Chapter 2
Level Sets and Tangent Spaces

Summary *Using systems of linear equations as a guide we discuss the significance of the implicit function theorem for level sets. We define the tangent space and the normal space at a point on a level set.*

We shall be concerned with many different aspects of surfaces, level sets and graphs in this book. In this chapter we consider the role of differentiability in the local structure of level sets. By considering the linear approximation of differentiable functions and standard results on solving systems of linear equations we see that level sets are locally graphs.

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, U open, $F = (f_1, \ldots, f_m)$, $C = (c_1, \ldots, c_m) \in \mathbb{R}^m$. We suppose that F is differentiable and consider the level set $F^{-1}(C) = \bigcap_{i=1}^m f_i^{-1}(c_i)$, i.e. the points $(x_1, \ldots, x_n) \in U$ which satisfy the equations

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= c_1 \\ &\vdots \\ f_m(x_1, \ldots, x_n) &= c_m. \end{aligned} \tag{2.1}$$

We have n unknowns, $x_1, \ldots, x_n$ and m equations. If each f_i is linear we have a *system of linear equations* and the rank of the matrix of coefficients gives information on the number of linearly independent solutions and procedures on how to identify a complete set of independent variables. The *implicit function theorem* says that this process is also valid *locally* for differentiable functions. The key, of course, is the fact that differentiable functions, by definition, enjoy a good local *linear* approximation.

Fix a point $P \in F^{-1}(C)$ and suppose $X \in \mathbb{R}^n$ is close to 0 and $P + X \in F^{-1}(C)$. Since F is differentiable,

$$F(P + X) = F(P) + F'(P)(X) + G(X)(X)$$

where $G : U \longrightarrow M_{m,n}$ and $G(X) \to 0$ as $X \to 0$. Since $F(P + X) = F(X) = C$,

$$F'(P)(X) \approx 0$$

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(where $\approx$ denotes approximately equal).

We assume from now on that $n \geq m$. We thus have something very close to the following system of *linear* equations

$$\begin{aligned} \frac{\partial f_1}{\partial x_1}(P)x_1 + \quad & \cdots + \frac{\partial f_1}{\partial x_n}(P)x_n = 0 \\ & \vdots \\ \frac{\partial f_m}{\partial x_1}(P)x_1 + \quad & \cdots + \frac{\partial f_m}{\partial x_n}(P)x_n = 0. \end{aligned} \tag{2.2}$$

The matrix of coefficients of this system of linear equations is

$$\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$$

i.e. $F'(P)$. From linear algebra we have

$$\begin{aligned} \mathrm{Rank}\left(F'(P)\right) = m \iff & \text{the } m \text{ rows of } F'(P) \text{ are linearly independent} \\ \iff & \text{there exist } m \text{ linearly independent columns} \\ & \text{in } F'(P) \\ \iff & F'(P) \text{ contains } m \text{ columns such that the} \\ & \text{associated } m \times m \text{ matrix has non-zero} \\ & \text{determinant} \\ \iff & \text{the space of solutions for the system (2.2)} \\ & \text{is } n - m \text{ dimensional.} \end{aligned}$$

Moreover, if any of these conditions are satisfied and we choose m columns which are linearly independent then the variables corresponding to the *remaining columns* can be taken as a *complete set of independent variables* and the full set of solutions coincides with the graph of a function of the independent variables. If the above conditions are satisfied we say that F has *full* or *maximum rank* at P.

As a simple example consider

$$\begin{aligned} 2x - y + z &= 0 \\ y - w &= 0 \end{aligned}$$

with matrix of coefficients

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The 2×2 matrix obtained by using the first two columns is $\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ and this has determinant $2 \neq 0$. Hence A has rank 2 and the two rows are linearly independent. Since the first two columns are linearly independent we can take the remaining two variables, z and w, as the independent variables. We have $y = w$, $2x = y - z = w - z$ and so $\{((w-z)/2, w, z, w) : z \in \mathbb{R},\ w \in \mathbb{R}\}$ is the solution set for this system of equations. We can write this in the form

$$\{(\phi(z, w), z, w) : (z, w) \in \mathbb{R}^2\}$$

where $\phi(z, w) = ((w-z)/2, w)$ and in this format the solution space is the *graph* of the function ϕ (see Example 1.7).

