

Chapter 2

Equilibrium

2.1 Equilibrium of Particle Systems

2.1.1 Introduction

It is well known that, in accordance with Newton's laws of motion, the effect of forces on bodies is to produce accelerated motion. For a body to remain at rest, then, it is necessary for the system of forces acting on it to be statically equivalent to the absence of forces. In this case, we say that the body is in equilibrium.

We will begin the study of equilibrium by focusing on a single particle. Then, we will proceed to consider systems of such particles, eventually going to the limit of such a system becoming a continuous body. The concept of work, both *actual* and *virtual*, will be introduced along the way.

By way of background, we recall the three laws of motion postulated by Sir Isaac Newton in his monumental work first published in 1687 under the title *Philosophiae Naturalis Principia Mathematica* (usually referred to simply as *Principia*). As we will establish shortly, Newton's laws are sufficient to describe the motion of particles, but need to be generalized to describe the motion of more general bodies.

Newton's First Law states that every particle remains in a state of rest or motion with constant velocity unless an external force acts on it. *Newton's Second Law* states that a particle of mass m acted upon by a total external force $\mathbf{F}$ undergoes change in its velocity $\mathbf{v}$ according to

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) . \quad (2.1)$$

Assuming, in addition, that the mass m is constant, it follows from (2.1) that

$$\mathbf{F} = m\mathbf{a} , \quad (2.2)$$

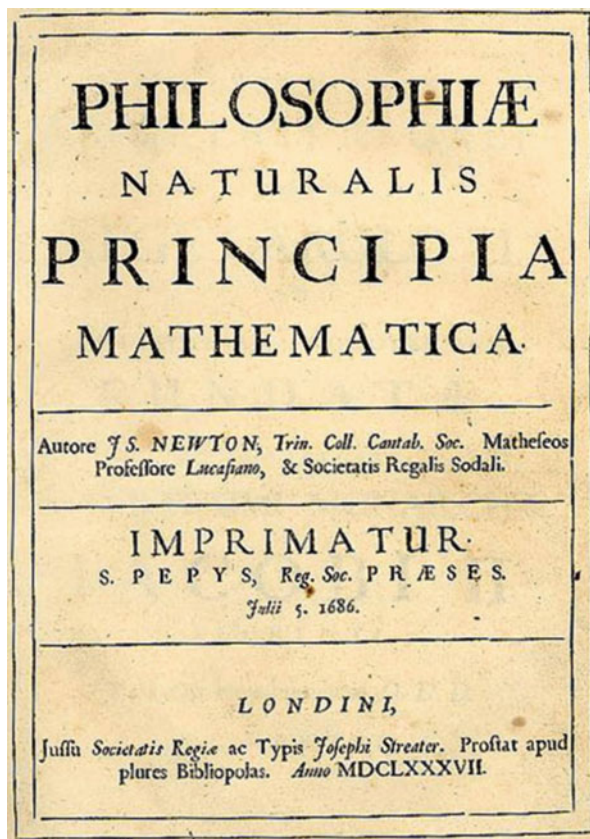


Figure 2.1. Title page of the *Principia*

where $\mathbf{a}$ is the acceleration of the particle. Equation (2.2) implies that a particle is at rest or moving at a uniform velocity if the resultant force on it is zero. This shows that the First Law may be regarded as a special case of the Second Law, as long as the mass of the particle is constant. *Newton's Third Law* states that the force $\mathbf{F}_{ij}$ acting on particle i due to its interaction with particle j is equal and opposite to the force $\mathbf{F}_{ji}$ acting on particle j due to its interaction with particle i , namely

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}. \quad (2.3)$$

Newton's Third Law is often referred to as the *action–reaction law*.

