

Chapter 2

Partially Ordered Sets

2.1 Introduction

We introduce the notion of a partially ordered set (poset) and define several types of special elements associated with partial orders. Two partially ordered sets receive special attention: the poset of real numbers and the poset of partitions of a finite set. Partially ordered sets serve as the starting point for the study of several algebraic structures in Chap. 11.

2.2 Partial Orders

The fundamental notion of this chapter is introduced next.

Definition 2.1 A partial order on a set S is a relation $\rho \subseteq S \times S$ that is reflexive, antisymmetric, and transitive. The pair (S, ρ) is referred to as a partially ordered set or, for short, a poset.

When $|S|$ is finite, we refer to poset (S, ρ) as a *finite poset*.

A *strict partial order*, or more simply, a *strict order* on S , is a relation $\rho \subseteq S \times S$ that is irreflexive and transitive.

Example 2.2 The identity relation on a set S , ι_S , is a partial order; this is often referred to as the *discrete partial order* on S . Also, the relation $\theta_S = S \times S$ is a partial order on S .

Example 2.3 The relation “ \leq ” on the set of partitions of a set $\text{PART}(S)$ introduced in Definition 1.110 is a partial order on the set $\text{PART}(S)$.

Example 2.4 The divisibility relation δ introduced in Example 1.27 is a partial order on \mathbb{N} since, as we have shown in Example 1.43, δ is reflexive, antisymmetric, and transitive.

For a poset (S, ρ) , we prefer to use the *infix notation*; that is, write $s \rho t$ instead of $(s, t) \in \rho$. Moreover, various partial orders have their traditional notations, which we favor. For example, the relation δ introduced in Example 1.27 is usually denoted by $|$. Therefore, we write $m | n$ to denote that $(m, n) \in \delta$. Whenever practical, for generic partially ordered sets, we denote their partial order relation by \leq . Generic strict partial orders will be denoted by $<$.

Example 2.5 The inclusion relation \subseteq is a partial order on the set of subsets $\mathcal{P}(S)$ of a set S .

Example 2.6 For $\mathbf{u}, \mathbf{v} \in \mathbf{Seq}(S)$ define $\mathbf{u} \leq_{\text{pref}} \mathbf{v}$ if \mathbf{u} is a prefix of \mathbf{v} . Clearly, $\mathbf{u} \leq_{\text{pref}} \mathbf{v}$ if and only if there exists $\mathbf{t} \in \mathbf{Seq}(S)$ such that $\mathbf{v} = \mathbf{ut}$. It is immediate that “ \leq_{pref} ” is a reflexive relation.

If $\mathbf{u} \leq_{\text{pref}} \mathbf{v}$ and $\mathbf{v} \leq_{\text{pref}} \mathbf{u}$ there exist $\mathbf{t}, \mathbf{t}' \in \mathbf{Seq}(S)$ such that $\mathbf{v} = \mathbf{ut}$ and $\mathbf{u} = \mathbf{vt}'$. This implies $\mathbf{u} = \mathbf{utt}'$. Thus, $\mathbf{tt}' = \mathbf{\lambda}$, so $\mathbf{t} = \mathbf{t}' = \mathbf{\lambda}$, which allows us to infer that $\mathbf{u} = \mathbf{v}$. This shows that “ \leq_{pref} ” is antisymmetric.

Finally, suppose that $\mathbf{u} \leq_{\text{pref}} \mathbf{v}$ and $\mathbf{v} \leq_{\text{pref}} \mathbf{w}$. We have $\mathbf{v} = \mathbf{ut}$ and $\mathbf{w} = \mathbf{vs}$ for some $\mathbf{s}, \mathbf{t} \in \mathbf{Seq}(S)$. This implies $\mathbf{w} = \mathbf{uts}$, which shows that $\mathbf{u} \leq_{\text{pref}} \mathbf{w}$. Thus, “ \leq_{pref} ” is indeed a partial order on $\mathbf{Seq}(S)$.

In a similar manner, it is possible to show that the relations

$$\begin{aligned} \leq_{\text{suff}} &= \{(\mathbf{u}, \mathbf{v}) \in (\mathbf{Seq}(S))^2 \mid \mathbf{v} = \mathbf{tu} \text{ for some } \mathbf{t} \in \mathbf{Seq}(S)\}, \\ \leq_{\text{infix}} &= \{(\mathbf{u}, \mathbf{v}) \in (\mathbf{Seq}(S))^2 \mid \mathbf{v} = \mathbf{tut}' \text{ for some } \mathbf{t}, \mathbf{t}' \in \mathbf{Seq}(S)\}, \end{aligned}$$

are partial orders on $\mathbf{Seq}(S)$ (exercise!).

If (S, \leq) is a poset and $T \subseteq S$, then (T, \leq_T) is also a poset, where $\leq_T = \leq \cap (T \times T)$ is the *trace of \leq on T* .

Every strict partial order is also asymmetric. Indeed, let $<$ be a strict partial order on S and assume that $x < y$. If $y < x$, then $x < x$ due to the transitivity of $<$, which contradicts the irreflexivity of $<$. This shows that $<$ is indeed asymmetric.

A strict partial order is not, in general, a partial order since strict partial orders are irreflexive, while partial orders are reflexive. The link between partial orders and strict partial orders is given next.

Theorem 2.7 Let \leq be a partial order on a set S and let $<$ be the relation $\leq - \iota_S$. The relation $<$ is a strict partial order on S .

If $<$ is a strict partial order on S , then the relation \leq defined as the union $< \cup \iota_S$ is a partial order on S .

Proof Since $\iota_S \cap < = \emptyset$, the relation $<$ is irreflexive.

To prove the transitivity of $<$, let $x, y, z \in S$ be such that $x < y$, $y < z$. Because of the transitivity of \leq , we have $x \leq z$. On the other hand, we also have $x \neq z$. Indeed, if we assume that $x = z$, then we would have both $z < y$ and $y < z$, which is impossible by the asymmetry of $<$. Therefore, $(x, z) \in \leq - \iota_S = <$, which implies the transitivity of $<$.

Let $<$ be a strict partial order and let \leq be the relation $< \cup \iota_S$. The reflexivity of \leq is immediate.

To show that \leq is antisymmetric, assume that $x \leq y$ and $y \leq x$. Because of the definition of \leq , we may have $x < y$ or $(x, y) \in \iota_S$ (that is, $x = y$). In the first case, we have a contradiction. Indeed, if $y < x$, this contradicts the asymmetry of $<$; if $(y, x) \in \iota_S$, we also have $(x, y) \in \iota_S$, and this contradicts the irreflexivity of $<$. Consequently, we must have $x = y$.

Let $x \leq y$ and $y \leq x$. We need to consider the following four cases.

- (i) If $x < y$, $y < z$, we have $x < z$ because of the transitivity of $<$. This implies $x \leq z$.
- (ii) If $(x, y) \in \iota_S$ and $y < z$, we have $x = y$; hence, $x < z$ and therefore $x \leq z$.
- (iii) If $x < y$ and $(y, z) \in \iota_S$, we follow an argument similar to the one used in the previous case.
- (iv) If $(x, y), (y, z) \in \iota_S$, we have $(x, z) \in \iota_S$ because of the transitivity of ι_S ; hence, $x \leq z$.

We proved that \leq is also transitive, and this concludes our argument.

Example 2.8 Consider the relation “ \leq ” on \mathbb{R} , which is a partial order. The strict partial order attached to it by the previous proposition is the relation “ $<$ ”.

A relation $\rho \subseteq S \times S$ is *acyclic* if $\rho^n \cap \iota_S = \emptyset$ for every $n \geq 1$.

Theorem 2.9 Every strict partial order is acyclic.

Proof Let ρ be a strict partial order relation on S . Its transitivity implies the existence of the descending sequence $\rho \supseteq \rho^2 \supseteq \dots \supseteq \rho^n \supseteq \dots$. Since ρ is irreflexive, we have $\rho \cap \iota_S = \emptyset$, and this implies $\rho^n \cap \iota_S = \emptyset$.

Next we introduce a graphical representation of partial orders.

Definition 2.10 Let (S, \leq) be a poset. The Hasse diagram of (S, \leq) is the digraph of the relation $< - (<)^2$, where “ $<$ ” is the strict partial order corresponding to \leq .

In view of the properties of acyclic relations discussed above, it is clear that the relation $< - (<)^2$ is acyclic; therefore, the Hasse diagram is always an acyclic directed graph. We will denote this relation by “ \prec ”.

Observe that $x < y$ if $x \neq y$, $x \leq y$, and there is no $u \in S$ such that $x \leq u$ and $u \leq y$. In other words, if $x < y$, then y covers x directly, without any intermediate elements.

The use of Hasse diagrams in representing posets is justified by the following statement.

Theorem 2.11 If \leq is a partial order on a finite set S , $<$ is the strict partial order corresponding to \leq , and $\theta = < - (<)^2$, then $\theta^* = \leq$.

Proof Let $x, y \in S$ such that $x \leq y$. If $x = y$, then we have $(x, y) \in \iota_S \subseteq \theta^*$.

Assume now that $x \leq y$ and $x \neq y$, which means that $x < y$. Consider the collection \mathcal{C}_{xy} of all sequences of elements of A that can be “interpolated” between x and y :

$$\mathcal{C}_{xy} = \{(\mathbf{s}(0), \dots, \mathbf{s}(n-1)) \mid x = \mathbf{s}(0), \mathbf{s}(n-1) = y, \text{ and} \\ \mathbf{s}(i) < \mathbf{s}(i+1) \text{ for } 0 \leq i \leq n-2, n \geq 2\}.$$

We have $\mathcal{C}_{xy} \neq \emptyset$ since the sequence (x, y) belongs to \mathcal{C}_{xy} . Furthermore, no sequence from \mathcal{C}_{xy} may contain a repetition. Since S is finite, \mathcal{C}_{xy} contains a finite number of sequences.

Let $(\mathbf{s}(0), \mathbf{s}(1), \dots, \mathbf{s}(m-1))$ be a sequence of maximal length from \mathcal{C}_{xy} , where $x = \mathbf{s}(0)$ and $y = \mathbf{s}(m-1)$.

Observe that for no pair $(\mathbf{s}(i), \mathbf{s}(i+1))$ can we have $\mathbf{s}(i) < z < \mathbf{s}(i+1)$ because, otherwise, the maximality of m would be contradicted. Therefore, $(\mathbf{s}(i), \mathbf{s}(i+1)) \in (< - (<)^2) = \theta$, and this shows that $(x, y) \in \theta^{m-1} \subseteq \theta^*$.

Conversely, if $(x, y) \in \theta^*$, there is $k \in \mathbb{N}$ such that $(x, y) \in \theta^k$, which means that there exists a sequence $(\mathbf{z}(0), \dots, \mathbf{z}(k))$ such that

$$x = \mathbf{z}(0), (\mathbf{z}(i), \mathbf{z}(i+1)) \in \theta \text{ for } 0 \leq i \leq k-1 \text{ and } y = \mathbf{z}(k).$$

This implies $\mathbf{z}(i) \leq \mathbf{z}(i+1)$; hence, $(x, y) \in (\leq)^k \subseteq \leq$ because of the transitivity of \leq .

The relation θ introduced in Theorem 2.11 is called the *transitive reduction* of the partial order ρ .

Example 2.12 The Hasse diagram of the poset $(\mathcal{P}(S), \subseteq)$, where $S = \{a, b, c\}$, is given in Fig. 2.1a.

Example 2.13 Consider the poset $(\{1, 2, 3, 4, 5, 6, 7, 8\}, \delta)$, where δ is the divisibility relation introduced in Example 1.27. Its Hasse diagram is shown in Fig. 2.1b.

