{P}(\mathbf{x}_n, t_n | \mathbf{x}_{n-1}, t_{n-1}; \dots; \mathbf{x}_0, t_0) = \mathcal{P}(\mathbf{x}_n, t_n | \mathbf{x}_{n-1}, t_{n-1}) \quad (2.7)$$

In Eq. (2.7), $\mathcal{P}(\mathbf{x}_n, t_n | \mathbf{x}_{n-1}, t_{n-1})$ is also called the transition probability. This assumption is very powerful and means everything can be defined in terms of a transition probability $\mathcal{P}(\mathbf{x}_n, t_n | \mathbf{x}_{n-1}, t_{n-1})$ and an initial probability $\mathcal{P}(\mathbf{x}_0, t_0)$. Thus, for example,

$$\mathcal{P}(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1; \mathbf{x}_0, t_0) = \mathcal{P}(\mathbf{x}_2, t_2 | \mathbf{x}_1, t_1) \mathcal{P}(\mathbf{x}_1, t_1 | \mathbf{x}_0, t_0) \mathcal{P}(\mathbf{x}_0, t_0) \quad (2.8)$$

Note that the process can be continuous or discontinuous, regardless of the nature of the variable \mathbf{X} . For example, the sample space of classical Brownian motion of a particle immersed in a collection of light molecules, with assumption of hard sphere collisions, by virtue of instant jumps, is not continuous in spite of the fact that the range of velocities is continuous.

2.3 The Chapman–Kolmogorov Equation

For a general stochastic process, we can write:

$$\mathcal{P}(\mathbf{x}_2, t_2 | \mathbf{x}_0, t_0) = \int \mathcal{P}(\mathbf{x}_2, t_2 | \mathbf{x}_1, t_1; \mathbf{x}_0, t_0) \mathcal{P}(\mathbf{x}_1, t_1 | \mathbf{x}_0, t_0) d\mathbf{x}_1 \quad (2.9)$$

Introducing the Markov property, i.e. Eq. (2.7) into Eq. (2.9), results in the celebrated Chapman–Kolmogorov (CK) equation:

$$\mathcal{P}(\mathbf{x}_2, t_2 | \mathbf{x}_0, t_0) = \int \mathcal{P}(\mathbf{x}_2, t_2 | \mathbf{x}_1, t_1) \mathcal{P}(\mathbf{x}_1, t_1 | \mathbf{x}_0, t_0) d\mathbf{x}_1 \quad (2.10)$$

¹ In this chapter, we intend to use the tensor notation for all variables except when a variable is used as an argument of a function, e.g. $\mathcal{P}(\mathbf{x})$, and when used as an integration variable, where using the tensor notation is confusing and leads to misinterpretation of the equation.

Equation (2.10) simply states that the probability of a process ending in state (\mathbf{x}_2, t_2) given the initial state (\mathbf{x}_0, t_0) is equal to the sum of all possible paths from $(\mathbf{x}_0, t_0) \rightarrow (\mathbf{x}_2, t_2)$. The CK equation is a complex nonlinear equation relating all conditional probabilities to each other. Therefore, it is convenient to derive a differential form of CK equation which is easier to handle and physically easier to interpret. For example, the differential form can be derived based on a trajectory point of view [2].

Now considering the time evolution of the expectation of a twice differentiable function and using the CK equation, Eq. (2.10), the differential CK equation [2, 4, 5] can be derived as

$$\begin{aligned} \frac{\partial \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)}{\partial t} = & - \sum_i \frac{\partial}{\partial x_i} (A_i(\mathbf{x}, t) \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)) \\ & + \sum_{ij} \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} (B_{ij}(\mathbf{x}, t) \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)) \\ & + \int (J(\mathbf{x} | \mathbf{z}, t) \mathcal{P}(\mathbf{z}, t | \mathbf{x}_0, t_0) - J(\mathbf{z} | \mathbf{x}, t) \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)) d\mathbf{z}. \end{aligned} \quad (2.11)$$

In the differential CK equation, Eq. (2.11), the first and second terms define the drift and diffusion processes, respectively, and the last integral defines a jump process. Jump processes will be discussed thoroughly in Sect. 2.6 and a pure drift process, which is also known as the deterministic process (or Liouville's Equation) and a diffusion process (or Fokker–Planck Equation), is discussed in Sect. 2.5. The differential CK equation simply states that having the initial distribution of some variable \mathbf{x} at time t_0 , the final distribution at any time in the future can be found using this equation.

We are sometimes interested in knowing the probability of a system at state \mathbf{x} at time t_0 to finish in a subset of states S . In this case, we want to know for every state \mathbf{x} at time $t < t_{\text{final}}$ what is the probability of ending up in the subset S . In this case, the differential CK equation can be restated as the backward CK equation:

$$\begin{aligned} \frac{\partial \mathcal{P}(\mathbf{x}_0, t_0 | \mathbf{x}, t)}{\partial t} = & - \sum_i A_i(\mathbf{x}, t) \frac{\partial}{\partial x_i} (\mathcal{P}(\mathbf{x}_0, t_0 | \mathbf{x}, t)) \\ & - \frac{1}{2} \sum_{ij} B_{ij}(\mathbf{x}, t) \frac{\partial^2}{\partial x_i \partial x_j} (\mathcal{P}(\mathbf{x}_0, t_0 | \mathbf{x}, t)) \\ & + \int (J(\mathbf{z} | \mathbf{x}, t) \mathcal{P}(\mathbf{x}_0, t_0 | \mathbf{x}, t) - J(\mathbf{x}_0, t_0 | \mathbf{z}, t) \mathcal{P}(\mathbf{x}_0, t_0 | \mathbf{z}, t)) d\mathbf{z}. \end{aligned} \quad (2.12)$$

Next, we will define the most basic stochastic process (Wiener process) and discuss the solution of forward CK equations for this simple case, and in Sect. 2.7, we provide a solution to the backward equation.