2.1.2 Particle Systems

Let us now consider a discrete system of N particles, each labeled with a subscript i ($i = 1, \dots, N$). Let the resultant force on the i -th particle be denoted $\mathbf{F}_i$. This force can be thought of as the vector sum of the forces exerted on this particle by each of the other particles in the system, $\mathbf{F}_{ij}$ ($j \neq i$), and the

external force $\mathbf{F}_{ie}$, as in Fig. 2.2. The external force $\mathbf{F}_{ie}$ is defined as the resultant of all forces acting on the particle i due to its interaction with all entities other than the other particles in the system. The preceding statement is expressed in mathematical terms as

$$\mathbf{F}_i = \mathbf{F}_{ie} + \sum_{j \neq i} \mathbf{F}_{ij}, \quad (2.4)$$

where

$$\sum_{j \neq i} = \sum_{j=1}^{i-1} + \sum_{j=i+1}^N \quad (2.5)$$

with the understanding that, when $i = 1$ (or, $i = N$), the first (or, second) term on the right-hand side of (2.5) vanishes. If a particle i is in equilibrium, so that $\mathbf{F}_i = \mathbf{0}$, then Eq. (2.4) implies that

$$\mathbf{F}_{ie} + \sum_{j \neq i} \mathbf{F}_{ij} = \mathbf{0}. \quad (2.6)$$

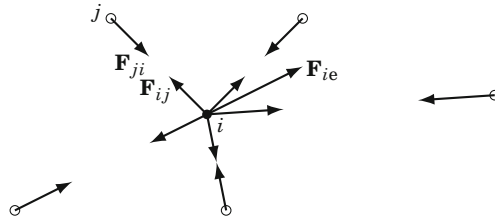


Figure 2.2. Forces on particle i belonging to a system of particles

If *each* particle i is in equilibrium, then, of course, the sum of the resultant forces acting on the whole system of particles is zero, namely

$$\sum_{i=1}^N \mathbf{F}_i = \mathbf{0}. \quad (2.7)$$

Appealing to (2.4), this sum is given by

$$\sum_{i=1}^N \mathbf{F}_i = \sum_{i=1}^N \mathbf{F}_{ie} + \sum_{i=1}^N \sum_{j \neq i} \mathbf{F}_{ij}. \quad (2.8)$$

But the double-sum term in (2.8) can be expressed with the aid of (2.5) as

$$\sum_{i=1}^N \sum_{j \neq i} \mathbf{F}_{ij} = \sum_{i=1}^N \sum_{j=1}^{i-1} \mathbf{F}_{ij} + \sum_{i=1}^N \sum_{j=i+1}^N \mathbf{F}_{ij}. \quad (2.9)$$

Further, the second double sum on the right-hand side of (2.9) can be rewritten, by interchanging the indices i and j , as $\sum_{j=1}^N \sum_{i=j+1}^N \mathbf{F}_{ji}$ or, equivalently, as

$\sum_{i=1}^N \sum_{j=1}^{i-1} \mathbf{F}_{ji}$, since either way it represents the summation over all particle pairs (i, j) such that $i > j$. Consequently, Eq. (2.9) may be rewritten as

$$\sum_{i=1}^N \sum_{j \neq i} \mathbf{F}_{ij} = \sum_{i=1}^N \sum_{j=1}^{i-1} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) . \quad (2.10)$$

In view of Newton's Third Law, as stated in Eq. (2.3), it follows that each term in parenthesis on the right-hand side of (2.10) vanishes, hence $\sum_{i=1}^N \sum_{j \neq i} \mathbf{F}_{ij} = \mathbf{0}$.

This, in turn, implies that Eq. (2.8) reduces to

$$\sum_{i=1}^N \mathbf{F}_i = \sum_{i=1}^N \mathbf{F}_{ie} . \quad (2.11)$$

Finally, then, a necessary condition for a system of particles to be in equilibrium is deduced from (2.7) and (2.11) as

$$\sum_{i=1}^N \mathbf{F}_{ie} = \mathbf{0} , \quad (2.12)$$

that is, the vector sum of the *external* forces on the system is zero.

Another property of forces due to the interaction between two particles is that they are typically directed along the line joining the particles, that is, they are *central*^{*} (see Fig. 2.3). Now, if the position vector of particle i (with respect to an arbitrary origin O) is $\mathbf{r}_i$, then the vector from particle i to particle j is $\mathbf{r}_j - \mathbf{r}_i$, and since the cross product of two parallel vectors is zero, it follows that $(\mathbf{r}_j - \mathbf{r}_i) \times \mathbf{F}_{ij} = \mathbf{0}$, as in Fig. 2.3. Now, Eq. (2.4) implies that

$$\sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_{ie} + \sum_{i=1}^N \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{ij} . \quad (2.13)$$

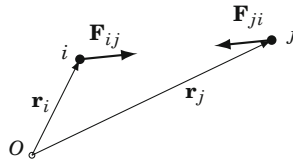


Figure 2.3. Central forces acting on particles i and j

^{*}Non-central forces are encountered in electromagnetism and in certain complex particle interactions which involve more than two particles at a time.

