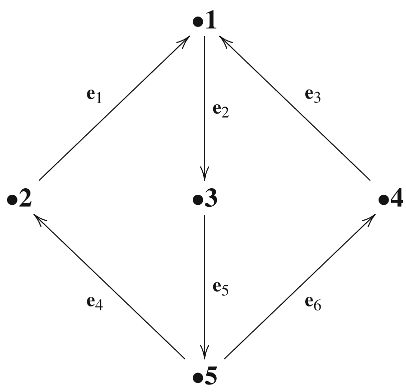
Chapter 2
Incidence Matrix

Let G be a graph with $V(G) = \{1, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Suppose each edge of G is assigned an orientation, which is arbitrary but fixed. The (*vertex-edge*) *incidence matrix* of G, denoted by $Q(G)$, is the $n \times m$ matrix defined as follows. The rows and the columns of $Q(G)$ are indexed by $V(G)$ and $E(G)$, respectively. The (i, j)-entry of $Q(G)$ is 0 if vertex i and edge e_j are not incident, and otherwise it is 1 or -1 according as e_j originates or terminates at i, respectively. We often denote $Q(G)$ simply by Q. Whenever we mention $Q(G)$ it is assumed that the edges of G are oriented.

Example 2.1 Consider the graph shown. Its incidence matrix is given by Q.



$$Q = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

R.B. Bapat, *Graphs and Matrices*, Universitext,
DOI 10.1007/978-1-4471-6569-9_2

2.1 Rank

For any graph G, the column sums of $Q(G)$ are zero and hence the rows of $Q(G)$ are linearly dependent. We now proceed to determine the rank of $Q(G)$.

Lemma 2.2 *If G is a connected graph on n vertices, then* rank $Q(G) = n - 1$.

Proof Suppose x is a vector in the left null space of $Q := Q(G)$, that is, $x'Q = 0$. Then $x_i - x_j = 0$ whenever $i \sim j$. It follows that $x_i = x_j$ whenever there is an ij-path. Since G is connected, x must have all components equal. Thus, the left null space of Q is at most one-dimensional and therefore the rank of Q is at least $n - 1$. Also, as observed earlier, the rows of Q are linearly dependent and therefore rank $Q \leq n - 1$. Hence, rank $Q = n - 1$. □

Theorem 2.3 *If G is a graph on n vertices and has k connected components then* rank $Q(G) = n - k$.

Proof Let $G_1, \ldots, G_k$ be the connected components of G. Then, after a relabeling of vertices (rows) and edges (columns) if necessary, we have

$$Q(G) = \begin{bmatrix} Q(G_1) & 0 & \cdots & 0 \\ 0 & Q(G_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & Q(G_k) \end{bmatrix}.$$

Since G_i is connected, rank $Q(G_i)$ is $n_i - 1$, where n_i is the number of vertices in G_i, $i = 1, \ldots, k$. It follows that

$$\begin{aligned} \operatorname{rank} Q(G) &= \operatorname{rank} Q(G_1) + \cdots + \operatorname{rank} Q(G_k) \\ &= (n_1 - 1) + \cdots + (n_k - 1) \\ &= n_1 + \cdots + n_k - k = n - k. \end{aligned}$$

This completes the proof.

Lemma 2.4 *Let G be a connected graph on n vertices. Then the column space of $Q(G)$ consists of all vectors $x \in \mathbb{R}^n$ such that $\sum_i x_i = 0$.*

Proof Let U be the column space of $Q(G)$ and let

$$W = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0 \right\}.$$

Then dim $W = n - 1$. Each column of $Q(G)$ is clearly in W and hence $U \subset W$. It follows by Lemma 2.2 that

$$n - 1 = \dim U \leq \dim W = n - 1.$$

Therefore, $\dim U = \dim W$. Thus, $U = W$ and the proof is complete. □

Lemma 2.5 *Let G be a graph on n vertices. Columns $j_1, \ldots, j_k$ of $Q(G)$ are linearly independent if and only if the corresponding edges of G induce an acyclic graph.*

