

## Chapter 2

# Further Properties of the Laplace Transform

### 2.1 Real Functions

Sometimes, a function  $F(t)$  represents a natural or engineering process that has no obvious starting value. Statisticians call this a *time series*. Although we shall not be considering  $F(t)$  as stochastic, it is nevertheless worth introducing a way of “switching on” a function. Let us start by finding the Laplace transform of a step function the name of which pays homage to the pioneering electrical engineer Oliver Heaviside (1850–1925). The formal definition runs as follows.

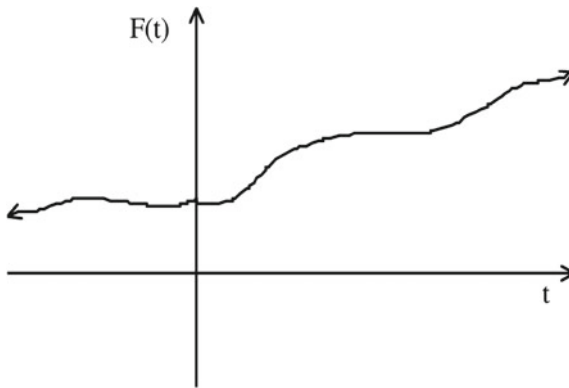
**Definition 2.1** *Heaviside’s unit step function, or simply the unit step function, is defined as*

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

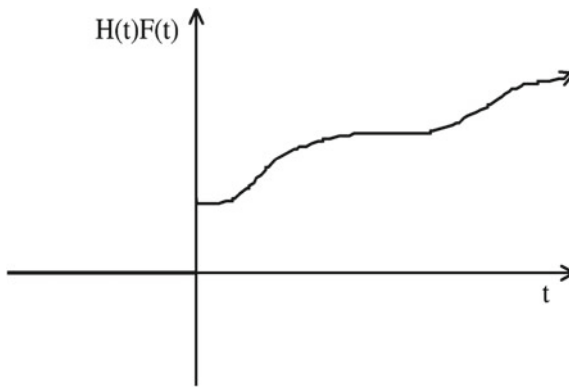
Since  $H(t)$  is precisely the same as 1 for  $t > 0$ , the Laplace transform of  $H(t)$  must be the same as the Laplace transform of 1, i.e.  $1/s$ . The switching on of an arbitrary function is achieved simply by multiplying it by the standard function  $H(t)$ , so if  $F(t)$  is given by the function shown in Fig. 2.1 and we multiply this function by the Heaviside unit step function  $H(t)$  to obtain  $H(t)F(t)$ , Fig. 2.2 results. Sometimes it is necessary to define what is called the *two sided* Laplace transform

$$\int_{-\infty}^{\infty} e^{-st} F(t) dt$$

which makes a great deal of mathematical sense. However the additional problems that arise by allowing negative values of  $t$  are severe and limit the use of the two sided Laplace transform. For this reason, the two sided transform will not be pursued here.



**Fig. 2.1**  $F(t)$ , a function with no well defined starting value



**Fig. 2.2**  $H(t)F(t)$ , the function is now zero before  $t = 0$

## 2.2 Derivative Property of the Laplace Transform

Suppose a differentiable function  $F(t)$  has Laplace transform  $f(s)$ , we can find the Laplace transform

$$\mathcal{L}\{F'(t)\} = \int_0^{\infty} e^{-st} F'(t) dt$$

of its derivative  $F'(t)$  through the following theorem.

### Theorem 2.1

$$\mathcal{L}\{F'(t)\} = \int_0^{\infty} e^{-st} F'(t) dt = -F(0) + sf(s).$$

**Proof** Integrating by parts once gives

$$\begin{aligned}\mathcal{L}\{F'(t)\} &= [F(t)e^{-st}]_0^\infty + \int_0^\infty se^{-st}F(t)dt \\ &= -F(0) + sf(s)\end{aligned}$$

where  $F(0)$  is the value of  $F(t)$  at  $t = 0$ . □

This is an important result and lies behind future applications that involve solving linear differential equations. The key property is that the transform of a derivative  $F'(t)$  does not itself involve a derivative, only  $-F(0) + sf(s)$  which is an algebraic expression involving  $f(s)$ . The downside is that the value  $F(0)$  is required. Effectively, an integration has taken place and the constant of integration is  $F(0)$ . Later, this is exploited further through solving differential equations. Later still in this text, partial differential equations are solved, and wavelets are introduced. Let us proceed here by finding the Laplace transform of the second derivative of  $F(t)$ . We also state this in the form of a theorem.

**Theorem 2.2** *If  $F(t)$  is a twice differentiable function of  $t$  then*

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0).$$

**Proof** The proof is unremarkable and involves integrating by parts twice. Here are the details.

$$\begin{aligned}\mathcal{L}\{F''(t)\} &= \int_0^\infty e^{-st}F''(t)dt \\ &= [F'(t)e^{-st}]_0^\infty + \int_0^\infty se^{-st}F'(t)dt \\ &= -F'(0) + [sF(t)e^{-st}]_0^\infty + \int_0^\infty s^2e^{-st}F(t)dt \\ &= -F'(0) - sF(0) + s^2f(s) \\ &= s^2f(s) - sF(0) - F'(0).\end{aligned}$$
□

The general result, proved by induction, is

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0)$$

where  $n$  is a positive integer. Note the appearance of  $n$  constants on the right hand side. This of course is the result of integrating this number of times.

This result, as we have said, has wide application so it is worth getting to know. Consider the result

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}.$$

Now,

$$\frac{d}{dt}(\sin(\omega t)) = \omega \cos(\omega t)$$

so using the formula

$$\mathcal{L}(F'(t)) = sf(s) - F(0)$$

with  $F(t) = \sin(\omega t)$  we have

$$\mathcal{L}\{\omega \cos(\omega t)\} = s \frac{\omega}{s^2 + \omega^2} - 0$$

so

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

another standard result.

Another appropriate quantity to find at this point is the determination of the value of the Laplace transform of

$$\int_0^t F(u) du.$$

First of all, the function  $F(t)$  must be integrable in such a way that

$$g(t) = \int_0^t F(u) du$$

is of exponential order. From this definition of  $g(t)$  it is immediately apparent that  $g(0) = 0$  and that  $g'(t) = F(t)$ . This latter result is called the fundamental theorem of the calculus. We can now use the result

$$\mathcal{L}\{g'(t)\} = s\bar{g}(s) - g(0)$$

to obtain

$$\mathcal{L}\{F(t)\} = f(s) = s\bar{g}(s)$$

where we have written  $\mathcal{L}\{g(t)\} = \bar{g}(s)$ . Hence

$$\bar{g}(s) = \frac{f(s)}{s}$$

which finally gives the result

$$\mathcal{L}\left(\int_0^t F(u) du\right) = \frac{f(s)}{s}.$$

The following result is also useful and can be stated in the form of a theorem.