Definition 2.14 Let (S, \leq) be a poset and let $K \subseteq S$. The set of upper bounds of the set K is the set $K^s = \{y \in S \mid x \leq y \text{ for every } x \in K\}$.

The set of lower bounds of the set K is the set $K^i = \{y \in S \mid y \leq x \text{ for every } x \in K\}$.

If $K^s \neq \emptyset$, we say that the set K is bounded above. Similarly, if $K^i \neq \emptyset$, we say that K is bounded below. If K is both bounded above and bounded below we will refer to K as a bounded set.

If $K^s = \emptyset$ ($K^i = \emptyset$), then K is said to be unbounded above (below).

Theorem 2.15 Let (S, \leq) be a poset and let U and V be two subsets of S . If $U \subseteq V$, then we have $V^i \subseteq U^i$ and $V^s \subseteq U^s$.

Also, for every subset T of S , we have $T \subseteq (T^s)^i$ and $T \subseteq (T^i)^s$.

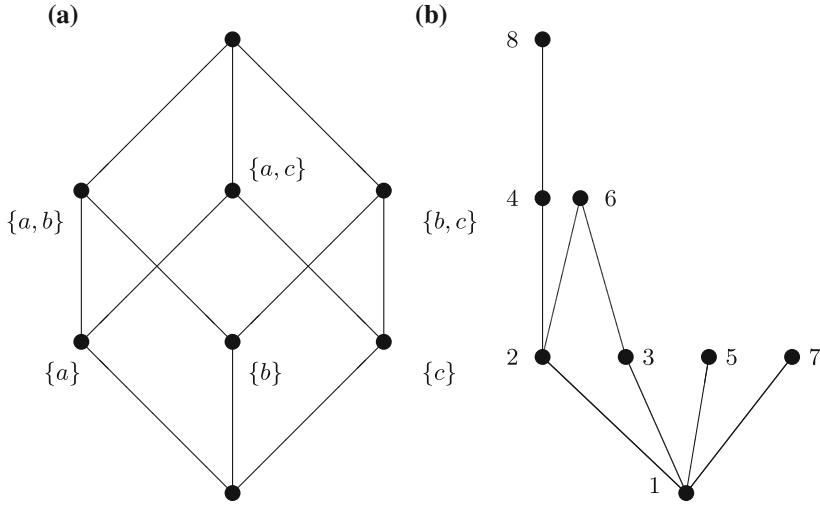


Fig. 2.1 Hasse diagrams. **a** The poset $(\mathcal{P}(S), \subseteq)$, and **b** The poset $(\{1, \dots, 8\}, \delta)$

Proof The argument for both statements of the theorem amounts to a direct application of Definition 2.14.

Note that for every subset T of a poset S , we have both

$$T^i = ((T^i)^s)^i \quad (2.1)$$

and

$$T^s = ((T^s)^i)^s. \quad (2.2)$$

Indeed, since $T \subseteq (T^i)^s$, by the first part of Theorem 2.15, we have $((T^s)^i)^s \subseteq T^s$. By the second part of the same theorem applied to T^s , we have the reverse inclusion $T^s \subseteq ((T^s)^i)^s$, which yields $T^s = ((T^s)^i)^s$.

Theorem 2.16 *For any subset K of a poset (S, ρ) , the sets $K \cap K^s$ and $K \cap K^i$ contain at most one element.*

Proof Suppose that $y_1, y_2 \in K \cap K^s$. Since $y_1 \in K$ and $y_2 \in K^s$, we have $(y_1, y_2) \in \rho$. Reversing the roles of y_1 and y_2 (that is, considering now that $y_2 \in K$ and $y_1 \in K^s$), we obtain $(y_2, y_1) \in \rho$. Therefore, we may conclude that $y_1 = y_2$ because of the antisymmetry of the relation ρ , which shows that $K \cap K^s$ contains at most one element.

A similar argument can be used for the second part of the proposition; we leave it to the reader.

Definition 2.17 Let (S, \leq) be a poset. The least (greatest) element of the subset K of S is the unique element of the set $K \cap K^i$ ($K \cap K^s$, respectively) if such an element exists.

If K is unbounded above, then it is clear that K has no greatest element. Similarly, if K is unbounded below, then K has no least element.

Applying Definition 2.17 to the set S , the least (greatest) element of the poset (S, \leq) is an element a of S such that $a \leq x$ ($x \leq a$, respectively) for all $x \in S$.

It is clear that if a poset has a least element u , then u is the unique minimal element of that poset. A similar statement holds for the greatest and the maximal elements.

Definition 2.18 Let (S, \leq) be a poset that has 0 as its least element. An atom of (S, \leq) is an element x of S such that $0 < x$.

If (S, \leq) is a poset that has 1 as its greatest element, then y is a co-atom of (S, \leq) if $y \leq 1$.

Example 2.19 For the poset introduced in Example 2.12, the greatest element is $\{a, b, c\}$, while the least element is \emptyset .

The atoms of this poset are $\{a\}$, $\{b\}$, $\{c\}$; its co-atoms are $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$.

Definition 2.20 The subset K of the poset (S, \leq) has a least upper bound u if $K^s \cap (K^s)^i = \{u\}$.

K has the greatest lower bound v if $K^i \cap (K^i)^s = \{v\}$.

We note that a set can have at most one least upper bound and at most one greatest lower bound. Indeed, we have seen above that for any set U the set $U \cap U^i$ may contain an element or be empty. Applying this remark to the set K^s , it follows that the set $K^s \cap (K^s)^i$ may contain at most one element, which shows that K may have at most one least upper bound. A similar argument can be made for the greatest lower bound.

If the set K has a least upper bound, we denote it by $\sup K$. The greatest lower bound of a set will be denoted by $\inf K$. These notations come from the terms *supremum* and *infimum* used alternatively for the least upper bound and the greatest lower bound, respectively.

Example 2.21 A two-element subset $\{m, n\}$ of (\mathbb{N}, δ) has both an infimum and a supremum. Indeed, let p be the least common multiple of m and n . Since $(n, p), (m, p) \in \delta$, it is clear that p is an upper bound of the set $\{m, n\}$. On the other hand, if k is an upper bound of $\{m, n\}$, then k is a multiple of both m and n . In this case, k must also be a multiple of p because otherwise we could write $k = pq + r$ with $0 < r < p$ by dividing k by p . This would imply $r = k - pq$; hence, r would be a multiple of both m and n because both k and p have this property. However, this would contradict the fact that p is the least multiple that m and n share! This shows that the least common multiple of m and n coincides with the supremum of the set $\{m, n\}$. Similarly, $\inf \{m, n\}$ equals the greatest common divisor m and n .

Example 2.22 Consider a set M and the poset $(\mathcal{P}(M), \subseteq)$. Let K and H be two subsets of M . The set $\{K, H\}$ has an infimum and a supremum. Indeed, let $L = K \cap H$. Clearly, $L \subseteq K$ and $L \subseteq H$, so L is a lower bound of the set $\{K, H\}$. Furthermore, if $J \subseteq K$ and $J \subseteq H$, then $J \subseteq L$ by the definition of the intersection. This proves that the infimum of $\{K, H\}$ is the intersection $K \cap H$. A similar argument shows that $K \cup H$ is the supremum of $\{K, H\}$.

In the previous two examples, any two-element subset of the poset has both a supremum and an infimum.

For a one-element subset $\{x\}$ of a poset (S, ρ) , we have $\sup\{x\} = \inf\{x\} = x$.

Definition 2.23 A minimal element of a poset (S, \leq) is an element $x \in S$ such that $\{x\}^i = \{x\}$. A maximal element of (S, \leq) is an element $y \in S$ such that $\{y\}^s = \{y\}$.

In other words, x is a minimal element of the poset (S, \leq) if there is no element less than or equal to x other than itself; similarly, x is maximal if there is no element greater than or equal to x other than itself.

The set of minimal elements of a poset (S, \leq) is denoted by $\text{MIN}(S, \leq)$; the set of maximal elements of this poset is denoted by $\text{MAX}(S, \leq)$.

Example 2.24 Not every subset of a poset has a least or a greatest element. Indeed, let $(\{2, 3, 4, 5, 6, 7, 8, \}, \delta)$ be a poset whose Hasse diagram is shown in Fig. 2.1b. It is easy to see that

$$\begin{aligned}\text{MIN}(\{2, 3, 4, 5, 6, 7, 8, \}, \delta) &= \{2, 3, 5, 7\}, \\ \text{MAX}(\{2, 3, 4, 5, 6, 7, 8, \}, \delta) &= \{5, 6, 7, 8\}.\end{aligned}$$

There is no least element and there is no largest element in this poset.

Theorem 2.25 Every finite nonempty subset K of a poset (S, \leq) has a minimal element and a maximal element.

Proof Suppose that $K = \{x_0, \dots, x_{n-1}\}$ for $n \geq 1$. Define the element $u_0 = x_0$ and

$$u_k = \begin{cases} x_k & \text{if } x_k < u_{k-1}, \\ u_{k-1} & \text{otherwise.} \end{cases}$$

Then, u_{n-1} is a minimal element. The proof of the existence of a maximal element of K is similar.

Next, we discuss a simple property of partially ordered sets that will allow us to obtain half of some of the arguments related to the properties of partial orders for free.

Theorem 2.26 Let ρ be a partial order on a set S . The inverse ρ^{-1} is also a partial order on the same set.

Proof Since $(x, x) \in \rho$ for every $x \in S$, it follows that $(x, x) \in \rho^{-1}$ for every $x \in S$, so ρ^{-1} is reflexive.

The antisymmetry of ρ^{-1} follows from $(\rho^{-1})^{-1} = \rho$ and because of the antisymmetry of ρ .

To prove the transitivity of ρ^{-1} , assume that $(x, y) \in \rho^{-1}$ and $(y, z) \in \rho^{-1}$. This means that $(y, x), (z, y) \in \rho$, and because of the transitivity of ρ , we obtain $(z, x) \in \rho$, so $(x, z) \in \rho^{-1}$, which proves that ρ^{-1} is transitive.

Definition 2.27 *The dual of the poset (S, ρ) is the poset (S, ρ^{-1}) .*

Concepts valid for a poset have a counterpart for their dual poset. For instance, x is an upper bound for the set K in the poset (S, ρ) if and only if x is a lower bound for K in the dual poset. Similarly, $t = \sup K$ in the poset (S, ρ) if and only if $t = \inf K$ in the dual poset. Similar pairs are minimal element and maximal element, infimum and supremum, etc.

If all concepts occurring in a statement about posets are replaced by their duals, we obtain the *dual statement*; the method of proving statements about posets is known as *dualization*. Furthermore, if a statement holds for a poset (S, ρ) , its dual holds for the dual poset (S, ρ^{-1}) . This allows us to formulate the following principle.

The Duality Principle for Posets: If a statement is true for all posets, then its dual is also true for all posets.

The validity of this principle follows from the fact that any poset can be regarded as the dual of some other poset. The duality principle allows us to simplify proofs of certain statements that concern posets. For statements involving both a concept and its dual we need to prove only half of the statement; the other half follows by applying the duality principle. For instance, once we prove the statement “any subset of a poset can have at most one least upper bound,” the dual statement “any subset of a poset can have at most one greatest lower bound” follows.

2.3 The Poset of Real Numbers

For the poset (\mathbb{R}, \leq) , it is possible to give more specific descriptions of the supremum and infimum of a subset when they exist.

