

Preface

The first edition of this book appeared in 1998 and was published by Vieweg (Braunschweig/Wiesbaden). Several years later, the book was sold out and no longer available. Some time ago, I discussed this situation jointly with Ulrike Schmickler-Hirzebruch from Vieweg (which is now Springer-Vieweg) and with Clemens Heine from Springer Germany, and we opted for a second edition published by Springer, this publisher being better linked to the English language market.

The book covers many algorithms for summation and integration, most of which have not changed much in the meantime and are still up-to-date. Fasenmyer's algorithm for definite summation ([Chap. 4](#)) is very old, nevertheless it is so easy to describe that it must be included for didactical reasons. Gosper's algorithm ([Chap. 5](#)) solves the problem of how to find a hypergeometric antidifference, and it is the starting point of Zeilberger's celebrated algorithm for definite summation ([Chap. 7](#)). The book also covers the differential counterpart of Zeilberger's summation algorithm ([Chap. 10](#)) as well as its integration counterparts ([Chaps. 11 and 12](#)), and Gosper's algorithm is the driving force for all these algorithms. Therefore, its description remained unchanged. The other mentioned algorithms are also still up-to-date. Therefore, the above chapters have been updated only cautiously. However, in most chapters, new developments are cited and suggestions for further reading are given. As in the first edition, in all chapters an introduction to the corresponding q -theory is given.

The situation is quite different concerning the following parts of the book. Multivariate hypergeometric summation was still unfeasible when the first edition was written. In the meantime, ideas by Wegschaider cleared the way. These newer developments are incorporated and illustrated in [Chap. 4](#), and the corresponding `multsum`-package is introduced. Furthermore, van Hoeij's algorithm has dramatically improved the efficiency of finding hypergeometric term solutions of holonomic recurrence equations over Petkovšek's original approach. Therefore, his ideas have been incorporated in [Chap. 8](#) so that the reader gets a clear impression of where the new efficiency comes from. Nevertheless, the presentation of Petkovšek's original algorithm has not been withdrawn since it is still interesting from a historical point of view. More decisively, the efficiency of van Hoeij's algorithm can only be understood by comparison with Petkovšek's approach. The chapter finishes with the Maple package `qFPS` which contains the

q -case of van Hoeij's algorithm. More details about operator factorization are given in [Chaps. 9](#) and [12](#). Finally, there were some new developments on discrete Rodrigues formulas, by my Ph.D. student Kornelia Fischer, which have been incorporated in [Chap. 13](#).

For the first edition I had selected Maple as the computer algebra system in which the algorithms were programmed and demonstrated. Moreover, these (and some more) algorithms were incorporated in the packages `hsum` (and `qsum` for the q -case). This selection has proven successful, and since the other packages mentioned (`multsum` and `qFPS`) are also written in Maple, Maple is still the best system to keep the book self-contained.

On the web resource www.hypergeometric-summation.org/ all the Maple packages

- `hsum.mpl` (programs for hypergeometric summation)
- `qsum.mpl` (programs for q -hypergeometric summation)
- `multsum.mpl` (programs for multiple hypergeometric summation)
- `qFPS.mpl` (contains the q -Petkovšek-van-Hoeij algorithm)

and further materials such as the book's Maple sessions are available. These packages are regularly updated to work with newer versions of Maple.

I would like to thank Mama Foupouagnigni, Jürgen Gerhard, Dieter Schmersau[†] and Glenn P. Tesler who had read the first edition very carefully and identified several errors that I could correct. Special thanks go to Mark van Hoeij for his warm hospitality when I visited him in November–December 2013 at Florida State University (FSU) in Tallahassee. He gave me important assistance, especially concerning [Chap. 9](#), about his brilliant algorithm. Also special thanks go to Torsten Sprenger who updated the `hsum` and `qsum` packages, contributed the `multsum` ([Chap. 4](#)) and `qFPS` packages ([Chap. 9](#)) and incorporated the `FormalPowerSeries` package, which is mentioned in [Chaps. 10](#) and [13](#), into Maple. Finally I am very grateful to Martin Muldoon who smoothed out the English of the manuscript again.