Proof Consider the edges $j_1, \ldots, j_k$ and suppose there is a cycle in the corresponding induced subgraph. Without loss of generality, suppose the columns $j_1, \ldots, j_p$ form a cycle. After relabeling the vertices if necessary, we see that the submatrix of $Q(G)$ formed by the columns $j_1, \ldots, j_p$ is of the form $\begin{bmatrix} B \\ 0 \end{bmatrix}$, where B is the $p \times p$ incidence matrix of the cycle formed by the edges $j_1, \ldots, j_p$. Note that B is a square matrix with column sums zero. Thus, B is singular and the columns $j_1, \ldots, j_p$ are linearly dependent. This proves the "only if" part of the lemma.

Conversely, suppose the edges $j_1, \ldots, j_k$ induce an acyclic graph, that is, a forest. If the forest has q components then clearly $k = n - q$, which by Theorem 2.3, is the rank of the submatrix formed by the columns $j_1, \ldots, j_k$. Therefore, the columns $j_1, \ldots, j_k$ are linearly independent.. □

2.2 Minors

A matrix is said to be *totally unimodular* if the determinant of any square submatrix of the matrix is either 0 or ± 1. It is easily proved by induction on the order of the submatrix that $Q(G)$ is totally unimodular as seen in the next result.

Lemma 2.6 *Let G be a graph with incidence matrix $Q(G)$. Then $Q(G)$ is totally unimodular.*

Proof Consider the statement that any $k \times k$ submatrix of $Q(G)$ has determinant 0 or ± 1. We prove the statement by induction on k. Clearly the statement holds for $k = 1$, since each entry of $Q(G)$ is either 0 or ± 1. Assume the statement to be true for $k - 1$ and consider a $k \times k$ submatrix B of $Q(G)$. If each column of B has a 1 and a -1, then $\det B = 0$. Also, if B has a zero column, then $\det B = 0$. Now suppose B has a column with only one nonzero entry, which must be ± 1. Expand the determinant of B along that column and use induction assumption to conclude that $\det B$ must be 0 or ± 1. □

Lemma 2.7 *Let G be a tree on n vertices. Then any submatrix of $Q(G)$ of order $n - 1$ is nonsingular.*

Proof Consider the submatrix X of $Q(G)$ formed by the rows $1, \ldots, n - 1$. If we add all the rows of X to the last row, then the last row of X becomes the negative of the last row of $Q(G)$. Thus, if Y denotes the submatrix of $Q(G)$ formed by the rows $1, \ldots, n - 2, n$, then $\det X = -\det Y$. Thus, if $\det X = 0$, then $\det Y = 0$.

Continuing this way we can show that if det $X = 0$ then each $(n-1) \times (n-1)$ submatrix of $Q(G)$ must be singular. In fact, we can show that if any one of the $(n-1)\times(n-1)$ submatrices of $Q(G)$ is singular, then all of them must be so. However, by Lemma 2.2, rank $Q(G) = n-1$ and hence at least one of the $(n-1) \times (n-1)$ submatrices of $Q(G)$ must be nonsingular. □

We indicate another argument to prove Lemma 2.7. Consider any $n-1$ rows of $Q(G)$. Without loss of generality, we may consider the rows $1, 2, \dots, n-1$, and let B be the submatrix of $Q(G)$ formed by these rows. Let x be a row vector of $n-1$ components in the row null space of B. Exactly as in the proof of Lemma 2.2, we may conclude that $x_i = 0$ whenever $i \sim n$, and then the connectedness of G shows that x must be the zero vector.

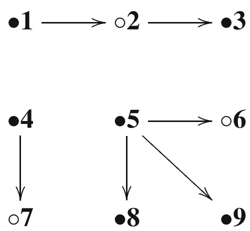
Lemma 2.8 *Let A be an $n \times n$ matrix and suppose A has a zero submatrix of order $p \times q$ where $p + q \geq n + 1$. Then* det $A = 0$.