Theorem 2.28 *If $T \subseteq \mathbb{R}$, then $u = \sup T$ if and only if u is an upper bound of T and, for every $\epsilon > 0$, there is $t \in T$ such that $u - \epsilon < t \leq u$.*

The number $v = \inf T$ if and only if v is a lower bound of T and, for every $\epsilon > 0$, there is $t \in T$ such that $v \leq t < v + \epsilon$.

Proof We prove only the first part of the theorem; the argument for the second part is similar and is left to the reader.

Suppose that $u = \sup T$; that is, $\{u\} = T^s \cup (T^s)^i$. Since $u \in T^s$, it is clear that u is an upper bound for T . Suppose that there is $\epsilon > 0$ such that no $t \in T$ exists such

that $u - \epsilon < t \leq u$. This means that $u - \epsilon$ is also an upper bound for T , and in this case u cannot be a lower bound for the set of upper bounds of T . Therefore, no such ϵ may exist.

Conversely, suppose that u is an upper bound of T and for every $\epsilon > 0$, there is $t \in T$ such that $u - \epsilon < t \leq u$. Suppose that u does not belong to $(K^s)^i$. This means that there is another upper bound u' of T such that $u' < u$. Choosing $\epsilon = u - u'$, we would have no $t \in T$ such that $u - \epsilon = u' < t \leq u$ because this would prevent u' from being an upper bound of T . This implies $u \in (K^s)^i$, so $u = \sup T$.

A very important axiom for the set \mathbb{R} is given next.

The Completeness Axiom for \mathbb{R} : If T is a nonempty subset of \mathbb{R} that is bounded above, then T has a supremum.

A statement equivalent to the Completeness Axiom for \mathbb{R} follows.

Theorem 2.29 *If T is a nonempty subset of \mathbb{R} that is bounded below, then T has an infimum.*

Proof Note that the set T^i is not empty. If $s \in T^i$ and $t \in T$, we have $s \leq t$, so the set T^i is bounded above. By the Completeness axiom $v = \sup T^i$ exists and $\{v\} = (T^i)^s \cap ((T^i)^s)^i = (T^i)^s \cap T^i$ by Equality (2.1). Thus, $v = \inf T$.

We leave to the reader to prove that Theorem 2.29 implies the Completeness Axiom for \mathbb{R} .

Another statement equivalent to the Completeness Axiom is the following.

Theorem 2.30 (Dedekind's Theorem) *Let U and V be nonempty subsets of \mathbb{R} such that $U \cup V = \mathbb{R}$ and $x \in U, y \in V$ imply $x < y$. Then, there exists $a \in \mathbb{R}$ such that if $x > a$, then $x \in V$, and if $x < a$, then $x \in U$.*

Proof Observe that $U \neq \emptyset$ and $V \subseteq U^s$. Since $V \neq \emptyset$, it means that U is bounded above, so by the Completeness Axiom $\sup U$ exists. Let $a = \sup U$. Clearly, $u \leq a$ for every $u \in U$. Since $V \subseteq U^s$, it also follows that $a \leq v$ for every $v \in V$.

If $x > a$, then $x \in V$ because otherwise we would have $x \in U$ since $U \cup V = \mathbb{R}$ and this would imply $x \leq a$. Similarly, if $x < a$, then $x \in U$.

Using the previously introduced notations, Dedekind's theorem can be stated as follows: if U and V are nonempty subsets of \mathbb{R} such that $U \cup V = \mathbb{R}$, $U^s \subseteq V$, $V^i \subseteq U$, then there exists a such that $\{a\}^s \subseteq V$ and $\{a\}^i \subseteq U$.

One can prove that Dedekind's theorem implies the Completeness Axiom. Indeed, let T be a nonempty subset of \mathbb{R} that is bounded above. Therefore $V = T^s \neq \emptyset$. Note that $U = (T^s)^i \neq \emptyset$ and $U \cup V = \mathbb{R}$. Moreover, $U^s = ((T^s)^i)^s = T^s = V$ and $V^i = (T^s)^i = U$. Therefore, by Dedekind's theorem, there is $a \in \mathbb{R}$ such that $\{a\}^s \subseteq V = T^s$ and $\{a\}^i \subseteq U = (T^s)^i$. Note that $a \in \{a\}^s \cap \{a\}^i \subseteq T^s \cap (T^s)^i$, which proves that $a = \sup T$.

By adding the symbols $+\infty$ and $-\infty$ to the set \mathbb{R} , one obtains the set $\hat{\mathbb{R}}$. The partial order \leq defined on \mathbb{R} can now be extended to $\hat{\mathbb{R}}$ by $-\infty \leq x$ and $x \leq +\infty$ for every $x \in \mathbb{R}$.

We also extend the addition and multiplication of reals to $\hat{\mathbb{R}}$ by

$$\begin{aligned}
 x + \infty &= +\infty + x = +\infty \text{ for } -\infty < x \leq +\infty, \\
 x - \infty &= -\infty + x = -\infty \text{ for } -\infty \leq x < +\infty, \\
 x \cdot \infty &= \infty \cdot x = \begin{cases} -\infty & \text{if } -\infty \leq x < 0, \\ 0 & \text{if } x = 0, \\ \infty & \text{if } 0 < x \leq +\infty \end{cases}, \\
 x \cdot (-\infty) &= -\infty \cdot x = \begin{cases} \infty & \text{if } -\infty \leq x < 0, \\ 0 & \text{if } x = 0, \\ -\infty & \text{if } 0 < x \leq +\infty \end{cases}, \\
 \frac{x}{+\infty} &= \frac{x}{-\infty} = 0 \text{ for } x \in \mathbb{R}.
 \end{aligned}$$

The operations $+\infty - \infty$ and $-\infty + \infty$ are undefined.

Note that, in the poset $(\hat{\mathbb{R}}, \leq)$, the sets T^i and T^s are nonempty for every $T \in \mathcal{P}(\hat{\mathbb{R}})$ because $-\infty \in T^i$ and $+\infty \in T^s$ for any subset T of $\hat{\mathbb{R}}$.

Theorem 2.31 *For every set $T \subseteq \hat{\mathbb{R}}$, both $\sup T$ and $\inf T$ exist in the poset $(\hat{\mathbb{R}}, \leq)$.*

Proof We present the argument for $\sup T$. If $\sup T$ exists in (\mathbb{R}, \leq) , then it is clear that the same number is $\sup T$ in $(\hat{\mathbb{R}}, \leq)$.

Assume now that $\sup T$ does not exist in (\mathbb{R}, \leq) . By the Completeness Axiom for \mathbb{R} , this means that the set T does not have an upper bound in (\mathbb{R}, \leq) . Therefore, the set of upper bounds of T in $(\hat{\mathbb{R}}, \leq)$ is $T^s = \{+\infty\}$. It follows immediately that in this case $\sup T = +\infty$ in $(\hat{\mathbb{R}}, \leq)$.

2.4 Chains and Antichains

The main notions of this section are introduced next.

Definition 2.32 *Let (S, \leq) be a poset. A chain of (S, \leq) is a subset T of S such that for every $x, y \in T$ such that $x \neq y$ we have either $x < y$ or $y < x$. If the set S is a chain, we say that (S, \leq) is a totally ordered set and the relation \leq is a total order.*

If $s \in \text{Seq}(S)$ (or $s \in \text{Seq}_\infty(S)$) and for every $i, j \in \mathbb{N}$ we have $s(i) < s(j)$ or $s(j) < s(i)$, we refer to the sequence s as a chain in S ; if $s(i) \leq s(j)$ or $s(j) \leq s(i)$ for every $i, j \in \mathbb{N}$, then we say that s is a multichain in (S, \leq) .

If $S = \{x_1, \dots, x_n\}$, the total order whose diagram is given in Fig. 2.2 is denoted by $\text{TO}(x_1, \dots, x_n)$.

Let (S, \leq) be a poset. The elements x, y of S are *incomparable* if we have neither $x \leq y$ nor $y \leq x$. This is denoted by $x \parallel y$. It is easy to see that “ \parallel ” is a symmetric and irreflexive relation. The set of pairs of incomparable elements of a poset (S, \leq) is

$$\text{INC}(S, \leq) = \{(x, y) \in S \times S \mid x \not\leq y \text{ and } y \not\leq x\}.$$

Fig. 2.2 Hasse diagram of a total order on $S = \{x_1, x_2, \dots, x_n\}$



Definition 2.33 An antichain of (S, \leq) is a subset U of S such that, for every two distinct elements $x, y \in U$, we have $x \parallel y$.

Example 2.34 The set of real numbers equipped with the usual partial order (\mathbb{R}, \leq) is a chain since, for every $x, y \in \mathbb{R}$, we have either $x \leq y$ or $y \leq x$.

Example 2.35 In the poset (\mathbb{N}, δ) , the set of all prime numbers is an antichain since if p and q are two distinct primes, we have neither $(p, q) \in \delta$ nor $(q, p) \in \delta$.

Example 2.36 If S is a finite set such that $|S| = n$, the set of subsets of S that contain k elements (for a fixed $k, k \leq |S|$) is an antichain in the poset $(\mathcal{P}(S), \subseteq)$ that contains $\binom{n}{k}$ elements.

Example 2.37 If (S, \leq) is a poset, then both $\text{MIN}(S, \leq)$ and $\text{MAX}(S, \leq)$ are maximal antichains of (S, \leq) (with respect to set inclusion).

Every finite chain of a poset has a least element and a greatest element. Indeed, by Theorem 2.25, a finite chain has a minimal element and a maximal element. Since the notions of minimal and maximal elements in a chain coincide with the notions of least element and largest element, respectively, it statement follows.

Definition 2.38 Let u, v be two elements of a poset (S, \leq) such that $u \leq v$. The interval determined by u and v is the set

$$[u, v] = \{x \in S \mid u \leq x \leq v\}.$$

Example 2.39 In the poset (\mathbb{N}, δ) we have $(3, 24) \in \delta$. The interval $[3, 24]$ is

$$[3, 24] = \{3, 6, 12, 24\}.$$

Not every poset is a chain, as shown in the next example.

Example 2.40 The poset $(\mathcal{P}(S), \subseteq)$ considered in Example 2.12 is not a chain; elements of $\mathcal{P}(S)$ such as $\{a, b\}$ and $\{b, c\}$ are incomparable.

The poset from Example 2.13 is not a chain since it contains incomparable elements (for instance, $4 \parallel 6$). However, the subset $\{1, 2, 4, 8\}$ is a chain, as can be easily seen. Thus, a poset (S, \leq) that is not a chain itself may very well contain subsets that are chains with respect to the trace of the partial order of the set itself.

Denote by $\text{CHAINS}(S)$ the set of chains of a poset (S, \leq) . We use the poset $(\text{CHAINS}(S), \subseteq)$, where the partial order relation is the set inclusion.

Theorem 2.41 *If $\{U_i \mid i \in I\}$ is a chain of the poset $(\text{CHAINS}(S), \subseteq)$ (that is, a chain of chains of (S, \leq)), then $\bigcup\{U_i \mid i \in I\}$ is itself a chain of (S, \leq) (that is, a member of $(\text{CHAINS}(S), \subseteq)$).*

Proof Let $x, y \in \bigcup\{U_i \mid i \in I\}$. There are $i, j \in I$ such that $x \in U_i$ and $y \in U_j$ and we have either $U_i \subseteq U_j$ or $U_j \subseteq U_i$. In the first case, we have either $x_i \leq x_j$ or $x_j \leq x_i$ because both x and y belong to the chain U_j . The same conclusion can be reached in the second case when both x and y belong to the chain U_i . So, in any case, x and y are comparable, which proves that $\bigcup\{U_i \mid i \in I\}$ is a chain of (S, \leq) .