Note that columns 1 and 3 are not linearly independent, since the corresponding 2×2 matrix $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ has zero determinant, and we cannot choose y and w (the remaining variables) as the independent variables.

Assuming that the rows of $F'(P)$ are linearly independent is equivalent to requiring that $\{\nabla f_1(P), \ldots, \nabla f_m(P)\}$ are linearly independent vectors in $\mathbb{R}^n$. The implicit function theorem says that with this condition we can solve the *non-linear* system of Eq. (2.1) near P and use the same method to identify a set of independent variables. The hypothesis of a *good* linear approximation in the definition of differentiable functions implies that the systems of Eqs. (2.1) and (2.2), are very close to one another.

We now state without proof the Implicit Function Theorem.

Theorem 2.1 (Implicit Function Theorem) *Let $F\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, denote a differentiable function, let $P \in U$ and suppose $F(P) = C$ and rank $(F'(P)) = m$ (for convenience we suppose that the final m columns of $F'(P)$ are linearly independent). If $P = (p_1, \ldots, p_n)$ let $P_1 = (p_1, \ldots, p_{n-m})$ and $P_2 = (p_{n-m+1}, \ldots, p_n)$ so that $P = (P_1, P_2)$. Then there exists an open set V in $\mathbb{R}^{n-m}$ containing P_1, a differentiable function $\phi\colon V \to \mathbb{R}^m$, an open subset W of U containing P such that $\phi(P_1) = P_2$ and*

$$F^{-1}(C) \cap W = \Big\{(X, \phi(X)) : X \in V\Big\} = \textit{graph } (\phi).$$

Thus locally every level set is a graph (Fig. 2.1).

Example 2.2 Let $F(x_1, x_2, x_3, x_4) = (x_1x_2, x_3x_4^2)$. We have

$$F'(x_1, x_2, x_3, x_4) = \begin{pmatrix} \nabla f_1(x_1, x_2, x_3, x_4) \\ \nabla f_2(x_1, x_2, x_3, x_4) \end{pmatrix} = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_4^2 & 2x_3x_4 \end{pmatrix}$$

where $f_1(x_1, x_2, x_3, x_4) = x_1x_2$ and $f_2(x_1, x_2, x_3, x_4) = x_3x_4^2$. Consider the level set $F^{-1}(2, -4)$ and the point $P = (1, 2, -1, 2) \in F^{-1}(2, -4)$. We have

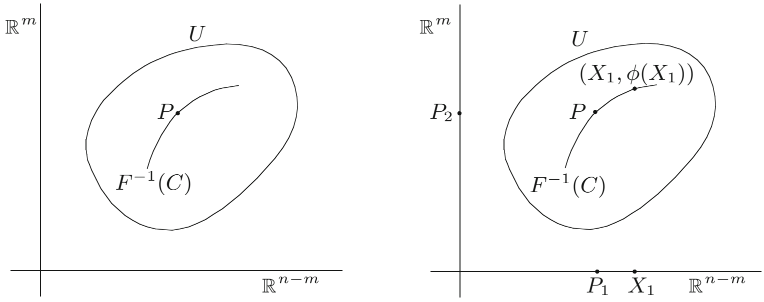

Fig. 2.1

$$F'(1, 2, -1, 2) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & -4 \end{pmatrix}.$$

If $\alpha(2, 1, 0, 0) + \beta(0, 0, 4, -1) = (0, 0, 0, 0)$ then $(2\alpha, \alpha, 4\beta, -\beta) = (0, 0, 0, 0)$ and hence $\alpha = \beta = 0$. This implies that the rows of $F'(1, 2, -1, 2)$ are linearly independent and $F'(P)$ has rank 2. This can be seen even more rapidly by finding a 2×2 submatrix with non-zero determinant, e.g. if we use columns 2 and 3 we get the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ with determinant $4 \neq 0$. It is easily checked that the following pairs of columns are linearly independent $(1, 3)$, $(1, 4)$, $(2, 3)$ and $(2, 4)$ while the pairs $(1, 2)$ and $(3, 4)$ are not linearly independent. Since columns 1 and 4 are linearly independent we know that the variables x_2 and x_3 can be chosen as a complete set of independent variables. Thus *we know* that x_1 and x_4 can be expressed as functions of x_2 and x_3 near the point $(1, 2, -1, 2)$. The implicit function theorem is *important* because it tells us that certain functions *exist* even though it does not show how to find them. In general, we would have to solve the system of Eq. (2.1) to find these functions and this may often be extremely difficult or even impossible in any reasonable fashion. In our rather simple situation we have the two equations

