Chapter 2
A Dance of Instruction with Construction in Mathematics Education

Norma Presmeg

Setting the Scene

Our field, our baby field that is brand new in comparison with the millennia for which mathematics has existed as a discipline, has seen some dramatic changes in its half-century of being a field in its own right, with its own journals and conferences. We have come a long way, even since the early 1980s, when "illuminative evaluation" (McCormick 1982) was slowly replacing or, initially at least, supplementing the psychometric experiments that used "subjects" (*people*) who were being taught mathematics. Before that period, in the old paradigm, no research that did not aim for objectivity by means of carefully controlled experiments and statistical analysis was considered scientific in our field. In connection with the research methods of this period, Krutetskii (1976) gave a pungent critique:

> It is hard to understand how theory or practice can be enriched by, for instance, the research of Kennedy, who computed, for 130 mathematically gifted adolescents, their scores on different kinds of test and studied the correlations between them, finding that in some cases it was significant and in others not. The process of solution did not interest the investigator. But what rich material could be provided by the process of mathematical thinking in 130 mathematically able adolescents! (p. 14)

Krutetskii's interview methods, in Soviet Russia, were in many ways a precursor to the qualitative methodologies that followed this early period. Slowly, the qualitative research paradigm gained credence. After all, we are dealing with human beings in their teaching and learning of mathematics, with all the complexities and uncertainties that that fact implies! Even Krutetskii (1976), aware as he was of individual differences, wrote of "perfect teaching methods" (p. 6), terminology that we might use more circumspectly today. With regard to useful and believable research (rather than reliable and valid experiments), initial crude attempts at quality control became strengthened. Thus, *triangulation* of various types (Stake 1995) was needed to ensure that research results and insights reported more than merely the researcher's opinions. We learned to go back and ask the mathematics teachers and their stu-

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dents whether they agreed with the results of our observations and interviews, in "member checks" that were a means of respondent validation. By the 1990s, such qualitative research was the prominent methodology, and it was in this climate that radical constructivism became the dominant theoretical framework for research in our field. Radical constructivism was salutary in its critique of the behaviorism that had preceded it. And this theoretical precedence leads me to the topic of this talk.

Construction and Instruction

I remember, in the early 1990s sitting on a stone seat in the garden of The Florida State University with Ernst von Glasersfeld and asking him about the status of *conventional* knowledge in mathematics education according to radical constructivism. It seemed obvious that attempts by teachers to give their students space to construct their ideas of mathematics in more personal ways (e.g., by discussion in groups) would lead to a kind of knowledge that could be more *meaningful* to learners in terms of their mathematical identities and ownership. It is *not* that some kinds of instruction lead to construction and others do not. What other ways of "appropriation of knowledge" (van Oers 2002) do we have than by construction? We are constructing even in the choice of what we make of a straightforward lecture as we sit and listen. We may listen, but what do we *hear*? It was concerns such as these, in part, that caused debates on whether or not radical constructivism was epistemological, and whether or not it made claims about the ontology of mathematical knowledge. Nell Noddings, in the 1990s, called it "post-epistemological" (Janvier 1996).

But to return to my conversation with Ernst von Glasersfeld in the garden, Ernst acknowledged that there are different kinds of knowledge, and that knowledge of conventions had a different status, belonging as it does to accidents of cultural historicity rather than to the logic of rational thinking. Even the ability to use conventional knowledge would entail construction by an individual; but telling by somebody who knows the convention (aurally or in written form) is required, simply because there is no logical necessity for this kind of knowledge, except perhaps in a historical sense. Why, for instance, do we have 360° in a complete revolution? 100 degrees would be much more convenient. Reporting on some of his work with Les Steffe, in one of his many publications during this period, von Glasersfeld (1994) gave a short synopsis of the radical constructivist position concerning early mathematics concepts such as number; and early mathematical learning is of particular relevance in this conference, although it is clear that mathematics learning between the poles of instruction and construction is an important topic at all levels.