Proof Without loss of generality, suppose the submatrix formed by the first p rows and the first q columns of A is the zero matrix. If $p \geq q$, then evaluating det A by Laplace expansion in terms of the first p rows we see that det $A = 0$. Similarly, if $p < q$, then by evaluating by Laplace expansion in terms of the first q columns, we see that det $A = 0$. □

We return to a general graph G, which is not necessarily a tree. Any submatrix of $Q(G)$ is indexed by a set of vertices and a set of edges. Consider a square submatrix B of $Q(G)$ with the rows corresponding to the vertices $i_1, \dots, i_k$ and the columns corresponding to the edges $e_{j_1}, \dots, e_{j_k}$. We call the object formed by these vertices and edges a *substructure* of G. Note that a substructure is not necessarily a subgraph, since one or both end-vertices of some of the edges may not be present in the substructure.

If we take a tree and delete one of its vertices, but not the incident edges, then the resulting substructure will be called a *rootless tree*. In view of Lemma 2.7, the incidence matrix of a rootless tree is nonsingular. Clearly, if we take a vertex-disjoint union of several rootless trees, then the incidence matrix of the resulting substructure is again nonsingular, since it is a direct sum of the incidence matrices of the individual rootless trees.

Example 2.9 The following substructure is a vertex-disjoint union of rootless trees. The deleted vertices are indicated as hollow circles.



The incidence matrix of the substructure is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and is easily seen to be nonsingular. Note that the rows of the incidence matrix are indexed by the vertices 1, 3, 4, 5, 8, and 9, respectively.

Let G be a graph with the vertex set $V(G) = \{1, 2, \ldots, n\}$ and the edge set $\{e_1, \ldots, e_m\}$. Consider a submatrix X of $Q(G)$ indexed by the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$. It can be seen that if X is nonsingular then it corresponds to a substructure which is a vertex-disjoint union of rootless trees. A sketch of the argument is as follows. Since X is nonsingular, it does not have a zero row or column. Then, after a relabeling of rows and columns if necessary, we may write

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & X_t \end{bmatrix}.$$

If any X_i is not square, then X must have a zero submatrix of order $p \times q$ with $p + q \geq k + 1$. It follows by Lemma 2.8, that $\det X = 0$ and X is singular. Hence, each X_i is a square matrix. Consider the substructure S_i corresponding to X_i. If S_i has a cycle then by Lemma 2.5 X_i is singular. If S_i is acyclic then since, it has an equal number of vertices and edges, it must be a rootless tree.

2.3 Path Matrix

Let G be a graph with the vertex set $V(G) = \{1, 2, \ldots, n\}$ and the edge set $E(G) = \{e_1, \ldots, e_m\}$. Given a path $\mathscr{P}$ in G, the incidence vector of $\mathscr{P}$ is an $m \times 1$ vector defined as follows. The entries of the vector are indexed by $E(G)$. If $e_i \in E(G)$ then the ith element of the vector is 0 if the path does not contain e_i. If the path contains e_i then the entry is 1 or -1, according as the direction of the path agrees or disagrees, respectively, with e_i.

Let G be a tree with the vertex set $\{1, 2, \ldots, n\}$. We identify a vertex, say n, as the root. The path matrix P_n of G (with reference to the root n) is defined as follows. The jth column of P_n is the incidence vector of the (unique) path from vertex j to n, $j = 1, \ldots, n - 1$.

Theorem 2.10 *Let G be a tree with the vertex set $\{1, 2, \ldots, n\}$. Let Q be the incidence matrix of G and let Q_n be the reduced incidence matrix obtained by deleting row n of Q. Then $Q_n^{-1} = P_n$.*

Proof Let $m = n - 1$. For $i \neq j$, consider the (i, j)-element of $P_n Q_n$, which is $\sum_{k=1}^{m} p_{ik} q_{kj}$. Suppose e_i is directed from x to y, and e_j is directed from w to z. Then $q_{kj} = 0$ unless $k = w$ or $k = z$. Thus,

$$\sum_{k=1}^{m} p_{ik} q_{kj} = p_{iw} q_{wj} + p_{iz} q_{zj}.$$

As $i \neq j$, we see that the path from w to n contains e_i if and only if the path from z to n contains e_i. Furthermore, when p_{iw} and p_{iz} are nonzero, they both have the same sign. Since $q_{wj} = 1 = -q_{zj}$, it follows that $\sum_{k=1}^{m} p_{ik} q_{kj} = 0$.