**Theorem 2.3** *If  $\mathcal{L}(F(t)) = f(s)$  then  $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$ , assuming that*

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

**Proof** Let  $G(t)$  be the function  $F(t)/t$ , so that  $F(t) = tG(t)$ . Using the property

$$\mathcal{L}\{tG(t)\} = -\frac{d}{ds}\mathcal{L}\{G(t)\}$$

we deduce that

$$f(s) = \mathcal{L}\{F(t)\} = -\frac{d}{ds}\mathcal{L}\left\{\frac{F(t)}{t}\right\}.$$

Integrating both sides of this with respect to  $s$  from  $s$  to  $\infty$  gives

$$\int_s^\infty f(u)du = \left[-\mathcal{L}\left\{\frac{F(t)}{t}\right\}\right]_s^\infty = \mathcal{L}\left\{\frac{F(t)}{t}\right\}\Big|_s = \mathcal{L}\left\{\frac{F(t)}{t}\right\}$$

since

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} \rightarrow 0 \text{ as } s \rightarrow \infty$$

which completes the proof. □

The function

$$Si(t) = \int_0^t \frac{\sin u}{u} du$$

defines the Sine Integral function which occurs in the study of optics. The formula for its Laplace transform can now be easily derived as follows.

Let  $F(t) = \sin t$  in the result

$$\mathcal{L}\left(\frac{F(t)}{t}\right) = \int_s^\infty f(u)du$$

to give

$$\begin{aligned} \mathcal{L}\left(\frac{\sin t}{t}\right) &= \int_s^\infty \frac{du}{1+u^2} \\ &= \left[\tan^{-1}(u)\right]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}(s) \\ &= \tan^{-1}\left(\frac{1}{s}\right). \end{aligned}$$

We now use the result

$$\mathcal{L}\left(\int_0^t F(u)du\right) = \frac{f(s)}{s}$$

to deduce that

$$\mathcal{L}\left(\int_0^t \frac{\sin u}{u} du\right) = \mathcal{L}\{Si(t)\} = \frac{1}{s} \tan^{-1}\left(\frac{1}{s}\right).$$

## 2.3 Heaviside's Unit Step Function

As promised earlier, we devote this section to exploring some properties of Heaviside's unit step function  $H(t)$ . The Laplace transform of  $H(t)$  has already been shown to be the same as the Laplace transform of 1, i.e.  $1/s$ . The Laplace transform of  $H(t - t_0)$ ,  $t_0 > 0$ , is a little more enlightening:

$$\mathcal{L}\{H(t - t_0)\} = \int_0^\infty H(t - t_0)e^{-st} dt.$$

Now, since  $H(t - t_0) = 0$  for  $t < t_0$  this Laplace transform is

$$\mathcal{L}\{H(t - t_0)\} = \int_{t_0}^\infty e^{-st} dt = \left[ -\frac{e^{-st}}{s} \right]_{t_0}^\infty = \frac{e^{-st_0}}{s}.$$

This result is generalised through the following theorem.

**Theorem 2.4 (Second Shift Theorem)** *If  $F(t)$  is a function of exponential order in  $t$  then*

$$\mathcal{L}\{H(t - t_0)F(t - t_0)\} = e^{-st_0} f(s)$$

where  $f(s)$  is the Laplace transform of  $F(t)$ .

**Proof** This result is proved by direct integration.

$$\begin{aligned} \mathcal{L}\{H(t - t_0)F(t - t_0)\} &= \int_0^\infty H(t - t_0)F(t - t_0)e^{-st} dt \\ &= \int_{t_0}^\infty F(t - t_0)e^{-st} dt \text{ (by definition of } H) \\ &= \int_0^\infty F(u)e^{-s(u+t_0)} du \text{ (writing } u = t - t_0) \\ &= e^{-st_0} f(s). \end{aligned}$$

This establishes the theorem. □

The only condition on  $F(t)$  is that it is a function that is of exponential order which means of course that it is free from singularities for  $t > t_0$ . The principal use of this theorem is that it enables us to determine the Laplace transform of a function that is switched on at time  $t = t_0$ . Here is a straightforward example.

**Example 2.1** Determine the Laplace transform of the sine function switched on at time  $t = 3$ .

**Solution** The sine function required that starts at  $t = 3$  is  $S(t)$  where

$$S(t) = \begin{cases} \sin t & t \geq 3 \\ 0 & t < 3. \end{cases}$$

We can use the Heaviside step function to write

$$S(t) = H(t - 3) \sin t.$$

The second shift theorem can then be used by utilising the summation formula

$$\sin t = \sin(t - 3 + 3) = \sin(t - 3) \cos(3) + \cos(t - 3) \sin(3)$$

so

$$\mathcal{L}\{S(t)\} = \mathcal{L}\{H(t - 3) \sin(t - 3)\} \cos(3) + \mathcal{L}\{H(t - 3) \cos(t - 3)\} \sin(3).$$

This may seem a strange step to take, but in order to use the second shift theorem it is essential to get the arguments of both the Heaviside function and the target function in the question the same; in this case  $(t - 3)$ . We can now use the second shift theorem directly to give

$$\mathcal{L}\{S(t)\} = e^{-3s} \cos(3) \frac{1}{s^2 + 1} + e^{-3s} \sin(3) \frac{s}{s^2 + 1}$$

or

$$\mathcal{L}\{S(t)\} = (\cos 3 + s \sin 3) e^{-3s} / (s^2 + 1).$$

## 2.4 Inverse Laplace Transform

Virtually all operations have inverses. Addition has subtraction, multiplication has division, differentiation has integration. The Laplace transform is no exception, and we can define the *Inverse* Laplace transform as follows.

**Definition 2.2** If  $F(t)$  has the Laplace transform  $f(s)$ , that is

$$\mathcal{L}\{F(t)\} = f(s)$$

then the inverse Laplace transform is defined by

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

and is unique apart from null functions.

Perhaps the most important property of the inverse transform to establish is its linearity. We state this as a theorem.