Using a simple manipulation, as earlier, the double sum on the right-hand side of (2.13) can again be rewritten as $\sum_{i=1}^N \sum_{j=1}^{i-1} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \mathbf{0}$, as in Exercise 2.1-3. Consequently, another necessary condition for the equilibrium of a particle system is that

$$\sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_{ie} = \mathbf{0}, \quad (2.14)$$

namely that the vector sum of the moments of the external forces on the system is zero.

As already stated, Eqs. (2.12) and (2.14) are *necessary* conditions for the system of particles to be in equilibrium. It is important to recognize here that for a system of particles to be in equilibrium, every subset of particles from the system must also be in equilibrium. If all such subsystems are in equilibrium, then the whole system is also in equilibrium. Therefore, a *sufficient* condition for the system of N particles to be in equilibrium is that Eqs. (2.12) and (2.14) hold for all subsystems comprising M particles ($M < N$) from the original system. For each particle i that belongs to such a subsystem, it should be understood that all forces acting on it due to its interaction with the $N - M$ particles that do not belong to the system must be treated as external forces. By the same token, interaction forces between particles in the subsystem are *internal forces* to the subsystem. Therefore, the distinction between internal and external forces depends crucially on the definition of the system under consideration.

Let us now consider a particle i that occupies a point defined by the vector $\mathbf{r}_i$ drawn to it from a fixed point O . If the particle changes its position such that it now occupies a new point given by vector $\mathbf{r}'_i$, then the displacement vector

$$\mathbf{u} = \mathbf{r}'_i - \mathbf{r}_i \quad (2.15)$$

is the change in position (or placement) of the particle, as in Fig. 2.4.

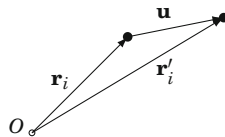


Figure 2.4. Displacement of a particle j

A particle system is *rigid* if the distances between all pairs (i, j) of particles remain unchanged. This means, in mathematical terms, that

$$(\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) = (\mathbf{r}'_i - \mathbf{r}'_j) \cdot (\mathbf{r}'_i - \mathbf{r}'_j), \quad (2.16)$$

where particles i, j occupy points corresponding to vectors $\mathbf{r}_i, \mathbf{r}_j$ and $\mathbf{r}'_i, \mathbf{r}'_j$ at two instances. Invoking (2.15), the condition (2.16) may be restated as

$$(\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) = [(\mathbf{r}_i + \mathbf{u}_i) - (\mathbf{r}_j + \mathbf{u}_j)] \cdot [(\mathbf{r}_i + \mathbf{u}_i) - (\mathbf{r}_j + \mathbf{u}_j)] . \quad (2.17)$$

In a later section, it will be argued how the equilibrium equations (2.12) and (2.14) for rigid particle systems can be used to derive corresponding equilibrium equations for rigid bodies.

2.1.3 An Extension to Continuous Bodies

In reality, any physical body is ultimately a system of particles (atoms or molecules), though these are not necessarily governed by classical Newtonian mechanics. In practice, however, it is convenient to treat bodies as if they were continua, allowing the use of integral and differential calculus in place of algebraic operations over huge numbers of particles. In this case, the results we have just derived for particle systems are applicable to continuous bodies by merely attaching the labels i ($i = 1, \dots, N$) to those points of the body at which forces are applied. We let $\mathbf{F}_i$ now denote the *external force* acting there (see Fig. 2.5); then Eqs. (2.12) and (2.14) reduce, respectively, to

$$\sum_{i=1}^N \mathbf{F}_i = \mathbf{0} \quad (2.18)$$

and

$$\sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0} . \quad (2.19)$$

Equation (2.19) may be appropriately modified if concentrated couples act on the body.