If $i = j$, then we leave it as an exercise to check that $\sum_{k=1}^{m} p_{ik} q_{ki} = 1$. This completes the proof. □

2.4 Integer Generalized Inverses

An integer matrix need not admit an integer g-inverse. A trivial example is a matrix with each entry equal to 2. Certain sufficient conditions for an integer matrix to have at least one integer generalized inverse are easily given. We describe some such conditions and show that the incidence matrix of a graph belongs to the class.

A square integer matrix is called *unimodular* if its determinant is ± 1.

Lemma 2.11 *Let A be an $n \times n$ integer matrix. Then A is nonsingular and admits an integer inverse if and only if A is unimodular.*

Proof If $\det A = \pm 1$, then $\dfrac{1}{\det A} \operatorname{adj} A$ is the integer inverse of A. Conversely, if A^{-1} exists and is an integer matrix, then from $AA^{-1} = I$ we see that $(\det A)(\det A^{-1}) = 1$ and hence $\det A = \pm 1$. □

The next result gives the well-known Smith normal form of an integer matrix.

Theorem 2.12 *Let A be an $m \times n$ integer matrix. Then there exist unimodular matrices S and T of order $m \times m$ and $n \times n$, respectively, such that*

$$SAT = \begin{bmatrix} \operatorname{diag}(z_1, \ldots, z_r) & 0 \\ 0 & 0 \end{bmatrix},$$

where $z_1, \ldots, z_r$ are positive integers (called the invariant factors of A) such that z_i divides z_{i+1}, $i = 1, 2, \ldots, r - 1$. Furthermore, $z_1 \ldots z_i = d_i$, where d_i is the greatest common divisor of all $i \times i$ minors of A, $i = 1, \ldots, \min\{m, n\}$.

In Theorem 2.12 suppose each $z_i = 1$. Then it is easily verified that $T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S$ is an integer g-inverse of A.

Note that if A is an integer matrix which has integer rank factorization $A = FH$, where F admits an integer left inverse F^- and H admits an integer right inverse H^-, then H^-F^- is an integer g-inverse of A.

We denote the column vector consisting of all 1 s by $\mathbf{1}$. The order of the vector will be clear from the context. Similarly the matrix of all 1 s will be denoted by J. We may indicate the $n \times n$ matrix of all 1 s by J_n as well.

In the next result we state the Smith normal form and an integer rank factorization of the incidence matrix explicitly.

Theorem 2.13 *Let G be a graph with vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $\{e_1, \dots, e_m\}$. Suppose the edges $e_1, \dots, e_{n-1}$ form a spanning tree of G. Let Q_1 be the submatrix of Q formed by the rows $1, \dots, n-1$ and the columns $e_1, \dots, e_{n-1}$. Let $q = m - n + 1$. Partition Q as follows:*

$$Q = \begin{bmatrix} Q_1 & Q_1 N \\ -\mathbf{1}' Q_1 & -\mathbf{1}' Q_1 N \end{bmatrix}.$$

Set

$$B = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} Q_1^{-1} & 0 \\ \mathbf{1}' & 1 \end{bmatrix}, \quad T = \begin{bmatrix} I_{n-1} & -N \\ 0 & I_q \end{bmatrix},$$

$$F = \begin{bmatrix} Q_1 \\ -\mathbf{1}' Q_1 \end{bmatrix}, \quad H = \begin{bmatrix} I_{n-1} & N \end{bmatrix}.$$

Then the following assertions hold:

(i) *B is an integer reflexive g-inverse of Q.*

(ii) *S and T are unimodular matrices.*

(iii) *$SQT = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$ is the Smith normal form of Q.*

(iv) *$Q = FH$ is an integer rank factorization of Q.*

The proof of Theorem 2.13 is by a simple verification and is omitted. Also note that F admits an integer left inverse and H admits an integer right inverse.