**Theorem 2.5** *The inverse Laplace transform is linear, i.e.*

$$\mathcal{L}^{-1}\{af_1(s) + bf_2(s)\} = a\mathcal{L}^{-1}\{f_1(s)\} + b\mathcal{L}^{-1}\{f_2(s)\}.$$

**Proof** Linearity is easily established as follows. Since the Laplace transform is linear, we have for suitably well behaved functions  $F_1(t)$  and  $F_2(t)$ :

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\} = af_1(s) + bf_2(s).$$

Taking the inverse Laplace transform of this expression gives

$$aF_1(t) + bF_2(t) = \mathcal{L}^{-1}\{af_1(s) + bf_2(s)\}$$

which is the same as

$$a\mathcal{L}^{-1}\{f_1(s)\} + b\mathcal{L}^{-1}\{f_2(s)\} = \mathcal{L}^{-1}\{af_1(s) + bf_2(s)\}$$

and this has established linearity of  $\mathcal{L}^{-1}\{f(s)\}$ . □

Another important property is uniqueness. It has been mentioned that the Laplace transform was indeed unique apart from null functions (functions whose Laplace transform is zero). It follows immediately that the inverse Laplace transform is also unique apart from the possible addition of null functions. These take the form of isolated values and can be discounted for all practical purposes.

As is quite common with inverse operations there is no systematic method of determining inverse Laplace transforms. The calculus provides a good example where there are plenty of systematic rules for differentiation: the product rule, the quotient rule, the chain rule. However by contrast there are no systematic rules for the inverse operation, integration. If we have an integral to find, we may try substitution or integration by parts, but there is no guarantee of success. Indeed, the integral may not be possible to express in terms of elementary functions. Derivatives that exist can always be found by using the rules; this is not so for integrals. The situation regarding the Laplace transform is not quite the same in that it may not be possible to find  $\mathcal{L}\{F(t)\}$  explicitly because it is an integral. There is certainly no guarantee of being able to find  $\mathcal{L}^{-1}\{f(s)\}$  and we have to devise various methods of trying so to



do. For example, given an arbitrary function of  $s$  there is no guarantee whatsoever that a function of  $t$  can be found that is its inverse Laplace transform. One necessary condition for example is that the function of  $s$  must tend to zero as  $s \rightarrow \infty$ . When we are certain that a function of  $s$  has arisen from a Laplace transform, there are techniques and theorems that can help us invert it. Partial fractions simplify rational functions and can help identify standard forms (the exponential and trigonometric functions for example), then there are the shift theorems which we have just met which extend further the repertoire of standard forms. Engineering texts spend a considerable amount of space building up a library of specific inverse Laplace transforms and to ways of extending these via the calculus. To a certain extent we need to do this too. Therefore we next do some reasonably elementary examples. Note that in Appendix B there is a list of some inverse Laplace transforms.

**Example 2.2** *Use partial fractions to determine*

$$\mathcal{L}^{-1} \left\{ \frac{a}{s^2 - a^2} \right\}$$

**Solution** Noting that

$$\frac{a}{s^2 - a^2} = \frac{1}{2} \left[ \frac{1}{s - a} - \frac{1}{s + a} \right]$$

gives straight away that

$$\mathcal{L}^{-1} \left\{ \frac{a}{s^2 - a^2} \right\} = \frac{1}{2} (e^{at} - e^{-at}) = \sinh(at).$$

The first shift theorem has been used on each of the functions  $1/(s - a)$  and  $1/(s + a)$  together with the standard result  $\mathcal{L}^{-1}\{1/s\} = 1$ . Here is another example.

**Example 2.3** *Determine the value of*

$$\mathcal{L}^{-1} \left\{ \frac{s^2}{(s + 3)^3} \right\}.$$

**Solution** Noting the standard partial fraction decomposition

$$\frac{s^2}{(s + 3)^3} = \frac{1}{s + 3} - \frac{6}{(s + 3)^2} + \frac{9}{(s + 3)^3}$$

we use the first shift theorem on each of the three terms in turn to give

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s^2}{(s + 3)^3} \right\} &= \mathcal{L}^{-1} \frac{1}{s + 3} - \mathcal{L}^{-1} \frac{6}{(s + 3)^2} + \mathcal{L}^{-1} \frac{9}{(s + 3)^3} \\ &= e^{-3t} - 6te^{-3t} + \frac{9}{2}t^2e^{-3t} \end{aligned}$$

where we have used the linearity property of the  $\mathcal{L}^{-1}$  operator. Finally, we do the following four-in-one example to hone our skills.

**Example 2.4** Determine the following inverse Laplace transforms

$$(a) \mathcal{L}^{-1} \frac{(s+3)}{s(s-1)(s+2)}; (b) \mathcal{L}^{-1} \frac{(s-1)}{s^2+2s-8}; (c) \mathcal{L}^{-1} \frac{3s+7}{s^2-2s+5}; (d) \mathcal{L}^{-1} \frac{e^{-7s}}{(s+3)^3}.$$

**Solution** All of these problems are tackled in a similar way, by decomposing the expression into partial fractions, using shift theorems, then identifying the simplified expressions with various standard forms.

(a) Using partial fraction decomposition and not dwelling on the detail we get

$$\frac{s+3}{s(s-1)(s+2)} = -\frac{3}{2s} + \frac{4}{3(s-1)} + \frac{1}{6(s+2)}.$$

Hence, operating on both sides with the inverse Laplace transform operator gives

$$\begin{aligned} \mathcal{L}^{-1} \frac{s+3}{s(s-1)(s+2)} &= -\mathcal{L}^{-1} \frac{3}{2s} + \mathcal{L}^{-1} \frac{4}{3(s-1)} + \mathcal{L}^{-1} \frac{1}{6(s+2)} \\ &= -\frac{3}{2} \mathcal{L}^{-1} \frac{1}{s} + \frac{4}{3} \mathcal{L}^{-1} \frac{1}{s-1} + \frac{1}{6} \mathcal{L}^{-1} \frac{1}{s+2} \end{aligned}$$

using the linearity property of  $\mathcal{L}^{-1}$  once more. Finally, using the standard forms, we get

$$\mathcal{L}^{-1} \left\{ \frac{s+3}{s(s-1)(s+2)} \right\} = -\frac{3}{2} + \frac{4}{3}e^t + \frac{1}{6}e^{-2t}.$$