$$x_1 x_2 = 2$$
$$x_3 x_4^2 = -4$$

and can find a solution. We have $x_1 = 2/x_2$ and since (x_1, x_2, x_3, x_4) is close to $(1, 2, -1, 2)$ we have x_2 close to 2 and the natural domain for x_2 from this equation is $x_2 > 0$. We have $x_4^2 = -4/x_3$ and since x_3 is close to -1 we take $x_3 < 0$. Hence $-4/x_3$ is positive and $x_4 = \pm\sqrt{-4/x_3}$. Since x_4 is close to 2 we take the positive square root. Thus the function ϕ, whose *existence is foretold* by the Implicit Function Theorem, has the form

$$\phi(x_2, x_3) = \left(\frac{2}{x_2}, +\sqrt{\frac{-4}{x_3}} \right)$$

on the open set $V = \{(x_2, x_3) : x_2 > 0,\ x_3 < 0\}$. After a rearrangement of the variables we get

$$\text{graph}\,(\phi) = \left\{\left(\frac{2}{x_2}, x_2, x_3, +\sqrt{\frac{-4}{x_3}}\right) : x_2 > 0,\ x_3 < 0\right\}$$

and, as expected, we have

$$\begin{aligned} F\left(\frac{2}{x_2}, x_2, x_3, +\sqrt{\frac{-4}{x_3}}\right) &= \left(\frac{2}{x_2}\cdot x_2, x_3\left(+\sqrt{\frac{-4}{x_3}}\right)^2\right) \\ &= \left(2, \frac{x_3(-4)}{x_3}\right) = (2, -4). \end{aligned}$$

An examination of the equations $x_1x_2 = 2$ and $x_3x_4^2 = -4$ shows that it is not possible to find, say x_3, as a function of x_1 and x_2 and thus we cannot, as expected, use x_1 and x_2 as the independent variables.

Example 2.3 Given the equations

$$x^2 - y^2 + u^2 + 2v^2 = 1 \tag{2.3}$$

$$x^2 + y^2 - u^2 - v^2 = 2 \tag{2.4}$$

we wish to find all (x, y, u, v) such that near (x, y, u, v), u and v can be expressed as differentiable functions of x and y and we also wish to find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ in terms of $x,\ y,\ u$ and v.

Let $F(x, y, u, v) = (x^2 - y^2 + u^2 + 2v^2, x^2 + y^2 - u^2 - v^2)$. Then (x, y, u, v) satisfies (2.3) and (2.4) if and only if $F(x, y, u, v) = (1, 2)$, i.e. if and only if $(x, y, u, v) \in F^{-1}((1, 2))$. We have

$$F'(x, y, u, v) = \begin{pmatrix} 2x & -2y & 2u & 4v \\ 2x & 2y & -2u & -2v \end{pmatrix}.$$

We require x and y as a complete set of independent variables and so must have linear independence of the third and fourth columns. Hence we wish to find the points (x, y, u, v) such that $\det\begin{pmatrix} 2u & 4v \\ -2u & -2v \end{pmatrix} = 4uv \neq 0$. This implies that any point (x, y, u, v) satisfying (2.3) and (2.4) with u and v both non-zero will be suitable. To compute $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ we could apply the chain rule to the equation

$$F\big(x, y, u(x, y), v(x, y)\big) = (1, 2)$$

or just use (2.3) and (2.4) which now read

$$x^2 - y^2 + u(x, y)^2 + 2v(x, y)^2 = 1$$
$$x^2 + y^2 - u(x, y)^2 - v(x, y)^2 = 2.$$

On differentiating we get

$$2x + 2u(x, y)\frac{\partial u}{\partial x} + 4v(x, y)\frac{\partial v}{\partial x} = 0 \tag{2.5}$$

$$2x - 2u(x, y)\frac{\partial u}{\partial x} - 2v(x, y)\frac{\partial v}{\partial x} = 0. \tag{2.6}$$

The method of differentiation used to obtain Eqs. (2.5) and (2.6) is often called *"implicit differentiation"* especially in one-variable calculus where level sets of the form $f(x, y) = 0$ are considered.

