

Chapter 2

Random Age Replacement Policies

It is well known in reliability theory: (1) When the failure time is exponential, an operating unit should be replaced only at failure and (2) an optimum age replacement is nonrandom [1, p. 86]. For such questions, we introduce the shortage and excess costs, and show that both a finite replacement time and a random replacement exist even for exponential failure times.

Suppose the unit works for a job with random working times. Then, the unit is replaced before failure at a planned time T or a random time Y , whichever occurs first, which is called *replacement first*. The expected cost rate is obtained and an optimum T_F^* which minimizes it is derived analytically. It is shown naturally that when the replacement costs at time T and time Y are the same, replacement first is not better than standard replacement in which the unit is replaced only at time T . However, if the replacement cost at time Y is lower than that at time T , then replacement first would be rather than standard replacement. We give an example and discuss numerically which replacement is better [2, 3].

If the replacement cost after failure is not so high, the unit should be working as long as possible before failure. Suppose the unit is replaced before failure at time T or at time Y , whichever occurs last, which is called *replacement last*. The expected cost rate is obtained and an optimum T_L^* which minimizes it is derived analytically. It is also shown that both replacement costs at time T and Y are the same, replacement last is not better than standard replacement. Furthermore, we compare two optimum policies for replacement first and last, and show theoretically and numerically, which is better than the other according to the ratio of replacement costs [4].

Finally, it may be wise to replace an operating unit after completion of its working time even if the replacement time T comes. Suppose the unit is replaced before failure at the first completion of working times over time T , which is called *replacement overtime*. Then, an optimum replacement time T_O^* is derived and is smaller than the other optimum times of standard replacement, and replacement first and last. When both costs of standard replacement and replacement overtime are the same, standard replacement is better than replacement overtime. However, when the two costs are different, we give an example and discuss numerically which replacement is better [5].

Throughout this chapter, suppose the unit with a failure time X ($0 < X < \infty$) deteriorates with age and fails according to a general distribution $F(t) \equiv \Pr\{X \leq t\}$ with finite mean $\mu \equiv \int_0^\infty \bar{F}(t) dt < \infty$, where $\bar{F}(t) \equiv 1 - F(t)$ for any distribution $F(t)$. When $F(t)$ has a density function $f(t) \equiv dF(t)/dt$, i.e., $F(t) \equiv \int_0^\infty f(u) du$, the failure rate $h(t) \equiv f(t)/\bar{F}(t)$ for $F(t) < 1$ is assumed to increase strictly from $h(0) = 0$ to $h(\infty) \equiv \lim_{t \rightarrow \infty} h(t)$, except Sect. 2.1.

2.1 Random Replacement

A unit is replaced at random time Y ($0 < Y \leq \infty$) or at failure, whichever occurs first, where Y is a random variable with a general distribution $G(t)$ and is independent of the failure time X . Then, the probability that the unit is replaced at random time Y is

$$\Pr\{Y \leq X\} = \int_0^\infty \bar{F}(t) dG(t),$$

and the probability that it is replaced at failure is

$$\Pr\{X \leq Y\} = \int_0^\infty \bar{G}(t) dF(t).$$

Thus, the mean time to replacement is

$$\int_0^\infty t \bar{F}(t) dG(t) + \int_0^\infty t \bar{G}(t) dF(t) = \int_0^\infty \bar{G}(t) \bar{F}(t) dt.$$

Therefore, using the theory of a renewal reward process [6, p. 77], [7, p. 56], the expected cost rate is [1, p. 86], [8, p. 247]

$$C_A(G) = \frac{c_F \int_0^\infty \bar{G}(t) dF(t) + c_R \int_0^\infty G(t) dF(t)}{\int_0^\infty \bar{G}(t) \bar{F}(t) dt}, \quad (2.1)$$

where c_R = replacement cost at random time Y , and c_F = replacement cost at failure with $c_F > c_R$, which was already given in (1.24).

It has been shown [1, p. 87] that (2.1) can be written as (Problem 1 in Sect. 2.5)

$$C_A(G) = \frac{\int_0^\infty Q(t) dG(t)}{\int_0^\infty S(t) dG(t)},$$

where

$$Q(t) \equiv c_F F(t) + c_R \bar{F}(t), \quad S(t) \equiv \int_0^t \bar{F}(u) du.$$

Suppose that there exists a minimum value T ($0 < T \leq \infty$) of $Q(t)/S(t)$. Because

$$\frac{Q(t)}{S(t)} \geq \frac{Q(T)}{S(T)},$$

it follows that

$$\int_0^\infty Q(t) dG(t) \geq \frac{Q(T)}{S(T)} \int_0^\infty S(t) dG(t).$$

So that,

$$C_A(G) \geq \frac{Q(T)}{S(T)} = C_A(G_T),$$

where $G_T(t)$ is the degenerate distribution placing unit mass at T , i.e., $G_T(t) \equiv 1$ for $t \geq T$ and $G_T(t) \equiv 0$ for $t < T$. If $T = \infty$ then the unit is replaced only at failure. Thus, the optimum policy is nonrandom.

An optimum time T_S^* for a standard age replacement with replacement time T ($0 < T \leq \infty$) which minimizes [1, p. 87], [8, p. 74]

$$C_S(T) \equiv \frac{Q(T)}{S(T)} = \frac{c_F F(T) + c_T \bar{F}(T)}{\int_0^T \bar{F}(t) dt} \quad (2.2)$$

satisfies

$$h(T) \int_0^T \bar{F}(t) dt - F(T) = \frac{c_T}{c_F - c_T},$$

or

$$\int_0^T \bar{F}(t) [h(T) - h(t)] dt = \frac{c_T}{c_F - c_T}, \quad (2.3)$$

and the resulting cost rate is

$$C_S(T_S^*) = (c_F - c_T)h(T_S^*), \quad (2.4)$$

where c_T = replacement cost at time T .

Example 2.1 (Random replacement for exponential failure and random times) When $F(t) = 1 - e^{-\lambda t}$ ($0 < \lambda < \infty$) and $G(t) = 1 - e^{-\theta t}$ ($0 \leq \theta < \infty$), the expected cost rate in (2.1) becomes a function of θ and is given by

$$C_A(\theta) = c_F\lambda + c_R\theta.$$

An optimum θ^* that minimizes $C_A(\theta)$ is $\theta^* = 0$ and $C_A(0) = c_F\lambda$, i.e., we should make no random maintenance. $\square$

Example 2.2 (Replacement for Uniform random time) Suppose that Y has a uniform distribution for the interval $[0, T]$ ($0 < T < \infty$), i.e., $G(t) \equiv t/T$ for $t \leq T$ and 1 for $t > T$. Then, the expected cost rate in (2.1) is a function of T and is given by

$$C_U(T) = \frac{c_F \int_0^T F(t) dt + c_R \int_0^T \bar{F}(t) dt}{\int_0^T (T - t) \bar{F}(t) dt}. \quad (2.5)$$

An optimum T_U^* which minimizes $C_U(T)$ will be discussed in Sect. 2.2.4. $\square$

2.1.1 Shortage and Excess Costs

Introduce the following two kinds of costs in Fig. 2.1 which will be more clearly defined in the scheduling problem of Chap. 7: If the unit would fail after time Y , then this causes a shortage cost $c_S(X - Y)$ because it might operate for a little more time [9, p. 83]. On the other hand, if the unit would fail before time Y , then this causes an excess cost $c_E(Y - X)$ due to its failure because it has failed at a little earlier time than Y , where $c_S(0) = c_E(0) \equiv 0$.

Under the above assumptions, the expected replacement cost is

$$\begin{aligned} C_1(G) &= \int_0^\infty \left[\int_0^t c_S(t - u) dG(u) \right] dF(t) + \int_0^\infty \left[\int_0^t c_E(t - u) dF(u) \right] dG(t) \\ &= \int_0^\infty \left[\int_0^t G(u) dc_S(u) \right] dF(t) + \int_0^\infty \left[\int_0^t F(u) dc_E(u) \right] dG(t) \\ &= \int_0^\infty G(t) \bar{F}(t) dc_S(t) + \int_0^\infty \bar{G}(t) F(t) dc_E(t). \end{aligned} \quad (2.6)$$

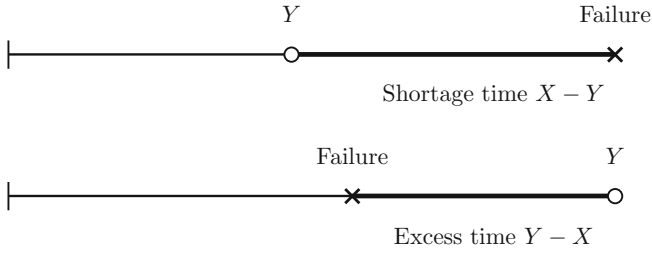


Fig. 2.1 Shortage and excess times of age replacement

Because the mean time to replacement is $\int_0^\infty \bar{G}(t) \bar{F}(t) dt$, the expected cost rate is

$$C_2(G) = \frac{\int_0^\infty G(t) \bar{F}(t) dc_S(t) + \int_0^\infty \bar{G}(t) F(t) dc_E(t)}{\int_0^\infty \bar{G}(t) \bar{F}(t) dt}. \quad (2.7)$$

When $c_S(t) = c_S t$, $c_E(t) = c_E t$, and $G(t) = 1 - e^{-\theta t}$ ($0 < \theta < \infty$), the expected costs $C_i(G)$ ($i = 1, 2$) are the functions of θ : From (2.6),

$$C_1(\theta) = c_S \mu + \frac{c_E}{\theta} - (c_S + c_E) \int_0^\infty e^{-\theta t} \bar{F}(t) dt. \quad (2.8)$$

Clearly,

$$C_1(0) \equiv \lim_{\theta \rightarrow 0} C_1(\theta) = \infty, \quad C_1(\infty) \equiv \lim_{\theta \rightarrow \infty} C_1(\theta) = c_S \mu.$$

Differentiating $C_1(\theta)$ with respect to θ and setting it equal to zero (Problem 2 in Sect. 2.5),

$$\int_0^\infty [1 - (1 + \theta t)e^{-\theta t}] dF(t) = \frac{c_E}{c_S + c_E}, \quad (2.9)$$

whose left-hand side increases with θ from 0 to 1.