The founders of theoretical edifices, such as von Glasersfeld, are thus aware of the contingencies and intricacies inherent in building theories. But Peirce (1992) had insight into what happens to such theories over time. He cast light on what he meant by continuity in his *law of mind*:

> Logical analysis applied to mental phenomena shows that there is but one law of mind, namely, that ideas tend to spread continuously and to affect certain others which stand to them in a peculiar relation of affectability. In this spreading they lose intensity, and especially the power of affecting others, but gain generality and become welded with other ideas. (Peirce 1992, p. 313)

Some followers of radical constructivism took the theory to be a prescription for instruction. The mantra became, "Teachers mustn't tell!" (I have an anecdote about a professor and her primary school mathematics education prospective teachers, who just smiled and moved on when her students decided in groups that doubling the length of a particular similar figure must, automatically, double the area.) It is to the credit of deep scholars in our field, such as Paul Cobb and Erna Yackel (e.g., Yackel and Cobb 1996) that they recognized even in the heyday of radical constructivism, that instruction has an indispensable role, and that there is a delicate blending of instruction and construction that is a fine-tuning of the teacher's craft. It is this blending that I am calling the *dance* of instruction with construction.

In an email conversation with Götz Krummheuer, it emerged that when we considered the metaphor of the dance in this regard, we were viewing different aspects of dance that had relevance. He was interested in the swirling motion as the dancers moved—and certainly there is movement if we are considering teachers and their pupils in interaction in a dynamic way that leads to deep contemplation of mathematical ideas and changes in cognition, ideally also with a positive affective component. I had been thinking more of dance involving canonical moves by people in interaction—although both aspects are relevant to instruction and construction in mathematics education. Within the set moves of a particular dance there is freedom, creativity, and vigor. Certainly, a dancer can decide to construct a different set of movements, and they may be harmonious and beautiful, but if they are too far from the set moves, that dancer cannot be considered to be doing that particular dance. As is the case with all metaphors, there are elements in which the source domain (in this case dance) resonates with the target domain (mathematics education), and this common structure constitutes the *ground* of the metaphor. But every metaphor also involves ways in which the source and target domains are different, and these constitute the *tension* of the metaphor (Presmeg 1997). The dance metaphor does not take into account that there is a knowledge differential between teachers and their students who are learning mathematics. Teachers know the conventions of reasoning and representation that are involved in the patterns of mathematical thinking: Students initially may not have this awareness. There is also thus a power differential involved. However, effective instruction can facilitate students' making of constructions that lie within the canons of mathematically accepted knowledge, and yet there is room for creativity and enjoyment. I present two examples of such instruction in the next sections.

An Example of the Dance

As an example of an effective dance, I would like to highlight the doctoral research of Andrejs Dunkels (1996) in Luleå in the north of Sweden, in the mid-1990s. But for the untimely and tragic death of Andrejs, it is likely that he would have been the very first mathematics *education* professor in Sweden, who was appointed at the University of Luleå in 2001. After establishing his credibility as a mathematician with publications in pure mathematics (which was a necessity in that academic climate), Andrejs set out to teach his section of an engineering calculus course in a way that was very different from the traditional lecture format. Of the 5 or 6 sections of the course, with students arranged in the sections according to their previous accomplishments, Andrejs chose a section for his research that was just one up from the bottom in the hierarchy (i.e., many of these students had experienced difficulty in mathematics courses previously). He collected baseline data, so that he could compare these data with the achievements of his class at the end of the course, using exploratory data analysis (EDA) as well as observations and interviews. Thus, the research design used mixed methods (quantitative and qualitative), prefiguring a balanced swing of the pendulum to methodologies that became more common in the 2000s.

How did Andrejs teach his class? Firstly, he arranged them in groups of four for ease of communication. Secondly, he told them in advance what would be the mathematical topic of a particular class session, and he expected them to read and try to make sense of the relevant material in the textbook of the course. Thirdly, they were expected to come to the session prepared to talk about their current constructions. Finally, in the session, he circulated among the groups, listened to their conversations, and answered their questions although not always directly; he sometimes answered a question by posing another question. He sometimes pointed the group in directions they had not considered—with suggestions, not as the all-knowing teacher, and without taking away their ownership and agency. He had instinctively mastered the difficult and delicate dance of instruction with construction.

At the end of the course, the statistical EDA revealed that his students had improved their accomplishments so significantly that their section was now almost at the top of the hierarchy, second only to one other section. But even more convincingly, the analysis of data from interviews with students showed that the *quality* of the mathematical knowledge the students had constructed had improved immeasurably. There was no longer memorization of *rules without reasons*; they knew *why* the rules worked, and above all, they experienced greater enjoyment of the mathematical content, and more self-confidence than previously. This doctoral research study thus provided convincing evidence, both quantitative and qualitative, of the efficacy of balancing instruction with construction in mathematics education.