Definition 2.42 *A well-ordered poset is a poset for which every nonempty subset has a least element.*

A well-ordered set is necessarily a chain. Indeed, consider the well-ordered set (S, \leq) and $x, y \in S$. Since the set $\{x, y\}$ must have a least element, we have either $x \leq y$ or $y \leq x$.

Example 2.43 The set of natural numbers is well-ordered. This property of natural numbers is known as the *well-ordering principle*.

(Well-Ordering Axiom) Given any set S , there is a binary relation ρ such that (S, ρ) is a well-ordered set.

The set (\mathbb{R}, \leq) is not well-ordered, despite the fact that it is a chain, since it contains subsets such as $(0, 1) = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$ that do not have a least element.

Definition 2.44 *Let “ $<$ ” be the strict partial order of the poset (S, \leq) . An infinite descending sequence in a poset (S, ρ) is an infinite sequence $s \in \text{Seq}_\infty(S)$ such that $s(n+1) < s(n)$ for all $n \in \mathbb{N}$.*

An infinite ascending sequence in a poset (S, ρ) is an infinite sequence $s \in \text{Seq}_\infty(S)$ such that $s(n) < s(n+1)$ for all $n \in \mathbb{N}$.

A poset with no infinite descending sequences is called Artinian. A poset with no infinite ascending sequences is called Noetherian.

Clearly, the range of every infinite ascending or descending sequence is a chain.

Example 2.45 The poset (\mathbb{N}, δ) is Artinian. Indeed, suppose that s is an infinite descending sequence of natural numbers. If $s(0) \neq 0$, then the natural number $s(0)$ has an infinite set of divisors $\{s(0), s(1), \dots\}$. If $s(0) = 0$, in view of the fact that any natural number is a divisor of 0, we obtain the impossibility of an infinite descending sequence by applying the same argument to $s(1)$. However, this poset is not Noetherian. For instance, the sequence $\mathbf{z} : \mathbb{N} \rightarrow \mathbb{N}$ defined by $\mathbf{z}(n) = 2^n$ for $n \in \mathbb{N}$ is an infinite ascending sequence.

A generalization of well-ordered posets is considered in the next definition.

Definition 2.46 A well-founded poset is a partially ordered set where every nonempty subset has a minimal element.

Since the least element of a subset is also a minimal element, it is clear that a well-ordered set is also well-founded. However, the inverse is not true; for instance, not every finite set is well-ordered.

Theorem 2.47 A poset (S, ρ) is well-founded if and only if it is Artinian.

Proof Let (S, ρ) be a well-founded poset, and suppose that \mathbf{s} is an infinite descending sequence in this poset. The set $T = \{\mathbf{s}(n) \mid n \in \mathbb{N}\}$ has no minimal element since, for every $\mathbf{s}(k) \in T$, we have $(\mathbf{s}(k+1), \mathbf{s}(k)) \in \rho_1$, which contradicts the well-foundedness of (S, ρ) .

Conversely, assume that (S, ρ) is Artinian; that is, there is no infinite descending sequence in (S, ρ) . Suppose that K is a nonempty subset of S without minimal elements. Let x_0 be an arbitrary element of K . Such an element exists since K is not empty. Since x_0 is not minimal, there is $x_1 \in K$ such that $(x_1, x_0) \in \rho$. Since x_1 is not minimal, there is $x_2 \in K$ such that $(x_2, x_1) \in \rho$, etc., and this construction can continue indefinitely. In this way, we can build an infinite descending sequence $\mathbf{s} : \mathbb{N} \rightarrow S$, where $\mathbf{s}(n) = x_n$ for $n \in \mathbb{N}$.

Theorem 2.47 implies immediately that any finite poset is well-founded.

Example 2.48 We will show that the poset $(\mathbb{N} \times \mathbb{N}, \preceq)$ is well-founded.

If $(m, n_0) \succ (m, n_1) \succ \dots$ is a descending chain of pairs having the same first component, then $n_0 > n_1 > \dots$ is a descending chain of natural numbers and such a chain is finite. Therefore, $(m, n_0) \succ (m, n_1) \succ \dots$ must be a finite chain.

Consider now an arbitrary descending chain,

$$(p_0, q_0) \succ (p_1, q_1) \succ \dots,$$

in $(\mathbb{N} \times \mathbb{N}, \preceq)$. We have $p_0 \geq p_1 \geq \dots$, and in this sequence we may have only finite “constant” fragments $p_k = p_{k+1} = \dots = p_{k+l}$. Therefore, the chain of the first components of the pairs of the sequence $(p_0, q_0) \succ (p_1, q_1) \succ \dots$ is ultimately decreasing, and this shows that the chain is finite. Thus, this poset is Artinian and therefore, by Theorem 2.47, it is well-founded.

Definition 2.49 A graded poset is a triple (S, \leq, h) , where (S, \leq) is a poset and $h : S \rightarrow \mathbb{N}$ is a function that satisfies the conditions:

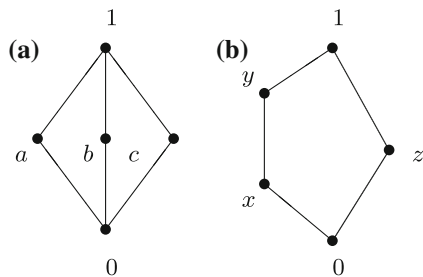
- (i) $x < y$ implies $h(x) < h(y)$ and
- (ii) y covers x implies $h(y) = h(x) + 1$,

for every $x, y \in S$. The function h is referred to as the grading function.

The set $L_k = \{x \in S \mid h(x) = k\}$ is called the k -th level of the poset (S, \leq, h) .

Example 2.50 Define the function $h : M_5 \rightarrow \mathbb{N}$ by $h(0) = 0$, $h(a) = h(b) = h(c) = 1$, and $h(1) = 2$. The triple (M_5, \leq, h) is a graded poset. Its levels are

Fig. 2.3 Hasse diagrams of posets **a** (M_5, \leq) and **b** (N_5, \leq)



$$\begin{aligned} L_0 &= \{0\}, \\ L_1 &= \{a, b, c\}, \\ L_2 &= \{1\}. \end{aligned}$$

Definition 2.51 Let (S, \leq) be a finite poset. The height of (S, \leq) , denoted by $\text{height}(S, \leq)$, is the maximal number of elements of a chain.

The width of (S, \leq) , $\text{width}(S, \leq)$, is the maximal number of elements of an antichain.

Example 2.52 Let $S = \{s_1, \dots, s_n\}$ be a finite set such that $|S| = n$. The poset $(\mathcal{P}(S), \subseteq)$ has height $n + 1$ since a maximal chain has the form $(\emptyset, \{s_{i_1}\}, \{s_{i_1}, s_{i_2}\}, \dots, S)$, where $(s_{i_1}, s_{i_2}, \dots, s_{i_n})$ is a list of the elements of S . Its width is $\binom{n}{\lfloor n/2 \rfloor}$.

Definition 2.53 Let (S, \leq) be a poset that has a least element denoted by 0.

The height of an element $x \in S$ (denoted by $\text{height}(x)$) is the least upper bound of the lengths of the chains of the form $0 < x_1 < \dots < x_k = x$.

If x is an atom of a poset that has the least element 0, then $\text{height}(x) = 1$.

Definition 2.54 A poset (S, \leq) satisfies the Jordan-Dedekind condition if all maximal chains between the same elements have the same finite length.

Example 2.55 The poset (M_5, \leq) whose Hasse diagram is shown in Fig. 2.3a satisfies the Jordan-Dedekind condition; the poset (N_5, \leq) shown in Fig. 2.3b fails this condition because it contains two maximal chains $0 < x < y < 1$ and $0 < z < 1$ of different lengths between 0 and 1.

The next theorem shows that for a poset that has finite chains the Jordan-Dedekind condition is equivalent to the fact that the poset is graded by its height function.

Theorem 2.56 Let (S, \leq) be a poset that has finite chains and has the least element 0. (S, \leq) satisfies the Jordan-Dedekind condition if and only if the following conditions are satisfied:

- (i) $x < y$ implies $\text{height}(x) < \text{height}(y)$, and

- (ii) y covers x implies $\text{height}(y) = \text{height}(x) + 1$
for every $x, y \in S$.

Proof If the height function satisfies the conditions of the theorem, then any maximal chain between the elements x and y has length $\text{height}(y) - \text{height}(x)$, so the Jordan-Dedekind condition is satisfied. Conversely, if the Jordan-Dedekind condition holds, then $\text{height}(x)$ is the length of any maximal chain between 0 and x and the conditions of the theorem follow immediately.

It is clear that if a finite poset (S, \leq) contains an antichain U such that $|U| = m$, then S is the union of at least m chains since no two elements of an antichain may belong to the same chain.

Theorem 2.57 (Dilworth's Theorem) *If (S, \leq) is a finite nonempty poset such that $\text{width}(S, \leq) = m$, then there is a partition of S into m chains.*

Proof The argument is by strong induction on $n = |S|$. If $n = 1$, then the statement holds trivially.

Suppose that the statement holds for sets with fewer than n elements, and let (S, \leq) be a poset with $|S| = n$.

Let C be a maximal chain in (S, \leq) . Two cases may occur:

- (i) If no antichain of $(S - C, \leq)$ has m elements, then, by the induction hypothesis, there exists a partition of $S - C$ into $m - 1$ chains, so there is a partition of S into m chains.
- (ii) If $S - C$ has an antichain $U = \{u_1, \dots, u_m\}$, define the sets UP_U and DOWN_U as

$$\begin{aligned}\text{UP}_U &= \{x \in S \mid x \geq u_i \text{ for some } u_i \in U\}, \\ \text{DOWN}_U &= \{x \in S \mid x \leq u_i \text{ for some } u_i \in U\}.\end{aligned}$$

Note that $S = \text{UP}_U \cup \text{DOWN}_U$ since otherwise S would contain an antichain with more than m elements. Since (S, \leq) is a finite poset, the chain C has a largest element t_1 and a smallest element t_0 . We have the strict inclusions $\text{UP}_U \subset S$ and $\text{DOWN}_U \subset S$ because $t_1 \notin \text{DOWN}_U$ and $t_0 \notin \text{UP}_U$. Thus, both DOWN_U and UP_U have fewer than n elements.

By the induction hypothesis, we can decompose both UP_U and DOWN_U as partitions of chains, $\text{UP}_U = \bigcup_{i=1}^m C_{\geq}^i$ and $\text{DOWN}_U = \bigcup_{i=1}^m C_{\leq}^i$, where $u_i \in C_{\geq}^i \cap C_{\leq}^i$. Note that u_i is the least element of C_{\geq}^i and the greatest element of C_{\leq}^i . Therefore, $C_{\geq}^i \cup C_{\leq}^i$ is a chain, which gives the desired result.

Next, we state a related statement using antichains.

Theorem 2.58 *If (S, \leq) is a finite nonempty poset such that $\text{height}(S, \leq) = m$, then there is a partition of S into m antichains.*

Proof We construct a sequence of finite posets (S_i, \leq_i) for $0 \leq i \leq k-1$. The first poset is $(S_0, \leq_0) = (S, \leq)$.

Suppose that we defined the nonempty poset (S_i, \leq_i) . Consider the antichain $U_{i+1} = \text{MAX}(S_i, \leq_i)$ and the poset (S_{i+1}, \leq_{i+1}) , where $S_{i+1} = S_i - U_{i+1}$ and $\leq_{i+1} = (\leq_i)_{S_{i+1}}$. The process halts when $S_k = S_{k-1} - U_k = \emptyset$. It is clear that the U_1, \dots, U_k are k pairwise disjoint antichains in (S, \leq) and that $S = \bigcup_{i=1}^k U_i$.