From (2.7),

$$C_2(\theta) = \frac{c_S \mu + c_E / \theta}{\int_0^\infty e^{-\theta t} \bar{F}(t) dt} - (c_S + c_E). \quad (2.10)$$

Clearly,

$$C_2(0) \equiv \lim_{\theta \rightarrow 0} C_2(\theta) = C_2(\infty) \equiv \lim_{\theta \rightarrow \infty} C_2(\theta) = \infty.$$

Differentiating $C_2(\theta)$ with respect to θ and setting it equal to zero,

$$\frac{\int_0^\infty [1 - (1 + \theta t)e^{-\theta t}]dF(t)}{\int_0^\infty te^{-\theta t}dF(t)} = \frac{c_E}{c_S\mu}, \quad (2.11)$$

whose left-hand side increases strictly with θ from 0 to ∞ . Therefore, there exists both finite and unique θ_i^* ($0 < \theta_i^* < \infty$) ($i = 1, 2$) which satisfies (2.9) and (2.11), respectively.

In particular, when $F(t) = 1 - e^{-\lambda t}$, (2.9) becomes

$$\left(\frac{\theta}{\theta + \lambda} \right)^2 = \frac{c_E}{c_S + c_E},$$

and (2.11) becomes

$$\left(\frac{\theta}{\lambda} \right)^2 = \frac{c_E}{c_S},$$

and hence, $\theta_1^* > \theta_2^*$ (Problem 3 in Sect. 2.5).

Next, when $c_S(t) = c_S t$, $c_E(t) = c_E t$, and $G(t) \equiv 0$ for $t < T$ and 1 for $t \geq T$, the expected costs $C_i(G)$ are the function of T , and from (2.6),

$$C_1(T) = c_S \int_T^\infty \bar{F}(t) dt + c_E \int_0^T F(t) dt.$$

Differentiating $C_1(T)$ with respect to T and setting it equal to zero,

$$F(T) = \frac{c_S}{c_S + c_E}. \quad (2.12)$$

The expected cost rate in (2.7) is

$$C_2(T) = \frac{c_S \int_T^\infty \bar{F}(t) dt + c_E \int_0^T F(t) dt}{\int_0^T \bar{F}(t) dt}.$$

Clearly, $C_2(0) = C_2(\infty) = \infty$. Differentiating $C_2(T)$ with respect to T and setting it equal to zero,

$$\frac{1}{\bar{F}(T)} \int_0^T [F(T) - F(t)] dt = \frac{\mu c_S}{c_E}, \quad (2.13)$$

whose left-hand side increases strictly with T from 0 to ∞ . Thus, there exists a finite and unique T_2^* ($0 < T_2^* < \infty$) which satisfies (2.13).

When $F(t) = 1 - e^{-\lambda t}$, T_1^* which satisfies (2.12) and T_2^* which satisfies (2.13) are given, respectively,

$$\begin{aligned} e^{\lambda T} - 1 &= \frac{c_S}{c_E}, \\ e^{\lambda T} - (1 + \lambda T) &= \frac{c_S}{c_E}, \end{aligned}$$

whose left-hand side agrees with (8.5) of [8, p. 204] for periodic inspection with exponential failure time. In this case, $T_1^* < T_2^*$. In particular case of $c_S = c_E$, $\lambda T_1^* = \log 2 = 0.693$ and $\lambda T_2^* = 1.15$.

2.1.2 Comparison of Age and Random Replacement Policies

When $G(t) = 1 - e^{-\theta t}$ and the failure rate $h(t)$ increases strictly to $h(\infty)$, we find an optimum θ^* which minimizes the expected cost rate in (2.1) given by

$$C_A(\theta) = \frac{c_R + (c_F - c_R) \int_0^\infty e^{-\theta t} dF(t)}{\int_0^\infty e^{-\theta t} \bar{F}(t) dt}. \quad (2.14)$$

Differentiating $C_A(\theta)$ with respect to θ and setting it equal to zero,

$$Q_1(\theta) \int_0^\infty e^{-\theta t} \bar{F}(t) dt - \int_0^\infty e^{-\theta t} dF(t) = \frac{c_R}{c_F - c_R}, \quad (2.15)$$

where $Q_1(\theta) \equiv \lim_{T \rightarrow \infty} Q_1(T; \theta)$ and for $0 < T \leq \infty$,

$$Q_1(T; \theta) \equiv \frac{\int_0^T t e^{-\theta t} dF(t)}{\int_0^T t e^{-\theta t} \bar{F}(t) dt} \leq h(T).$$

First, note from (5) of Appendix A.1 that $h(0) \leq Q_1(T; \theta) \leq h(T)$ and $Q_1(T; \theta)$ increases strictly with T from $h(0)$ to

$$Q_1(\theta) = \frac{\int_0^\infty t e^{-\theta t} dF(t)}{\int_0^\infty t e^{-\theta t} \bar{F}(t) dt}.$$

Furthermore, $Q_1(T; \theta)$ decreases strictly with θ from

$$Q_1(T; 0) \equiv \lim_{\theta \rightarrow 0} Q_1(T; \theta) = \frac{\int_0^T t dF(t)}{\int_0^T t \bar{F}(t) dt}$$

to $h(0) = 0$.

From the above results, the left-hand side of (2.15) decreases strictly with θ from

$$Q_1(0)\mu - 1 = \frac{\mu \int_0^\infty t dF(t)}{\int_0^\infty t \bar{F}(t) dt} - 1 = \frac{2\mu \int_0^\infty t dF(t)}{\int_0^\infty t^2 dF(t)} - 1 = \frac{2\mu^2}{\mu^2 + \sigma^2} - 1$$

to 0, where $\sigma^2 \equiv \int_0^\infty (t - \mu)^2 dF(t)$. Therefore, if $Q_1(0) > c_F/[\mu(c_F - c_R)]$, i.e., $2\mu^2/(\mu^2 + \sigma^2) > c_F/(c_F - c_R)$, then there exists an optimum θ^* ($0 < \theta^* < \infty$) which satisfies (2.15), and the resulting cost rate is

$$C_A(\theta^*) = (c_F - c_R)Q_1(\theta^*). \quad (2.16)$$

Note that $Q_1(\theta)$ plays the same role as the failure rate $h(t)$ in standard replacement.

We have already known that if $c_T \leq c_R$, then standard replacement with replacement time T is better than random one in Sect. 2.1. When $c_T > c_R$ and $Q_1(0) > c_F/[\mu(c_F - c_R)]$, we compare the expected cost rates $C_S(T)$ in (2.2) and $C_A(\theta)$ in (2.14). We derive a replacement cost $\hat{c}_R$ in which two optimum cost rates $C_S(T_S^*)$ in (2.4) and $C_A(\theta^*)$ in (2.16) are the same. First, we compute T_S^* ($0 < T_S^* < \infty$) which satisfies (2.3) for c_T and c_F , and $C_S(T_S^*)$ in (2.4). Next, we compute $\hat{c}_R$ which satisfies

$$\begin{aligned} Q_1(\theta) \int_0^\infty e^{-\theta t} \bar{F}(t) dt + \int_0^\infty (1 - e^{-\theta t}) dF(t) &= \frac{c_F}{c_F - \hat{c}_R}, \\ (c_F - c_T)h(T_S^*) &= (c_F - \hat{c}_R)Q_1(\theta), \end{aligned}$$

i.e., we firstly obtain $\hat{\theta}$ which satisfies

$$\frac{1}{Q_1(\hat{\theta})} \int_0^\infty (1 - e^{-\hat{\theta} t}) dF(t) + \int_0^\infty e^{-\hat{\theta} t} \bar{F}(t) dt = \frac{1}{h(T_S^*)} \frac{c_F}{c_F - c_T}, \quad (2.17)$$

and using $\hat{\theta}$, we compute $\hat{c}_R$ which satisfies

$$\frac{c_F - \hat{c}_R}{c_F - c_T} = \frac{h(T_S^*)}{Q_1(\hat{\theta})}. \quad (2.18)$$

When $T_S^* = \infty$, $\hat{\theta} = 0$ and $C_S(\infty) = C_A(0) = c_F/\mu$, and hence, $\hat{c}_R = c_T$.

Example 2.3 (Random replacement for gamma failure time) Suppose that the failure time has a gamma distribution with order k , i.e., $F(t) = \sum_{j=k}^{\infty} [(\lambda t)^j / j!] e^{-\lambda t}$ ($k = 2, 3, \dots$), $f(t) = [\lambda(\lambda t)^{k-1} / (k-1)!] e^{-\lambda t}$, $\mu = k/\lambda$ and $h(t) = [\lambda(\lambda t)^{k-1} / (k-1)!] / \sum_{j=0}^{k-1} [(\lambda t)^j / j!]$, which increases strictly with t from 0 to λ . Then, if $k > c_F / (c_F - c_T)$, then an optimum T_S^* ($0 < T_S^* < \infty$) satisfies uniquely

$$\frac{\lambda(\lambda T)^{k-1} / (k-1)!}{\sum_{j=0}^{k-1} [(\lambda T)^j / j!]} \sum_{j=0}^{k-1} \int_0^T \frac{(\lambda t)^j}{j!} e^{-\lambda t} dt + \sum_{j=0}^{k-1} \frac{(\lambda T)^j}{j!} e^{-\lambda T} = \frac{c_F}{c_F - c_T},$$

and the resulting cost rate is

$$C_S(T_S^*) = (c_F - c_T) \frac{\lambda(\lambda T_S^*)^{k-1} / (k-1)!}{\sum_{j=0}^{k-1} [(\lambda T_S^*)^j / j!]}.$$

On the other hand, when $F(t)$ is a gamma distribution,

$$\begin{aligned} \int_0^{\infty} t e^{-\theta t} dF(t) &= \frac{k}{\theta + \lambda} \left(\frac{\lambda}{\theta + \lambda} \right)^k, \\ \int_0^{\infty} t e^{-\theta t} \bar{F}(t) dt &= \frac{1}{\lambda^2} \sum_{j=1}^k j \left(\frac{\lambda}{\theta + \lambda} \right)^{j+1}, \\ Q_1(\theta) &= \frac{k\lambda[\lambda/(\theta + \lambda)]^{k-1}}{\sum_{j=1}^k j[\lambda/(\theta + \lambda)]^{j-1}}, \end{aligned}$$

which decreases strictly with θ from $2\lambda/(k+1)$ to 0. If $2k/(k+1) > c_F/(c_F - c_R)$, then from (2.15), an optimum θ^* ($0 < \theta^* < \infty$) satisfies uniquely

$$\frac{kA^{k-1}}{\sum_{j=1}^k jA^{j-1}} \sum_{j=1}^k A^j - A^k = \frac{c_R}{c_F - c_R},$$

where $A \equiv \lambda/(\theta + \lambda)$, and the resulting cost rate is

$$C_A(\theta^*) = (c_F - c_R) \frac{k\lambda[\lambda/(\theta^* + \lambda)]^{k-1}}{\sum_{j=1}^k j[\lambda/(\theta^* + \lambda)]^{j-1}}.$$

Next, we compute a replacement cost $\hat{c}_R$ for random replacement in which both expected cost rates of age and random replacement are the same. Using T_S^* , we compute $\hat{\theta}$ which satisfies (2.17), and using both T_S^* and $\hat{\theta}$, we compute $\hat{c}_R/c_F$ from (2.18). If $2k/(k+1) \leq c_F/(c_F - c_R)$, then $T_S^* = 1/\theta^* = \infty$, and hence, $\hat{c}_R = c_T$.