2.4 Wiener Process

The Wiener process is the most fundamental of continuous time stochastic processes and is essentially a diffusion process with $A_i = 0$, $B_{ij} = 1$ and $J = 0$. A standard Wiener process on the interval $[0, T]$ is a random variable $W(t)$ that depends continuously on $t \in [0, T]$ and satisfies the following conditions: (i) the increments of the Wiener process are distributed according to a normal (Gaussian) distribution; (ii) trajectories of $W(t)$ are continuous but cannot be differentiated; (iii) Increments of $W(t)$ are stationary and independent; and (iv) trajectories are of unbounded variation in finite time intervals. Numerical solution to the SDEs is out of the scope of this book and can be found elsewhere [5]. However, simulating a Wiener process is straightforward by using the property (i), and one only needs to sample from a normal distribution and add this to the current state of the system using the relation $W_{i+1} = W_i + dW_i$ and scale the results. The Wiener process physically translates to a pure Brownian motion without any frictional coefficient defined by [6, 7]

$$\frac{dU_{p,x}}{dt} = W_{p,x}(t), \quad (2.13)$$

with initial condition:

$$U_{p,x}|_{t=0} = U_{p,x0}, \quad (2.14)$$

where $W_{p,x}$ is a random rapidly fluctuating force per unit mass exerted on the particle p , due to collisions with other smaller particles. This type of differential equation with a stochastic function on one side is known as the Langevin equation, c.f. [1, 2]. The right-hand side of Eq. (2.13) and so the Langevin equation can include other terms, such as a friction ($\mu U_{p,x}$) term [6].

Figure 2.1 shows a sample path of such motion simulated by integrating Eq. (2.13) directly. The pure stochastic motion without any friction is shown with $\mu = 0$ in this figure. Evidently, the solution deviates from the initial conditions significantly. However, setting the frictional coefficient to $\mu = -1$ acts as a restoring force, and the solution only vibrates around the initial value which physically translates to a random motion but with a bounded molecular velocity and is a better model for the physical phenomenon.

The solution of the forward differential CK equation for the Wiener process is given in Fig. 2.2. For the forward equation, the proper initial condition is $\mathcal{P}(\mathbf{x}, t_0 | \mathbf{x}_0, t_0) = \delta(\mathbf{x} - \mathbf{x}_0)$, which simply means that we know the solution at $t = 0$ with probability one. Figure 2.3 shows the effect of the restoring force, which prevents the PDF from rapid evolution, and the value of the velocity remains around

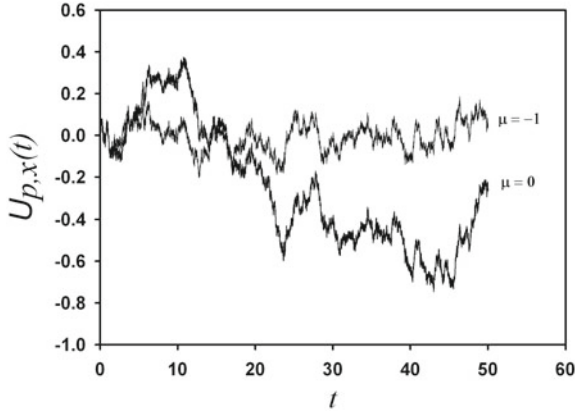


Fig. 2.1 1D Brownian motion—two sample path of the Brownian motion with and without friction coefficient μ . Friction force acts as a restoring force that keeps the velocity deviations bounded

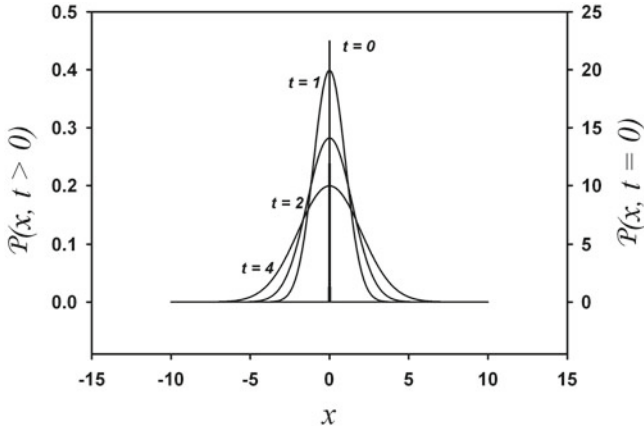


Fig. 2.2 Solution of the forward CK equation for a Wiener process—the initial condition is known with probability one and thus is a scaled delta function at $t = 0$, and the PDF then evolves and becomes flatter in time, increasing the uncertainty in the solution

the initial conditions with probability one. This process with a linear drift added to the Wiener process is also known as Ornstein–Uhlenbeck process. Figure 2.4 shows the changes in the standard deviation, σ , of the PDF in time: smaller standard deviation corresponds to the smaller probability of occurrence of velocities far from the mean, while large values show that the extreme velocities are more probable. This simple example clarifies the connection between the trajectory and PDF point of view and also shows why the trajectory point of view resolves more information than the PDF point of view.

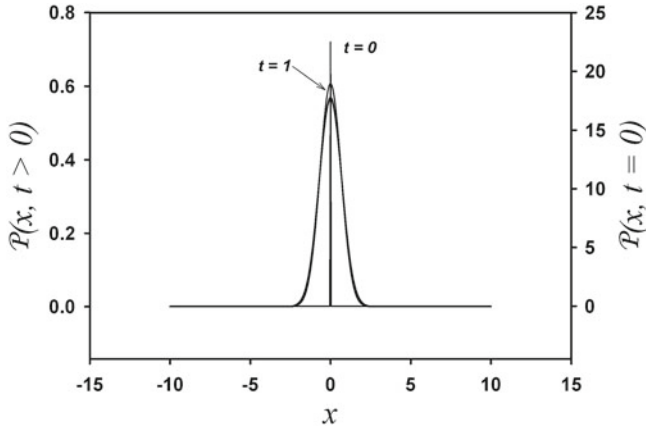


Fig. 2.3 Solution of the forward CK equation for the Brownian motion—a Brownian motion with $\mu = -1$. Due to the certainty of the initial condition, it takes the form of the delta function. In this case, restoring force keeps the solution around a mean and the PDF remains constant for $t > 2$