(b) The expression

$$\frac{s-1}{s^2+2s-8}$$

is factorised to

$$\frac{s-1}{(s+4)(s-2)}$$

which, using partial fractions is

$$\frac{1}{6(s-2)} + \frac{5}{6(s+4)}.$$

Therefore, taking inverse Laplace transforms gives

$$\mathcal{L}^{-1} \frac{s-1}{s^2+2s-8} = \frac{1}{6}e^{2t} + \frac{5}{6}e^{-4t}.$$

(c) The denominator of the rational function

$$\frac{3s + 7}{s^2 - 2s + 5}$$

does not factorise. In this case we use completing the square and standard trigonometric forms as follows:

$$\frac{3s + 7}{s^2 - 2s + 5} = \frac{3s + 7}{(s - 1)^2 + 4} = \frac{3(s - 1) + 10}{(s - 1)^2 + 4}.$$

So

$$\begin{aligned}\mathcal{L}^{-1} \frac{3s + 7}{s^2 - 2s + 5} &= 3\mathcal{L}^{-1} \frac{(s - 1)}{(s - 1)^2 + 4} + 5\mathcal{L}^{-1} \frac{2}{(s - 1)^2 + 4} \\ &= 3e^t \cos(2t) + 5e^t \sin(2t).\end{aligned}$$

Again, the first shift theorem has been used.

(d) The final inverse Laplace transform is slightly different. The expression

$$\frac{e^{-7s}}{(s - 3)^3}$$

contains an exponential in the numerator, therefore it is expected that the second shift theorem will have to be used. There is a little “fiddling” that needs to take place here. First of all, note that

$$\mathcal{L}^{-1} \frac{1}{(s - 3)^3} = \frac{1}{2} t^2 e^{3t}$$

using the first shift theorem. So

$$\mathcal{L}^{-1} \frac{e^{-7s}}{(s - 3)^3} = \begin{cases} \frac{1}{2} (t - 7)^2 e^{3(t-7)} & t > 7 \\ 0 & 0 \leq t \leq 7. \end{cases}$$

Of course, this can succinctly be expressed using the Heaviside unit step function as

$$\frac{1}{2} H(t - 7) (t - 7)^2 e^{3(t-7)}.$$

We shall get more practice at this kind of inversion exercise, but you should try your hand at a few of the exercises at the end.

## 2.5 Limiting Theorems

In many branches of mathematics there is a necessity to solve differential equations. Later chapters give details of how some of these equations can be solved by using Laplace transform techniques. Unfortunately, it is sometimes the case that it is not possible to invert  $f(s)$  to retrieve the desired solution to the original problem. Numerical inversion techniques are possible and these can be found in some software packages, especially those used by control engineers. Insight into the behaviour of the solution can be deduced without actually solving the differential equation by examining the asymptotic character of  $f(s)$  for small  $s$  or large  $s$ . In fact, it is often very useful to determine this asymptotic behaviour without solving the equation, even when exact solutions are available as these solutions are often complex and difficult to obtain let alone interpret. In this section two theorems that help us to find this asymptotic behaviour are investigated.

**Theorem 2.6 (Initial Value)** *If the indicated limits exist then*

$$\lim_{t \rightarrow 0} F(t) = \lim_{s \rightarrow \infty} sf(s).$$

(The left hand side is  $F(0)$  of course, or  $F(0+)$  if  $\lim_{t \rightarrow 0} F(t)$  is not unique.)

**Proof** We have already established that

$$\mathcal{L}\{F'(t)\} = sf(s) - F(0). \quad (2.1)$$

However, if  $F'(t)$  obeys the usual criteria for the existence of the Laplace transform, that is  $F'(t)$  is of exponential order and is piecewise continuous, then

$$\begin{aligned} \left| \int_0^\infty e^{-st} F'(t) dt \right| &\leq \int_0^\infty |e^{-st} F'(t)| dt \\ &\leq \int_0^\infty e^{-st} e^{Mt} dt \\ &= -\frac{1}{M-s} \rightarrow 0 \text{ as } s \rightarrow \infty. \end{aligned}$$

Thus letting  $s \rightarrow \infty$  in Eq.(2.1) yields the result. □

**Theorem 2.7 (Final Value)** *If the limits indicated exist, then*

$$\lim_{t \rightarrow \infty} F(t) = \lim_{s \rightarrow 0} sf(s).$$

**Proof** Again we start with the formula for the Laplace transform of the derivative of  $F(t)$

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t) dt = sf(s) - F(0) \quad (2.2)$$

this time writing the integral out explicitly. The limit of the integral as  $s \rightarrow 0$  is

$$\begin{aligned} \lim_{s \rightarrow 0} \int_0^\infty e^{-st} F'(t) dt &= \lim_{s \rightarrow 0} \lim_{T \rightarrow \infty} \int_0^T e^{-st} F'(t) dt \\ &= \lim_{s \rightarrow 0} \lim_{T \rightarrow \infty} \{e^{-sT} F(T) - F(0)\} \\ &= \lim_{T \rightarrow \infty} F(T) - F(0) \\ &= \lim_{t \rightarrow \infty} F(t) - F(0). \end{aligned}$$

Thus we have, using Eq. (2.2),

$$\lim_{t \rightarrow \infty} F(t) - F(0) = \lim_{s \rightarrow 0} sf(s) - F(0)$$

from which, on cancellation of  $-F(0)$ , the theorem follows.  $\square$

Since the improper integral converges independently of the value of  $s$  and all limits exist (a priori assumption), it is therefore correct to have assumed that the order of the two processes (taking the limit and performing the integral) can be exchanged. (This has in fact been demonstrated explicitly in this proof.)

Suppose that the function  $F(t)$  can be expressed as a power series as follows

$$F(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots .$$

If we assume that the Laplace transform of  $F(t)$  exists,  $F(t)$  is of exponential order and is piecewise continuous. If, further, we assume that the power series for  $F(t)$  is absolutely and uniformly convergent the Laplace transform can be applied term by term

$$\begin{aligned} \mathcal{L}\{F(t)\} &= f(s) = \mathcal{L}\{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots\} \\ &= a_0 \mathcal{L}\{1\} + a_1 \mathcal{L}\{t\} + a_2 \mathcal{L}\{t^2\} + \cdots + a_n \mathcal{L}\{t^n\} + \cdots \end{aligned}$$

provided the transformed series is convergent. Using the standard form

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

the right hand side becomes

$$\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \cdots + \frac{n!a_n}{s^{n+1}} + \cdots .$$

Hence

$$f(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \cdots + \frac{n!a_n}{s^{n+1}} + \cdots.$$