Table 2.1 Optimum T_S^* , $1/\theta^*$, $1/\hat{\theta}$, $\hat{c}_R/c_F$, and $\hat{c}_R/c_T$ when $F(t) = \sum_{j=k}^{\infty} (t^j/j!)e^{-t}$

c_T/c_F or c_R/c_F	$k = 2$					$k = 3$				
	T_S^*	$1/\theta^*$	$1/\hat{\theta}$	$\hat{c}_R/c_F$	$\hat{c}_R/c_T$	T_S^*	$1/\theta^*$	$1/\hat{\theta}$	$\hat{c}_R/c_F$	$\hat{c}_R/c_T$
0.01	0.157	0.051	0.030	0.006	0.600	0.357	0.195	0.137	0.005	0.500
0.02	0.233	0.062	0.044	0.012	0.600	0.468	0.299	0.204	0.010	0.500
0.05	0.412	0.335	0.207	0.031	0.620	0.697	0.540	0.348	0.026	0.520
0.1	0.680	0.793	0.430	0.064	0.640	0.984	0.955	0.557	0.053	0.530
0.2	1.306	4.017	1.104	0.129	0.645	1.512	2.462	1.042	0.109	0.545

In other words, if c_F approaches to c_T and c_R then both T_S^* and $1/\theta^*$ become large and $\hat{c}_R/c_T$ approaches to 1.

Table 2.1 presents T_S^* , $1/\theta^*$, $1/\hat{\theta}$, $\hat{c}_R/c_F$ and $\hat{c}_R/c_T$ for c_T/c_F or c_R/c_F when $F(t) = \sum_{j=k}^{\infty} (t^j/j!)e^{-t}$ ($k = 2, 3$). From the comparison results among T_S^* , $1/\theta^*$ and $1/\hat{\theta}$, $T_S^* > 1/\theta^*$ for small c_T/c_F or c_R/c_F , however, $T_S^* < 1/\theta^*$ for large ones. Note that $1/\hat{\theta} < 1/\theta^*$ and has the same variation trend with $1/\theta^*$. It is also shown that $\hat{c}_R/c_F$ decreases with k and increases with c_T/c_F or c_R/c_F . From the numerical values of $\hat{c}_R/c_T$, we can find that if how much $\hat{c}_R$ is less than c_T , the expected costs of standard and random replacements are the same. Taking $k = 2$ for example, when $\hat{c}_R$ is a little higher than 60 % of c_T , we should adopt random replacement. $\square$

2.2 Random Replacement Policies

Using age and random replacement policies introduced in Sect. 2.1, we define three new policies of replacement first, replacement last, and replacement overtime, and discuss their optimum policies which minimize the expected cost rates. Furthermore, we compare the three policies with standard replacement.

2.2.1 Replacement First

When the failure rate $h(t)$ increases strictly to $h(\infty)$, we consider an age replacement policy in which the unit is replaced at time T , Y or at failure, whichever occurs first, i.e., at time $\min\{T, X, Y\}$. This is called *replacement first*. It is assumed that T ($0 < T \leq \infty$) is constant, and X and Y are independent random variables with the respective distributions $F(t)$ and $G(t)$. Then, the probability that the unit is replaced at time T is

$$\Pr\{Y > T, X > T\} = \bar{G}(T)\bar{F}(T), \quad (2.19)$$

the probability that it is replaced at random time Y is

$$\Pr\{Y \leq T, Y \leq X\} = \int_0^T \bar{F}(t) dG(t), \quad (2.20)$$

and the probability that it is replaced at failure is

$$\Pr\{X \leq T, X \leq Y\} = \int_0^T \bar{G}(t) dF(t), \quad (2.21)$$

where (2.19) + (2.20) + (2.21) = 1. Thus, the mean time to replacement is

$$T \bar{G}(T) \bar{F}(T) + \int_0^T t \bar{F}(t) dG(t) + \int_0^T t \bar{G}(t) dF(t) = \int_0^T \bar{G}(t) \bar{F}(t) dt. \quad (2.22)$$

Therefore, the expected cost rate is

$$C_F(T) = \frac{c_T + (c_F - c_T) \int_0^T \bar{G}(t) dF(t) + (c_R - c_T) \int_0^T \bar{F}(t) dG(t)}{\int_0^T \bar{G}(t) \bar{F}(t) dt}, \quad (2.23)$$

where c_F = replacement cost at failure, and c_T and c_R are the respective replacement costs at time T and random time Y with $c_F > c_T$ and $c_F > c_R$. Clearly,

$$\begin{aligned} C_F(0) &\equiv \lim_{T \rightarrow 0} C_F(T) = \infty, \\ C_F(\infty) &\equiv \lim_{T \rightarrow \infty} C_F(T) = \frac{c_F - (c_F - c_R) \int_0^\infty \bar{F}(t) dG(t)}{\int_0^\infty \bar{G}(t) \bar{F}(t) dt}, \end{aligned}$$

which agrees with (2.1). First, suppose that $Y = \infty$, i.e., $\bar{G}(t) \equiv 1$ for $t \geq 0$. Then, the unit cannot be replaced at random time Y , and the expected cost rate is

$$C_S(T) = \frac{c_T + (c_F - c_T) F(T)}{\int_0^T \bar{F}(t) dt}, \quad (2.24)$$

which agrees with (2.2). Therefore, if $h(\infty) > c_F/[\mu(c_F - c_T)]$, then there exists an optimum T_S^* ($0 < T_S^* < \infty$) which satisfies (2.3), and the resulting cost rate is given in (2.4)

Next, when $c_T = c_R$, the expected cost rate is, from (2.23),

$$C_F(T) = \frac{c_T + (c_F - c_T) \int_0^T \overline{G}(t) dF(t)}{\int_0^T \overline{G}(t) \overline{F}(t) dt}. \quad (2.25)$$

We find an optimum T_F^* which minimizes $C_F(T)$. Differentiating $C_F(T)$ with respect to T and setting it equal to zero,

$$h(T) \int_0^T \overline{G}(t) \overline{F}(t) dt - \int_0^T \overline{G}(t) dF(t) = \frac{c_T}{c_F - c_T}, \quad (2.26)$$

whose left-hand side increases strictly from 0 because $h(t)$ increases strictly to $h(\infty)$. Thus, if

$$h(\infty) \int_0^\infty \overline{G}(t) \overline{F}(t) dt - \int_0^\infty \overline{G}(t) dF(t) > \frac{c_T}{c_F - c_T},$$

then there exists a finite and unique T_F^* ($0 < T_F^* < \infty$) which satisfies (2.26), and the resulting cost rate is

$$C_F(T_F^*) = (c_F - c_T)h(T_F^*). \quad (2.27)$$

In addition, (2.26) is written as

$$\int_0^T \overline{G}(t) \overline{F}(t) [h(T) - h(t)] dt = \frac{c_T}{c_F - c_T}. \quad (2.28)$$

In particular, when $G(t) = 1 - e^{-\theta t}$ ($0 < \theta < \infty$), (2.28) becomes

$$\int_0^T e^{-\theta t} \overline{F}(t) [h(T) - h(t)] dt = \frac{c_T}{c_F - c_T}. \quad (2.29)$$

Thus, an optimum T_F^* increases with θ from T_S^* given in (2.3), i.e., decreases with mean random time $1/\theta$, and $T_F^* > T_S^*$ (Problem 4 in Sect. 2.5). This means that if $1/\theta$ increases, then the unit should be replaced mainly at time T , and hence, T_F^* decreases. In addition, if

$$h(\infty) \frac{1 - F^*(\theta)}{\theta} - F^*(\theta) > \frac{c_T}{c_F - c_T},$$

then there exists a finite and unique T_F^* ($0 < T_F^* < \infty$) which satisfies (2.29), where $F^*(s)$ is the LS (Laplace-Stieltjes) transform of $F(t)$, i.e., $F^*(s) \equiv \int_0^\infty e^{-st} dF(t)$ for $Re(s) > 0$.

Furthermore, from (2.3) and (2.28),

$$h(T) \int_0^T \bar{F}(t) dt - F(T) \geq \int_0^T \bar{G}(t) \bar{F}(t) [h(T) - h(t)] dt$$

follows that $T_F^* \geq T_S^*$. So that, comparing (2.4) with (2.27), $C_S(T_S^*) \leq C_F(T_F^*)$. Thus, standard replacement with only time T is better than replacement first, as already shown in Sect. 2.1. However, if the replacement cost c_R at time Y would be lower than c_T , then replacement first might be rather than standard replacement.

It is assumed from the above discussions that $c_T > c_R$. Differentiating $C_F(T)$ in (2.23) with respect to T and setting it equal to zero,

$$\begin{aligned} (c_F - c_T) \left[h(T) \int_0^T \bar{G}(t) \bar{F}(t) dt - \int_0^T \bar{G}(t) dF(t) \right] \\ - (c_T - c_R) \left[r(T) \int_0^T \bar{G}(t) \bar{F}(t) dt - \int_0^T \bar{F}(t) dG(t) \right] = c_T, \end{aligned} \quad (2.30)$$

where $r(t) \equiv g(t)/\bar{G}(t)$ and $g(t)$ is a density function of $G(t)$. If $r(t)$ decreases with t , then the left-hand side of (2.30) increases strictly with T from 0 to

$$\begin{aligned} (c_F - c_T) \int_0^\infty \bar{G}(t) \bar{F}(t) [h(\infty) - h(t)] dt \\ - (c_T - c_R) \int_0^\infty \bar{G}(t) \bar{F}(t) [r(\infty) - r(t)] dt. \end{aligned} \quad (2.31)$$

Thus, if (2.31) is greater than c_T , then there exists a finite and unique T_F^* ($0 < T_F^* < \infty$) which satisfies (2.30), and the resulting cost rate is

$$C_F(T_F^*) = (c_F - c_T)h(T_F^*) - (c_T - c_R)r(T_F^*). \quad (2.32)$$

In particular, when $G(t) = 1 - e^{-\theta t}$, i.e., $r(t) = \theta$, (2.30) agrees with (2.29).