These are just two linear equations in $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ which are easily solved to give

$$\frac{\partial u}{\partial x} = \frac{3x}{u} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{-2x}{v}.$$

Notice how we need u and v to be both non-zero. In most cases of this type it is not possible to find *explicit* formulae for u and v by solving equations similar to (2.3) and (2.4)—hence the *implicit* in the implicit function theorem. So although in general we cannot find explicit formulae for the dependent variables in terms of the independent variables we can find the partial derivatives in terms of the independent and dependent variables. We choose this particular example because we are able to verify our formulae for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$. Adding (2.3) and (2.4) gives $2x^2 + v^2 = 3$ and hence $v = \pm(3 - 2x^2)^{1/2}$ where we take the appropriate sign depending on the value of v. We have

$$\frac{\partial v}{\partial x} = \pm\frac{1}{2}(3 - 2x^2)^{-1/2}(-4x) = \frac{-2x}{\pm(3 - 2x^2)^{1/2}} = \frac{-2x}{v}.$$

Subtracting (2.4) from (2.3) gives

$$-2y^2 + 2u^2 + 3v^2 = -1 = -2y^2 + 2u^2 + 3(3 - 2x^2)$$

i.e. $2u^2 = 2y^2 + 6x^2 - 10$ and $u = \pm(3x^2 + y^2 - 5)^{1/2}$. Hence

$$\frac{\partial u}{\partial x} = \pm\frac{1}{2}(3x^2 + y^2 - 5)^{-1/2} \cdot 6x = \frac{3x}{\pm(3x^2 + y^2 - 5)^{1/2}} = \frac{3x}{u}$$

and this agrees with our earlier calculation.

We now return to the general situation and let

$$F = (f_1, \ldots, f_m) : U \text{ (open) } \subset \mathbb{R}^n \to \mathbb{R}^m$$

denote a differentiable function which has full rank at the point P in U. Let $F(P) = C$. We have

$$\begin{aligned} F^{-1}(C) &= \{X : F(X) = C\} = \{X : F(P + X - P) = C\} \\ &= \{X : F(P) + F'(P)(X - P) + G(X)(X - P) = C\} \\ &\approx \{X : F'(P)(X - P)\} = 0\} \\ &= \big\{X : \langle \nabla f_i(P), X - P\rangle = 0,\ i = 1, \ldots, m\big\} \end{aligned}$$

where $G : U \longrightarrow M_{m,n}$ and $G(X) \longrightarrow 0$ as $X \longrightarrow P$.

The set

$$\big\{X \in \mathbb{R}^n : F'(P)(X - P) = 0\big\} = P + \big\{X \in \mathbb{R}^n : F'(P)(X) = 0\big\}$$

is the closest linear approximation to $F^{-1}(C)$ near P and we call it the *tangent space* to $F^{-1}(C)$ at P. Since

$$F'(P) = \left(\frac{\partial f_i}{\partial x_j}(P)\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

the set of all X satisfying the equation $F'(P)(X) = 0$ is precisely the set of all solutions $\{x_1, \ldots, x_n\}$ of the system of *homogeneous* linear equations (2.2) that we encountered earlier and, as F has full rank at P, this space of solutions forms an $(n - m)$–dimensional subspace of $\mathbb{R}^n$. Since

$$\big\{X \in \mathbb{R}^n : F'(P)(X) = 0\big\} = \big\{X \in \mathbb{R}^n : \langle \nabla f_i(P), X\rangle = 0,\ i = 1, \ldots, m\big\}$$

the tangent space consists of the vectors which are perpendicular to the gradients of the component functions *transferred* to the point P (see Fig. 2.2). Normal vectors are perpendicular to tangent vectors and it is thus natural to define the *normal space* to the level set at P as

$$P + \left\{\sum_{i=1}^{m} \lambda_i \nabla f_i(P) : \lambda_i \in \mathbb{R}\right\}.$$

If the tangent space or normal space is two-dimensional we use the term *tangent plane* and *normal plane* respectively and if it is one-dimensional we use *tangent line* and *normal line* respectively. The tangent space and normal space are both translates of vector subspaces of $\mathbb{R}^n$ to the point P. The tangent space is the subspace which fits closest to the level set of F at P while the normal space is the set of directions which are—roughly speaking—perpendicular to the surface near P.