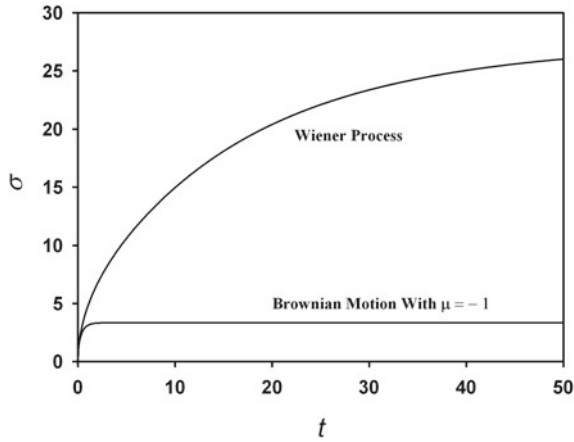
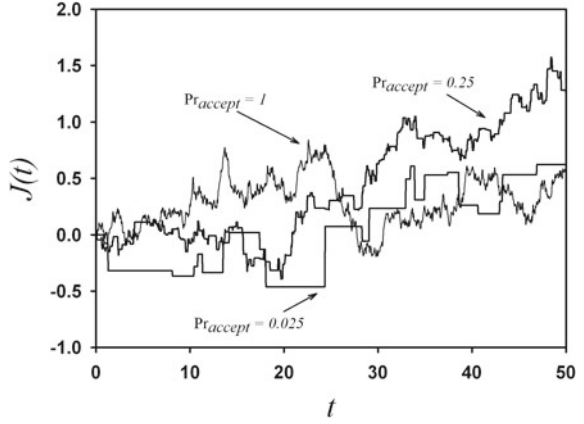


Fig. 2.4 Comparison between standard deviations—changes in standard deviation of the PDF of the Wiener process and Brownian motion with $\mu = -1$ for $t \in [0, 50]$. It increases significantly for the Wiener process, while it rapidly reaches a small constant value for the Brownian motion with $\mu = -1$

2.5 Diffusion Process

A diffusion process is another subset of Markov process applicable to many physical systems where the sample path is continuous. In this case, J should be zero in the differential CK equation, Eq. (2.11), as this term represents discontinuities, thereby reducing the complexity of the equation describing this process. Thus, the final form

Fig. 2.5 Jump process—a typical jump process (labelled by $\text{Pr}_{\text{accept}} = 0.025$). By increasing the acceptance rate and reducing the size of each jump, we can always approximate a diffusion process by a jump process. Note the similarities between the graph for $\text{Pr}_{\text{accept}} = 1$ and Fig. 2.1



of the equation, which is commonly referred to as the Fokker–Planck equation, becomes

$$\begin{aligned} \frac{\partial \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)}{\partial t} = & - \sum_i \frac{\partial}{\partial x_i} (A_i(\mathbf{x}, t) \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)) \\ & + \sum_{ij} \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} (B_{ij}(\mathbf{x}, t) \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)). \end{aligned} \quad (2.15)$$

2.6 Jump Process

Consider the case where $A_i(\mathbf{x}, t) = B_{ij}(\mathbf{x}, t) = 0$, differential form of CK equation, Eq. (2.11), reduces to

$$\frac{\partial \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)}{\partial t} = \int (J(\mathbf{x} | \mathbf{z}, t) \mathcal{P}(\mathbf{z}, t | \mathbf{x}_0, t_0) - J(\mathbf{z} | \mathbf{x}, t) \mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0)) d\mathbf{z}. \quad (2.16)$$

Equation (2.16) is usually known as the master equation. It should be noted that the master equation can also be interpreted as a simple gain–loss of probabilities such that the first term in the integral is the gain due to transition from other states and the second is the loss due to transition to other states. It can be said that every diffusion process can be approximated by a jump process. This means that in the limit of infinitely small jump sizes, the master equation becomes a Fokker–Planck equation. Figure 2.5 shows the sample path of a jump process and this scaling assumption (see also [2]).

The equivalence between the jump process and diffusion process can also be demonstrated analytically by deriving the Fokker–Planck equation from the master equation. To show this, without loss of generality, we assume a univariate stochastic process. Furthermore, we need to define the increments of a stochastic process. The increment defines the behaviour of the process over small time periods. The increment Δ_s of a general stochastic process $X(t)$ in a positive time interval s is defined by

$$\Delta_s X(t) = X(t + s) - X(t), \quad \text{for } s > 0, \quad (2.17)$$

and

$$dX(t) = \lim_{s \rightarrow 0} [X(t + s) - X(t)] = \lim_{s \rightarrow 0} [\Delta_s X(t)]. \quad (2.18)$$

Here, s is positive; hence, the increment is considered to be forward in time. A process can be considered to be the sum of increments.

$$X(t_N) = X(t_0) + \sum_{k=1}^N \Delta_{t_k - t_{k-1}} X(t_{k-1}). \quad (2.19)$$

The PDF of the increment $\Delta_s X(t)$ conditional on $X(t) = x$ is denoted as $g(\hat{x}; s, x, t)$ where \hat{x} represents the phase space increment. If s is taken to be $t_2 - t_1$, then $X(t_1)$ can be expressed as

$$X(t_1) = X(t_2) - \Delta_s X(t_1), \quad (2.20)$$

hence, we can rewrite

$$\mathcal{P}(x_2; t_1 + s | x_1, t_1) = g(\hat{x}; s, x_2 - \hat{x}, t_1). \quad (2.21)$$

Now, the CK equation (2.10) is rewritten

$$\mathcal{P}(x_2; t_1 + s | x_0, t_0) = \int g(\hat{x}; s, x_2 - \hat{x}, t_1) \mathcal{P}(x_2 - \hat{x}; t_1 | x_0, t_0) d\hat{x}, \quad (2.22)$$

or

$$\mathcal{P}(x; t_1 + s | x_0, t_0) = \int g(\hat{x}; s, x - \hat{x}, t_1) \mathcal{P}(x - \hat{x}; t_1 | x_0, t_0) d\hat{x}. \quad (2.23)$$

In the CK equation, both g and p on the right-hand side involve the argument $x - \hat{x}$. Using a Taylor series expansion,² we have

² Taylor series expansion of $f(x)$ about a point $x - a$ is given by

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^n(a)}{n!}(x - a)^n + \cdots.$$

$$\begin{aligned}
\mathcal{P}(x; t_1 + s | x_0, t_0) &= \mathcal{P}(x_1; t_1 | x_0, t_0) \\
&+ \int \sum_{n=1}^{\infty} \frac{(-\hat{x})^n}{n!} \frac{\partial^n}{\partial x^n} [g(\hat{x}; s, x_1, t_1) \mathcal{P}(x_1; t_1 | x_0, t_0)] d\hat{x}.
\end{aligned} \tag{2.24}$$