2.2.2 Replacement Last

Suppose that the unit is replaced at time T ($0 \leq T < \infty$) or at time Y , whichever occurs last, or at failure, i.e., it is replaced at time $\max\{T, Y\}$ or at failure, whichever occurs first. This is called *replacement last*. Then, the probability that the unit is replaced at time T is

$$\Pr\{Y \leq T < X\} = G(T)\bar{F}(T), \quad (2.33)$$

the probability that it is replaced at time Y is

$$\Pr\{T < Y < X\} = \int_T^\infty \bar{F}(t) dG(t), \quad (2.34)$$

and the probability that it is replaced at failure is

$$\Pr\{X \leq T \text{ or } T < X \leq Y\} = F(T) + \int_T^\infty \bar{G}(t) dF(t), \quad (2.35)$$

where (2.33) + (2.34) + (2.35) = 1. Thus, the mean time to replacement is

$$\begin{aligned} & TG(T)\bar{F}(T) + \int_T^\infty t\bar{F}(t) dG(t) + \int_0^T t dF(t) + \int_T^\infty t\bar{G}(t) dF(t) \\ &= \int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t)\bar{F}(t) dt. \end{aligned} \quad (2.36)$$

Therefore, the expected cost rate is

$$C_L(T) = \frac{c_F - (c_F - c_T)G(T)\bar{F}(T) - (c_F - c_R) \int_T^\infty \bar{F}(t) dG(t)}{\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t)\bar{F}(t) dt}, \quad (2.37)$$

where c_F , c_T , and c_R are given in (2.23). Clearly,

$$\begin{aligned} C_L(0) &\equiv \lim_{T \rightarrow 0} C_L(T) = \frac{c_F - (c_F - c_R) \int_0^\infty \bar{F}(t) dG(t)}{\int_0^\infty \bar{G}(t)\bar{F}(t) dt} = C_F(\infty), \\ C_L(\infty) &\equiv \lim_{T \rightarrow \infty} C_L(T) = \frac{c_F}{\mu} = C_S(\infty), \end{aligned}$$

which is the expected cost rate when the unit is replaced only at failure. In particular, when $Y = 0$, i.e., $G(t) \equiv 1$ for $t \geq 0$, $C_L(T)$ agrees with $C_S(T)$ in (2.2).

Next, when $c_T = c_R$, the expected cost rate is, from (2.37),

$$C_L(T) = \frac{c_F - (c_F - c_T) \int_T^\infty G(t) dF(t)}{\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t) \bar{F}(t) dt}. \quad (2.38)$$

We find an optimum T_L^* which minimizes $C_L(T)$. Differentiating $C_L(T)$ with respect to T and setting it equal to zero,

$$h(T) \left[\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t) \bar{F}(t) dt \right] - \left[1 - \int_T^\infty G(t) dF(t) \right] = \frac{c_T}{c_F - c_T}, \quad (2.39)$$

whose left-hand side increases strictly to $\mu h(\infty) - 1$. Thus, if $h(\infty) > c_F / [\mu(c_F - c_T)]$, then there exists a finite and unique T_L^* ($0 \leq T_L^* < \infty$) which satisfies (2.39), and the resulting cost rate is

$$C_L(T_L^*) = (c_F - c_T) h(T_L^*). \quad (2.40)$$

Note that if a finite T_F^* in (2.26) exists, then both finite T_S^* in (2.3) and T_L^* in (2.39) exist. In particular, when $G(t) = 1 - e^{-\theta t}$, (2.39) becomes

$$\int_0^T \bar{F}(t) [h(T) - h(t)] dt - \int_T^\infty e^{-\theta t} \bar{F}(t) [h(t) - h(T)] dt = \frac{c_T}{c_F - c_T}.$$

Thus, an optimum T_L^* decreases with θ to T_S^* given in (2.3), i.e., increases with a mean random time $1/\theta$, and $T_L^* > T_S^*$ (Problem 5 in Sect. 2.5).

Furthermore, from (2.3) and (2.39),

$$\begin{aligned} h(T) \int_0^T \bar{F}(t) dt - F(T) &\geq h(T) \left[\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t) \bar{F}(t) dt \right] \\ &\quad + \int_T^\infty G(t) dF(t) - 1, \end{aligned}$$

follows that $T_L^* \geq T_S^*$. So that, comparing (2.4) with (2.40), $C_S(T_S^*) \leq C_L(T_L^*)$. Thus, standard replacement is better than replacement last, as shown similarly in Sect. 2.2.1. However, if the replacement cost c_R at time Y would be lower than c_T , then replacement last might be rather than standard replacement.

It is assumed from the above discussions that $c_T > c_R$. Differentiating $C_L(T)$ in (2.37) with respect to T and setting it equal to zero,

$$\begin{aligned} (c_F - c_T) & \left\{ h(T) \left[\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t) \bar{F}(t) dt \right] - F(T) \right. \\ & \left. - \int_T^\infty \bar{G}(t) dF(t) \right\} + (c_T - c_R) \left\{ \tilde{r}(T) \left[\int_0^T \bar{F}(t) dt \right. \right. \\ & \left. \left. + \int_T^\infty \bar{G}(t) \bar{F}(t) dt \right] + \int_T^\infty \bar{F}(t) dG(t) \right\} = c_T, \end{aligned} \quad (2.41)$$

where $\tilde{r}(t) \equiv g(t)/G(t)$. Denoting left-hand side of (2.41) by $L_1(T)$,

$$L_1'(T) = [(c_F - c_T)h'(T) + (c_T - c_R)\tilde{r}'(T)] \left[\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t) \bar{F}(t) dt \right],$$

$$L_1(\infty) = (c_F - c_T)[h(\infty)\mu - 1] + (c_T - c_R)\tilde{r}(\infty)\mu.$$

Therefore, if $(c_F - c_T)h(t) + (c_T - c_R)\tilde{r}(t)$ increases strictly with t and $L_1(\infty) > c_T$, i.e., $(c_F - c_T)h(\infty) + (c_T - c_R)\tilde{r}(\infty) > c_F/\mu$ then there exists a finite and unique T_L^* ($0 \leq T_L^* < \infty$) which satisfies (2.41). Note that when $G(t) = 1 - e^{-\theta t}$, $\tilde{r}(t)$ decreases from ∞ to 0. In this case, if $h(\infty) > c_F/[\mu(c_F - c_T)]$ then a finite T_L^* to satisfy (2.41) exists.

2.2.3 Replacement Overtime

It might be wise to replace practically the unit at the completion of its working time even if T comes because it continues to work for some job. Suppose in the age replacement that the unit is replaced before failure at the first completion of random times Y_j ($j = 1, 2, \dots$) over time T ($0 \leq T < \infty$), where Y_j is independent and has an identical distribution $G(t) \equiv \Pr\{Y_j \leq j\}$ with finite mean $1/\theta$ in Fig. 2.2, where $G^{(j)}(t)$ ($j = 1, 2, \dots$) is the j -fold Stieltjes convolution of $G(t)$ with itself, and $G^{(0)}(t) \equiv 1$ for $t \geq 0$. This is called *replacement overtime* [5].

The probability that the unit is replaced before failure at the first completion of working times over time T is

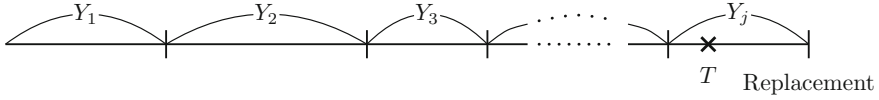


Fig. 2.2 Replacement over time T

$$\sum_{j=0}^{\infty} \int_0^T \left[\int_{T-t}^{\infty} \bar{F}(t+u) dG(u) \right] dG^{(j)}(t), \quad (2.42)$$

the probability that it is replaced at failure before time T is

$$\sum_{j=0}^{\infty} \int_0^T [G^{(j)}(t) - G^{(j+1)}(t)] dF(t) = F(T), \quad (2.43)$$

and the probability that it is replaced at failure after time T is

$$\sum_{j=0}^{\infty} \int_0^T \left\{ \int_{T-t}^{\infty} [F(t+u) - F(T)] dG(u) \right\} dG^{(j)}(t), \quad (2.44)$$

where (2.42) + (2.43) + (2.44) = 1 (Problem 6 in Sect. 2.5). Thus, the mean time to replacement is

$$\begin{aligned} & \sum_{j=0}^{\infty} \int_0^T \left[\int_{T-t}^{\infty} (t+u) \bar{F}(t+u) dG(u) \right] dG^{(j)}(t) + \int_0^T t dF(t) \\ & + \sum_{j=0}^{\infty} \int_0^T \left\{ \int_{T-t}^{\infty} \left[\int_T^{t+u} v dF(v) \right] dG(u) \right\} dG^{(j)}(t), \\ & = \int_0^T \bar{F}(t) dt + \sum_{j=0}^{\infty} \int_0^T \left[\int_T^{\infty} \bar{G}(u-t) \bar{F}(u) du \right] dG^{(j)}(t). \end{aligned} \quad (2.45)$$

Therefore, the expected cost rate is

$$C_O(T) = \frac{c_F - (c_F - c_R) \sum_{j=0}^{\infty} \int_0^T \left[\int_{T-t}^{\infty} \bar{F}(t+u) dG(u) \right] dG^{(j)}(t)}{\int_0^T \bar{F}(t) dt + \sum_{j=0}^{\infty} \int_0^T \left[\int_T^{\infty} \bar{G}(u-t) \bar{F}(u) du \right] dG^{(j)}(t)}, \quad (2.46)$$

where c_R = replacement cost over time T and c_F is given in (2.23).

In particular, when $G(t) = 1 - e^{-\theta t}$,

$$C_O(T) = \frac{c_F - (c_F - c_R) \int_T^\infty \bar{F}(t) \theta e^{-\theta(t-T)} dt}{\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{F}(t) e^{-\theta(t-T)} dt}. \quad (2.47)$$

Clearly,

$$C_O(0) \equiv \lim_{T \rightarrow 0} C_O(T) = \frac{c_F - (c_F - c_R) \int_0^\infty \bar{F}(t) \theta e^{-\theta t} dt}{\int_0^\infty \bar{F}(t) e^{-\theta t} dt} = C_F(\infty) = C_L(0),$$

$$C_O(\infty) \equiv \lim_{T \rightarrow \infty} C_O(T) = \frac{c_F}{\mu} = C_S(\infty) = C_L(\infty).$$

We find an optimum T_O^* which minimizes $C_O(T)$ in (2.47). Differentiating $C_O(T)$ with respect to T and setting it equal to zero,

$$\tilde{Q}_0(T; \theta) \int_0^T \bar{F}(t) dt - F(T) = \frac{c_R}{c_F - c_R}, \quad (2.48)$$

where

$$\tilde{Q}_0(T; \theta) \equiv \frac{\int_T^\infty e^{-\theta t} dF(t)}{\int_T^\infty e^{-\theta t} \bar{F}(t) dt} \geq h(T).$$

First, note from (7) of Appendix A.1 that when the failure rate $h(t)$ increases strictly, $\tilde{Q}_0(T; \theta)$ is greater than $h(T)$ and increases strictly to $h(\infty)$, and increases with θ to $h(T)$ for $0 \leq T < \infty$.