Since no two members of an antichain may belong to the same chain and S contains a chain having m elements, it follows that any partition of S into antichains requires at least m antichains. Therefore, we have $m \leq k$, which means that we need to show only that $k \leq m$.

To prove that $k \leq m$, we construct a chain $x_1 < x_2 < \dots < x_k$ in the poset (S, \leq) beginning with x_k . Choose x_k to be an arbitrary element of U_k . If $x_j \in U_j$ for $i \leq j \leq k$, then choose $x_{i-1} \in U_{i-1}$ such that $x_i < x_{i-1}$. This choice is possible because otherwise $x_i \in U_{i-1} = \text{MAX}(S_{i-1}, \leq_{i-1})$, which is contradictory because $x_i \in U_i$. This proves that $\{x_1, \dots, x_k\}$ is a chain, so $\text{height}(S, \leq) = m \geq k$.

2.5 Poset Product

Let I be a set, and (S, ρ) be a poset. A partial order ρ is defined on the set of functions $I \rightarrow S$ as $(f, g) \in \rho$ if $(f(i), g(i)) \in \rho$ for every $i \in I$ for $f, g : I \rightarrow S$.

The relation ρ on $I \rightarrow S$. We verify only the antisymmetry and leave for the reader the proofs of the reflexivity and transitivity. Assume that $(f, g), (g, f) \in \rho$ for $f, g : I \rightarrow S$. We have $(f(i), g(i)) \in \rho$ and $(g(i), f(i)) \in \rho$ for every $i \in I$. Therefore, taking into account the antisymmetry of ρ , we obtain $f(i) = g(i)$ for all $i \in I$; hence, $f = g$, which proves the antisymmetry of ρ .

For a set of functions $F \subseteq I \rightarrow S$, define the subset $F(i)$ of S as $S(i) = \{f(i) \mid f \in F\}$ for $i \in I$.

Theorem 2.59 *The subset F of the poset $(I \rightarrow S, \rho)$ has a supremum if and only if $\sup F(i)$ exists for every $i \in I$ in the poset (S, ρ) .*

Proof Suppose that $\sup F(i)$ exists for every $i \in I$ in the poset (S, ρ) . Define the mapping $g : I \rightarrow S$ by $g(i) = \sup F(i)$ for every $i \in I$. We claim that g is $\sup F$.

If $f \in F$, then $(f(i), g(i)) \in \rho$ for every $i \in I$ because of the definition of g . This shows that $(f, g) \in \rho$; hence, g is an upper bound of F . Let h be an upper bound of F . For every $f \in F$, we have $(f(i), h(i)) \in \rho$ for $i \in I$. The definition of g implies $(g(i), h(i)) \in \rho$ for $i \in I$; hence, $g = \sup F$.

Conversely, assume that $k = \sup F$ exists in the poset $(I \rightarrow S, \rho)$. We prove that $k(i)$ is $\sup F(i)$ for every $i \in I$ in the poset (S, ρ) .

The definition of k implies that, for every $f \in F$, we have $(f, k) \in \rho$; that is, $(f(i), k(i)) \in \rho$ for every $i \in I$. Therefore, $k(i)$ is an upper bound of the set $F(i)$ for every $i \in I$.

Let l_i be an upper bound for $F(i)$ for $i \in I$. Define the function $l : I \rightarrow S$ as $l(i) = l_i$ for $i \in I$. Clearly, l is an upper bound of the set F in the poset $(I \rightarrow S, \rho)$,

and therefore $(k, l) \in \rho$. This, in turn, means that $(k(i), l(i)) = (k(i), l_i) \in \rho$, which shows that $\sup F(i)$ exists and is equal to $k(i)$.

Definition 2.60 *The product of the posets $\{(S_i, \leq_i) \mid i \in I\}$ is the poset (D, \leq) , where $D = \prod_{i \in I} S_i$ and “ \leq ” is the partial order introduced above on D . When $I = \{1, \dots, n\}$, the product will be denoted by*

$$(S_1, \leq_1) \times \cdots \times (S_n, \leq_n)$$

or by $\prod_{i \in I} (S_i, \leq_i)$.

Theorem 2.61 *Let $\{(S_i, \leq_i) \mid i \in I\}$ be a family of partially ordered sets. If $H \subseteq \prod_{i \in I} S_i$, then in the product poset, $\sup H$ ($\inf H$) exists if and only if $\sup p_i(H)$ ($\inf p_i(H)$, respectively) exists for every $i \in I$. Moreover, if $y = \sup H$ ($y = \inf H$), then $p_i(y) = \sup p_i(H)$ ($p_i(y) = \inf p_i(H)$) for every $i \in I$.*

Proof Assume that $y_i = \sup p_i(H)$ exists for every $i \in I$. We need to prove that the element y of $\prod_{i \in I} S_i$ defined by $p_i(y) = y_i$ is $\sup H$.

Consider an arbitrary element $z \in H$. Since $p_i(z) \in p_i(H)$, we have $p_i(z) \leq_i y_i$, that is, $p_i(z) \leq_i p_i(y)$ for every $i \in I$. This means that $z \leq y$, which shows that y is an upper bound of H .

Suppose now that v is an arbitrary upper bound of H . To show that y is $\sup H$, we need to prove that y is the least upper bound of H ; that is, $y \leq v$ or, equivalently, $p_i(y) \leq_i p_i(v)$ for every $i \in I$.

If v is an upper bound of H , then $p_i(v)$ is an upper bound of $p_i(H)$. Since $p_i(y) = y_i = \sup p_i(H)$, we obtain immediately $p_i(y) \leq_i p_i(v)$ for every $i \in I$.

Conversely, suppose that $\sup H$ exists. Let $y = \sup H$ and let $y_i = p_i(y)$ for every $i \in I$. We have $x_i \in p_i(H)$ if there is $x \in H$ such that $p_i(x) = x_i$. Since $x \leq y$, it follows that $x_i \leq_i p_i(y)$, which shows that $p_i(y)$ is an upper bound for $p_i(H)$.

Let w_i be an arbitrary upper bound of $p_i(H)$ for every $i \in I$. There is $w \in \prod_{i \in I} S_i$ such that $p_i(w) = w_i$, and we have $y \leq w$ because w is an upper bound for H . Consequently, $p_i(y) \leq_i p_i(w)$, and this means that $y_i = \sup p_i(H)$ for every $i \in I$.

The statement for \inf follows by dualization.

Another kind of partial order that can be introduced on $S_1 \times \cdots \times S_n$ is defined next.

Theorem 2.62 *For $f, g \in S_1 \times \cdots \times S_n$, define $f \preceq g$ if $f = g$ or if there is k , $1 \leq k \leq n$, such that $f(k) \neq g(k)$, $f(i) = g(i)$ for $1 \leq i < k$ and $f(k) <_k g(k)$.*

The relation \preceq is a partial order on $S_1 \times \cdots \times S_n$.

Proof The relation \preceq is obviously reflexive. Suppose now that $f \preceq g$ and $g \preceq f$ and that $f \neq g$. There are $k, h \in \mathbb{N}$ such that $f(i) = g(i)$ for $1 \leq i < k$, $f(k) <_k g(k)$, and $f(i) = g(i)$ for $1 \leq i < h$, $f(h) <_h g(h)$. If $k < h$, this leads to a contradiction since we cannot have $f(k) <_k g(k)$ and $f(k) = g(k)$. The case $h < k$ also results

Fig. 2.4 Hasse diagrams of **a** $(\{0, 1\}, \leq)$ and **b** $(\{0, 1\}^2, \leq)$

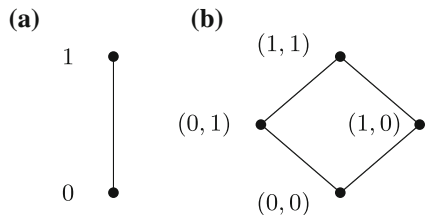
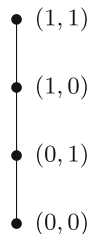


Fig. 2.5 Hasse diagram of $(\{0, 1\}^2, \preceq)$



in a contradiction. For $k = h$, the previous supposition implies $f(k) <_k g(k)$ and $g(k) <_k f(k)$, which is contradictory because “ $<_k$ ” is a strict partial order.

Assume that $f \preceq g$ and $g \preceq l$ and that $f \neq g$, $g \neq l$. There are $k, h \in \mathbb{N}$ such that $f(i) = g(i)$ for $1 \leq i < k$, $f(k) <_k g(k)$, and $g(i) = l(i)$ for $1 \leq i < h$, $g(h) <_h l(h)$. Define p as being the least of the numbers k, h . For $1 \leq i < p$, we have $f(i) = g(i) = l(i)$. In addition, we have $f(p) \preceq_p l(p)$. Three cases may occur:

1. $f(p) = g(p)$ and $g(p) <_p l(p)$ (when $k > h$),
2. $f(p) <_p g(p)$ and $g(p) = l(p)$ (when $k < h$), and
3. $f(p) <_p g(p)$ and $g(p) <_p l(p)$ (when $k = h$).

If $f = l$, then we have $f \preceq l$. Therefore, we can assume that $f \neq l$. In the first two cases mentioned above, this would imply immediately $f \preceq l$ because of the fact that $f(p) <_p l(p)$. The same conclusion can be reached in the third case because of the transitivity of the strict partial order $<_p$.

We refer to the partial order “ \preceq ” as the *lexicographic partial order* on $S_1 \times \cdots \times S_n$.

Let $\{(S_i, \leq_i) \mid 1 \leq i \leq n\}$ be a family of totally ordered posets. The product poset $\prod_{i=1}^n (S_i, \leq_i)$ is not necessarily a total order; however, the lexicographic product $(S_1 \times \cdots \times S_n, \preceq)$ is a total order (see Exercise 24).

Example 2.63 Consider the totally ordered set $(\{0, 1\}, \leq)$, whose Hasse diagram is given in Fig. 2.4a. The Hasse diagram of the poset $(S \times S, \leq)$ is shown in Fig. 2.4b.

On the other hand, the Hasse diagram of the poset $(\{0, 1\}^2, \preceq)$ given in Fig. 2.5 shows that “ \preceq ” is a total order on $\{0, 1\}^2$.

If $S_1 = \cdots = S_n = S$, then we obtain the poset $(\text{Seq}_n(S), \preceq)$.

2.6 Functions and Posets

Let (S, \leq) and (T, \leq) be two posets.

Definition 2.64 A morphism between (S, \leq) and (T, \leq) or a monotonic mapping between (S, \leq) and (T, \leq) is a mapping $f : S \longrightarrow T$ such that $u, v \in S$ and $u \leq v$ imply $f(u) \leq f(v)$.

A mapping $g : S \longrightarrow T$ is antimonotonic if $u, v \in S$ and $u \leq v$ imply $g(u) \geq g(v)$.

The mapping f is strictly monotonic if $u < v$ implies $f(u) < f(v)$, where “ $<$ ” is the strict partial order associated with the partial order “ \leq ”.

Note that $g : S \longrightarrow T$ is antimonotonic if and only if g is a monotonic mapping between the poset (S, \leq) and the dual (T, \geq) of the poset (T, \leq) .