Rearranging and dividing through by s give

$$\begin{aligned}
&\frac{1}{s} [\mathcal{P}(x; t_1 + s | x_0, t_0) - \mathcal{P}(x_1; t_1 | x_0, t_0)] \\
&= \frac{1}{s} \int \sum_{n=1}^{\infty} \frac{(-\hat{x})^n}{n!} \frac{\partial^n}{\partial x_n} [g(\hat{x}; s, x_1, t_1) \mathcal{P}(x_1; t_1 | x_0, t_0)] d\hat{x},
\end{aligned} \tag{2.25}$$

and taking the limit $s \rightarrow 0$ term by term gives

$$\lim_{s \rightarrow 0} \frac{1}{s} [\mathcal{P}(x; t_1 + s | x_0, t_0) - \mathcal{P}(x_1; t_1 | x_0, t_0)] = \frac{\partial}{\partial x} \mathcal{P}(x; t | x_0, t_0), \tag{2.26}$$

for the left-hand side and

$$\begin{aligned}
&\lim_{s \rightarrow 0} \frac{1}{s} \int \sum_{n=1}^{\infty} \frac{(-\hat{x})^n}{n!} \frac{\partial^n}{\partial x_n} [g(\hat{x}; s, x_1, t_1) \mathcal{P}(x_1; t_1 | x_0, t_0)] d\hat{x} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x_n} \left[\lim_{s \rightarrow 0} \frac{1}{s} \left\{ \int \hat{x} g(\hat{x}; s, x_1, t_1) d\hat{x} \right\} \mathcal{P}(x_1; t_1 | x_0, t_0) \right] \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x_n} [B_n(x, t) \mathcal{P}(x_1; t_1 | x_0, t_0)],
\end{aligned} \tag{2.27}$$

for the right-hand side. Combining terms gives us the Kramers–Moyal equation

$$\frac{\partial}{\partial x} \mathcal{P}(x; t | x_0, t_0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x_n} [B_n(x, t) \mathcal{P}(x_1; t_1 | x_0, t_0)]. \tag{2.28}$$

At this stage, it is still assumed that the parameters B_n exist for all $n > 0$. The Kramers–Moyal equation is valid for general stochastic processes. If $t > t_0$, then we have the initial condition

$$\mathcal{P}(x; t | x_0, t_0) = \delta(x - x_0), \tag{2.29}$$

and for a diffusion process, we know that $B_n = 0$ for $n \geq 3$. Given these conditions, the Kramers–Moyal equation simplifies to give the Fokker–Planck equation

$$\frac{\partial}{\partial t} \mathcal{P}(x; t | x_0, t_0) = -\frac{\partial}{\partial x} [a(x, t) \mathcal{P}(x; t | x_0, t_0)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [b(x, t)^2 \mathcal{P}(x; t | x_0, t_0)]. \quad (2.30)$$

The Fokker–Planck equation describes the evolution of the transitional PDF $\mathcal{P}(x; t | x_0, t_0)$ for a stochastic diffusion process. To obtain the evolution equation of the marginal PDF $\mathcal{P}(x; t)$, simply multiply by $\mathcal{P}(x_0; t_0)$ and integrate over x_0 to give

$$\frac{\partial}{\partial x} \mathcal{P}(x; t) = -\frac{\partial}{\partial x} [a(x, t) \mathcal{P}(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [b(x, t)^2 \mathcal{P}(x, t)]. \quad (2.31)$$

This equation is for the stochastic process $X(t)$ with drift and diffusion coefficients $a(x; t)$ and $b(x; t)^2$, respectively.

2.7 Stochastic Differential Equations

A stochastic differential equation (SDE) is a differential equation in which one or more terms are stochastic processes resulting in a solution which itself is a stochastic process. A simple SDE would be that of a Brownian motion introduced in Sect. 2.4. Here, we restrict attention to the relations between SDEs and the Fokker–Planck equation and will not discuss the SDE theory in detail.

A general stochastic differential equation also known as Langevin equation has the form:

$$dZ_i(t) = A_i(\mathbf{Z}(t), t)dt + B_{ij}(\mathbf{Z}(t), t)dW_j(t), \quad (2.32)$$

where W_j are a set of independent Wiener processes. It is important to notice that both the drift vector A_i and the diffusion matrix B_{ij} are functions of state vector variables $Z_i(t)$. To get from this equation to the corresponding Fokker–Planck equation, one should consider the time development of an arbitrary $f(\mathbf{Z}(t))$, and using the rules of Itô calculus, it is easy to show [2, 3] that the corresponding Fokker–Planck equation in n -dimensional sample space is

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\partial}{\partial z_i} [A_i(\mathbf{z}; t) \mathcal{P}] + \frac{1}{2} \frac{\partial^2}{\partial z_i \partial z_j} [(\mathbf{B}\mathbf{B}^T)_{ij}(\mathbf{z}; t) \mathcal{P}]. \quad (2.33)$$

In the above equation, \mathbf{B}^T is the transpose of \mathbf{B} . If the diffusion matrix $B_{ij} = 0$, then the SDE reduces to the deterministic process:

$$dZ_i(t) = A_i(\mathbf{Z}(t), t)dt \quad (2.34)$$

With corresponding Liouville equation,

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\partial}{\partial z_i} [A_i(\mathbf{z}; t) \mathcal{P}]. \quad (2.35)$$