From the above results, if $h(\infty) > c_F/[\mu(c_F - c_R)]$, then there exists a finite and unique T_O^* ($0 < T_O^* < \infty$) which satisfies (2.48), and the resulting cost rate is (Problem 7 in Sect. 2.5)

$$C_O(T_O^*) = (c_F - c_R) \tilde{Q}_0(T_O^*; \theta) = \frac{c_R + (c_F - c_R) F(T_O^*)}{\int_0^{T_O^*} \bar{F}(t) dt}. \quad (2.49)$$

Because $\tilde{Q}_0(T; \theta)$ decreases strictly with θ from $\bar{F}(T)/\int_T^\infty \bar{F}(t) dt$ to $h(T)$, T_O^* increases with θ to T_S^* given in (2.3) when $c_T = c_R$ and $T_O^* > T_S^*$. Standard replacement is better than replacement overtime. This is proved easily from (2.2) and (2.49) because T_S^* minimizes the right-hand side of (2.49).

2.2.4 Replacement of Uniform Random Time

We find an optimum T_U^* which minimizes $C_U(T)$ in (2.5). Clearly,

$$C_U(0) \equiv \lim_{T \rightarrow 0} C_U(T) = \lim_{T \rightarrow 0} C_S(T) = \infty,$$

$$C_U(\infty) \equiv \lim_{T \rightarrow \infty} C_U(T) = \lim_{T \rightarrow \infty} C_S(T) = \frac{c_F}{\mu}.$$

Differentiating $C_U(T)$ with respect to T and setting it equal to zero,

$$Q_1(T) \int_0^T \bar{F}(t) dt - F(T) = \frac{c_R}{c_F - c_R}, \quad (2.50)$$

where

$$Q_1(T) \equiv Q_1(T; 0) = \frac{\int_0^T t dF(t)}{\int_0^T t \bar{F}(t) dt} \leq h(T).$$

First, note that when the failure rate $h(t)$ increases strictly, from (5) of Appendix A.1, $Q_1(T)$ is less than $h(T)$ and increases strictly to

$$Q_1(\infty) = \frac{2 \int_0^\infty t dF(t)}{\int_0^\infty t^2 dF(t)} = \frac{2\mu}{\sigma^2 + \mu^2},$$

where $\sigma^2 \equiv V\{X\} = \int_0^\infty (t - \mu)^2 dF(t)$. Differentiating the left-hand side of (2.50) with respect to T ,

$$\frac{\bar{F}(T)}{[\int_0^T t \bar{F}(t) dt]^2} \int_0^T (T - t) \bar{F}(t) dt \int_0^T t \bar{F}(t) [h(T) - h(t)] dt > 0.$$

Thus, the left-hand side of (2.50) increases strictly from 0 to $\mu Q_1(\infty) - 1$. Therefore, if $Q_1(\infty) > c_F/[\mu(c_F - c_R)]$ then there exists a finite and unique T_U^* ($0 < T_U^* < \infty$) which satisfies (2.50), and the resulting cost rate is

$$C_U(T_U^*) = (c_F - c_R) Q_1(T_U^*) = \frac{c_R + (c_F - c_R) F(T_U^*)}{\int_0^{T_U^*} \bar{F}(t) dt}. \quad (2.51)$$

Because $Q_1(T) < h(T)$ for $0 < T < \infty$, $T_U^* > T_S^*$ when $c_T = c_R$, and hence, standard replacement is better than this replacement.

2.3 Comparisons of Replacement Times

We compare replacement first, replacement last, and replacement overtime analytically and numerically.

2.3.1 Comparison of Replacement First and Last

Compare the expected cost rates $C_F(T)$ in (2.25) and $C_L(T)$ in (2.38). It is assumed that the failure rate $h(t)$ increases strictly from 0 to ∞ , i.e., $h(\infty) = \infty$. Then, there exists both finite and unique T_F^* ($0 < T_F^* < \infty$) and T_L^* ($0 < T_L^* < \infty$) which satisfies (2.26) and (2.39), respectively. First, compare the left-hand sides of (2.26) and (2.39): Denote

$$\begin{aligned} L_2(T) &\equiv h(T) \left[\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{G}(t) \bar{F}(t) dt \right] - \left[1 - \int_T^\infty G(t) dF(t) \right] \\ &\quad - \left[h(T) \int_0^T \bar{G}(t) \bar{F}(t) dt - \int_0^T \bar{G}(t) dF(t) \right] \\ &= \int_0^T G(t) \bar{F}(t) [h(T) - h(t)] dt - \int_T^\infty \bar{G}(t) \bar{F}(t) [h(t) - h(T)] dt. \end{aligned}$$

Clearly,

$$\begin{aligned} L_2(0) &\equiv \lim_{T \rightarrow 0} L_2(T) = - \int_0^\infty \bar{G}(t) dF(t) < 0, \quad L_2(\infty) \equiv \lim_{T \rightarrow \infty} L_2(T) = \infty, \\ L'_2(T) &= h'(T) \left[\int_0^T G(t) \bar{F}(t) dt + \int_T^\infty \bar{G}(t) \bar{F}(t) dt \right] > 0. \end{aligned}$$

Thus, there exists a finite and unique T_A ($0 < T_A < \infty$) which satisfies $L_2(T) = 0$. We set that

$$L(T_A) \equiv h(T_A) \int_0^{T_A} \bar{G}(t) \bar{F}(t) dt - \int_0^{T_A} \bar{G}(t) dF(t). \quad (2.52)$$

Then, it is shown from (2.26) to (2.39) that if $L(T_A) \geq c_T/(c_F - c_T)$, i.e., $c_F/c_T \geq 1 + 1/L(T_A)$, then $T_F^* \leq T_L^*$, and hence, from (2.27) and (2.40), replacement first is better than replacement last. Conversely, if $L(T_A) < c_T/(c_F - c_T)$, then $T_F^* > T_L^*$ and replacement last is better than replacement first. This means that if the ratio of c_T/c_F is greater than $L(T_A)/[1 + L(T_A)]$, i.e., the replacement cost c_F is nearly to cost c_T , replacement last is better than replacement first.

Table 2.2 Optimum T_F^* , T_L^* , T_A and $L(T_A)$ when $F(t) = 1 - e^{-t^2}$

c_T	$1/\theta = 0.1$		$1/\theta = 0.3$		$1/\theta = 0.5$		$1/\theta = 0.7$		$1/\theta = 1.0$		$1/\theta = \infty$
$c_F - c_T$	T_F^*	T_L^*	T_F^*	T_L^*	T_F^*	T_L^*	T_F^*	T_L^*	T_F^*	T_L^*	T_S^*
0.01	0.12	0.13	0.11	0.23	0.10	0.31	0.10	0.36	0.10	0.41	0.10
0.03	0.24	0.18	0.19	0.26	0.18	0.33	0.18	0.38	0.18	0.42	0.17
0.05	0.35	0.23	0.26	0.29	0.24	0.35	0.24	0.40	0.23	0.44	0.22
0.07	0.45	0.27	0.31	0.32	0.29	0.37	0.28	0.42	0.28	0.46	0.26
0.10	0.61	0.32	0.39	0.36	0.36	0.41	0.35	0.44	0.34	0.48	0.32
0.30	1.62	0.56	0.80	0.57	0.69	0.59	0.65	0.61	0.62	0.63	0.56
0.50	2.64	0.74	1.19	0.74	0.97	0.75	0.89	0.76	0.84	0.77	0.74
0.70	3.66	0.89	1.57	0.89	1.24	0.89	1.12	0.90	1.04	0.91	0.89
1.00	5.19	1.09	2.14	1.09	1.64	1.09	1.45	1.10	1.33	1.10	1.09
T_A	0.13		0.33		0.46		0.55		0.65		
$L(T_A)$	0.011		0.077		0.155		0.228		0.325		

Example 2.4 (Replacement for Weibull failure time) Suppose that the failure time X has a Weibull distribution $F(t) = 1 - \exp(-t^2)$ and a random working time Y has an exponential distribution $G(t) = 1 - e^{-\theta t}$. Table 2.2 presents optimum T_F^* , T_L^* , $L(T_A)$, and T_S^* which satisfy (2.26), (2.39), (2.52) and (2.3), respectively, for $1/\theta$ and $c_T/(c_F - c_T)$. When $1/\theta = \infty$, T_F^* becomes equal to T_S^* .

Table 2.2 indicates that both T_F^* and T_L^* increase with $c_T/(c_F - c_T)$, i.e., decrease with c_F/c_T . When c_F/c_T increases, the replacement time should be smaller to prevent a high replacement cost. In other words, the unit can work longer as c_F/c_T becomes smaller. Furthermore, replacement last is much better than replacement first when c_F/c_T becomes smaller, especially for small $1/\theta$. For example, when $c_T/(c_F - c_T) = 0.7$, i.e., $c_F/c_T = 17/7$, and $1/\theta = 0.1$, $T_L^* = 0.89$ is much less than $T_F^* = 3.66$.

When $L(T_A) \geq c_T/(c_F - c_T)$, $T_F^* \leq T_L^*$ and replacement first is better than replacement last, and conversely, when $L(T_A) < c_T/(c_F - c_T)$, $T_F^* > T_L^*$ and replacement last is better than replacement first. For example, when $1/\theta = 0.1$, $L(T_A) = 0.011$, and hence, $T_F^* = 0.12 < T_L^* = 0.13$ for $c_T/(c_F - c_T) = 0.01$, and $T_L^* = 0.18 < T_F^* = 0.24$ for $c_T/(c_F - c_T) = 0.03$.