The Purported "epistemological paradox"

An issue that is relevant at this point is the oft-quoted paradox of instruction and construction (e.g., Simon 1995) that students can actively work only with what they have *already* constructed: How then is new knowledge possible? I shall argue shortly that there really is no paradox; the seeming paradox hinges on a false dichotomy. However, let me first give an example of a related phenomenon from my own research on ethnomathematics. I asked students in a masters-level course in mathematics education to take a personally meaningful cultural activity, and to construct mathematics from it. I gave examples from ethnomathematics literature and my own experiences to show them how to use several steps of semiotic chaining (Presmeg 2006a) to build connections between a cultural activity and mathematical ideas suitable for teaching at some level in a mathematics classroom. The process is akin in many ways to the horizontal mathematization, followed by vertical mathematization, used by the Freudenthal group (e.g., Treffers 1993; Gravemeijer 1994) in *Realistic Mathematics Education* (RME). The students in my course took ownership of the project, and the activities they chose were diverse and personally meaningful to them. However, it was evident that the mathematical ideas that students recognized in their chosen cultural activities depended heavily on what mathematics they already knew. For example, Vivienne, a primary school teacher, did not recognize the hyperbola that resulted when she analyzed the gear ratios and distances traveled by her mountain bicycle: Vivienne called the graph “a nice curve.” In contrast, David constructed a “dihedral group of order 4” when he analyzed the symmetries of a tennis court: He was a teacher of college-level number theory. And in the data there were many more examples of this phenomenon. How then might teachers use the connections of horizontal mathematization to facilitate students’ construction of *new* mathematical ideas? This question might be particularly vexing for a teacher who feels under pressure to ‘cover’ the topics listed in a mathematics syllabus.

I can do no more here (the topic has been addressed in several papers or book chapters, e.g., Presmeg 1998, 2007) than to report that the ethnomathematics course had the effect of broadening participants’ beliefs about the *nature of mathematics*, which was no longer seen as a “bunch of rules to be memorized” (initial student characterization of what mathematics is), with or without understanding. Many students expressed in reflective journals that after the course they saw mathematics as inherent in patterns and regularities that they could identify also in their daily lives and activities. This change of beliefs prefigures what Tony Brown (2011) is accomplishing in his “weekly session centred on broadening the students’ perceptions of mathematics and of how mathematics might be taught” (p. 18). Brown does not use the conceptual framework of semiotics, but the contemporary theoretical lenses of Zizek and Badiou, in his work, but the aim of his teaching resonates with a dance of instruction with construction.

To return to the so-called *learning paradox,* as I hinted, there really is no paradox at all if mathematics education is reconceptualized as a dance of construction with instruction. The crux of the matter is the relationship between the constructions made by an individual, and the broader societal context, the culture in which established mathematical ideas reside: These might be characterized as Karl Popper’s (1974, 1983) worlds 2 and 3, respectively. Radford (2012) has trenchantly pointed out that the seeming dilemma results from what he calls the “antinomies” in epistemological views that we have accepted: “Unfortunately, we have become used to thinking that either students construct their own knowledge or knowledge is imposed upon them” (p. 4). As he points out, this conception is a misleading oversimplification. Radford poses the paradox in terms of *emancipation* in mathematics education rather than in terms of construction, but the ideas are relevant to both. He points out that the antinomies reside in two epistemological ideas: “First, knowledge is something that subjects *make*. Second, the making of knowledge must be carried out free from authority” (p. 102, italics in original). What is problematic is the relationship between freedom and truth. Radford points out convincingly that the paradox results from “a subjectivist view of the world espoused by modernity (a world thought of as made and known through the individual’s deeds) and the cultural regimes of reason and truth that precede the individual’s own activity” (p. 104). Although Radford does not cast it in these terms, it is the mistaken notion that Popper’s worlds 2 and 3 are colliding. But all individual constructions (world 2) are made in the *context* of a cultural milieu (world 3). This relationship is inescapable. Seen in this light, the paradox disappears, and this relationship has its practical manifestation in a delicate blending of freedom and truth, a dance of instruction with construction. It is not necessary for the teacher’s role to conform to an irreducible and contradictory dichotomy of “the sage on the stage” versus “the guide on the side,” because elements of both these metaphors are evident in the dance, as the following example illustrates.

Blending Popper’s Worlds in the Teaching of Trigonometry

I would like to present here an instance of teaching high school trigonometry that uses the dance of instruction and construction to the fullest, thereby—at least in some measure—resolving the apparent paradox suggested in the previous section.