This is a completely deterministic system, i.e. if $z_i(\mathbf{x}_0, t)$ is the solution to Eq. (2.34) with initial conditions $z_i(\mathbf{x}_0, t_0) = x_{i,0}$, then $\mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0) = \delta(\mathbf{x} - \mathbf{z}(\mathbf{x}_0, t))$ is the solution to Eq. (2.35) with initial conditions $\mathcal{P}(\mathbf{x}, t_0 | \mathbf{x}_0, t_0) = \delta(\mathbf{x} - \mathbf{x}_0)$. The proof is best obtained by direct substitution of

$$\mathcal{P}(\mathbf{x}, t | \mathbf{x}_0, t_0) = \delta(\mathbf{x} - \mathbf{z}(\mathbf{x}_0, t)), \quad (2.36)$$

into the RHS of Eq. (2.35) (note that x_i is the independent variables):

$$\begin{aligned} -\frac{\partial}{\partial x_i} [A_i(\mathbf{x}; t) \delta(\mathbf{x} - \mathbf{z}(\mathbf{x}_0, t))] &= -\frac{\partial}{\partial x_i} [A_i(\mathbf{z}; t) \delta(\mathbf{x} - \mathbf{z}(\mathbf{x}_0, t))] \\ &= -A_i(\mathbf{z}; t) \frac{\partial}{\partial x_i} [\delta(\mathbf{x} - \mathbf{z}(\mathbf{x}_0, t))]. \end{aligned} \quad (2.37)$$

The property of delta function is used in moving from first to second line and the fact that A_i is not a function of \mathbf{x} anymore, to move from second to third line. Also substituting Eq. (2.36) into the LHS of Eq. (2.35) and using the chain rule and again using the properties of the delta function give

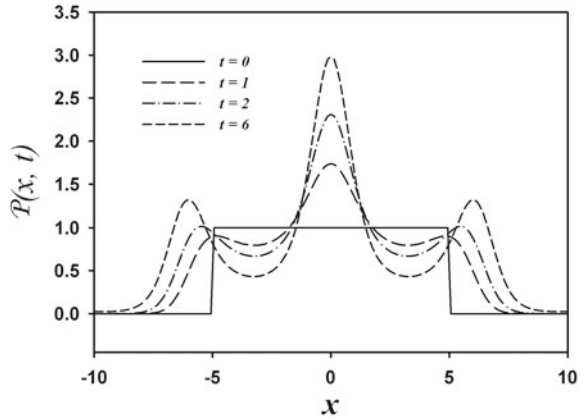
$$\frac{\partial}{\partial t} [\delta(\mathbf{x} - \mathbf{z}(\mathbf{x}_0, t))] = -\frac{\partial}{\partial x_i} [\delta(\mathbf{x} - \mathbf{z}(\mathbf{x}_0, t))] \frac{dz_i(t)}{dt}. \quad (2.38)$$

Now using $dZ_i(t) = A_i(\mathbf{Z}(t), t)dt$ in Eq. (2.38) and changing to phase space notation Eq. (2.37) are restored.

Using an approach based on the Fokker–Planck equation provides a unique opportunity to add many physical phenomena to the model very conveniently, and also, information about the model can be inferred at this stage. For example, it is well known that inhomogeneous turbulence results in non-Gaussian PDFs [8] and this behaviour can easily be included in the model using a nonlinear drift term.

Figure 2.6 shows the evolution of the PDF of a SDE derived by adding the nonlinear drift $\sin(X_t)$ to a Wiener process. An appropriate final condition for the backward CK equation can be the box function $\mathbb{1}_{[-5,5]}$. The solution shows that the non-Gaussian behaviour is easily captured using this nonlinear drift term, and the probability of a solution starting at any previous time ending up in state $\mathbb{1}_{[-5,5]}$ is the solution to the backward equation.

Fig. 2.6 Solution of the backward CK equation—a nonlinear drift term $\sin(X_t)$ produces a non-Gaussian behaviour of the PDF showing the benefits of modelling at a mesoscopic level



2.8 Fluid–Particle Systems

In this section, we show how the results of the probability theory and stochastic calculus presented in the previous section can be used to derive the various equations for modelling poly-dispersed particulate flows. Characterization of individual particles (solid or fluid) requires detailed knowledge of particle position, velocity, size, shape, rotation and temperature at a specified time. The equations derived in this manner are not ordinary PDEs, but are PDEs in a hyperspace of several dimensions, and this makes the direct solution to these equations a formidable task [9]. Therefore, it is necessary to reduce the dimensionality of the problem and make some simplifying assumptions to reduce the solution cost and also the model development process. We assume throughout the rest of this review that

- All particles are spherical, and hence, particles are only characterized by their diameter.
- The state vector of particle i consists of its position, velocity and diameter $(\mathbf{X}^{(i)}, \mathbf{U}^{(i)}, \phi^{(i)})$.
- Particles are non-interacting and can be treated as point processes, which implies a particle with no spatial extent. In Sect. 4.2, we discussed that here, we are interested in turbulence–particle interaction, and thus, particle–particle interactions are discarded in the following discussion. However, it should be obvious at this point that these forces can be included as a jump process in our general framework. This can be done by defining $W(\mathbf{x}|\mathbf{z}, t) = \lambda(\mathbf{z}, t)g(\mathbf{x}|\mathbf{z}, t)$ where $\lambda(\mathbf{z}, t)$ is the probability of occurrence of a jump at state \mathbf{z} and $g(\mathbf{x}|\mathbf{z}, t)$ is the probability of a jump to have a specified amplitude in going from state \mathbf{z} to state \mathbf{x} . This is exactly the process we used to produce Fig. 2.5; however, the problem is how to model g and λ based on the underlying physical phenomena [4]. Williams and Crane [10] studied the particle collision rate in turbulent flows for both solid particles and bubbles and could be a good starting point for such models. For a discussion of

multiparticle statistics in the context of kinetic theory for dense particulate flows, see [11–14].

- We deal only with Markov-type processes, which implies no dependence on past events.
- The fluid particle may be considered to be a small element of fluid with characteristic length scale smaller than the Kolmogorov length scale, $\eta = (\nu^3/\langle\varepsilon\rangle)^{1/4}$, but much larger than the molecular free path.

2.8.1 Definition of Distribution Function

When seeking a statistical description of a turbulent particle-laden flow, the starting point should be the nature of the phase space distribution function which describes the state of an ensemble of particles. By analogy with the kinetic theory of gases [15], it is possible to describe the ensemble by either a Klimontovich approach or a Liouville approach [16].