Optimum T_F^* decreases to T_S^* with $1/\theta$ and T_L^* increases from T_S^* with $1/\theta$, because the unit is replaced before failure at time $\min\{T_F^*, Y\}$ for replacement first and at time $\max\{T_L^*, Y\}$ for replacement last. Furthermore, replacement first is better than replacement last as $1/\theta$ becomes larger. For example, when $c_T/(c_F - c_T) = 0.10$, if $1/\theta < 0.3$, then replacement last is better than replacement first, and if $1/\theta > 0.5$, then replacement first is better than replacement last. This means that if $1/\theta$ becomes larger, the unit has to be replaced mainly at a planned time T . So that, replacement first is better than replacement last, because the unit cannot be replaced before a random time Y for replacement last. It is of interest that when $1/\theta = 1.0$ and $c_T/(c_F - c_T) = 0.70$, $T_L^* = 0.91 < 1/\theta = 1.0 < T_F^* = 1.04$, and optimum replacement times are equal nearly to $1/\theta$. $\square$

2.3.2 Comparisons of Replacement Overtime, First, and Last

Compare the expected cost rate $C_S(T)$ in (2.2) for standard replacement and $C_O(T)$ in (2.47) when $c_T = c_R$. It has been already shown that $T_O^* < T_S^*$ and when $h(\infty) = \infty$, both finite T_O^* and T_S^* exist. Furthermore, because T_S^* is an optimum solution of minimizing $C_S(T)$ in (2.2), $C_O(T_O^*)$ is greater than $C_S(T_S^*)$ from (2.49), i.e., standard replacement is better than replacement overtime. If $c_R < c_T$, then replacement overtime might be rather than standard replacement, as shown in Sect. 2.1. In this case, we would compute numerically $C_S(T_S^*)$ in (2.4) and $C_O(T_O^*)$ in (2.49), and compare them.

Example 2.5 (Comparison of replacements) We compute $\widehat{c}_{RO}$ in which both expected costs of standard replacement and replacement overtime are the same, when $c_R < c_T$, $G(t) = 1 - e^{-\theta t}$, and $F(t) = 1 - \exp(-t^2)$. First, compute T_S^* from (2.3), and the expected cost rate $C_S(T_S^*)$ from (2.4), where T_S^* have been already given in Table 2.2. Next, compute $\widehat{T}_O$ and $\widehat{c}_{RO}$ which satisfy the simultaneous equations:

$$\begin{aligned} (c_F - \widehat{c}_{RO})\widetilde{Q}_0(\widehat{T}_O; \theta) &= (c_F - c_T)h(T_S^*), \\ \widetilde{Q}_0(\widehat{T}_O; \theta) \int_0^{\widehat{T}_O} \overline{F}(t) dt + \overline{F}(\widehat{T}_O) &= \frac{c_F}{c_F - \widehat{c}_{RO}}, \end{aligned}$$

where

$$h(T) = 2T, \quad \widetilde{Q}_0(T; \theta) = \frac{\int_T^\infty e^{-\theta t} 2t e^{-t^2} dt}{\int_T^\infty e^{-\theta t} e^{-t^2} dt}.$$

That is, for given $c_T/(c_F - c_T)$ and T_S^* , we compute $\widehat{T}_O$ which satisfies

$$\int_0^{\widehat{T}_O} \overline{F}(t) dt + \frac{\overline{F}(\widehat{T}_O)}{\widetilde{Q}_0(\widehat{T}_O; \theta)} = \frac{c_F}{c_F - c_T} \frac{1}{h(T_S^*)},$$

and using $\widehat{T}_O$ and T_S^* , compute $\widehat{c}_{RO}$ which satisfies

$$(c_F - \widehat{c}_{RO})\widetilde{Q}_0(\widehat{T}_O; \theta) = (c_F - c_T)h(T_S^*).$$

Table 2.3 presents the replacement cost $\widehat{c}_{RO}$ when $C_S(T_S^*) = C_O(\widehat{T}_O)$. When $1/\theta = 0.5$, $1/\theta = 1.0$, and $c_T/(c_F - c_T) \leq 0.10$, there exists no $\widehat{c}_{RO}$, i.e., standard replacement is better than replacement overtime for any cost c_R . Table 2.3 indicates that $\widehat{c}_{RO}/c_T$ increases with $c_T/(c_F - c_T)$ from 0 to 1 and decreases with $1/\theta$. For

Table 2.3 Random replacement cost $\widehat{c}_{\text{RO}}$ when $C_S(T_S^*) = C_O(\widehat{T}_O)$

$\frac{c_T}{c_F - c_T}$	$1/\theta = 0.1$		$1/\theta = 0.5$		$1/\theta = 1.0$	
	$\widehat{c}_{\text{RO}}/c_F$	$\widehat{c}_{\text{RO}}/c_T$	$\widehat{c}_{\text{RO}}/c_F$	$\widehat{c}_{\text{RO}}/c_T$	$\widehat{c}_{\text{RO}}/c_F$	$\widehat{c}_{\text{RO}}/c_T$
0.01	0.0002	0.0240	–	–	–	–
0.03	0.0198	0.6804	–	–	–	–
0.05	0.0386	0.8105	–	–	–	–
0.07	0.0567	0.8660	–	–	–	–
0.10	0.0825	0.9073	–	–	–	–
0.30	0.2240	0.9708	0.1445	0.6262	0.0303	0.1312
0.50	0.3277	0.9831	0.2665	0.7944	0.1833	0.5498
0.70	0.4070	0.9883	0.3579	0.8691	0.2941	0.7143
1.00	0.4961	0.9921	0.4592	0.9183	0.4139	0.8277

example, when $1/\theta = 0.1$, $\widehat{c}_{\text{RO}}$ becomes 0 as $c_T \rightarrow 0$ and becomes equal to c_T as $c_T \rightarrow c_F$ (Problem 8 in Sect. 2.5). $\square$

Furthermore, when $c_T = c_R$ and $G(t) = 1 - e^{-\theta t}$, we compare replacement overtime with the other replacements: Comparing (2.26) with (2.48),

$$\begin{aligned}
& \widetilde{Q}_0(T; \theta) \int_0^T \overline{F}(t) dt - F(T) - h(T) \int_0^T e^{-\theta t} \overline{F}(t) dt + \int_0^T e^{-\theta t} dF(t) \\
&= [\widetilde{Q}_0(T; \theta) - h(T)] \int_0^T \overline{F}(t) dt + \int_0^T (1 - e^{-\theta t}) \overline{F}(t) [h(T) - h(t)] dt > 0,
\end{aligned}$$

and hence, $T_O^* < T_F^*$. In addition, from (2.27) and (2.49), and noting that $\widetilde{Q}_0(T; \theta) > h(T)$, if $\widetilde{Q}_0(T_O^*; \theta) < h(T_F^*)$, then replacement overtime is better than replacement first, and *vice versa*.

Comparing (2.39) with (2.48).

$$\begin{aligned}
& \widetilde{Q}_0(T; \theta) \int_0^T \overline{F}(t) dt - F(T) - h(T) \left[\int_0^T \overline{F}(t) dt + \int_T^\infty e^{-\theta t} \overline{F}(t) dt \right] \\
&+ 1 - \int_T^\infty (1 - e^{-\theta t}) dF(t) \\
&= [\widetilde{Q}_0(T; \theta) - h(T)] \int_0^T \overline{F}(t) dt + \int_T^\infty e^{-\theta t} \overline{F}(t) [h(t) - h(T)] dt > 0,
\end{aligned}$$

and hence, $T_O^* < T_L^*$. Thus, if $\tilde{Q}_0(T_O^*; \theta) < h(T_L^*)$, then replacement overtime is better than replacement last, and *vice versa* (Problem 9 in Sect. 2.5).

For $0 < \theta < \infty$ and $0 < T < \infty$, we can obtain the following inequalities, using (11) of Appendix A.1 (Problem 10 in Sect. 2.5):

$$\frac{\int_T^\infty t dF(t)}{\int_T^\infty t \bar{F}(t) dt} > \frac{\int_T^\infty e^{-\theta t} dF(t)}{\int_T^\infty e^{-\theta t} \bar{F}(t) dt} > h(T) > \frac{\int_0^T t dF(t)}{\int_0^T t \bar{F}(t) dt} > \frac{\int_0^T e^{-\theta t} dF(t)}{\int_0^T e^{-\theta t} \bar{F}(t) dt}. \quad (2.53)$$

Compared (2.3) with (2.48) and (2.50) when $c_R = c_T$, $T_U^* > T_S^* > T_O^*$, and $C_U(T_U^*) > C_S(T_S^*)$ and $C_O(T_O^*) > C_S(T_S^*)$. Thus, compared (2.49) with (2.51), if $\tilde{Q}_0(T_O^*; \theta) < Q_1(T_U^*)$ then replacement overtime is better than replacement with uniform random time, and *vice versa* (Problem 11 in Sect. 2.5).

2.4 Nth Working Time

It is assumed that Y_j ($j = 1, 2, \dots$) is the j th working time of a job in Fig. 2.2, and is independent and has an identical distribution $G(t) \equiv \Pr\{Y_j \leq t\}$ with finite mean $1/\theta$ ($0 < \theta < \infty$). That is, the unit works for a job with a renewal process according to a general distribution $G(t)$. Then, the probability that the unit works exactly j times in $[0, t]$ is $G^{(j)}(t) - G^{(j+1)}(t)$, where $G^{(j)}(t)$ ($j = 1, 2, \dots$) denotes the j -fold Stieltjes convolution of $G(t)$ with itself and $G^{(0)}(t) \equiv 1$ for $t \geq 0$. In addition, when $G^{(j)}(t)$ has a density function $g^{(j)}(t)$, i.e., $g^{(j)}(t) \equiv dG^{(j)}(t)/dt$, $r_j(t) \equiv g^{(j)}(t)/[1 - G^{(j)}(t)]$. Note that $r_j(t)dt$ represents the probability that the unit completes the j th work in $[t, t + dt]$, given that it operates for the j th number of working times at time t . For the above job with random working times, we take up the following two policies of replacement first and last.

2.4.1 Replacement First

Suppose that the unit is replaced before failure at a planned time T ($0 < T \leq \infty$) or at a planned number N ($N = 1, 2, \dots$) of working times, whichever occurs first. Then, the probability that the unit is replaced at time T is

$$\bar{F}(T)[1 - G^{(N)}(T)], \quad (2.54)$$

the probability that it is replaced at number N is

$$\int_0^T \bar{F}(t) dG^{(N)}(t), \quad (2.55)$$

and the probability that it is replaced at failure is

$$\int_0^T [1 - G^{(N)}(t)] dF(t), \quad (2.56)$$

where note that (2.54) + (2.55) + (2.56) = 1. Thus, the mean time to replacement is

$$\begin{aligned} & T \bar{F}(T) [1 - G^{(N)}(T)] + \int_0^T t \bar{F}(t) dG^{(N)}(t) + \int_0^T t [1 - G^{(N)}(t)] dF(t) \\ &= \int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt. \end{aligned} \quad (2.57)$$

Therefore, the expected cost rate is

$$C_F(T, N) = \frac{c_T + (c_F - c_T) \int_0^T [1 - G^{(N)}(t)] dF(t) + (c_R - c_T) \int_0^T \bar{F}(t) dG^{(N)}(t)}{\int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt}, \quad (2.58)$$

where c_F = replacement cost at failure, c_T = replacement cost at time T , and c_R = replacement cost at number N for any $N \geq 1$ with $c_F > c_T$ and $c_F > c_R$. By replacing $G(t)$ with $G^{(N)}(t)$ formally in (2.23), (2.58) is also obtained easily (Problem 12 in Sect. 2.5).