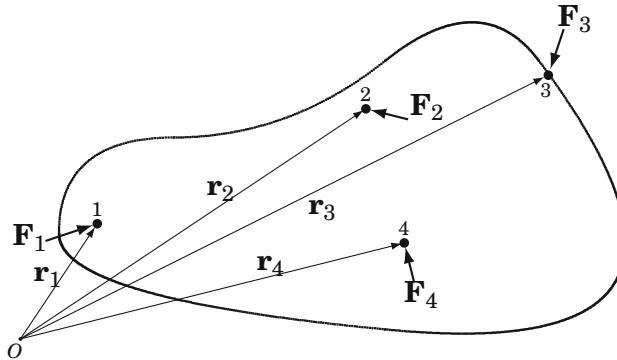


Figure 2.5. Discrete forces on a continuous body

In the case of forces that are not applied at discrete points but are distributed over volume (such as gravity) or area (such as pressure), the sums can be replaced by integrals, and, once the resultant of such a force distribution is found, it can be applied (instead of the distributed force) as a discrete force at an appropriate point of the body, as discussed in Sect. 1.5. Thus, for example, the gravitational force that is distributed throughout the body can be replaced by the body's weight acting at the center of mass. Similarly, any combination of discrete forces can be replaced by its resultant. Such a replacement does not affect the equilibrium of the body.

Equations (2.18) and (2.19) are often referred to as the equilibrium form of *Euler's laws*, because Euler[†] was the first to recognize the independence of force and moment equilibrium in continuous bodies.

2.1.4 Work and Power

If the point of application of a force $\mathbf{F}$ (whether acting on a particle or a point of a continuous body) moves from an initial position given by $\mathbf{r}$ to one given



Figure 2.6. Leonhard Euler

by $\mathbf{r} + d\mathbf{r}$ (where $d\mathbf{r}$ is an infinitesimal displacement), then the *infinitesimal work* done by the force is $dW = \mathbf{F} \cdot d\mathbf{r}$. If the final position of the point is $\mathbf{r}'$, then the *work* done by the force is

$$W = \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{F} \cdot d\mathbf{r}, \quad (2.20)$$

as shown in Fig. 2.7. Generally, the force $\mathbf{F}$ need not remain constant throughout the displacement of the point, but if it does, then the work in Eq. (2.20)

[†]Leonhard Euler (1707–1783) was a Swiss mathematician and physicist.

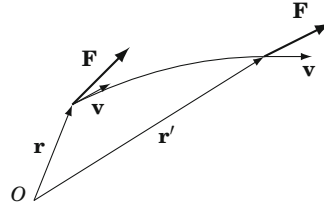


Figure 2.7. Work done by a force acting on a point

is just $\mathbf{F} \cdot (\mathbf{r}' - \mathbf{r})$. Work, as defined here, is also called *actual work* in order to distinguish it from *virtual work*, to be defined shortly.

If the point of application of the force $\mathbf{F}$ is moving with velocity $\mathbf{v}$, then the infinitesimal displacement $d\mathbf{r}$ during a time increment dt is $\mathbf{v}dt$, and the infinitesimal work done is $dW = \mathbf{F} \cdot \mathbf{v}dt$. The rate of work, or *power*, is therefore

$$\Pi = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}. \quad (2.21)$$

If the force and velocity vectors are parallel, with components F and v , respectively, along their common axis, then the power is just Fv .

The definition of power in (2.21) is frequently used to identify a force (or force-like) quantity and a displacement (or displacement-like) quantity as *conjugate*. In this sense, a force is conjugate to the displacement of the point on which it acts. If the body's motion is limited to rotation about an axis, then, in the course of a rotation by an infinitesimal angle $d\theta$, the displacement of any point in the body has the scalar value $r d\theta$ (where r is the distance of the point from the axis of rotation), while its direction is perpendicular to both the axis and the line from the point to the axis, as shown in Fig. 2.8. Now, if a force $\mathbf{F}$ is acting at the point, then the component of the force that is parallel to that displacement, multiplied by r , is just the moment M of the force about the axis. Consequently, $dW = M d\theta$ and, if the angular velocity (in radians