**Example 2.5** *Demonstrate the initial and final value theorems using the function  $F(t) = e^{-t}$ . Expand  $e^{-t}$  as a power series, evaluate term by term and confirm the legitimacy of term by term evaluation.*

**Solution**

$$\begin{aligned}\mathcal{L}\{e^{-t}\} &= \frac{1}{s+1} \\ \lim_{t \rightarrow 0} F(t) &= F(0) = e^{-0} = 1 \\ \lim_{s \rightarrow \infty} sf(s) &= \lim_{s \rightarrow \infty} \frac{s}{s+1} = 1.\end{aligned}$$

This confirms the initial value theorem. The final value theorem is also confirmed as follows:-

$$\begin{aligned}\lim_{t \rightarrow \infty} F(t) &= \lim_{t \rightarrow \infty} e^{-t} = 0 \\ \lim_{s \rightarrow 0} sf(s) &= \lim_{s \rightarrow 0} \frac{s}{s+1} = 0.\end{aligned}$$

The power series expansion for  $e^{-t}$  is

$$\begin{aligned}e^{-t} &= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots + (-1)^n \frac{t^n}{n!} \\ \mathcal{L}\{e^{-t}\} &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \cdots + \frac{(-1)^n}{s^{n+1}} \\ &= \frac{1}{s} \left(1 + \frac{1}{s}\right)^{-1} = \frac{1}{s+1}.\end{aligned}$$

Hence the term by term evaluation of the power series expansion for  $e^{-t}$  gives the right answer. This is not a proof of the series expansion method of course, merely a verification that the method gives the right answer in this instance.

## 2.6 The Impulse Function

There is a whole class of “functions” that, strictly, are not functions at all. In order to be a function, an expression has to be defined for all values of the variable in the specified range. When this is not so, then the expression is not a function because it is

not well defined. It may not seem at all sensible for us to bother with such creatures, in that if a function is not defined at a certain point then what use is it? However, if a “function” instead of being well defined possesses some global property, then it indeed does turn out to be worth considering such pathological objects. Of course, having taken the decision to consider such objects, strictly there needs to be a whole new mathematical language constructed to deal with them. Notions such as adding them together, multiplying them, performing operations such as integration cannot be done without preliminary mathematics. The general consideration of this kind of object forms the study of *generalised functions* (see Jones 1966 or Lighthill 1970) which is outside the scope of this text. For our purposes we introduce the first such function which occurred naturally in the field of electrical engineering and is the so called impulse function. It is sometimes called Dirac’s  $\delta$  function after the pioneering theoretical physicist P.A.M. Dirac (1902–1984). It has the following definition which involves its integral. This has not been defined properly, but if we write the definition first we can then comment on the integral.

**Definition 2.3** *The Dirac- $\delta$  function  $\delta(t)$  is defined as having the following properties*

$$\delta(t) = 0 \quad \forall t, t \neq 0 \quad (2.3)$$

$$\int_{-\infty}^{\infty} h(t)\delta(t)dt = h(0) \quad (2.4)$$

for any function  $h(t)$  continuous in  $(-\infty, \infty)$ .

We shall see in the next paragraph that the Dirac- $\delta$  function can be thought of as the limiting case of a top hat function of unit area as it becomes infinitesimally thin but infinitely tall, i.e. the following limit

$$\delta(t) = \lim_{T \rightarrow \infty} T_p(t)$$

where

$$T_p(t) = \begin{cases} 0 & t \leq -1/T \\ \frac{1}{2}T & -1/T < t < 1/T \\ 0 & t \geq 1/T. \end{cases}$$

The integral in the definition can then be written as follows:

$$\int_{-\infty}^{\infty} h(t) \lim_{T \rightarrow \infty} T_p(t)dt = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} h(t)T_p(t)dt$$

provided the limits can be exchanged which of course depends on the behaviour of the function  $h(t)$  but this can be so chosen to fulfil our needs. The integral inside the limit exists, being the product of continuous functions, and its value is the area under the curve  $h(t)T_p(t)$ . This area will approach the value  $h(0)$  as  $T \rightarrow \infty$  by the following argument. For sufficiently large values of  $T$ , the interval  $[-1/T, 1/T]$

will be small enough for the value of  $h(t)$  not to differ very much from its value at the origin. In this case we can write  $h(t) = h(0) + \epsilon(t)$  where  $|\epsilon(t)|$  is in some sense small and tends to zero as  $T \rightarrow \infty$ . The integral thus can be seen to tend to  $h(0)$  as  $T \rightarrow \infty$  and the property is established.

Returning to the definition of  $\delta(t)$  strictly, the first condition is redundant; only the second is necessary, but it is very convenient to retain it. Now as we have said,  $\delta(t)$  is not a true function because it has not been defined for  $t = 0$ .  $\delta(0)$  has no value. Equivalent conditions to Eq. (2.4) are:-

$$\int_{0-}^{\infty} h(t)\delta(t)dt = h(0)$$

and

$$\int_{-\infty}^{0+} h(t)\delta(t)dt = h(0).$$

These follow from a similar argument as before using a limiting definition of  $\delta(t)$  in terms of the top hat function. In this section, wherever the integral of a  $\delta$  function (or later related “derivatives”) occurs it will be assumed to involve this kind of limiting process. The details of taking the limit will however be omitted.

Let us now look at a more visual approach. As we have seen algebraically in the last paragraph  $\delta(t)$  is sometimes called the impulse function because it can be thought of as the shape of Fig. 2.3, the top hat function if we let  $T \rightarrow \infty$ . Of course there are many shapes that will behave like  $\delta(t)$  in some limit. The top hat function is one of the simplest to state and visualise. The crucial property is that the area under this top hat function is unity for all values of  $T$ , so letting  $T \rightarrow \infty$  preserves this property. Diagrammatically, the Dirac- $\delta$  or impulse function is represented by an arrow as in Fig. 2.4 where the length of the arrow is unity. Using Eq. (2.4) with  $h \equiv 1$  we see that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

which is consistent with the area under  $\delta(t)$  being unity.