In particular, when the unit is replaced only at time T ,

$$C_S(T) \equiv \lim_{N \rightarrow \infty} C_F(T, N) = \frac{c_F - (c_F - c_T) \bar{F}(T)}{\int_0^T \bar{F}(t) dt}, \quad (2.59)$$

which agrees with (2.2), and when $N = 1$,

$$\begin{aligned} C_F(T) &\equiv C_F(T, 1) \\ &= \frac{c_T + (c_F - c_T) \int_0^T \bar{G}(t) dF(t) + (c_R - c_T) \int_0^T \bar{F}(t) dG(t)}{\int_0^T \bar{G}(t) \bar{F}(t) dt}, \end{aligned} \quad (2.60)$$

which agrees with (2.23).

Furthermore, when the unit is replaced only at number N ,

$$C_F(N) \equiv \lim_{T \rightarrow \infty} C_F(T, N) = \frac{c_F - (c_F - c_R) \int_0^\infty \bar{F}(t) dG^{(N)}(t)}{\int_0^\infty [1 - G^{(N)}(t)] \bar{F}(t) dt} \quad (N = 1, 2, \dots). \quad (2.61)$$

In particular, when each working time is constant at T_0 , i.e., $G(t) \equiv 0$ for $t < T_0$, 1 for $t \geq T_0$, $G^{(N)}(t) = 0$ for $t < NT_0$, 1 for $t \geq NT_0$, and hence, the expected cost rate in (2.61) is

$$C_F(N) = \frac{c_F - (c_F - c_R) \bar{F}(NT_0)}{\int_0^{NT_0} \bar{F}(t) dt}, \quad (2.62)$$

which agrees with the discrete age replacement in (9.1) of [8, p. 236].

2.4.1.1 Optimum N^*

We find an optimum N^* which minimizes $C_F(N)$ in (2.61). From the inequality $C_F(N+1) - C_F(N) \geq 0$,

$$Q_N \int_0^\infty [1 - G^{(N)}(t)] \bar{F}(t) dt - \int_0^\infty [1 - G^{(N)}(t)] dF(t) \geq \frac{c_R}{c_F - c_R}, \quad (2.63)$$

where $Q_N \equiv \lim_{T \rightarrow \infty} Q_N(T)$ and for $0 < T \leq \infty$,

$$Q_N(T) \equiv \frac{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)] dF(t)}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)] \bar{F}(t) dt}.$$

It is easily proved that if Q_N increases strictly to Q_∞ , then the left-hand side of (2.63) also increases strictly to $\mu Q_\infty - 1$. Thus, if $Q_\infty > c_F / [\mu(c_F - c_R)]$, then there exists a finite and unique minimum N^* ($1 \leq N^* < \infty$) which satisfies (2.63).

In particular, when $G(t) = 1 - e^{-\theta t}$, i.e., $G^{(N)}(t) = \sum_{j=N}^\infty [(\theta t)^j / j!] e^{-\theta t}$, from (3) of Appendix A.1,

$$Q_N(T; \theta) = \frac{\int_0^T (\theta t)^N e^{-\theta t} dF(t)}{\int_0^T (\theta t)^N e^{-\theta t} \bar{F}(t) dt} \leq h(T) \quad (2.64)$$

increases strictly with N to $h(T)$ and hence, Q_N increases strictly to $h(\infty)$. Therefore, if $h(\infty) > c_F / [\mu(c_F - c_R)]$, then there exists a finite and unique minimum N^* ($1 \leq N^* < \infty$) which satisfies (2.63).

2.4.1.2 Optimum T_F^* and N_F^*

We find optimum T_F^* and N_F^* which minimize $C_F(T, N)$ in (2.58). Differentiating $C_F(T, N)$ with respect to T and setting it equal to zero,

$$(c_F - c_T) \left\{ h(T) \int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt - \int_0^T [1 - G^{(N)}(t)] dF(t) \right\} \\ - (c_T - c_R) \left\{ r_N(T) \int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt - \int_0^T \bar{F}(t) dG^{(N)}(t) \right\} = c_T,$$

i.e.,

$$\int_0^T [1 - G^{(N)}(t)] \bar{F}(t) \{ (c_F - c_T)[h(T) - h(t)] \\ - (c_T - c_R)[r_N(T) - r_N(t)] \} dt = c_T, \quad (2.65)$$

where $r_N(t) \equiv g^{(N)}(t)/[1 - G^{(N)}(t)]$ ($N = 1, 2, \dots$) and $r_0(t) \equiv 0$. From the inequality $C_F(T, N+1) - C_F(T, N) \geq 0$,

$$\int_0^T [1 - G^{(N)}(t)] \bar{F}(t) \left\{ (c_F - c_T)[Q_N(T) - h(t)] \right. \\ \left. + (c_T - c_R) \left\{ \frac{\int_0^T \bar{F}(t) d[G^{(N)}(t) - G^{(N+1)}(t)]}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)] \bar{F}(t) dt} + r_N(t) \right\} \right\} dt \geq c_T. \quad (2.66)$$

In addition, substituting (2.65) for (2.66), (2.66) becomes

$$(c_F - c_T)[Q_N(T) - h(T)] \\ + (c_T - c_R) \left\{ \frac{\int_0^T \bar{F}(t) d[G^{(N)}(t) - G^{(N+1)}(t)]}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)] \bar{F}(t) dt} + r_N(T) \right\} \geq 0. \quad (2.67)$$

Thus, when $c_T \leq c_R$, there does not exist finite optimum N_F^* for $T > 0$ because $Q_N(T) \leq h(T)$, i.e., $N_F^* = \infty$. In this case, the unit should be replaced only at time T .

Next, assume that $G(t) = 1 - e^{-\theta t}$, i.e., $G^{(N)}(t) = \sum_{j=N}^{\infty} [(\theta t)^j / j!] e^{-\theta t}$, and $c_T > c_R$. Then, (2.67) is

$$(c_F - c_T)[Q_N(T; \theta) - h(T)] + (c_T - c_R) \left[Q_N(T; \theta) + r_N(T) + \frac{(\theta T)^N e^{-\theta T} \bar{F}(T)}{\int_0^T (\theta t)^N e^{-\theta t} \bar{F}(t) dt} \right] \geq 0, \quad (2.68)$$

where $Q_N(T; \theta)$ is given in (2.64) and

$$r_N(T) = \frac{\theta(\theta T)^{N-1}/(N-1)!}{\sum_{j=0}^{N-1} [(\theta T)^j/j!]} \quad (N = 1, 2, \dots).$$

Recalling that $Q_N(T; \theta)$ increases with N to $h(T)$ and $r_N(T)$ decreases with N to 0 from (1) and (3) of Appendix A.1, there exists a finite N_F^* ($1 \leq N_F^* < \infty$) which satisfies (2.68) for $T > 0$.

Furthermore, when $G(t) = 1 - e^{-\theta t}$, the left-hand side of (2.65) goes to

$$\int_0^\infty [1 - G^{(N)}(t)] \bar{F}(t) \{ (c_F - c_T)[h(\infty) - h(t)] - (c_T - c_R)[\theta - r_N(t)] \} dt, \quad (2.69)$$

as $T \rightarrow \infty$, because $r_N(T)$ increases with T to θ . Therefore, if (2.69) is greater than c_T , then there exists a finite T_F^* ($0 < T_F^* < \infty$) which satisfies (2.65). It can be clearly seen that if $h(\infty) = \infty$, then (2.69) becomes ∞ . In this case, the resulting cost rate is

$$C_F(T_F^*, N_F^*) = (c_F - c_T)h(T_F^*) - (c_T - c_R)r_{N_F^*}(T_F^*). \quad (2.70)$$

When $N_F^* = \infty$, $r_\infty(T) = 0$ and the expected cost rate is given in (2.27).

Example 2.6 (Replacement for Weibull failure time) Table 2.4 presents optimum N_F^* which minimizes $C_F(N)$ in (2.61), and T_F^* and N_F^* which minimize $C_F(T, N)$ in (2.58) when $G(t) = 1 - e^{-t}$ and $F(t) = 1 - \exp[-(t/10)^2]$. This indicates that optimum N_F^* given in (2.63) is constant for different c_T . Optimum T_F^* given in (2.65) increases, and N_F^* given in (2.68) decreases with c_T , however, is almost constant except for $N_F^* = \infty$, and becomes equal to N_F^* in (2.63) as c_T becomes larger. $\square$

2.4.2 Replacement Last

Suppose that the unit is replaced before failure at time T ($0 \leq T < \infty$) or at number N ($N = 0, 1, 2, \dots$), whichever occurs last. Then, the probability that the unit is replaced at time T is

$$\bar{F}(T)G^{(N)}(T), \quad (2.71)$$

the probability that it is replaced at number N is

Table 2.4 Optimum T_F^* and N_F^* when $G(t) = 1 - e^{-t}$, $F(t) = 1 - e^{-(t/10)^2}$, and $c_F = 100$

c_T	$c_R = 10$					$c_R = 15$				
	N_F^*	$C_F(N_F^*)$	T_F^*	N_F^*	$C_F(T_F^*, N_F^*)$	N_F^*	$C_F(N_F^*)$	T_F^*	N_F^*	$C_F(T_F^*, N_F^*)$
10	4	24.94	3.36	∞	6.06	5	34.98	3.36	∞	6.06
11	4	24.94	3.64	6	6.31	5	34.98	3.55	∞	6.32
12	4	24.94	4.05	5	6.50	5	34.98	3.74	∞	6.57
13	4	24.94	4.68	4	6.62	5	34.98	3.91	∞	6.81
14	4	24.94	5.14	4	6.69	5	34.98	4.09	∞	7.04
15	4	24.94	5.64	4	6.73	5	34.98	4.26	∞	7.25
16	4	24.94	6.17	4	6.77	5	34.98	4.49	8	7.45
17	4	24.94	6.73	4	6.79	5	34.98	4.92	6	7.61
18	4	24.94	7.32	4	6.80	5	34.98	5.26	6	7.72
19	4	24.94	7.95	4	6.81	5	34.98	5.96	5	7.79
20	4	24.94	8.60	4	6.82	5	34.98	6.45	5	7.83

$$\int_T^\infty \bar{F}(t) dG^{(N)}(t), \quad (2.72)$$

and the probability that it is replaced at failure is

$$F(T) + \int_T^\infty [1 - G^{(N)}(t)] dF(t) = 1 - \int_T^\infty G^{(N)}(t) dF(t), \quad (2.73)$$

where note that (2.71) + (2.72) + (2.73) = 1. Thus, the mean time to replacement is

$$\begin{aligned} T\bar{F}(T)G^{(N)}(T) + \int_T^\infty t\bar{F}(t)dG^{(N)}(t) + \int_0^T t dF(t) + \int_T^\infty t[1 - G^{(N)}(t)]dF(t) \\ = \int_0^T \bar{F}(t) dt + \int_T^\infty \bar{F}(t)[1 - G^{(N)}(t)] dt. \end{aligned} \quad (2.74)$$

Therefore, the expected cost rate is

$$C_L(T, N) = \frac{c_F - (c_F - c_T) \int_T^\infty G^{(N)}(t) dF(t) + (c_R - c_T) \int_T^\infty \bar{F}(t) dG^{(N)}(t)}{\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt}. \quad (2.75)$$

By replacing $G(t)$ with $G^{(N)}(t)$ formally in (2.37), (2.75) is also obtained easily (Problem 13 in Sect. 2.5). Compared to $C_F(T, N)$ in (2.58) when $c_T = c_R$, both numerator and denominator are larger than those in (2.58). In particular, when $N = 0$,

$C_L(T, 0)$ agrees with $C_S(T)$ in (2.24), and when $T = 0$, $C_L(0, N)$ agrees with $C_F(N)$ in (2.61).