The Liouville approach defines a fine-grained density function conditioned on the total number of particles at a particular time $N_p(t)$. Considering the particulate phase only, where each particle is characterized by its position $\mathbf{X}^{(i)}$, velocity $\mathbf{U}^{(i)}$ and diameter $\Phi^{(i)}$, then the Liouville distribution function is defined as

$$\begin{aligned} f'(\phi_{p,1}, \mathbf{u}_{p,1}, \mathbf{x}_{p,1}, \dots, \phi_{p,k}, \mathbf{u}_{p,k}, \mathbf{x}_{p,k}; t | N_p(t) = k) \\ \equiv \prod_{i=1}^k \delta[\phi_p - \Phi_p^{(i)}(t)] \delta[\mathbf{u}_p - \mathbf{U}_p^{(i)}(t)] \delta[\mathbf{x}_p - \mathbf{X}_p^{(i)}(t)], \end{aligned} \quad (2.39)$$

and the Liouville PDF is simply the ensemble of the fine-grained distribution function

$$\begin{aligned} \mathcal{P}(\phi_{p,1}, \mathbf{u}_{p,1}, \mathbf{x}_{p,1}, \dots, \phi_{p,k}, \mathbf{u}_{p,k}, \mathbf{x}_{p,k}; t | N_p(t) = k) \\ = \left\langle \prod_{i=1}^k \delta[\phi_p - \Phi_p^{(i)}(t)] \delta[\mathbf{u}_p - \mathbf{U}_p^{(i)}(t)] \delta[\mathbf{x}_p - \mathbf{X}_p^{(i)}(t)] \right\rangle. \end{aligned} \quad (2.40)$$

Upon closer examination of Eqs. 2.39 and 2.40, it can be seen that the Liouville distribution is a multipoint distribution that characterizes all joint events in the ensemble. The information contained in the Liouville distribution is vast and clearly not suitable for a tractable description when deriving an engineering model. An alternative to this distribution is the Klimontovich distribution, which is defined as

$$f'(\phi_p, \mathbf{u}_p, \mathbf{x}_p; t) \equiv \sum_{i=1}^{N_p} \delta[\phi_p - \Phi_p^{(i)}(t)] \delta[\mathbf{u}_p - \mathbf{U}_p^{(i)}(t)] \delta[\mathbf{x}_p - \mathbf{X}_p^{(i)}(t)], \quad (2.41)$$

and represents the number density of the particles in the phase space. The ensemble averages of the Klimontovich distribution function produce a droplet distribution function [16]

$$\mathcal{P}(\phi_p, \mathbf{u}_p, \mathbf{x}_p; t) \equiv \left\langle \sum_{i=1}^{N_p} \delta[\phi_p - \Phi_p^{(i)}(t)] \delta[\mathbf{u}_p - \mathbf{U}_p^{(i)}(t)] \delta[\mathbf{x}_p - \mathbf{X}_p^{(i)}(t)] \right\rangle, \quad (2.42)$$

which does not strictly represent a PDF because it does not integrate to unity over the phase space. The Klimontovich distribution function clearly contains less information (no multiparticle events) than the Liouville equation and serves as a more feasible starting point for the statistical description. In kinetic theory, there exists a simple relationship between the two descriptions (after a number of assumptions), which leads to the famous BBGKY hierarchy. However, as described by Subramaniam [16, 17], several key differences between the kinetic and particulate flows complicate the issue here. For a detailed discussion of these differences and their implications, the reader is referred to [16, 17]. For the purposes of the framework developed in this chapter, a Klimontovich distribution will be used. The distribution function will be of the form

$$\begin{aligned} \mathcal{P}(\Psi_p, \mathbf{u}_p, \mathbf{x}_p, \Psi_f, \mathbf{u}_f, \mathbf{x}_f; t) \equiv & \left\langle \sum_{i=1}^{N_p} \delta[\psi_p - \Psi_p^{(i)}(t)] \delta[\mathbf{u}_p - \mathbf{U}_p^{(i)}(t)] \delta[\mathbf{x}_p - \mathbf{X}_p^{(i)}(t)] \right. \\ & \left. \delta[\psi_f - \Psi_f^{(i)}(t)] \delta[\mathbf{u}_f - \mathbf{U}_f^{(i)}(t)] \delta[\mathbf{x}_f - \mathbf{X}_f^{(i)}(t)] \right\rangle, \end{aligned} \quad (2.43)$$

where the phase space contains both the fluid and the particle. In the most general sense, the fluid and particle positions do not have to coincide ($\mathbf{X}_p^{(i)} \neq \mathbf{X}_f^{(i)}$) in Eq. 2.43 and the distribution function represents a two-point/two-particle (fluid and solid) Lagrangian distribution. However, the ultimate aim of the following analysis is to derive a single-point Eulerian model for both phases mapped onto a single spatial point via a consistency relation. Therefore, it seems more appropriate to term the following framework as a single-point, two-particle description. These descriptions are further discussed in the following sections.

2.8.2 Eulerian and Lagrangian Two-Point, Two-Particle Description

There are two possible viewpoints of the fluid–particle system. A Lagrangian point of view describes the probability of finding two particles (fluid and discrete) at a given state $\mathbf{Z}_{fp} = (\mathbf{X}_f, \mathbf{U}_f, \Psi_f, \mathbf{X}_p, \mathbf{U}_p, \Psi_p)$ where \mathbf{X} , \mathbf{U} and Ψ are position, velocity

and an arbitrary property vectors, respectively.³ The Lagrangian PDF is defined by $\mathcal{P}_{fp}^L(\mathbf{y}_f, \mathbf{u}_f, \boldsymbol{\psi}_f, \mathbf{y}_p, \mathbf{u}_p, \boldsymbol{\psi}_p; t)$, and the probability of finding a fluid/particle pair in the range $[\mathbf{u}_k, \mathbf{u}_k + d\mathbf{u}_k]$, $[\mathbf{y}_k, \mathbf{y}_k + d\mathbf{y}_k]$, $[\boldsymbol{\psi}_k, \boldsymbol{\psi}_k + d\boldsymbol{\psi}_k]$ at time t (where $k = f$ for fluid and $k = p$ for particle) is simply given by

$$\mathcal{P}_{fp}^L(\mathbf{y}_f, \mathbf{u}_f, \boldsymbol{\psi}_f, \mathbf{y}_p, \mathbf{u}_p, \boldsymbol{\psi}_p; t) d\mathbf{y}_f d\mathbf{u}_f d\boldsymbol{\psi}_f d\mathbf{y}_p d\mathbf{u}_p d\boldsymbol{\psi}_p, \quad (2.44)$$