Example 2.65 Consider a set M , the poset $(\mathcal{P}(M), \subseteq)$, and the functions $f, g : (\mathcal{P}(M))^2 \longrightarrow \mathcal{P}$, defined by $f(K, H) = K \cup H$ and $g(K, H) = K \cap H$, for $K, H \in \mathcal{P}(M)$. If the Cartesian product is equipped with the product partial order, then both f and g are monotonic. Indeed, if $(K_1, H_1) \subseteq (K_2, H_2)$, we have $K_1 \subseteq K_2$ and $H_1 \subseteq H_2$, which implies that

$$f(K_1, H_1) = K_1 \cup H_1 \subseteq K_2 \cup H_2 = f(K_2, H_2).$$

The argument for g is similar, and it is left to the reader.

Example 2.66 Let $\{(S_i, \rho_i) \mid i \in I\}$ be a collection of posets and let

$$\left(\prod_{i \in I} S_i, \rho \right)$$

be the product of these posets. The projections $p_i : \prod_{i \in I} S_i \longrightarrow S_i$ are monotonic mappings, as the reader will easily verify.

Example 2.67 Let (M, ρ) be an arbitrary poset. Any function $f : S \longrightarrow M$ is monotonic when considered between the posets (S, ι_S) and (M, ρ) .

Theorem 2.68 Let $(P, \leq), (R, \leq), (S, \leq)$ be three posets and let $f : P \longrightarrow R$, $g : R \longrightarrow S$ be two monotonic mappings. The mapping $gf : P \longrightarrow S$ is also monotonic.

Proof Let $x, y \in P$ be such that $x \leq y$. In view of the monotonicity of f , we have $f(x) \leq f(y)$, and this implies $(g(f(x)) \leq g(f(y)))$ because of the monotonicity of g . Therefore, gf is monotonic.

Let (P, \leq) and (R, \leq) be two posets. For a monotonic function $f : P \longrightarrow R$, the quotient set, $P/\ker(f)$ can also be organized as a poset. Indeed, if $[x], [y] \in$

$P/\ker(f)$, then we define $[x] \leq [y]$ if $f(x) \leq f(y)$. This partial order on $P/\ker(f)$ is well-defined because if $x' \in [x]$ and $y' \in [y]$, we have $(f(x'), f(y')) = (f(x), f(y))$.

Theorem 2.69 *The mapping $g : P \longrightarrow P/\ker(f)$ defined by $g(x) = [x]$ for $x \in P$ is a monotonic mapping between the posets (P, \leq) and $(P/\ker(f), \leq)$.*

Proof The argument is straightforward, and it is left to the reader as an exercise.

Let $f : S \longrightarrow T$ be a monotonic bijection between the posets (S, \leq) and (T, \leq) . As we have seen in Chapter 1, the inverse f^{-1} is also a bijection. Nevertheless, the inverse is not necessarily monotonic, as follows from the next example.

Example 2.70 Let (M_5, \leq) and (N_5, \leq) be the posets whose Hasse diagrams are given in Fig. 2.3, and consider the mapping $f : M_5 \longrightarrow N_5$ defined by $f(0) = 0$, $f(a) = y$, $f(b) = x$, $f(c) = z$, and $f(1) = 1$. The inverse bijection f^{-1} is not monotonic because we have $x \leq y$ in (N_5, \leq) and $(f^{-1}(x), f^{-1}(y)) = (b, a)$ and $b \not\leq a$ in (M_5, \leq) .

Let (R, \leq) and (S, \leq) be two posets. The previous considerations justify the following definition.

Definition 2.71 *A poset isomorphism between the posets (R, \leq) and (S, \leq) is a monotonic bijective mapping $f : R \longrightarrow S$ for which the inverse mapping f^{-1} is also monotonic.*

If a poset isomorphism exists between the posets (P, \leq) and (S, \leq) , then we refer to these posets as isomorphic.

Example 2.72 Let $\{p_1, p_2, \dots, p_n\}$ be the first n primes, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. Let $m = p_1 \cdots p_n$ be their product and let D_m be the set of all divisors of m . Consider an arbitrary set $A = \{a_1, \dots, a_n\}$ having n elements.

The posets $(\mathcal{P}(A), \subseteq)$ and (D_m, δ) are isomorphic. Indeed, define the mapping $f : \mathcal{P}(A) \longrightarrow D_m$ by $f(\emptyset) = 1$ and $f(\{a_{i_1}, \dots, a_{i_k}\}) = p_{i_1} \cdots p_{i_k}$.

The mapping f is bijective. Indeed, for any divisor h of m , we have $h = p_{i_1} \cdots p_{i_k}$ and therefore $h = f(\{a_{i_1}, \dots, a_{i_k}\})$, which shows that f is surjective.

If $f(\{a_{i_1}, \dots, a_{i_k}\}) = f(\{a_{j_1}, \dots, a_{j_l}\})$, then $p_{i_1} \cdots p_{i_k} = p_{j_1} \cdots p_{j_l}$. This gives $k = l$ and $i_1 = j_1, \dots, i_k = j_k$; hence, $\{a_{i_1}, \dots, a_{i_k}\} = \{a_{j_1}, \dots, a_{j_l}\}$, which proves that f is injective.

The mapping f is monotonic because if $\{a_{i_1}, \dots, a_{i_k}\} \subseteq \{a_{j_1}, \dots, a_{j_l}\}$,

$$\{i_1, \dots, i_k\} \subseteq \{j_1, \dots, j_l\},$$

and this means that the number $p_{i_1} \cdots p_{i_k}$ divides $p_{j_1} \cdots p_{j_l}$.

The inverse mapping $g : D_m \longrightarrow \mathcal{P}(A)$ is also monotonic; we leave the argument to the reader.

Monotonic functions map chains to chains, as we show next.

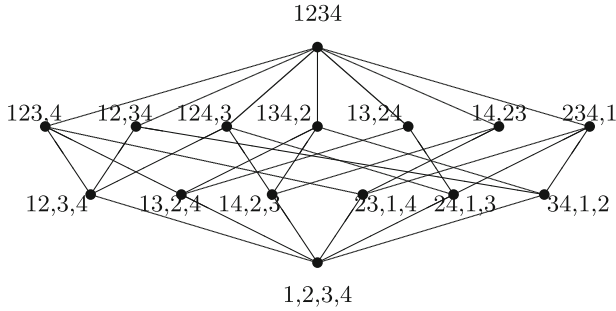


Fig. 2.6 The Hasse diagram of $(\text{PART}(\{1, 2, 3, 4\}), \leq)$

Theorem 2.73 Let (P, \leq) and (R, \leq) be two posets and $f : P \rightarrow R$ be a monotonic function. If $L \subseteq P$ is a chain in (P, \leq) , then $f(L)$ is a chain in (R, \leq) .

Proof Let $u, v \in f(L)$ be two elements of $f(L)$. There exist $x, y \in L$ such that $f(x) = u$ and $f(y) = v$. Since L is a chain, we have either $x \leq y$ or $y \leq x$. In the former case, the monotonicity of f implies $u \leq v$; in the latter situation, we have $v \leq u$.

2.7 The Poset of Equivalences and the Poset of Partitions

In Definition 1.110 we introduced the relation “ \leq ” on $\text{PART}(S)$ and we examined the relationships that exists between equivalences and partitions on a set. It is easy to verify that this is a partial order relation on $\text{PART}(S)$. Thus, the pair $(\text{PART}(S), \leq)$ is a poset.

Example 2.74 The Hasse diagram of $(\text{PART}(\{1, 2, 3, 4\}), \leq)$ is given in Fig. 2.6.

To simplify this figure, we represent each nonempty subset of $\{1, 2, 3, 4\}$ as an increasing set of its elements and omit the outer braces; for instance, instead of $\{1, 2, 3\}$, we write 123.

The poset $(\text{PART}(S), \leq)$ has α_S as its first element and ω_S as its largest.

Theorem 2.75 Let $\pi, \sigma \in \text{PART}(S)$ such that $\pi \leq \sigma$. The partition σ covers the partition π if and only if there exists a block C of σ that is the union of two blocks B and B' of π and every block of σ that is distinct of C is a block of π .

Proof Suppose that σ is a partition that covers the partition π . Since $\pi \leq \sigma$, every block of σ is a union of blocks of π . Suppose that there exists a block E of σ that is the union of more than two blocks of π ; that is, $E = \bigcup \{B_i \mid i \in I\}$, where $|I| \geq 3$, and let $B_{i_1}, B_{i_2}, B_{i_3}$ be three blocks of π included in E . Consider the partitions

$$\begin{aligned}\sigma_1 &= \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1}, B_{i_2}, B_{i_3}\}, \\ \sigma_2 &= \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1} \cup B_{i_2}, B_{i_3}\}.\end{aligned}$$

It is easy to see that $\pi \leq \sigma_1 < \sigma_2 < \sigma$, which contradicts the fact that σ covers π . Thus, each block of σ is the union of at most two blocks of π .

Suppose that σ contains two blocks C' and C'' that are unions of two blocks of π , namely $C' = B_{i_0} \cup B_{i_1}$ and $C'' = B_{i_2} \cup B_{i_3}$. Define the partitions

$$\begin{aligned}\sigma' &= \{C \in \sigma \mid C \notin \{C', C''\}\} \cup \{C', B_{i_2}, B_{i_3}\}, \\ \sigma'' &= \{C \in \sigma \mid C \notin \{C', C''\}\} \cup \{B_{i_1}, B_{i_2}, C''\}.\end{aligned}$$

Since $\pi < \sigma', \sigma'' < \sigma$, this contradicts the fact that σ covers π . Thus, we obtain the conclusion of the theorem.

We introduced the equivalence ρ_π that can be built from a partition π and the partition π_ρ that consists of the equivalence classes of ρ . Furthermore, in Corollary 1.114 we noted that $\rho = \rho_{\pi_\rho}$ and $\pi = \pi_{\rho_\pi}$. These observations can be strengthened in the framework of posets.

Theorem 2.76 *The posets $(EQ(S), \subseteq)$ and $(PART(S), \leq)$ are isomorphic.*

Proof Let $f : EQ(S) \rightarrow PART(S)$ be the mapping defined by $f(\rho) = S/\rho$. We need to show that f is a monotonic bijective mapping and that its inverse mapping f^{-1} is also monotonic.

The bijectivity of f follows immediately from the remarks that precede the theorem. Let ρ_0, ρ_1 be two equivalences such that $\rho_0 \subseteq \rho_1$ and let $S/\rho_0 = \{B_i \mid i \in I\}$, $S/\rho_1 = \{C_j \mid j \in J\}$. Let B_i be a block in S/ρ_0 and assume that $B_i = [x]_{\rho_0}$. We have $y \in B_i$ if and only if $(x, y) \in \rho_0$, so $(x, y) \in \rho_1$. Therefore, $y \in [x]_{\rho_1}$, which shows that every block $B \in S/\rho_0$ is included in a block $C \in \rho_1$. This shows that $f(\rho_0) \leq f(\rho_1)$, so f is indeed monotonic. We leave to the reader the proof of monotonicity for f^{-1} .

Theorem 2.77 *Let $\{\rho_i \mid i \in I\} \subseteq EQ(S)$ be a collection of equivalences. Then, $\inf\{\rho_i \mid i \in I\} = \bigcap_{i \in I} \rho_i$.*

Proof By Theorem 1.170, $\rho = \bigcap_{i \in I} \rho_i$ is the closure of the family of equivalences $\{\rho_i \mid i \in I\}$. It is clear that if $\xi \in EQ(S)$ and $\rho_i \subseteq \xi$ for $i \in I$, then $\rho \subseteq \xi$.