2.5 Moore–Penrose Inverse

We now turn our attention to the Moore–Penrose inverse Q^+ of Q. We first prove some preliminary results. The next result is the well-known fact that the null space of A^+ is the same as that of A' for any matrix A. We include a proof.

Lemma 2.14 *If A is an $m \times n$ matrix, then for an $n \times 1$ vector x, $Ax = 0$ if and only if $x'A^+ = 0$.*

Proof If $Ax = 0$ then $A^+Ax = 0$ and hence $x'(A^+A)' = 0$. Since A^+A is symmetric, it follows that $x'A^+A = 0$. Hence, $x'A^+AA^+ = 0$, and it follows that $x'A^+ = 0$. The converse follows since $(A^+)^+ = A$. □

Lemma 2.15 *If G is connected, then $I - QQ^+ = \frac{1}{n}J$.*

Proof Note that $(I - QQ^+)Q = 0$. Thus, any row of $I - QQ^+$ is in the left null space of Q. Since G is connected, the left null space of Q is spanned by the vector $\mathbf{1}'$. Thus, any row of $I - QQ^+$ is a multiple of any other row. Since $I - QQ^+$ is symmetric, it follows that all the elements of $I - QQ^+$ are equal to a constant. The constant must be nonzero, since Q cannot have a right inverse. Now using the fact that $I - QQ^+$ is idempotent, it follows that it must equal $\frac{1}{n}J$. □

Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Suppose the edges $e_1, \ldots, e_{n-1}$ form a spanning tree of G. Partition Q as follows:

$$Q = \begin{bmatrix} U & V \end{bmatrix},$$

where U is $n \times (n-1)$ and V is $n \times (m-n+1)$. Also, let Q^+ be partitioned as

$$Q^+ = \begin{bmatrix} X \\ Y \end{bmatrix},$$

where X is $(n-1) \times n$ and Y is $(m-n+1) \times n$.

There exists an $(n-1) \times (m-n+1)$ matrix D such that $V = UD$. By Lemma 2.14 it follows that $Y = D'X$. Let $M = I - \frac{1}{n}J$. By Lemma 2.15

$$M = QQ^+ = UX + VY = UX + UDD'X = U(I + DD')X.$$

Thus, for any i, j,

$$U_i(I + DD')X^j = M(i, j),$$

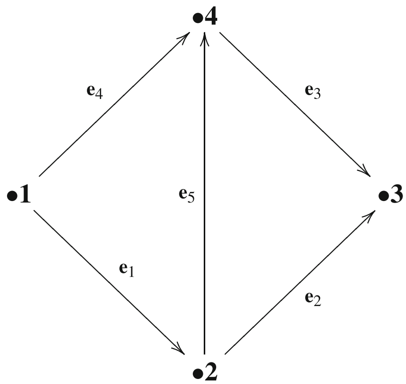
where U_i is U with row i deleted, and X^j is X with column j deleted.

By Lemma 2.7, U_i is nonsingular. Also, DD' is positive semidefinite and thus $I + DD'$ is nonsingular. Therefore, $U_i(I + DD')$ is nonsingular and

$$X^j = (U_i(I + DD'))^{-1}M(i, j).$$

Once X^j is determined, the jth column of X is obtained using the fact that $Q^+\mathbf{1} = 0$. Then Y is determined, since $Y = D'X$.

We illustrate the above method of calculating Q^+ by an example. Consider the graph



with the incidence matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}.$$

Fix the spanning tree formed by $\{e_1, e_2, e_3\}$. Then $Q = \begin{bmatrix} U & V \end{bmatrix}$ where U is formed by the first three columns of Q. Observe that $V = UD$, where

$$D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Set $i = j = 4$. Then $Q^+ = \begin{bmatrix} X \\ Y \end{bmatrix}$ where

$$X^4 = (U_4(I + DD'))^{-1}M(4,4) = \frac{1}{8}\begin{bmatrix} 3 & -2 & -1 \\ 1 & 2 & -3 \\ 1 & 0 & -3 \end{bmatrix}.$$