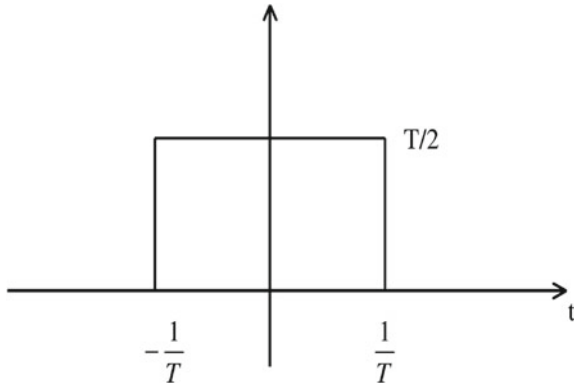
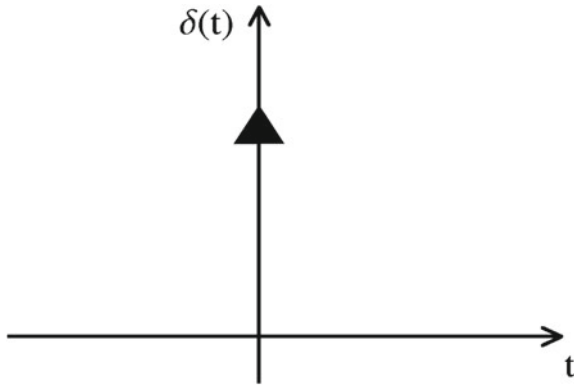
We now ask ourselves what is the Laplace transform of  $\delta(t)$ ? Does it exist? We suspect that it might be 1 for Eq. (2.4) with  $h(t) = e^{-st}$ , a perfectly valid choice of  $h(t)$  gives

$$\int_{-\infty}^{\infty} \delta(t)e^{-st}dt = \int_{0-}^{\infty} \delta(t)e^{-st}dt = 1.$$

However, we progress with care. This is good advice when dealing with generalised functions. Let us take the Laplace transform of the top hat function  $T_p(t)$  defined mathematically by

$$T_p(t) = \begin{cases} 0 & t \leq -1/T \\ \frac{1}{2}T & -1/T < t < 1/T \\ 0 & t \geq 1/T. \end{cases}$$



**Fig. 2.3** The “top hat” function**Fig. 2.4** The Dirac- $\delta$  function

The calculation proceeds as follows:-

$$\begin{aligned}
 \mathcal{L}\{T_p(t)\} &= \int_0^{\infty} T_p(t)e^{-st} dt \\
 &= \int_0^{1/T} \frac{1}{2}T e^{-st} dt \\
 &= \left[ -\frac{T}{2s} e^{-st} \right]_0^{1/T} \\
 &= \left[ \frac{T}{2s} - \frac{T}{2s} e^{-s/T} \right].
 \end{aligned}$$

As  $T \rightarrow \infty$ ,

$$e^{-s/T} \approx 1 - \frac{s}{T} + O\left(\frac{1}{T^2}\right)$$

hence

$$\frac{T}{2s} - \frac{T}{2s}e^{-s/T} \approx \frac{1}{2} + O\left(\frac{1}{T}\right)$$

which  $\rightarrow \frac{1}{2}$  as  $T \rightarrow \infty$ .

In Laplace transform theory it is usual to define the impulse function  $\delta(t)$  such that

$$\mathcal{L}\{\delta(t)\} = 1.$$

This means reducing the width of the top hat function so that it lies between 0 and  $1/T$  (not  $-1/T$  and  $1/T$ ) and increasing the height from  $\frac{1}{2}T$  to  $T$  in order to preserve unit area. Clearly the difficulty arises because the impulse function is centred on  $t = 0$  which is precisely the lower limit of the integral in the definition of the Laplace transform. Using 0- as the lower limit of the integral overcomes many of the difficulties.

The function  $\delta(t - t_0)$  represents an impulse that is centred on the time  $t = t_0$ . It can be considered to be the limit of the function  $K(t)$  where  $K(t)$  is the displaced top hat function defined by

$$K(t) = \begin{cases} 0 & t \leq t_0 - 1/2T \\ \frac{1}{2}T & t_0 - 1/2T < t < t_0 + 1/2T \\ 0 & t \geq t_0 + 1/2T \end{cases}$$

as  $T \rightarrow \infty$ . The definition of the delta function can be used to deduce that

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)dt = h(t_0)$$

and that, provided  $t_0 > 0$

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}.$$

Letting  $t_0 \rightarrow 0$  leads to

$$\mathcal{L}\{\delta(t)\} = 1$$

a correct result. Another interesting result can be deduced almost at once and expresses mathematically the property of  $\delta(t)$  to pick out a particular function value, known to engineers as the *filtering property*. Since

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)dt = h(t_0)$$

with  $h(t) = e^{-st} f(t)$  and  $t_0 = a \geq 0$  we deduce that

$$\mathcal{L}\{\delta(t - a)f(t)\} = e^{-as} f(a).$$

Mathematically, the impulse function has additional interest in that it enables insight to be gained into the properties of discontinuous functions. From a practical point of view too there are a number of real phenomena that are closely approximated by the delta function. The sharp blow from a hammer, the discharge of a capacitor or even the sound of the bark of a dog are all in some sense impulses. All of this provides motivation for the study of the delta function.

One property that is particularly useful in the context of Laplace transforms is the value of the integral

$$\int_{-\infty}^t \delta(u - u_0) du.$$

This has the value 0 if  $u_0 > t$  and the value 1 if  $u_0 < t$ . Thus we can write

$$\int_{-\infty}^t \delta(u - u_0) du = \begin{cases} 0 & 0 < u_0 \\ 1 & t > u_0 \end{cases}$$

or

$$\int_{-\infty}^t \delta(u - u_0) du = H(t - u_0)$$

where  $H$  is Heaviside's unit step function. If we were allowed to differentiate this result, or to put it more formally to use the fundamental theorem of the calculus (on functions one of which is not really a function, a second which is not even continuous let alone differentiable) then one could write that " $\delta(u - u_0) = H'(u - u_0)$ " or state that "the impulse function is the derivative of the Heaviside unit step function". Before the pure mathematicians send out lynching parties, let us examine these loose notions. Everywhere except where  $u = u_0$  the statement is equivalent to stating that the derivative of unity is zero, which is obviously true. The additional information in the albeit loose statement in quotation marks is a quantification of the nature of the unit jump in  $H(u - u_0)$ . We know the gradient there is infinite, but the nature of it is embodied in the second integral condition in the definition of the delta function, Eq. (2.4). The subject of *generalised functions* is introduced through this concept and the interested reader is directed towards the texts by Jones and Lighthill. All that will be noted here is that it is possible to define a whole string of derivatives  $\delta'(t)$ ,  $\delta''(t)$ , etc. where all these derivatives are zero everywhere except at  $t = 0$ . The key to keeping rigorous here is the property

$$\int_{-\infty}^{\infty} h(t)\delta(t)dt = h(0).$$

The "derivatives" have analogous properties, viz.

$$\int_{-\infty}^{\infty} h(t)\delta'(t)dt = -h'(0)$$

and in general

$$\int_{-\infty}^{\infty} h(t)\delta^{(n)}(t)dt = (-1)^n h^{(n)}(0).$$

Of course, the function  $h(t)$  will have to be appropriately differentiable. Now the Laplace transform of this  $n$ th derivative of the Dirac delta function is required. It can be easily deduced that

$$\int_{-\infty}^{\infty} e^{-st}\delta^{(n)}(t)dt = \int_{0-}^{\infty} e^{-st}\delta^{(n)}(t)dt = s^n.$$

Notice that for all these generalised functions, the condition for the validity of the initial value theorem is violated, and the final value theorem although perfectly valid is entirely useless. It is time to do a few examples.