Definition 2.78 *Let S be a set and let $\rho, \tau \in EQ(S)$. A (ρ, τ) -alternating sequence that joins x to y is a sequence (s_0, s_1, \dots, s_n) such that $x = s_0$, $y = s_n$, $(s_i, s_{i+1}) \in \rho$ for every even i and $(s_i, s_{i+1}) \in \tau$ for every odd i , where $0 \leq i \leq n-1$.*

lemma 2.79 *Let S be a set and let $\rho, \tau \in EQ(S)$. If s and z are two (ρ, τ) -alternating sequences joining x to y and y to z , respectively, then there exists a (ρ, τ) -alternating sequence that joins x to z .*

Proof Let (s_0, \dots, s_n) be a (ρ, τ) -alternating sequences joining x to y and (w_0, \dots, w_m) a (ρ, τ) -alternating sequences joining y to z , where $x = s_0$, $s_n = w_0 = y$ and $w_m = z$. If $(s_{n-1}, s_n) \in \tau$, then the sequence $(s_0, \dots, s_n, w_1, \dots, w_m)$ is a (ρ, τ) -alternating sequence joining x to z . Otherwise, that is, if $(s_{n-1}, s_n) \in \rho$, then taking into account the reflexivity of τ we have $(s_n, w_0) = (s_n, s_n) \in \tau$. In this case, $(s_0, \dots, s_n, s_n, w_1, \dots, w_m)$ is a (ρ, τ) -alternating sequence joining x to z .

Theorem 2.80 *Let S be a set and let $\rho, \tau \in EQ(S)$. If ξ is the relation that consists of all pairs $(x, y) \in S \times S$ that can be joined by a (ρ, τ) -alternating sequence, then $\xi = \sup\{\rho, \tau\}$.*

Proof It is easy to verify that ξ is indeed an equivalence relation. Note that we have both $\rho \subseteq \xi$ and $\tau \subseteq \xi$. Indeed, if $(x, y) \in \rho$, then (x, y, y) is a (ρ, τ) -alternating sequence joining x to y . If $(x, y) \in \tau$, then (x, x, y) is the needed alternating sequence.

Let $\zeta \in EQ(S)$ such that $\rho \subseteq \zeta$ and $\tau \subseteq \zeta$. If $(x, y) \in \xi$, and (s_0, s_1, \dots, s_n) is a (ρ, τ) -alternating sequence such that $x = s_0$, $y = s_n$, then each pair (s_i, s_{i+1}) belongs to ζ . By the transitivity property, $(x, y) \in \zeta$, so $\xi \subseteq \zeta$. This implies that $\xi = \sup\{\rho, \tau\}$.

By Theorem 2.76, if $\pi, \sigma \in PART(S)$ both the infimum and the supremum of the set $\{\pi, \sigma\}$ exist and their description follows from the corresponding results that refer to the equivalence relations. Namely, if $\pi, \sigma \in PART(S)$, where $\pi = \{B_i \mid i \in I\}$ and $\sigma = \{C_j \mid j \in J\}$, the partition $\inf\{\pi, \sigma\}$ exists and is given by

$$\inf\{\pi, \sigma\} = \{B_i \cap C_j \mid i \in I, j \in J \text{ and } B_i \cap C_j \neq \emptyset\}.$$

The partition $\inf\{\pi, \sigma\}$ will be denoted by $\pi \wedge \sigma$.

A block of the partition $\sup\{\pi, \sigma\}$, denoted by $\pi \vee \sigma$, is an equivalence class of the equivalence $\theta = \sup\{\rho_\pi \wedge \rho_\sigma\}$. We have $y \in [x]_\theta$ if there exists a sequence $(s_0, \dots, s_n) \in \mathbf{Seq}(S)$ such that $x = s_0$, $s_n = y$ and successive sets $\{s_i, s_{i+1}\}$ are included, alternatively, in a block of π or in a block of σ . More intuitive descriptions of $\sup\{\pi, \sigma\}$ and $\inf\{\pi, \sigma\}$ is given in Sect. 10.4 of Chap. 10.

2.8 Posets and Zorn's Lemma

A statement equivalent to a fundamental principle of set theory known as the Axiom of Choice is Zorn's lemma stated below.

Zorn's Lemma: If every chain of a poset (S, \leq) has an upper bound, then S has a maximal element.

Theorem 2.81 *The following three statements are equivalent for a poset (S, \leq) :*

- (i) *If every chain of (S, \leq) has an upper bound, then S has a maximal element (Zorn's Lemma).*

- (ii) If every chain of (S, \leq) has a least upper bound, then S has a maximal element.
- (iii) S contains a chain that is maximal with respect to set inclusion (Hausdorff maximality principle).

Proof (i) implies (ii) is immediate.

(ii) implies (iii): Let $(\text{CHAINS}(S), \subseteq)$ be the poset of chains of S ordered by set inclusion. By Theorem 2.41, every chain $\{U_i \mid i \in I\}$ of the poset $(\text{CHAINS}(S), \subseteq)$ has a least upper bound $\bigcup \{U_i \mid i \in I\}$ in the poset $(\text{CHAINS}(S), \subseteq)$. Therefore, by (ii), $(\text{CHAINS}(S), \subseteq)$ has a maximal element that is a chain of (S, \leq) that is maximal with respect to set inclusion.

(iii) implies (i): Suppose that S contains a chain W that is maximal with respect to set inclusion and that every chain of (S, \leq) has an upper bound. Let w be an upper bound of W .

If $w \in W$, then w is a maximal element of S . Indeed, if this were not the case, then S would contain an element t such that $w < t$ and $W \cup \{t\}$ would be a chain that would strictly include W .

If $w \notin W$, then $W \cup \{w\}$ would be a chain strictly including W , which, again, would contradict the maximality of W . Thus, w is a maximal element of (S, \leq) .

Denote by $\text{PORD}(S)$ the collection of partial order relations on the set S .

Definition 2.82 Let $\rho, \rho' \in \text{PORD}(S)$. The partial order ρ' is an extension of ρ if $(x, y) \in \rho$ implies $(x, y) \in \rho'$. Equivalently, we shall say that ρ' extends ρ .

An important consequence of Zorn's lemma is the next statement, which shows that any partial order defined on a set can be extended to a total order on the same set.

Theorem 2.83 (Szpilrajn's Theorem) Let (S, \leq) be a poset. There is a total order \leq' on S that is an extension of \leq .

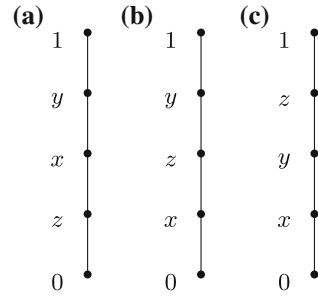
Proof Let $\text{PORD}(S, \leq)$ be the set of partial order relations that can be defined on the set S and contain the relation " \leq "; clearly, the relation " \leq " itself is a member of $\text{PORD}(S, \leq)$. We will apply Zorn's lemma to the poset $(\text{PORD}(S, \leq), \subseteq)$.

Let $\mathcal{R} = \{\rho_i \mid i \in I\}$ be a chain of $(\text{PORD}(S, \leq), \subseteq)$; that is, a chain of partial orders ρ_i relative to set inclusion such that $x \leq y$ implies $(x, y) \in \rho_i$ for every $i \in I$ and all $x, y \in S$. We claim that the relation $\rho = \bigcup \mathcal{R}$ is a partial order on S .

Indeed, since $\iota_S \subseteq \leq \subseteq \rho_i$ for $i \in I$ we have $\iota_S \subseteq \rho$, so ρ is a reflexive relation. To prove that ρ is antisymmetric let $x, y \in S$ be two elements such that $(x, y) \in \rho$ and $(y, x) \in \rho$. By the definition of ρ , there exist $i, j \in I$ such that $(x, y) \in \rho_i$ and $(y, x) \in \rho_j$. Since \mathcal{R} is a chain, we have either $\rho_i \subseteq \rho_j$ or $\rho_j \subseteq \rho_i$. In the first case, both (x, y) and (y, x) belong to ρ_j , so $x = y$ because of the antisymmetry of ρ_j ; in the second case, the same conclusion follows because (x, y) and (y, x) belong to ρ_i . Thus, ρ is indeed antisymmetric.

We leave it to the reader to prove the transitivity of ρ . Thus, ρ is a partial order that includes " \leq ", and the arbitrary chain \mathcal{R} has an upper bound. By Zorn's lemma

Fig. 2.7 Hasse diagrams of three total orders on the set $\{0, x, y, z, 1\}$



the poset $(\text{ORD}(S, \leq), \subseteq)$ has a maximal element \leq' . We now prove that \leq' is a total order.

Suppose that (u, v) and (v, u) are two distinct ordered pairs of elements of S such that $u \not\leq' v$ and $v \not\leq' u$. We show that this supposition leads to a contradiction.

Let \leq_1 be the relation on S given by

$$\begin{aligned} \leq_1 = & \{(x, y) \in S \times S \mid x \leq' y\} \cup \{(u, v)\} \\ & \cup \{(z, v) \in S \times \{v\} \mid z \leq' v\} \cup \{(u, t) \in \{u\} \times S \mid u \leq' t\}. \end{aligned}$$

Since $\iota_S \subseteq \leq' \subseteq \leq_1$, it follows that \leq_1 is reflexive.

To prove the antisymmetry of \leq_1 , suppose that $p \leq_1 q$ and $q \leq_1 p$. Since $v \not\leq' u$, it follows that $(p, q) \neq (u, v)$. Thus, the following cases may occur:

- (i) If $p \leq' q$ and $q \leq' p$, then $p = q$ by the antisymmetry of \leq' .
- (ii) If $p = u$, we have $u \leq_1 q$ and $q \leq_1 u$. By the definition of \leq_1 , this implies $u \leq' q$ and $q \leq' u$, respectively, so $q = u = p$.
- (iii) If $q = v$, we have $p \leq_1 v$ and $v \leq_1 p$, which imply $p \leq' v$ and $v \leq' p$, respectively. Thus, $p = v = q$.

We leave the proof of transitivity for " \leq_1 " to the reader.

Note that \leq' is strictly included in \leq_1 because $u \not\leq' v$. This contradicts the maximality of the partial order \leq' , so \leq' must be a total order.

Example 2.84 Consider the poset (N_5, \leq) introduced in Example 2.55. The posets (N_5, \leq_i) , where $1 \leq i \leq 3$ whose Hasse diagrams are shown in Fig. 2.7a–c are such that $\leq \subset \leq_i$ and \leq_i is a total order for $1 \leq i \leq 3$. Also, it is easy to see that we have actually $\leq = \leq_1 \cap \leq_3$.

Corollary 2.85 Let (S, \leq) be a poset and let x and y be two incomparable elements in (S, \leq) . There exists a total order \leq' on S that extends \leq such that $x \leq' y$ and a total order \leq'' that extends \leq such that $y \leq'' x$.

Proof This statement follows immediately from Szpilrajn's theorem.