The finalization of this project was made possible by a sabbatical from the University of Kassel, and by the Alexander von Humboldt Foundation who financed my stay in the USA by awarding an alumni research scholarship. I am very grateful for this invaluable support.

Last but not least, I thank Ulrike Schmickler-Hirzebruch from Vieweg, Clemens Heine from Springer Germany, and Lynn Brandon from Springer London, for their good collaboration and for making this second edition possible.

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Wolfram Koepf

Preface to the First Edition

The current book is the result of a lecture course that I gave at the Free University, Berlin, during the spring semester 1995. This course was influenced by the remarkable book *Concrete Mathematics* by Graham, Knuth, and Patashnik, and by the interesting lecture notes *Identities and Their Computer Proofs* by Herbert Wilf [Wilf93]. In the meantime, these notes have appeared together with other material in the book $A = B$ by Petkovšek, Wilf, and Zeilberger [PWZ96].

In contrast to the books just mentioned, it is my objective to present the material by giving more detailed advice on implementation. Furthermore, I wished to cover not only material about recurrence equations but also about differential equations, not only about sums but also about integrals, and finally not only the hypergeometric case but also its q -analogue.

In the current book, up-to-date algorithmic techniques for summation are described in detail, and worked out using Maple programs. With Maple release V.4 and higher, some of these tools are available through Maple's `sum` command and `sumtools` package, by an implementation that I incorporated in the Maple library prior to my lecture course. In this book, readers are invited to implement the algorithms step by step. This will give them a detailed insight into the structure of the algorithms under consideration, and will enable them to solve quite involved problems.

The book covers Gosper's algorithm for indefinite hypergeometric summation and Zeilberger's algorithm for definite hypergeometric summation, as well as the WZ method and extensions of these algorithms. Petkovšek's decision procedure for hypergeometric term solutions of holonomic recurrence equations completes the picture on the summation topic.

By an analogous technique, differential equations, derivative rules, and similar identities for sums can be generated, and a chapter on this topic is included. An equivalent theory of hyperexponential integration, both indefinite and definite, which was given by Almkvist and Zeilberger, completes the book.

The combination of all results considered gives work with orthogonal polynomials and (hypergeometric type) special functions a solid algorithmic foundation. Hence, many examples from this very active field are given.

Although multiple sums are briefly mentioned in Chapter 4, I have not covered the algorithmic theory of multisums, integral sums, etc., which was developed by Wilf and Zeilberger. Instead, by many examples I show how the one-dimensional theory can be applied successfully to double sums and integral sums, in particular to sums and integrals involving orthogonal polynomials.

The book contains many worked examples of the algorithms considered, and Maple implementations of them are presented. Furthermore, a lot of exercises encourage the readers to do their own implementations in Maple, and to study the topics included in detail. Exercises that demand Maple implementations are marked by a diamond (\diamond).

In all chapters, an introduction to the corresponding q -theory is given, whereas in the hypergeometric case, the algorithms are rigorously presented and detailed proofs of the statements are given, in the q -case we state only the results, indicate their proofs, present Maple implementations, and give examples and exercises.

A basic knowledge of a programming language such as Pascal or C should be sufficient to understand the Maple programs and to solve the corresponding exercises since all major Maple procedures that are used are briefly described. On the other hand, a deeper familiarity with Maple might help the reader to understand the code in more detail.

I could have presented the algorithms in pseudo code, without giving preference to a particular computer algebra system. On the other hand, an implementation in an existing and widely distributed computer algebra system makes the algorithms ready for execution, and therefore fills them with life. As a result, every student can execute all the examples no matter how complicated they may be.

Hence I had to decide on one of the major systems. Of the most important general purpose systems, Axiom [JS93], Macsyma [Macsyma], Maple [Char-et-al91]–[Char-et-al92], Mathematica [Wolfram96] and REDUCE [Hearn95], undoubtedly Maple and Mathematica have the largest audiences, since they are accessible at most universities and research institutions.