We find optimum T_L^* and N_L^* which minimize $C_L(T, N)$ in (2.75). Differentiating $C_L(T, N)$ with respect to T and setting it equal to zero,

$$\begin{aligned} (c_F - c_T) & \left\{ h(T) \left[\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt \right] \right. \\ & \left. + \int_T^\infty G^{(N)}(t) dF(t) \right\} + (c_T - c_R) \left\{ \tilde{r}_N(T) \left[\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt \right] \right. \\ & \left. + \int_T^\infty \bar{F}(t) dG^{(N)}(t) \right\} = c_F, \end{aligned} \quad (2.76)$$

where $\tilde{r}_N(T) \equiv g^{(N)}(T)/G^{(N)}(T)$. From the inequality $C_L(T, N+1) - C_L(T, N) \geq 0$,

$$\begin{aligned} (c_F - c_T) & \left\{ \tilde{Q}_N(T) \left[\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt \right] \right. \\ & \left. + \int_T^\infty G^{(N)}(t) dF(t) \right\} + (c_T - c_R) \left\{ \tilde{Q}_N(T) \left[\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt \right] \right. \\ & \left. + \int_T^\infty \bar{F}(t) dG^{(N)}(t) \right\} \geq c_F. \end{aligned} \quad (2.77)$$

Furthermore, substituting (2.76) for (2.77), (2.77) becomes

$$(c_F - c_T)[\tilde{Q}_N(T) - h(T)] + (c_T - c_R)[\tilde{Q}_N(T) - \tilde{r}_N(T)] \geq 0, \quad (2.78)$$

where

$$\tilde{Q}_N(T) \equiv \frac{\int_T^\infty [G^{(N)}(t) - G^{(N+1)}(t)] dF(t)}{\int_T^\infty [G^{(N)}(t) - G^{(N+1)}(t)] \bar{F}(t) dt} \geq h(T).$$

Thus, if $c_R \geq c_T$ and

$$\tilde{r}_N(T) \geq \tilde{Q}_N(T), \quad (2.79)$$

then $C_L(T, N+1) - C_L(T, N) \geq 0$ for all $N \geq 0$ and $T > 0$, i.e., $N_L^* = 0$. In this case, the unit should be replaced only at time T .

Example 2.7 (Replacement for Weibull failure time) Table 2.5 presents optimum T_L^* and N_L^* which minimize $C_L(T, N)$ in (2.75) when $G(t) = 1 - e^{-t}$ and $F(t) = 1 - \exp[-(t/10)^2]$.

Table 2.5 Optimum T_L^* and N_L^* when $G(t) = 1 - e^{-t}$, $F(t) = 1 - e^{-(t/10)^2}$, and $c_F = 100$

c_T	$c_R = 10$			$c_R = 15$		
	T_L^*	N_L^*	$C_L(T_L^*, N_L^*)$	T_L^*	N_L^*	$C_L(T_L^*, N_L^*)$
10	3.36	0	6.06	3.37	0	6.07
11	3.55	0	6.32	3.56	0	6.33
12	3.70	1	6.57	3.73	0	6.58
13	3.72	2	6.79	3.92	0	6.81
14	0	4	6.82	4.09	0	7.04
15	0	4	6.82	4.26	0	7.25
16	0	4	6.82	4.44	0	7.45
17	0	4	6.82	4.59	1	7.64
18	0	4	6.82	4.68	2	7.82
19	0	4	6.82	0	5	7.91
20	0	4	6.82	0	5	7.91

This indicates that optimum T_L^* given in (2.76) and N_L^* given in (2.78) increase with c_T except $T_L^* = 0$, N_L^* is constant for $T_L^* = 0$. Clearly, $N_L^* = 0$ for $c_R \geq c_T$. Compared to Table 2.4, the expected cost $C_L(T_L^*, N_L^*)$ is a little greater than $C_F(T_F^*, N_F^*)$. $\square$

2.4.3 Replacement with Constant Time

Suppose that when $c_T = c_R < c_F$, a planned replacement time T ($0 < T < \infty$) is fixed. It would be estimated from the above discussions that if $T \leq T_S^*$ then the unit should be replaced only at time T . From (2.58), we find an optimum N_F^* of replacement first for a given T , which minimizes the expected cost rate

$$C_F(N; T) = \frac{c_T + (c_F - c_T) \int_0^T [1 - G^{(N)}(t)] dF(t)}{\int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt} \quad (N = 1, 2, \dots). \quad (2.80)$$

From the inequality $C_F(N+1; T) - C_F(N; T) \geq 0$, i.e., setting by $c_T = c_R$ in (2.66),

$$Q_N(T) \int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt - \int_0^T [1 - G^{(N)}(t)] dF(t) \geq \frac{c_T}{c_F - c_T}, \quad (2.81)$$

where $Q_N(T)$ is given in (2.63). From (2.3), (2.81) is rewritten as

$$\begin{aligned}
Q_N(T) &= \int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt - \int_0^T [1 - G^{(N)}(t)] dF(t) \\
&\geq h(T_S^*) \int_0^{T_S^*} \bar{F}(t) dt - F(T_S^*).
\end{aligned} \tag{2.82}$$

Recalling that when $G(t) = 1 - e^{-\theta t}$, $Q_N(T; \theta)$ in (2.64) increases strictly with N to $h(T)$, the left-hand side of (2.82) also increases strictly with N to

$$h(T) \int_0^T \bar{F}(t) dt - F(T).$$

Thus, if $T \leq T_S^*$, then there does not exist any N which satisfies (2.82), i.e., $N_F^* = \infty$. Conversely, if $T > T_S^*$, then there exists a finite and unique minimum N_F^* which satisfies (2.81) when $G(t) = 1 - e^{-\theta t}$.

Next, from (2.75), we find an optimum N_L^* of replacement last for a given T which minimizes the expected cost rate

$$C_L(N; T) = \frac{c_F - (c_F - c_T) \int_T^\infty G^{(N)}(t) dF(t)}{\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt} \quad (N = 0, 1, 2, \dots). \tag{2.83}$$

From the inequality $C_L(N+1; T) - C_L(N; T) \geq 0$, i.e., setting by $c_T = c_R$ in (2.77),

$$h(T) \left[\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt \right] + \int_T^\infty G^{(N)}(t) dF(t) \geq \frac{c_F}{c_F - c_T}. \tag{2.84}$$

From (2.3) to (2.84) is rewritten as

$$\begin{aligned}
h(T) &\left[\mu - \int_T^\infty \bar{F}(t) G^{(N)}(t) dt \right] + \int_T^\infty G^{(N)}(t) dF(t) \\
&\geq h(T_S^*) \int_0^{T_S^*} \bar{F}(t) dt + \bar{F}(T_S^*).
\end{aligned} \tag{2.85}$$

When $G(t) = 1 - e^{-\theta t}$, the left-hand side of (2.85) increases strictly with N from $\tilde{Q}_0(T; \theta) \int_0^T \bar{F}(t) dt + \bar{F}(T)$ to $\mu h(\infty)$ from (6) of Appendix A.1. Thus, if $T \geq T_S^*$ then $N_L^* = 0$, i.e., the unit is replaced only at time T . Conversely, if $T < T_S^*$ and $h(\infty) = \infty$, then there exists a finite and unique minimum N_L^* which satisfies (2.85) (Problem 14 in Sect. 2.5). This means that if $T > T_S^*$ then we should adopt replacement first, and conversely, if $T < T_S^*$ then we should adopt replacement last.

2.5 Problems

- 1 Show that $Q(t) \equiv c_1 F(t) + c_R \bar{F}(t)$ and $S(t) \equiv \int_0^t \bar{F}(u) du$.
- 2 Derive (2.9) and (2.11).
- 3 Prove that $\theta_1^* > \theta_2^*$.
- 4 Prove that T_F^* increases with θ from T_S^* .
- 5 Prove that T_L^* decreases with θ to T_S^* .
- 6 Prove that (2.42) + (2.43) + (2.44) = 1.
- 7 Show that if $h(\infty) > c_F/[\mu(c_F - c_R)]$ then there exists a finite and unique T_O^* which satisfies (2.48), and $C_O(T_O^*)$ is given in (2.49).
- 8 Compute $\hat{c}_{RO}/c_F$ and $\hat{c}_{RO}/c_T$ in Table 2.3.
- 9 Show that if $\bar{Q}_0(T_O^*; \theta) < h(T_F^*)$ and $\bar{Q}_0(T_O^*; \theta) < h(T_L^*)$, then replacement overtime is better than both replacement first and last.
- 10 Prove that

$$\frac{\int_0^T t^\alpha dF(t)}{\int_0^T t^\alpha \bar{F}(t) dt} \quad \text{and} \quad \frac{\int_T^\infty t^\alpha dF(t)}{\int_T^\infty t^\alpha \bar{F}(t) dt} \quad (2.86)$$

increases with α ($0 < \alpha < \infty$).

- 11 Give numerical examples of $C_O(T_O^*)$ and $C_U(T_U^*)$, and compare them.
- *12 When the unit is replaced at number N before time T , and it is replaced at the first completion of working time over time T before the N th time, obtain the expected cost rate and discuss an optimum policy.
- *13 When the unit is replaced at time T after the N th working time, and it is replaced at the first completion of working time over time T before the N th time, obtain the expected cost rate and discuss an optimum policy.
- 14 Make a numerical example of optimum N_F^* and N_L^* for a given T when $F(t) = 1 - \exp(-t^2)$.

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