Sue Brown (2005) carried out a powerful dissertation study in which she analyzed high school students’ understanding of connections among trigonometric definitions (particularly of sine and cosine) that move from right triangles to the coordinate plane and unit circle, and then to definitions that establish sine and cosine as functions. Following this research (which involved quantitative as well as qualitative methods), she and I set out to examine further, pedagogy that might facilitate the students’ constructions of such connections in trigonometry. In this postdoctoral phase, I served as the researcher in Sue’s trigonometry class in the spring of 2006, in Chicago, USA. The research question was as follows: *How may teaching facilitate students’ construction of connections among registers in learning the basic concepts of trigonometry?* The main goal in Sue’s trigonometry class was to foster skill in converting among signs as students build up comprehensive knowledge of trigonometry concepts.

The methodology of this teaching experiment included cycles of joint reflection based on interviews with students, followed by further teaching. Early in our collaboration, Sue listed ways in which she tried to facilitate connected knowledge in her class—actions that were confirmed in my observations of her lessons, and in documents such as tests and quizzes. In the analysis of data, her list was compared with the connections constructed—or the lack of connections—by four students in a series of six interviews conducted with each student at intervals during the semester. The four students were purposively chosen by the teacher in collaboration with the researcher to ensure a range of learning styles and proficiency.

Some of Sue's facilitative principles that have the intent of helping students to move freely and flexibly among trigonometric registers are summarized as follows:

- Connecting old knowledge with new, starting with the "big ideas," providing contexts that demand the use of trigonometry, allowing ample time, and moving into complexity slowly
- Connecting visual and nonvisual registers, e.g., numerical, algebraic, and graphical signs, and requiring or encouraging students to make these connections in their classwork, homework, tests, and quizzes
- Supplementing problems with templates that make it easy for students to draw and use a sketch, or asking students to interpret diagrams that are given
- Providing contextual ("real world") signs that have an iconic relationship with trigonometric principles, e.g., a model of a boom crane that rotates through an angle θ, $0^{o} < \theta < 180°$, on a half plane
- Providing memorable summaries in diagram form, which have the potential of becoming for the students prototypical images of trigonometric objects, because these inscriptions are sign vehicles for these objects
- Providing or requiring students to construct static or dynamic computer simulations of trigonometric principles and their connections, in many cases giving a sense of physical motion; and
- Using metaphors that are sometimes based on the students' contextual experiences, e.g., a bow tie and the boom crane, for trigonometric ratios in the unit circle.

An analysis of the complete corpus of data in terms of Sue's full list (abridged here) assessed the effectiveness of these principles in accomplishing their goal, at least for the four students who were interviewed (Presmeg 2006b). On the surface, Sue's list appears to relate to the *instruction* pole of the dance; however, it was her long experience of students' *constructions*—informed also by her intensive doctoral research—that formed the foundation for her principles of instruction in this list. And many instances were present of ways that Sue incorporated idiosyncratic constructions of students in her teaching. An example of this inclusion is the bow tie metaphor, which was introduced in class by Sue, but originated in interviews with students in a task in which they were finding the sine of angles in the second and third quadrants. Sue's pedagogy provides an illustration of principles that alternate flexibly and sensitively between instruction and construction in learning trigonometry.

Some Conclusions

In this introduction to the topic of *a mathematics education Perspective On Early Mathematics learning between the poles of instruction and construction (POEM)*, I have introduced a brief overview of the way our field has moved from a behaviorist emphasis on instruction, to an opposite concern with pupils' constructions, and further to the realization that instruction and construction can mutually constitute each other in a fine-tuning awareness that I have called a dance. Other writers have used different terminology, although the ideas resonate with the notions of construction and instruction: Hewitt (2012) makes the distinction between arbitrary and necessary knowledge, which he characterizes as knowledge that has the function of assisting memory and knowledge that is necessary in educating awareness of the accepted canons of a discipline, respectively. In any case, learning mathematics involves not only becoming aware of conventions and standards of the mathematics that has been accepted as such through the ages but also making sense of the logic of these canons in a personally and individually meaningful way.

I tried to initiate conversations on the topic with reference to two examples: one in a university-level calculus class and the other in a high school trigonometry class. I look forward to examples our colleagues will present in early childhood teaching and learning of mathematics. But I hope the cases presented here exemplify my belief that the topic is important at all levels of learning mathematics and that attention to this topic is required at both theoretical and empirical levels, the former, for example, with regard to the so-called paradoxes of our field and the latter in the day-to-day lives of teachers and students.

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