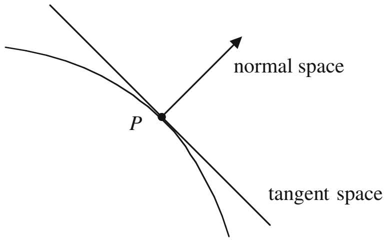

Fig. 2.2

In $\mathbb{R}^n$ there are various ways of presenting lines, planes, etc. The *normal form* consists of a description as the set of points satisfying a set of equations while the *parametric form* is in terms of independent variables and this, as we shall see in Chaps. 7 and 10, is almost a parametrization of the space.

Example 2.4 Let S denote the set of all points in $\mathbb{R}^3$ which satisfy the equation $x^2 + 2y^2 - 5z^2 = 1$. We wish to find the tangent space and the normal line at the point $(2, -1, 1)$ on S. Let $f(x, y, z) = x^2 + 2y^2 - 5z^2$. Then $S = f^{-1}(1)$. The tangent plane at $P = (x_0, y_0, z_0)$ in normal form is

$$\Big\{(x, y, z) : \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0\Big\}$$
$$= \Big\{(x, y, z) : (x - x_0)\frac{\partial f}{\partial x}(P) + (y - y_0)\frac{\partial f}{\partial y}(P) + (z - z_0)\frac{\partial f}{\partial z}(P) = 0\Big\}.$$

Now $\nabla f(x, y, z) = (2x, 4y, -10z)$ and $\nabla f(2, -1, 1) = (4, -4, -10)$ and so the tangent plane at $(2, -1, 1)$ is

$$\Big\{(x, y, z) : (x - 2)4 + (y + 1)(-4) + (z - 1)(-10) = 0\Big\}$$
$$= \Big\{(x, y, z) : 2x - 2y - 5z = 1\Big\}.$$

The normal line at (x_0, y_0, z_0) is the line through (x_0, y_0, z_0) in the direction of $\nabla f(x_0, y_0, z_0)$. In our case we have, in parametric form, the normal line

$$\Big\{(2, -1, 1) + t(4, -4, -10) : t \in \mathbb{R}\Big\} = \Big\{(2 + 4t, -1 - 4t, 1 - 10t) : t \in \mathbb{R}\Big\}.$$

To change this into normal form let $x = 2 + 4t$, $y = -1 - 4t$ and $z = 1 - 10t$. Solving for t we get

$$\frac{x - 2}{4} = \frac{y + 1}{-4} = \frac{z - 1}{-10} = t$$

and obtain the normal form

$$\Big\{(x, y, z) : \frac{x - 2}{4} = \frac{y + 1}{-4} = \frac{z - 1}{-10}\Big\}.$$

To find the tangent plane in parametric form we must find two linearly independent vectors which are perpendicular to $\nabla f(x_0, y_0, z_0)$. Applying the observation that (x, y) and $(-y, x)$ are perpendicular vectors in $\mathbb{R}^2$ to different pairs of coordinates in $\mathbb{R}^3$ we see easily that

$$(4, -4, -10) \cdot (4, 4, 0) = 0$$

and

$$(4, -4, -10) \cdot (10, 0, 4) = 0$$

and, moreover, $(4, 4, 0)$ and $(10, 0, 4)$ are linearly independent. In parametric form the tangent space at $(2, -1, 1)$ is

$$\begin{aligned}(2, -1, 1) &+ \Big\{x(4, 4, 0) + y(10, 0, 4) : x, y \in \mathbb{R}\Big\}\\ &= \Big\{(2 + 4x + 10y, -1 + 4x, 1 + 4y) : x, y \in \mathbb{R}\Big\}.\end{aligned}$$

Example 2.5 We wish to find the normal plane and the tangent line to the set of points satisfying

$$x^2 + y^2 - 2z^2 = 2 \quad \text{and} \quad xyz = 2$$

at the point $(\sqrt{2}, \sqrt{2}, 1)$.