which conforms to the usual normalization constraint, i.e. $\int \mathcal{P}_{fp}^L d\mathbf{z}_{fp} = 1$. An Eulerian (field) point of view describes the probability of finding the fluid–particle mixture in a given state $\mathbf{z}_{fp} = (\mathbf{u}_f, \boldsymbol{\psi}_f, \mathbf{u}_p, \boldsymbol{\psi}_p)$ at two fixed points $(\mathbf{x}_f, \mathbf{x}_p)$ and fixed time. Here, \mathbf{x} , \mathbf{u} and $\boldsymbol{\psi}$ have the same meaning as in their Lagrangian counterparts, and distinction is made by using upper- and lower-case letters. Correspondingly, the probability of finding the system (at time t and position $\mathbf{x}_f, \mathbf{x}_p$) in the given state in the range $[\mathbf{u}_k, \mathbf{u}_k + d\mathbf{u}_k]$, $[\mathbf{x}_k, \mathbf{x}_k + d\mathbf{x}_k]$, $[\boldsymbol{\psi}_k, \boldsymbol{\psi}_k + d\boldsymbol{\psi}_k]$ is

$$\mathcal{P}_{fp}^E(\mathbf{u}_f, \boldsymbol{\psi}_f, \mathbf{u}_p, \boldsymbol{\psi}_p; \mathbf{x}_f, \mathbf{x}_p, t) d\mathbf{u}_f d\boldsymbol{\psi}_f d\mathbf{u}_p d\boldsymbol{\psi}_p. \quad (2.45)$$

However, $\int \mathcal{P}_{fp}^E d\mathbf{z}_{fp} < 1$, because in the Lagrangian point of view, positions are included in the state vector and we know that particle with the velocity \mathbf{u}_k and the property $\boldsymbol{\psi}_k$ is in position \mathbf{x}_k with some probability; therefore, after the integration, all particles are guaranteed to be counted. Whereas in the Eulerian point of view, we are observing the system at two fixed points \mathbf{x}_p and \mathbf{x}_f ; hence, the particle (or fluid) position is not a property of the system. Consequently, it is possible that the point \mathbf{x}_p , where we are expecting a discrete particle in any state $(\mathbf{u}_p, \boldsymbol{\psi}_p)$, be actually occupied by a fluid particle. Here, after the integration, some fluid or discrete particles may not be counted and the integration result will be less than unity.

2.8.3 One-Point Description and Consistency Relations

The above description is referred to as a two-point, two-particle description where the ‘two-particle’ term is added to emphasize that the two points kept in the definition of the PDF correspond to one-fluid and one-discrete-particle location. The PDF (Fokker–Planck) equation derived for this system is at least a 12-dimensional system of equations, even if only positions and velocities considered. To reduce the dimensionality, we define a one-point description with probability densities that for Eulerian description become

$$\mathcal{P}_f^E(\mathbf{u}_f, \boldsymbol{\psi}_f; \mathbf{x}_f, t) = \int \mathcal{P}_{fp}^E(\mathbf{u}_f, \boldsymbol{\psi}_f, \mathbf{u}_p, \boldsymbol{\psi}_p; \mathbf{x}_f, \mathbf{x}_p, t) d\mathbf{x}_p d\mathbf{u}_p d\boldsymbol{\psi}_p, \quad (2.46)$$

³ Ψ can be a combination of different scalars; therefore, we use a vector notation for it. However, here, we only consider the diameters, and consequently, this actually is only a scalar.

$$\mathcal{P}_p^E(\mathbf{u}_p, \psi_p; \mathbf{x}_p, t) = \int \mathcal{P}_{fp}^E(\mathbf{u}_f, \psi_f, \mathbf{u}_p, \psi_p; \mathbf{x}_f, \mathbf{x}_p, t) d\mathbf{x}_f d\mathbf{u}_f d\psi_f. \quad (2.47)$$

This is a one-point, two-particle description, which means that we are considering the probability of finding a fluid and a discrete particle separately at two different fixed points at a fixed time, which obviously contains less information than the two-point, two-particle description where we considered the joint probability of finding a fluid and a discrete particle. This can also be mathematically justified by noting that the marginal PDFs (one-point PDFs) can always be constructed from the joint PDFs by integration, while the reverse operation is not always possible.

It should also be noted that in a conventional two-point description of a single fluid, the two different stochastic particles represent two different realizations of the flow. Thus, two fluid particles with different characteristics can exist in the same position at the same time. In the present case, the stochastic particles are two real particles (fluid and discrete) and for the Eulerian PDF, we have

$$\mathcal{P}_{fp}^E(\mathbf{u}_f, \psi_f, \mathbf{u}_p, \psi_p; \mathbf{x}_f = \mathbf{x}, \mathbf{x}_p = \mathbf{x}, t) = 0 \quad (2.48)$$

Note that a similar constraint also applies to the corresponding Lagrangian PDF. Using Eqs. (2.46) and (2.47), Eq. (2.48) results in

$$\int \mathcal{P}_f^E(\mathbf{u}_f, \psi_f; \mathbf{x}_f, t) d\mathbf{u}_f d\psi_f + \int \mathcal{P}_p^E(\mathbf{u}_p, \psi_p; \mathbf{x}_p, t) d\mathbf{u}_p d\psi_p = 1, \quad (2.49)$$

This constraint has a rather simple physical interpretation: it simply states that the whole domain is filled with the volume of all fluid particles with any velocity \mathbf{u}_f and scalar property vector ψ_f plus the volume of the discrete particles with any velocity \mathbf{u}_p and scalar property vector ψ_p . Equations (2.46) and (2.47) can be normalized by defining normalization factors $\alpha_f(\mathbf{x}, t)$ and $\alpha_p(\mathbf{x}, t)$ for \mathcal{P}_f^E and \mathcal{P}_p^E , respectively, which can be interpreted as phase volume fraction and should sum to unity.