The last column of X is found using the fact that the row sums of X are zero. Then $Y = D'X$. After these calculations we see that

$$Q^+ = \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & -2 & -1 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 0 & -3 & 2 \\ 3 & 0 & -1 & -2 \\ 0 & 2 & 0 & -2 \end{bmatrix}.$$

2.6 0–1 Incidence Matrix

We now consider the incidence matrix of an undirected graph. Let G be a graph with $V(G) = \{1, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. The (vertex-edge) incidence matrix of G, which we denote by $M(G)$, or simply by M, is the $n \times m$ matrix defined as follows. The rows and the columns of M are indexed by $V(G)$ and $E(G)$, respectively. The (i, j)-entry of M is 0 if vertex i and edge e_j are not incident, and otherwise it is 1. We often refer to M as the 0–1 incidence matrix for clarity. The proof of the next result is easy and is omitted.

Lemma 2.16 *Let C_n be the cycle on the vertices $\{1, \ldots, n\}$, $n \geq 3$, and let M be its incidence matrix. Then* $\det M$ *equals* 0 *if n is even and* 2 *if n is odd.*

Lemma 2.17 *Let G be a connected graph with n vertices and let M be the incidence matrix of G. Then the rank of M is $n - 1$ if G is bipartite and n otherwise.*

Proof Suppose $x \in \mathbb{R}^n$ such that $x'M = 0$. Then $x_i + x_j = 0$ whenever the vertices i and j are adjacent. Since G is connected it follows that $|x_i| = \alpha$, $i = 1, \ldots, n$, for some constant α. Suppose G has an odd cycle formed by the vertices $i_1, \ldots, i_k$. Then going around the cycle and using the preceding observations we find that $\alpha = -\alpha$ and hence $\alpha = 0$. Thus, if G has an odd cycle then the rank of M is n.

Now suppose G has no odd cycle, that is, G is bipartite. Let $V(G) = X \cup Y$ be a bipartition. Orient each edge of G giving it the direction from X to Y and let Q be the corresponding $\{0, 1, -1\}$-incidence matrix. Note that Q is obtained from M by multiplying the rows corresponding to the vertices in Y by -1. Consider the columns $j_1, \ldots, j_{n-1}$ corresponding to a spanning tree of G and let B be the submatrix formed by these columns. By Lemma 2.7 any $n - 1$ rows of B are linearly independent and (since rows of M and Q coincide up to a sign) the corresponding rows of M are also linearly independent. Thus, rank $M \geq n - 1$.

Let $z \in \mathbb{R}^n$ be the vector with z_i equal to 1 or -1 according as i belongs to X or to Y, respectively. Then it is easily verified that $z'M = 0$ and thus the rows of M are linearly dependent. Thus, rank $M = n - 1$ and the proof is complete. □

A connected graph is said to be *unicyclic* if it contains exactly one cycle. We omit the proof of the next result, since it is based on arguments as in the oriented case.

Lemma 2.18 *Let G be a graph and let R be a substructure of G with an equal number of vertices and edges. Let N be the incidence matrix of R. Then N is nonsingular if and only if R is a vertex-disjoint union of rootless trees and unicyclic graphs with the cycle being odd.*

We summarize some basic properties of the minors of the incidence matrix of an undirected graph.

Let M be the 0–1 incidence matrix of the graph G with n vertices. Let N be a square submatrix of M indexed by the vertices and edges, which constitute a substructure denoted by R. If N has a zero row or a zero column then, clearly, $\det N = 0$. This case corresponds to R having an isolated vertex or an edge with both endpoints missing. We assume this not to be the case.

Let R be the vertex-disjoint union of the substructures $R_1, \ldots, R_k$. After a relabeling of rows and columns if necessary, we have

$$N = \begin{bmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & N_k \end{bmatrix},$$

where N_i is the incidence matrix of R_i, $i = 1, \ldots, k$.

If N_i is not square for some i, then using Lemma 2.8, we conclude that N is singular. Thus, if R_i has unequal number of vertices and edges for some i then $\det N = 0$.