**Example 2.6** *Determine the inverse Laplace transform*

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\}$$

*and interpret the  $F(t)$  obtained.*

**Solution** Writing

$$\frac{s^2}{s^2+1} = 1 - \frac{1}{s^2+1}$$

and using the linearity property of the inverse Laplace transform gives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\} &= \mathcal{L}^{-1}\{1\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \delta(t) - \sin t.\end{aligned}$$

This function is sinusoidal with a unit impulse at  $t = 0$ .

Note the direct use of the inverse  $\mathcal{L}^{-1}\{1\} = \delta(t)$ . This arises straight away from our definition of  $\mathcal{L}$ . It is quite possible for other definitions of Laplace transform to give the value  $\frac{1}{2}$  for  $\mathcal{L}\{\delta(t)\}$  (for example). This may worry those readers of a pure mathematical bent. However, as long as there is consistency in the definitions of the delta function and the Laplace transform and hence its inverse, then no inconsistencies arise. The example given above will always yield the same answer  $\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\} = \delta(t) - \sin t$ . The small variations possible in the definition of the Laplace transform around  $t = 0$  do not change this. Our definition, viz.

$$\mathcal{L}\{F(t)\} = \int_{0-}^{\infty} e^{-st} F(t) dt$$

remains the most usual.

**Example 2.7** Find the value of  $\mathcal{L}^{-1} \left\{ \frac{s^3}{s^2 + 1} \right\}$ .

**Solution** Using a similar technique to the previous example we first see that

$$\frac{s^3}{s^2 + 1} = s - \frac{s}{s^2 + 1}$$

so taking inverse Laplace transforms using the linearity property once more yields

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s^3}{s^2 + 1} \right\} &= \mathcal{L}^{-1}\{s\} - \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} \\ &= \delta'(t) - \cos t \end{aligned}$$

where  $\delta'(t)$  is the first derivative of the Dirac- $\delta$  function which was defined earlier.

Notice that the first derivative formula:

$$\mathcal{L}\{F'(t)\} = sf(s) - F(0)$$

with  $F'(t) = \delta'(t) - \cos t$  gives

$$\mathcal{L}\{\delta'(t) - \cos t\} = \frac{s^3}{s^2 + 1} - F(0)$$

which is indeed the above result apart from the troublesome  $F(0)$ .  $F(0)$  is of course not defined. Care indeed is required if standard Laplace transform results are to be applied to problems containing generalised functions. When in doubt, the best advice is to use limit definitions of  $\delta(t)$  and the like, and follow the mathematics through carefully, especially the swapping of integrals and limits. The little book by Lighthill is full of excellent practical advice.

## 2.7 Periodic Functions

We begin with a very straightforward definition that should be familiar to everyone:

**Definition 2.4** If  $F(t)$  is a function that obeys the rule

$$F(t) = F(t + \tau)$$

for some real  $\tau$  for all values of  $t$  then  $F(t)$  is called a periodic function with period  $\tau$ .

Periodic functions play a very important role in many branches of engineering and applied science, particularly physics. One only has to think of springs or alternating current present in household electricity to realise their prevalence. Here, a theorem on the Laplace transform of periodic functions is introduced, proved and used in some illustrative examples.

**Theorem 2.8** *Let  $F(t)$  have period  $T > 0$  so that  $F(t) = F(t + T)$ . Then*

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t) dt}{1 - e^{-sT}}.$$

**Proof** Like many proofs of properties of Laplace transforms, this one begins with its definition then evaluates the integral by using the periodicity of  $F(t)$

$$\begin{aligned} \mathcal{L}\{F(t)\} &= \int_0^\infty e^{-st} F(t) dt \\ &= \int_0^T e^{-st} F(t) dt + \int_T^{2T} e^{-st} F(t) dt \\ &\quad + \int_{2T}^{3T} e^{-st} F(t) dt + \cdots + \int_{(n-1)T}^{nT} e^{-st} F(t) dt + \cdots \end{aligned}$$

provided the series on the right hand side is convergent. This is assured since the function  $F(t)$  satisfies the condition for the existence of its Laplace transform by construction. Consider the integral

$$\int_{(n-1)T}^{nT} e^{-st} F(t) dt$$

and substitute  $u = t - (n - 1)T$ . Since  $F$  has period  $T$  this leads to

$$\int_{(n-1)T}^{nT} e^{-st} F(t) dt = e^{-s(n-1)T} \int_0^T e^{-su} F(u) du \quad n = 1, 2, \dots$$

which gives

$$\begin{aligned} \int_0^\infty e^{-st} F(t) dt &= (1 + e^{-sT} + e^{-2sT} + \cdots) \int_0^T e^{-st} F(t) dt \\ &= \frac{\int_0^T e^{-st} F(t) dt}{1 - e^{-sT}} \end{aligned}$$

on summing the geometric progression. This proves the result.  $\square$

Here is an example of using this theorem.

**Example 2.8** A rectified sine wave is defined by the expression

$$F(t) = \begin{cases} \sin t & 0 < t < \pi \\ -\sin t & \pi < t < 2\pi \end{cases}$$

$$F(t) = F(t + 2\pi)$$

determine  $\mathcal{L}\{F(t)\}$ .