Let $F(x, y, z) = (x^2 + y^2 - 2z^2, xyz)$. Then the set of points which satisfy the above equations form the level set $F^{-1}(2, 2)$. We have

$$F'\left(\sqrt{2}, \sqrt{2}, 1\right) = \begin{pmatrix} 2\sqrt{2} & 2\sqrt{2} & -4 \\ \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}.$$

The final two columns form a 2×2 matrix with non-zero determinant and hence F has full rank at $(\sqrt{2}, \sqrt{2}, 1)$. The normal plane at $(\sqrt{2}, \sqrt{2}, 1)$ has parametric form

$$\begin{aligned}&\left(\sqrt{2}, \sqrt{2}, 1\right) + \left\{x\left(2\sqrt{2}, 2\sqrt{2}, -4\right) + y\left(\sqrt{2}, \sqrt{2}, 2\right) : x,\ y \in \mathbb{R}\right\}\\ &= \left\{\left(\sqrt{2} + 2\sqrt{2}x + \sqrt{2}y, \sqrt{2} + 2\sqrt{2}x + \sqrt{2}y, 1 - 4x + 2y\right) : x,\ y \in \mathbb{R}\right\}.\end{aligned}$$

To find the normal plane in normal form we must find a non-zero vector perpendicular to both $\left(2\sqrt{2}, 2\sqrt{2}, -4\right)$ and $\left(\sqrt{2}, \sqrt{2}, 2\right)$. The cross product (see Chaps. 6 and 7) of the two given vectors is of the required type but we take a first-principles approach here. This amounts to finding (a, b, c) such that $\left(\sqrt{2}, \sqrt{2}, 2\right) \cdot (a, b, c) = 0$ and $\left(2\sqrt{2}, 2\sqrt{2}, -4\right) \cdot (a, b, c) = 0$. We thus have to solve the system of equations

$$\sqrt{2}a + \sqrt{2}b + 2c = 0$$
$$\sqrt{2}a + \sqrt{2}b - 2c = 0.$$

Subtracting we see that $c = 0$ and we can take $a = -b = 1$. Hence $(1, -1, 0)$ is a suitable vector. The normal plane (in normal form) is

$$\begin{aligned}&\Big\{(x, y, z) : \Big(x - \sqrt{2}, y - \sqrt{2}, z - 1\Big) \cdot (1, -1, 0) = 0\Big\}\\ &= \Big\{(x, y, z) : x - \sqrt{2} - y + \sqrt{2} = 0\Big\}\\ &= \Big\{(x, y, z) : x - y = 0\Big\}.\end{aligned}$$

The tangent line in normal form is

$$\begin{aligned}&\Big\{(x, y, z) : 2\sqrt{2}x + 2\sqrt{2}y - 4z = 4,\ \sqrt{2}x + \sqrt{2}y + 2z = 6\Big\}\\ &= \Big\{(x, y, z) : x + y = 2\sqrt{2},\ z = 1\Big\}.\end{aligned}$$

From this we see that the tangent line in parametric form is

$$\Big\{\Big(t, 2\sqrt{2} - t, 1\Big) : t \in \mathbb{R}\Big\}.$$

Exercises

2.1. Let $F_i : \mathbb{R}^4 \to \mathbb{R}^i$

$$\begin{aligned}F_1(x_1, x_2, x_3, x_4) &= x_1^2 - x_2^2,\ P_1 = (1, 2, 0, -1)\\ F_2(x_1, x_2, x_3, x_4) &= (x_1^2 - x_2^2, x_3^2 - x_4^2),\ P_2 = (1, 0, 2, -1)\\ F_3(x_1, x_2, x_3, x_4) &= (x_1^2 - x_2^2, x_3^2 - x_4^2, x_4^2 - x_1^2),\ P_3 = (1, 2, 3, 4).\end{aligned}$$

Calculate $F_i'(X)$ for $X \in \mathbb{R}^4$ and find all X such that F_i has full rank at X. When F_i has full rank find all subsets of $\{x_1, x_2, x_3, x_4\}$ which can be taken as complete sets of independent variables. If $F_i(P_i) = C_i$ and F_i has full rank at P_i find a function $\phi_i : \mathbb{R}^{4-i} \to \mathbb{R}^i$ such that $F_i^{-1}(C_i) = \text{graph}(\phi_i)$ near P_i.