I wished to write my code as near as possible to the mathematical description of the corresponding algorithms, and since the latter depend heavily on the fast symbolic solution of (sometimes very complicated) systems of linear equations, the poor performance of Mathematica's `Solve` command for linear systems (see [PS95]) supported my decision to choose Maple. Furthermore, Maple is much friendlier with respect to user information (e.g., the `infolevel` routine).

Readers who use one of these systems can access some of the algorithms considered:

Axiom	The <code>sum</code> command contains an implementation of Gosper's algorithm.
Macsyma	The <code>sum</code> command contains an implementation of Gosper's algorithm written by Gosper.

- Maple** Maple's `sum` command contains an implementation of Gosper's algorithm, completely rewritten by the author for Maple V.4. There are implementations of Zeilberger ([Zeilberger91b], [PWZ96]), and Koornwinder [Koornwinder93] of Zeilberger's algorithm; Almkvist and Zeilberger [AZ91] implemented the continuous version. Maple V.4's `sumtools` package was written by the author [Koe96] and contains an implementation of Zeilberger's algorithm. In the present book, structured implementations of Gosper's algorithm, Zeilberger's algorithm, Petkovšek's algorithm and their q -analogues are developed. Salvy and Zimmermann's Generating Functions package `gfun` [SZ94] and Chyzak's `Mgfun` package [Chyzak94] are also strongly connected with the algorithms developed in the current book.
- Mathematica** Implementations of Gosper's and Zeilberger's algorithms were done by Paule and Schorn [PS95], and Petkovšek implemented his algorithm and the corresponding q -version ([Petkovšek92], [PWZ96], and [APP98]). Also Paule and Riese [PR97] implemented the q -analog of Zeilberger's algorithm. A package on multidimensional summation is due to Wegschaider [Wegschaider97].¹
- REDUCE** Gosper's and Zeilberger's algorithms are accessible by an implementation of Koepf and Stölting [Koe95b]; Böing and Koepf [BK97] implemented the q -analogs of Gosper's and Zeilberger's algorithm.

The Maple programs for the current book are discussed in detail in the text. Some of the implementations are even printed in the book. The programs are collected in the package `hsum` and can be obtained from the URL <http://www.hypergeometric-summation.org/>. Worksheets containing the examples given in the text, as well as Maple solutions of the exercises are available at the same URL. The corresponding q -analogs of Gosper's, Zeilberger's and Petkovšek's algorithms are implemented in the package `qsum` [BK99], written by Harald Böing, and can be obtained from the same site.

The present book is designed for use in the framework of a seminar. In seminars at German universities, every participating student is asked to present a lecture about a certain topic. The arrangement of the book makes the division into lectures easy. Each chapter covers a certain subtopic which can be presented by one or two students. Obviously, the book is also suitable for a lecture course in this area since it was written in connection with such a course presented by the author. Special notational conventions used in the book are defined at their first occurrence, and are listed in the *List of Symbols*.

¹ On the web see <http://www.math.upenn.edu/~wilf/progs.html> and <http://www.risc.jku.at/research/combinat/software/>.

I would like to thank Peter Deuffhard, who introduced me to the study of this topic, for his support and encouragement. Furthermore, I thank Martin Grötschel, without whose support the final version would not have been possible. Thanks go to Herbert Melenk for his advice on Gröbner bases, and for his excellent REDUCE implementation [MA94]. Due to his severe bicycling accident, the paper [MK95] is still unfinished. Also, I am very grateful for the warm hospitality of the ETH Zürich, where I visited to install my code in the Maple library, and especially to Mike Monagan, who headed the installation. Furthermore, thanks go to Tom Koornwinder for his implementation `zeilb` which was the starting point of my Maple implementations, and to Harald Böing who did some extensions of the implementations of this book that are covered in the `hsum` package as well as the *q*-implementation under my supervision. A few of the exercises have been collected by Torsten Thiele, and Lisa Temme corrected some of my English language mistakes. I am very grateful to Martin Muldoon who smoothed out the English of the final manuscript and to Harald Böing for the final proofreading.

Last but not least, I thank Ulrike Schmickler-Hirzebruch from Vieweg as well as the editor of the current book series Martin Aigner for their good collaboration and for making this project happen.

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