2.8.4 Mass Density Function

Our ultimate goal is to derive the field (Eulerian) equations of moments of each phase by writing the Fokker–Planck equation of Eulerian distribution functions and taking the expectation (denoted by $\langle \cdot \rangle$) of a desired quantity. To do this, we need to derive the relation between the Lagrangian and Eulerian MDFs. Using Lagrangian marginal $\mathcal{P}_f^L(\mathbf{y}_f, \mathbf{u}_f, \psi_f; t)$ and $\mathcal{P}_p^L(\mathbf{y}_p, \mathbf{u}_p, \psi_p; t)$, we can define the MDF by

$$F_f^L(\mathbf{y}_f, \mathbf{u}_f, \psi_f; t) = M_f(t) \mathcal{P}_f^L(\mathbf{y}_f, \mathbf{u}_f, \psi_f; t), \quad (2.50)$$

and

$$F_p^L(\mathbf{y}_p, \mathbf{u}_p, \psi_p; t) = M_p(t) \mathcal{P}_p^L(\mathbf{y}_p, \mathbf{u}_p, \psi_p; t). \quad (2.51)$$

In these equations, F_k^L , with $k = f$ or p for fluid and particle, respectively, can be interpreted as the probable mass of fluid or particle at the given state $\mathbf{Z}_k = (\mathbf{y}_k, \mathbf{u}_k, \psi_k)$. M_k is the normalization constant for F_k or the total mass of phase k which physically can be calculated by $\int_{\mathcal{V}_f} \rho_f(\mathbf{x}_f, t) d\mathbf{x}_f$, \mathcal{V}_f being the total fluid volume, for the fluid phase, and $\sum_{i=1}^{N_p} m_{p,i}$, for the particle phase. Using these relations, we can define a two-point Lagrangian MDF as

$$\mathcal{F}_{fp}^L(\mathbf{y}_f, \mathbf{u}_f, \psi_f, \mathbf{y}_p, \mathbf{u}_p, \psi_p; t) = M_f(t) M_p(t) \mathcal{P}_{fp}^L(\mathbf{y}_f, \mathbf{u}_f, \psi_f, \mathbf{y}_p, \mathbf{u}_p, \psi_p; t) \quad (2.52)$$

Corresponding marginals can easily be calculated as $\mathcal{F}_f^L = M_p F_f^L$ and $\mathcal{F}_p^L = M_f F_p^L$ by integrating Eq. (2.52). At this point, it is possible to define the Eulerian fluid–particle mass density function by [4, 18]:

$$\begin{aligned} \mathcal{F}_{fp}^E(\mathbf{u}_f, \psi_f, \mathbf{u}_p, \psi_p; \mathbf{x}_f, \mathbf{x}_p, t) \\ = \mathcal{F}_{fp}^L(\mathbf{y}_f = \mathbf{x}_f, \mathbf{u}_f, \psi_f, \mathbf{y}_p = \mathbf{x}_p, \mathbf{u}_p, \psi_p; t) \\ = \int \mathcal{F}_{fp}^L(\mathbf{y}_f, \mathbf{u}_f, \psi_f, \mathbf{y}_p, \mathbf{u}_p, \psi_p; t) \delta(\mathbf{x}_f - \mathbf{y}_f) \delta(\mathbf{x}_p - \mathbf{y}_p) d\mathbf{y}_f d\mathbf{y}_p, \end{aligned} \quad (2.53)$$

The marginal or one-point Eulerian MDFs are simply derived by integration of either marginal Lagrangian MDF using delta functions or just by integrating the Eulerian fluid–particle MDF over the fluid or particle phase space as follows:

$$\mathcal{F}_f^E(\mathbf{u}_f, \psi_f; \mathbf{x}_f, t) = M_p(t) F_f^E(\mathbf{u}_f, \psi_f; \mathbf{x}_f, t) \quad (2.54)$$

and

$$\mathcal{F}_p^E(\mathbf{u}_p, \psi_p; \mathbf{x}_p, t) = M_f(t) F_p^E(\mathbf{u}_p, \psi_p; \mathbf{x}_p, t) \quad (2.55)$$

The normalization constant for the two-point Lagrangian MDF, Eq. (2.52), is defined by $M_p M_f$. Thus, upon integrating \mathcal{F}_{fp}^L , we get the mass of the fluid multiplied by the mass of the discrete phase. This is done for mathematical convenience, and as a consequence, mixed indices such as $M_p F_f^L$ appear in the marginals of both two-point Lagrangian MDFs and two-point Eulerian MDFs, which are merely mathematical objects derived from Eqs. (2.50) and (2.51). However, the importance of these definitions should not be underestimated because physical quantities such as expected densities and the probabilities of presence of phases, α_f and α_p , can conveniently be derived from these quantities by simple integrations:

$$\alpha_f(\mathbf{x}, t) \langle \rho_f \rangle(\mathbf{x}, t) = M_f^{-1}(t) \int \mathcal{F}_{fp}^E(\mathbf{u}_f, \psi_f, \mathbf{u}_p, \psi_p; \mathbf{x}, \mathbf{x}_p, t) d\mathbf{x}_p d\mathbf{u}_f d\psi_f d\mathbf{u}_p d\psi_p \quad (2.56)$$

$$\alpha_p(\mathbf{x}, t)(\rho_p)(\mathbf{x}, t) = M_f^{-1}(t) \int \mathcal{F}_{fp}^E(\mathbf{u}_f, \psi_f, \mathbf{u}_p, \psi_p; \mathbf{x}_f, \mathbf{x}, t) d\mathbf{x}_f d\mathbf{u}_f d\psi_f d\mathbf{u}_p d\psi_p \quad (2.57)$$

$$\alpha_f(\mathbf{x}, t) = M_p^{-1}(t) \int \rho_f^{-1} \mathcal{F}_{fp}^E(\mathbf{u}_f, \psi_f, \mathbf{u}_p, \psi_p; \mathbf{x}, \mathbf{x}_p, t) d\mathbf{x}_p d\mathbf{u}_f d\psi_f d\mathbf{u}_p d\psi_p \quad (2.58)$$

$$\alpha_p(\mathbf{x}, t) = M_f^{-1}(t) \int \rho_p^{-1} \mathcal{F}_{fp}^E(\mathbf{u}_f, \psi_f, \mathbf{u}_p, \psi_p; \mathbf{x}_f, \mathbf{x}, t) d\mathbf{x}_f d\mathbf{u}_f d\psi_f d\mathbf{u}_p d\psi_p \quad (2.59)$$

These can easily be expressed in terms of marginals of \mathcal{F}_{fp}^E , i.e. F_f^E and F_p^E , by simple integration and using Eqs. (2.54) and (2.55).

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