If R_i is unicyclic for some i, with the cycle being even, then $\det N = 0$. This follows easily from Lemma 2.16.

Now suppose each N_i is square. Then each R_i is either a rootless tree or is unicyclic with the cycle being odd. In the first case, $\det N_i = \pm 1$ while in the second case $\det N_i = \pm 2$. Note that $\det N = \prod_{i=1}^{k} \det N_i$, Thus, in this case $\det N = \pm 2^{\omega_1(R)}$, where $\omega_1(R)$ is the number of substructures $R_1, \ldots, R_k$ that are unicyclic.

The concept of a substructure will not be needed extensively henceforth. It seems essential to use the concept if one wants to investigate minors of incidence matrices. We have not developed the idea rigorously and have tried to use it informally.

2.7 Matchings in Bipartite Graphs

Lemma 2.19 *Let G be a bipartite graph. Then the 0–1 incidence matrix M of G is totally unimodular.*

Proof The proof is similar to that of Lemma 2.6. Consider the statement that any $k \times k$ submatrix of M has determinant 0 or ± 1. We prove the statement by induction on k. Clearly the statement holds for $k = 1$, since each entry of M is either 0 or 1. Assume the statement to be true for $k-1$ and consider a $k \times k$ submatrix B of M. If B has a zero column, then $\det B = 0$. Suppose B has a column with only one nonzero entry, which must be 1. Expand the determinant of B along that column and use the induction assumption to conclude that $\det B$ must be 0 or ± 1. Finally, suppose each column of B has two nonzero entries. Let $V(G) = X \cup Y$ be the bipartition of G. The sum of the rows of B corresponding to the vertices in X must equal the sum of the rows of B corresponding to the vertices in Y. In fact both these sums will be $\mathbf{1}'$. Therefore, B is singular in this case and $\det B = 0$. This completes the proof. □

Recall that a matching in a graph is a set of edges, no two of which have a vertex in common. The matching number $\nu(G)$ of the graph G is defined to be the maximum number of edges in a matching of G.

We need some background from the theory of linear inequalities and linear programming in the following discussion.

Let G be a graph with $V(G) = \{1, \ldots, n\},\ E(G) = \{e_1, \ldots, e_m\}$. Let M be the incidence matrix of G. Note that a 0–1 vector x of order $m \times 1$ is the incidence vector of a matching if and only if it satisfies $Mx \leq \mathbf{1}$. Consider the linear programming problem:

$$\max\{\mathbf{1}'x\} \text{ subject to } x \geq 0, \quad Mx \leq \mathbf{1}. \tag{2.1}$$

In order to solve (2.1) we may restrict attention to the basic feasible solutions, which are constructed as follows. Let rank $M = r$. Find a nonsingular $r \times r$ submatrix B of M and let $y = B^{-1}\mathbf{1}$. Set the subvector of x corresponding to the rows in B equal to y and set the remaining coordinates of x equal to 0. If the x thus obtained satisfies $x \geq 0,\ Mx \leq \mathbf{1}$, then it is called a basic feasible solution. With this terminology and notation we have the following.

Lemma 2.20 *Let G be a bipartite graph with incidence matrix M. Then there exists a 0–1 vector z which is a solution of* (2.1).

Proof By Lemma 2.19, M is totally unimodular and hence for any nonsingular submatrix B of M, B^{-1} is an integral matrix. By the preceding discussion, a basic feasible solution of $x \geq 0,\ Mx \leq \mathbf{1}$ has only integral coordinates. Hence there is a nonnegative, integral vector z which solves (2.1). Clearly if a coordinate of z is >1, then z cannot satisfy $Mz \leq \mathbf{1}$. Hence z must be a 0–1 vector. □

A vertex cover in a graph is a set of vertices such that each edge in the graph is incident to one of the vertices in the set. The covering number $\tau(G)$ of the graph G is defined to be the minimum number of vertices in a vertex cover of G.