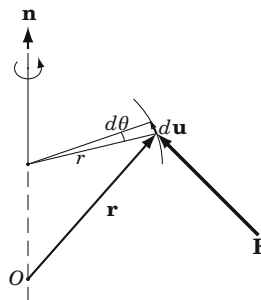


Figure 2.8. Force at a point in a body rotating about an axis

per unit time) is $\omega = d\theta/dt$, then the power is $M\omega$. In this case, we say that the moment M is conjugate to the angle of rotation θ . To obtain this result in terms of vectors, we let $\mathbf{n}$ be the unit vector along the axis such that the rotation about it follows the right-hand rule. If $\mathbf{r}$ is the radius vector to a given point from a point O on the axis, then the velocity of motion of the given point is $\mathbf{v} = \omega\mathbf{n} \times \mathbf{r}$, and the power is accordingly

$$\Pi = \mathbf{F} \cdot \mathbf{v} = \mathbf{F} \cdot \omega\mathbf{n} \times \mathbf{r} = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})\omega = M\omega, \quad (2.22)$$

where $M = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})$ is the component parallel to the axis of the moment vector (about O) of the force $\mathbf{F}$.

2.1.5 Virtual Work

Another way of expressing the equilibrium of a particle system—and by extension of any body—is by means of the *principle of virtual work*, which is introduced in this section.

Consider a system of particles that are subject to *constraints* in the sense that they are not free to displace arbitrarily, either with respect to one another (*internal constraints*) or with respect to a fixed base (*external constraints*). Simple examples are shown in Fig. 2.9. Specifically, Fig. 2.9a shows an internal constraint in the form of a rigid link connecting two particles, so that their movement must be such that the distance between them remains constant and equal to d . (That is, they form a rigid particle system, as previously defined.) Figure 2.9b shows the contours of a surface on which the particle must remain as it moves, representing an external constraint.

A displacement of the i -th particle that does not violate the constraints is called a *virtual[‡] displacement*; it may be written as $\mathbf{r}_i^* - \mathbf{r}_i$, where $\mathbf{r}_i$ is the current position vector and $\mathbf{r}_i^*$ is any other position vector that does not violate the constraints. If, for every particle, $\mathbf{r}_i^*$ is *close* to $\mathbf{r}_i$, then the virtual displacement may be regarded as infinitesimal and is usually denoted $\delta\mathbf{r}_i$. It



Figure 2.9. Simple constraints: (a) internal (b) external

[‡]The word *virtual* here means “possible,” and a virtual displacement is thus a possible displacement but not necessarily the actual one.

follows from its definition that the δ operator is distributive: we can write $\delta(\mathbf{r}_i - \mathbf{r}_j)$ for $\delta\mathbf{r}_i - \delta\mathbf{r}_j$, since $(\mathbf{r}_i^* - \mathbf{r}_i) - (\mathbf{r}_j^* - \mathbf{r}_j) = (\mathbf{r}_i^* - \mathbf{r}_j^*) - (\mathbf{r}_i - \mathbf{r}_j)$.

As we argued above, a particle system is in equilibrium if and only if every subsystem thereof is in equilibrium, and therefore, if and only if each particle is in equilibrium. Thus, if the total force acting on the i -th particle is given by Eq. (2.4), and the particle is in equilibrium, then Eq. (2.6) implies that the *virtual work* on the particle (that is, the work done on it in the course of a virtual displacement), denoted δW_i , is also zero:

$$\delta W_i = \mathbf{F}_i \cdot \delta \mathbf{r}_i = \left(\mathbf{F}_{ie} + \sum_{j \neq i} \mathbf{F}_{ij} \right) \cdot \delta \mathbf{r}_i = 0. \quad (2.23)$$

The total virtual work on the particle system is $\delta W = \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i$ and is also zero, that is,

$$\delta W = \sum_i^N \left[\left(\mathbf{F}_{ie} + \sum_{j \neq i} \mathbf{F}_{ij} \right) \cdot \delta \mathbf{r}_i \right] = 0. \quad (2.24)$$