**Solution** The graph of  $F(t)$  is shown in Fig. 2.5. The function  $F(t)$  actually has period  $\pi$ , but it is easier to carry out the calculation as if the period was  $2\pi$ . Additionally we can check the answer by using the theorem with  $T = \pi$ . With  $T = 2\pi$  we have from Theorem 2.8,

$$\mathcal{L}\{F(t)\} = \frac{\int_0^{2\pi} e^{-st} F(t) dt}{1 - e^{-sT}}$$

where the integral in the numerator is evaluated by splitting into two as follows:-

$$\int_0^{2\pi} e^{-st} F(t) dt = \int_0^{\pi} e^{-st} \sin t dt + \int_{\pi}^{2\pi} e^{-st} (-\sin t) dt.$$

Now, writing  $\Im\{\}$  to denote the imaginary part of the function in the brace we have

$$\begin{aligned} \int_0^{\pi} e^{-st} \sin t dt &= \Im \left\{ \int_0^{\pi} e^{-st+it} dt \right\} \\ &= \Im \left[ \frac{1}{i-s} e^{-st+it} \right]_0^{\pi} \\ &= \Im \left\{ \frac{1}{i-s} (e^{-s\pi+i\pi} - 1) \right\} \\ &= \Im \left\{ \frac{1}{s-i} (1 + e^{-s\pi}) \right\}. \end{aligned}$$

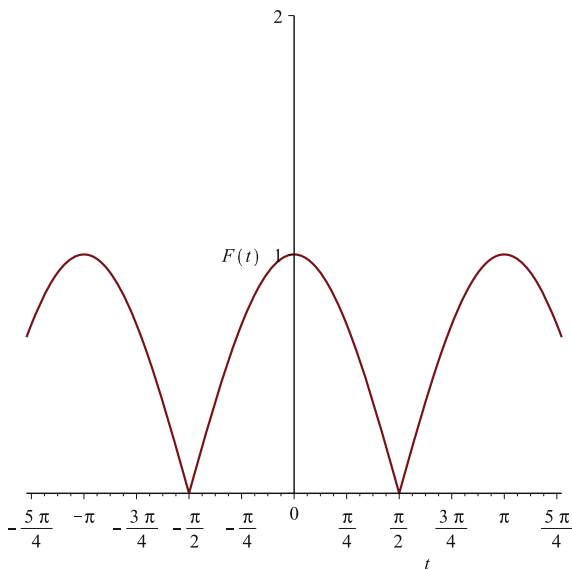
So

$$\int_0^{\pi} e^{-st} \sin t dt = \frac{1 + e^{-\pi s}}{1 + s^2}.$$

Similarly,

$$\int_{\pi}^{2\pi} e^{-st} \sin t dt = -\frac{e^{-2\pi s} + e^{-\pi s}}{1 + s^2}.$$

Hence we deduce that



**Fig. 2.5** The graph of  $F(t)$

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \frac{(1 + e^{-\pi s})^2}{(1 + s^2)(1 - e^{-2\pi s})} \\ &= \frac{1 + e^{-\pi s}}{(1 + s^2)(1 - e^{-\pi s})}.\end{aligned}$$

This is precisely the answer that would have been obtained if Theorem 2.8 had been applied to the function

$$F(t) = \sin t \quad 0 < t < \pi \quad F(t) = F(t + \pi).$$

We can therefore have some confidence in our answer.

## 2.8 Exercises

1. If  $F(t) = \cos(at)$ , use the derivative formula to re-establish the Laplace transform of  $\sin(at)$ .
2. Use Theorem 2.1 with

$$F(t) = \int_0^t \frac{\sin u}{u} du$$

to establish the result.



$$\mathcal{L}\left\{\frac{\sin(at)}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

3. Prove that

$$\mathcal{L}\left\{\int_0^t \int_0^v F(u) du dv\right\} = \frac{f(s)}{s^2}.$$

4. Find

$$\mathcal{L}\left\{\int_0^t \frac{\cos(au) - \cos(bu)}{u} du\right\}.$$

5. Determine

$$\mathcal{L}\left\{\frac{2 \sin t \sinh t}{t}\right\}.$$

6. Prove that if  $\bar{f}(s)$  indicates the Laplace transform of a piecewise continuous function  $f(t)$  then

$$\lim_{s \rightarrow \infty} \bar{f}(s) = 0.$$

7. Determine the following inverse Laplace transforms by using partial fractions

$$(a) \frac{2(2s+7)}{(s+4)(s+2)}, \quad s > -2 \quad (b) \frac{s+9}{s^2-9},$$

$$(c) \frac{s^2+2k^2}{s(s^2+4k^2)}, \quad (d) \frac{1}{s(s+3)^2},$$

$$(e) \frac{1}{(s-2)^2(s+3)^3}.$$

8. Verify the initial value theorem, for the two functions

$$(a) 2 + \cos t \text{ and}$$

$$(b) (4+t)^2.$$

9. Verify the final value theorem, for the two functions

$$(a) 3 + e^{-t} \text{ and}$$

$$(b) t^3 e^{-t}.$$

10. Given that

$$\mathcal{L}\{\sin(\sqrt{t})\} = \frac{k}{s^{3/2}} e^{-1/4s}$$

use  $\sin x \sim x$  near  $x = 0$  to determine the value of the constant  $k$ . (You will need the table of standard transforms Appendix B.)

11. By using a power series expansion, determine (in series form) the Laplace transforms of  $\sin(t^2)$  and  $\cos(t^2)$ .

12.  $P(s)$  and  $Q(s)$  are polynomials, the degree of  $P(s)$  is less than that of  $Q(s)$  which is  $n$ . Use partial fractions to prove the result

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

where  $\alpha_k$  are the  $n$  distinct zeros of  $Q(s)$ .

13. Find the following Laplace transforms:

(a)

$$H(t - a)$$

(b)

$$f_1 = \begin{cases} t + 1 & 0 \leq t \leq 2 \\ 3 & t > 2 \end{cases}$$

(c)

$$f_2 = \begin{cases} t + 1 & 0 \leq t \leq 2 \\ 6 & t > 2 \end{cases}$$

(d) the derivative of  $f_1(t)$ .

14. Find the Laplace transform of the triangular wave function:

$$F(t) = \begin{cases} t & 0 \leq t < c \\ 2c - t & c \leq t < 2c \end{cases}$$

$$F(t + 2c) = F(t).$$

15. Find the Laplace transform of the generally placed top hat function:

$$F(t) = \begin{cases} \frac{1}{h} & a \leq t < a + h \\ 0 & \text{otherwise} \end{cases}.$$

Hence deduce the Laplace transform of the Dirac- $\delta$  function  $\delta(t - a)$  where  $a > 0$  is a real constant.

An Introduction to Laplace Transforms and Fourier  
Series

Dyke, P.

2014, XV, 318 p. 66 illus., 10 illus. in color., Softcover

ISBN: 978-1-4471-6394-7