As before, let G be a graph with $V(G) = \{1, \ldots, n\}$, $E(G) = \{e_1, \ldots, e_m\}$. Let M be the incidence matrix of G. Note that a 0–1 vector x of order $n \times 1$ is the incidence vector of a vertex cover if and only if it satisfies $M'x \geq \mathbf{1}$. Consider the linear programming problem:

$$\min\{\mathbf{1}'x\} \text{ subject to } x \geq 0, \quad M'x \geq \mathbf{1} \tag{2.2}$$

The proof of the next result is similar to that of Lemma 2.20 and hence is omitted.

Lemma 2.21 *Let G be a bipartite graph with the incidence matrix M. Then there exists a 0–1 vector z which is a solution of* (2.2).

The following result is the well-known König–Egervary theorem, which is central to the matching theory of bipartite graphs.

Theorem 2.22 *If G is a bipartite graph then $\nu(G) = \tau(G)$.*

Proof Let M be the incidence matrix of G. The linear programming problems (2.1) and (2.2) are dual to each other and their feasibility is obvious. Hence, by the duality theorem, their optimal values are equal. As discussed earlier, the optimal values of the two problems are $\nu(G)$ and $\tau(G)$, respectively. Hence it follows that $\nu(G) = \tau(G)$. □

Exercises

1. Let G be an oriented graph with the incidence matrix Q, and let B be a $k \times k$ submatrix of Q which is nonsingular. Show that there is precisely one permutation σ of $1, \ldots, k$ for which the product $b_{1\sigma(1)} \ldots b_{k\sigma(k)}$ is nonzero. (The property holds for the 0–1 incidence matrix as well.)
2. Let G be a connected graph with $V(G) = \{1, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Suppose the edges of G are oriented, and let Q be the incidence matrix. Let y be an $n \times 1$ vector with one coordinate 1, one coordinate -1, and the remaining coordinates zero. Show that there exists an $m \times 1$ vector x with coordinates $0, \pm 1$ such that $Qx = y$. Give a graph-theoretic interpretation.
3. Let each edge of K_n be given an orientation and let Q be the incidence matrix. Determine Q^+.
4. Let M be the 0–1 incidence matrix of the graph G. Show that if M is totally unimodular then G is bipartite.
5. Let A be an $n \times n$ 0–1 matrix. Show that the following conditions are equivalent:

   (i) For any permutation σ of $1, \ldots, n$, $a_{1\sigma(1)} \ldots a_{n\sigma(n)} = 0$.
   (ii) A has a zero submatrix of order $r \times s$ where $r + s = n + 1$.

Biggs [Big93] and Godsil and Royle [GR01] are essential references for the material related to this chapter as well as that in Chaps 3–6. Relevant references for various sections are as follows: Sect. 2.3: [Bap02], Sect. 2.4: [BHK81], Sect. 2.5: [Ij65], Sect. 2.6: [GKS95], Sect. 2.7: [LP86].

References and Further Reading

[Bap02] Bapat, R.B., Pati, S.: Path matrices of a tree. J. Math. Sci. **1**, 46–52 (2002). New Series (Delhi)
[BHK81] Bevis, J.H., Hall, F.J., Katz, I.J.: Integer generalized inverses of incidence matrices. Linear Algebra Appl. **39**, 247–258 (1981)
[Big93] Biggs, N.: Algebraic Graph Theory, 2nd edn. Cambridge University Press, Cambridge (1993)
[GR01] Godsil, C., Royle, G.: Algebraic Graph Theory, Graduate Texts in Mathematics. Springer, New York (2001)
[GKS95] Grossman, J.W., Kulkarni, D., Schochetman, I.E.: On the minors of an incidence matrix and its Smith normal form. Linear Algebra Appl. **218**, 213–224 (1995)

[Ij65] Ijiri, Y.: On the generalized inverse of an incidence matrix. J. Soc. Ind. Appl. Math. **13**(3), 827–836 (1965)

[LP86] Lovász, L., Plummer, M.D.: Matching Theory, Annals of Discrete Mathematics. North-Holland, Amsterdam (1986)

http://www.springer.com/978-1-4471-6568-2

Graphs and Matrices
Bapat, R.
2014, XI, 193 p. 32 illus., Softcover
ISBN: 978-1-4471-6568-2