2.2. If $u(x, y)$ and $v(x, y)$ are defined by the equations $u \cos v = x$ and $u \sin v = y$ find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ by

(i) finding explicit formulae for u and v
(ii) using implicit differentiation.

2.3. Let $F(x_1, x_2, x_3, x_4) = (x_1^2 x_2^2, x_1 x_2 x_3, x_4^2)$. Find $F'(1, 2, 3, 4)$. Let $A = F'(1, 2, 3, 4)$. Display the system of equations $AX = 0$. Solve this system of equations and find a basis for the space of solutions. Using your set of solutions find the tangent space to the level set of F at $(1, 2, 3, 4)$.

2.4. (a) Find in normal and parametric form the normal line and the tangent plane to the surface $z = xe^y$ at the point $(1, 0, 1)$.
(b) The surfaces $x^2 + y^2 - z^2 = 1$ and $x + y + z = 5$ intersect in a curve Γ. Find the equation in parametric form of the tangent line to Γ at the point $(1, 2, 2)$.

2.5. Find the equation of the plane passing through the points $(1, 2, 3)$ and $(4, 5, 6)$ which is perpendicular to the plane $7x + 8y + 9z = 10$.

2.6. Find the equation of the tangent plane to $\sqrt{x} + \sqrt{y} + \sqrt{z} = 4$ at the point $(1, 4, 1)$.

2.7. Find the tangent planes at $(1/\sqrt{2}, 1/4, 1/4)$ and $(\sqrt{3}/2, 0, 1/4)$ to the ellipsoid $x^2 + 4y^2 + 4z^2 = 1$. Find the line of intersection of these two planes. Show that this line is tangent to the sphere $x^2 + y^2 + z^2 = k$ for exactly one value of k and find this value.

2.8. Find the coordinates of the four points where the hyperbola $x^2 - y^2 = 1$ and the ellipse $x^2 + 2y^2 = 4$ intersect. Sketch to scale both curves and their tangents at the points of intersection of these tangent lines. If (a, b), $a > b > 0$, is one of these points show that the tangent line to the hyperbola at (a, b) coincides with the normal line to the ellipse at (a, b). Show that the tangent lines to the hyperbola at the four points of intersection enclose a parallelogram and find its area.

2.9. Find the direction of the normal line at the point $(1, 1, 4)$ to the paraboloid $z = x^2 + y^2 + 2$. Find the tangent plane in normal form at this point. Show that the normal line meets the paraboloid again at the point $(-5/4, -5/4, 41/8)$. If θ is the angle between the normal line through the point $(1, 1, 4)$ and the normal line through the point $(-5/4, -5/4, 41/8)$ show that $\sin\theta = 1/\sqrt{3}$.

2.10. Consider the subset S of $\mathbb{R}^3$ which lies above the (x, y)–plane and which is characterised by the property:

$$p \in S \iff \begin{array}{l}\text{the distance from } p \text{ to the } xy\text{-plane is}\\ \text{the logarithm of its distance to the } z\text{-axis.}\end{array}$$

Describe S as a level set and as a graph. Find the normal line and the tangent plane to S at the point $(1, -1, \log 2/2)$.

2.11. Let $S_1 = \{(x, y, z) \in \mathbb{R}^3 : y = f(x)\}$ denote a cylinder and let S_2 denote the level set $z^2 + 2zx + y = 0$. If S_1 is tangent to S_2 at all points of contact find f.

2.12. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ denote two non-zero vectors in $\mathbb{R}^2$ making angles θ_1 and θ_2, respectively, with the positive x-axis. Show that $\cos\theta_1 = a_1/\|\mathbf{a}\|$ and $\sin\theta_1 = a_2/\|\mathbf{a}\|$. Show that $\theta_2 - \theta_1$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and, by expanding $\cos(\theta_2 - \theta_1)$, prove that $\cos(\theta_2 - \theta_1) = \mathbf{a} \cdot \mathbf{b}/\|\mathbf{a}\| \cdot \|\mathbf{b}\|$. Prove the same result for arbitrary vectors in $\mathbb{R}^n$.

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