Exercises and Supplements

1. Define the relation \leq on the set \mathbb{N}^n by $(p_1, \dots, p_n) \leq (q_1, \dots, q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$. Prove that (\mathbb{N}^n, \leq) is a partially ordered set.
2. Prove that acyclicity is a hereditary property; this means that if a relation $\sigma \subseteq S \times S$ is acyclic and $\theta \subseteq \sigma$, then θ is also acyclic.
3. Let $f : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ and $g : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ be the functions defined by $f(x) = e^x$ for $x \in \mathbb{R}$ and $g(x) = \ln x$ for $x \in \mathbb{R}_{>0}$. Prove that f and g are mutually inverse isomorphisms between the posets (\mathbb{R}, \geq) and $(\mathbb{R}_{>0}, \geq)$.
4. Let S and T be two sets and let \sqsubseteq be the relation on $S \rightsquigarrow T$ defined by $f \sqsubseteq g$ if $\text{Dom}(f) \subseteq \text{Dom}(g)$ and $f(s) = g(s)$ for every $s \in \text{Dom}(f)$. Prove that \sqsubseteq is a partial order on $S \rightsquigarrow T$.
5. Prove that a binary relation ρ on a set S is a strict partial order on S if and only if it is irreflexive, transitive, and antisymmetric.
6. Let (S, \leq) be a poset. An *order ideal* is a subset I of S such that $x \in I$ and $y \leq x$ implies $y \in I$. If $\mathcal{I}(S, \leq)$ is the collection of order ideals of (S, \leq) , prove that $\mathcal{K} \subseteq \mathcal{I}(S, \leq)$ implies $\bigcap \mathcal{K} \in \mathcal{I}(S, \leq)$. Further, argue that $S \in \mathcal{I}(S, \leq)$.
7. Let (S, \leq) be a poset. An *order filter* is a subset F of S such that $x \in F$ and $y \geq x$ implies $y \in F$. If $\mathcal{F}(S, \leq)$ is the collection of order filters of (S, \leq) , prove that $\mathcal{K} \subseteq \mathcal{F}(S, \leq)$ implies $\bigcap \mathcal{K} \in \mathcal{F}(S, \leq)$. Further, show that $S \in \mathcal{F}(S, \leq)$.
8. Let (S, \leq) be a finite poset. Prove that S contains at least one maximal and at least one minimal element.
9. Let (S, \leq) be a finite poset, where $S = \{x_1, \dots, x_n\}$. Construct the sequence of posets $((S_1, \leq_1), (S_2, \leq_2), \dots)$ as follows. Let $(S_1, \leq_1) = (S, \leq)$. For $1 \leq i \leq n$, choose x_{p_i} to be the first element of S_i in the sequence $\mathbf{s} = (x_1, \dots, x_n)$ that is minimal in (S_i, \leq) . Define $S_{i+1} = S_i - \{x_{p_i}\}$ and $\leq_{i+1} = \leq_i \cap (S_{i+1} \times S_{i+1})$. Prove that the sequence $(x_{p_1}, \dots, x_{p_n})$ is a total order on S that extends the partial order \leq .
10. Let S be an infinite set and let (\mathcal{C}, \subseteq) be the partially ordered set of its cofinite sets. Prove that for every $U, V \in \mathcal{C}$ both $\sup\{U, V\}$ and $\inf\{U, V\}$ exist.
11. Does the poset of partial functions $(S \rightsquigarrow T, \sqsubseteq)$ introduced in Exercise 4 have a least element?
12. Let (S, \leq) be a poset and let U and V be two subsets of S such that $U \subseteq V$. Prove that if both $\sup U$ and $\sup V$ exist, then $\sup U \leq \sup V$. Prove that if both $\inf U$ and $\inf V$ exist, then $\inf V \leq \inf U$.
13. Prove that the Completeness Axiom of \mathbb{R} implies that for any positive real numbers x, y there exists $n \in \mathbb{N}$ such that $nx > y$ (Archimedes' property of \mathbb{R}).
14. Suppose that S and T are subsets of \mathbb{R} that are bounded above. Prove that $S \cup T$ is bounded above and $\sup S \cup T = \max\{\sup S, \sup T\}$.
15. Let π and σ be two partitions of a finite set S . Prove that $|\pi| + |\sigma| \leq |\pi \wedge \sigma| + |\pi \vee \sigma|$.
16. Prove that if π is a partition of a set S and $|\pi| = k$, then there are $\binom{k}{2}$ partitions that cover π .

Fig. 2.8 The Hasse diagram of the standard example

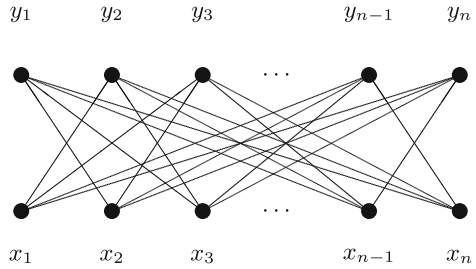
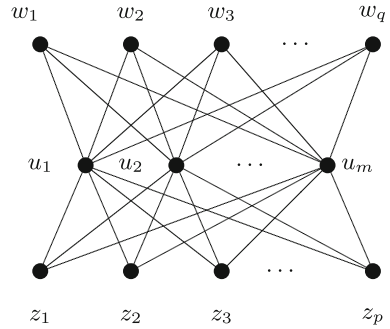


Fig. 2.9 Hasse diagram of the poset $T_{m,p,q}$



17. Let (S, \leq) be a poset. Prove that if a chain in S has at most p elements and an antichain has at most q elements, then $|S| \leq pq$.
18. Let (S, \leq) be a poset. Prove that (S, \leq) is a chain if and only if for every subset T of S both $\sup T$ and $\inf T$ exist and $\{\sup T, \inf T\} \subseteq T$.

Let (S, \leq) be a poset. A *realizer* of (S, \leq) is a family of total orders on S , $\mathcal{R} = \{\leq_i \mid i \in I\}$ such that

$$\leq = \bigcap \{\leq_i \mid i \in I\}.$$

If (S, \leq) is a finite poset, the *dimension* of (S, \leq) is the smallest size d of a realizer of (S, \leq) . The dimension of a finite poset (S, \leq) is denoted by $\dim(S, \leq)$.

19. Let $S = \{x_1, \dots, x_n\}$ be a finite set. Prove that the discrete partial order ι_S on S has dimension 2.

Solution: Consider the total order $\leq_1 = \text{TO}(x_1, \dots, x_n)$ and its dual $\leq_2 = \text{TO}(x_n, \dots, x_2, x_1)$. Note that $(x, x') \in \leq_1 \cap \leq_2$ if and only if $x = x'$; that is, if and only if $(x, x') \in \iota_S$.

20. Let (S_n, \leq) be the poset whose Hasse diagram is given in Fig. 2.8, where $S_n = \{x_1, \dots, x_n, y_1, \dots, y_n\}$. This poset was introduced in [1] and is known as the *standard example*. Prove that $\dim(S_n, \leq) = n$.
21. Consider the poset $(T_{p,m,q}, \leq)$, whose Hasse diagram is given in Fig. 2.9. The set $T_{p,m,q}$ consists of three sets of pairwise incomparable elements $\{z_1, \dots, z_p\}$, $\{u_1, \dots, u_m\}$, and $\{w_1, \dots, w_q\}$ such that $z_i < u_j < w_k$ for every $1 \leq i \leq p$,

- $1 \leq j \leq m$, and $1 \leq k \leq q$. Prove that if at least one of the numbers p, m, q is greater than 1, then $\dim(T_{p,m,q}, \leq) = 2$.
22. Prove that the set of partial order relations on a set S is a closure system on the set $S \times S$.
23. Prove that the transitive closure of an acyclic relation is a strict partial order.
24. Prove that if $\{(S_i, \leq_i) \mid 1 \leq i \leq n\}$ is a family of totally ordered posets, then the lexicographic product $(S_1 \times \cdots \times S_n, \preceq)$ is a total order.
25. Let (S_1, \leq_1) and (S_2, \leq_2) be two posets and let $f : S_1 \rightarrow S_2$ be a monotonic mapping. Prove that if S_2 has a least element 0, then $f^{-1}(0)$ is an order filter of S_1 , and if S_2 has a greatest element 1, then $f^{-1}(1)$ is an order ideal of S_1 .
26. Let (S, \leq) be a poset. Define the mapping $f_{\leq} : S \rightarrow \mathcal{P}(S)$ by $f_{\leq}(x) = \{y \in S \mid x < y\}$.
- (a) Prove that f_{\leq} is an antimonotonic mapping between the posets (S, \leq) and $(\mathcal{P}(S), \subseteq)$.
- (b) If C is a chain in (S, \leq) , prove that $f_{\leq}(C)$ is a chain in $(\mathcal{P}(S), \subseteq)$.
- (c) Let (S, \leq) and (S, \leq') be two posets defined on the set S . Prove that $f_{\leq \cap \leq'}(x) = f_{\leq}(x) \cap f_{\leq'}(x)$ for every $x \in S$.
27. In the proof of Szpilrajn's theorem, we introduced the set of partial orders that extend the partial order " \leq ". The inclusion between relations defines a partial order on $PORD(S, \leq)$. We saw that the maximal elements of $PORD(S, \leq)$ are total orders on S and that the least element of $PORD(S, \leq)$ is the relation \leq itself.

Let (S, \leq) be a poset. Prove that there exists a collection of total orders $\{\leq_i \mid i \in I\}$ on S such that $\leq = \bigcap_{i \in I} \leq_i$.

Solution: If \leq is itself a total order, then the desired collection of total orders consists of \leq itself. Suppose therefore that \leq is not total, and let $INC(S, \leq)$ be the set of all pairs of incomparable elements of (S, \leq) .

For each pair $(x, y) \in INC(S, \leq)$, consider the total orders \leq'_{xy} and \leq''_{xy} that extend \leq such that $x \leq'_{xy} y$ and $y \leq''_{xy} x$. Clearly,

$$\leq \subseteq \bigcap \{\leq'_{xy} \cap \leq''_{xy} \mid (x, y) \in INC(S, \leq)\}.$$

Suppose that $\bigcap \{\leq'_{xy} \cap \leq''_{xy} \mid (x, y) \in INC(S, \leq)\}$ contains a pair of elements $(r, s) \in INC(S, \leq)$. Then, we have both $r \leq'_{rs} s$ and $r \leq''_{rs} s$. Since $s \leq''_{rs} r$, this would imply $r = s$ by the antisymmetry of \leq''_{rs} . This, however, contradicts the incomparability of (r, s) in (S, \leq) . Thus, for any pair $(u, v) \in \bigcap \{\leq'_{xy} \cap \leq''_{xy} \mid (x, y) \in INC(S, \leq)\}$, we have $u \leq v$ or $v \leq u$, which shows that

$$\leq = \bigcap \{\leq'_{xy} \cap \leq''_{xy} \mid (x, y) \in INC(S, \leq)\}.$$

A poset (S, \leq) is *locally finite* if every interval $[x, y]$ of S is a finite set.

28. Prove that the poset (\mathbb{N}, \leq) is locally finite.

29. Let S be a finite set. Prove that the poset $(\mathbf{Seq}(S), \leq_{inf})$, where \leq_{inf} is the partial order introduced in Example 2.6, is locally finite.

Let (P, \leq) , and (Q, \leq) be two posets. Their product is the poset $(P \times Q, \leq)$ where $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

30. Let (P, \leq) , and (Q, \leq) be two posets. Prove that $(P \times Q, \leq)$ is locally finite if and only if both (P, \leq) and (Q, \leq) are locally finite.
31. Prove that if (P, \leq) and (Q, \leq) are graded posets by the grading functions h and g , respectively, then $(P \times Q, \leq)$ is graded by the function f defined by $f(p, q) = h(p)g(q)$ for $(p, q) \in P \times Q$.
32. Let $\zeta : S \times S \rightarrow \mathbb{R}$ be the Riemann function of a locally finite poset (S, \leq) , and let ζ^k be the product $\zeta * \zeta * \cdots * \zeta$, which contains k ζ factors, where $k \in \mathbb{N}$. Prove that:

- (a) $\zeta^2(x, y) = |[x, y]|$ if $x \leq y$.
- (b) $\zeta^k(x, y)$ gives the number of multichains of length k that can be interpolated between x and y .

Bibliographical Comments

There is a vast body of literature dealing with posets and their applications and a substantial number of references that focus on combinatorial study of posets. Among these we mention [2–5].

Two very useful references are [6] and [7].

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