The double sum involving the $\mathbf{F}_{ij}$ can now be manipulated similarly to what was done in the derivation of Eqs. (2.12) and (2.14) and can accordingly be rewritten as

$$\sum_{i=1}^N \sum_{j=1}^{i-1} \mathbf{F}_{ij} \cdot \delta(\mathbf{r}_i - \mathbf{r}_j). \quad (2.25)$$

The virtual work on the particle system can be therefore expressed as

$$\delta W = \delta W_{\text{ext}} + \delta W_{\text{int}}^*, \quad (2.26)$$

where the *external virtual work* is

$$\delta W_{\text{ext}} = \sum_i^N \mathbf{F}_{ie} \cdot \delta \mathbf{r}_i \quad (2.27)$$

and the *internal virtual work* is

$$\delta W_{\text{int}}^* = \sum_{i=1}^N \sum_{j=1}^{i-1} \mathbf{F}_{ij} \cdot \delta(\mathbf{r}_i - \mathbf{r}_j). \quad (2.28)$$

The reason for the asterisk in the designation δW_{int}^* is that in solid mechanics the convention is to define the internal virtual work as

$$\delta W_{\text{int}} = \sum_{i=1}^N \sum_{j=1}^{i-1} \mathbf{F}_{ji} \cdot \delta(\mathbf{r}_i - \mathbf{r}_j), \quad (2.29)$$

that is (since $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$), $\delta W_{\text{int}} = -\delta W_{\text{int}}^*$, and therefore the principle of virtual work takes the form

$$\delta W_{\text{ext}} = \delta W_{\text{int}} , \quad (2.30)$$

which will be used in later chapters. Equation (2.30) states that for a system of particles that is in equilibrium, the virtual work done by the external forces equals the virtual work done by the internal forces.

For continuous bodies, in view of the extension discussed in Sect. 2.1.3, the equation for the external virtual work is

$$\delta W_{\text{ext}} = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i . \quad (2.31)$$

The internal work δW_{int} depends on internal forces in the continuous body, which will be discussed in Chap. 4.

Example 2.1.1 (Rigid particle system):

The rigidity of a particle system, as we have defined it earlier in this section, implies that $|\mathbf{r}_i - \mathbf{r}_j|$, for any pair of particles i and j , does not change. In other words, it is an internal constraint requiring that the virtual displacements satisfy $(\mathbf{r}_i - \mathbf{r}_j) \cdot \delta(\mathbf{r}_i - \mathbf{r}_j) = 0$. But since the interparticle forces $\mathbf{F}_{ij}$ are parallel to $\mathbf{r}_i - \mathbf{r}_j$, it follows that each member of the double sum in Eq. (2.28) must be zero, and hence in a rigid system the internal virtual work is identically zero. Consequently, for a rigid particle system in equilibrium,

$$\delta W = \delta W_{\text{ext}} = 0 .$$

The preceding equation applies to any rigid body in equilibrium, with the external virtual work defined as in (2.31).

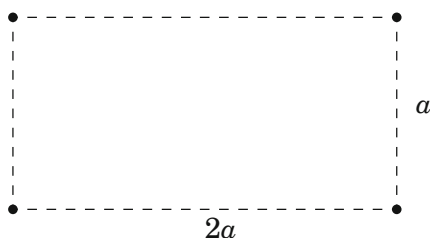
It is often convenient to specify the configuration of a body by means of a minimal set variables q_k that are not necessarily Cartesian coordinates; such variables are called *generalized coordinates* and may include, in particular, angles of rotation. The position of any particle is then given as $\mathbf{r}_i = \mathbf{r}_i(q_k)$. The notion of virtual displacement can be extended to these generalized coordinates, and if the similarly defined δq_k are likewise assumed to be infinitesimal, then the chain rule of differential calculus may be used to find the $\delta \mathbf{r}_i$, and the right-hand side of Eq. (2.31) becomes

$$\sum_i \mathbf{F}_i \cdot \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_k \left(\sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k = \sum_k Q_k \delta q_k , \quad (2.32)$$

where Q_k is known as the *generalized force* conjugate to the generalized coordinate q_k . Thus, if a generalized coordinate represents a displacement (rotation), then the corresponding generalized force is the conjugate force (moment).

Exercises

- 2.1-1.** Three particles are located on the vertices of an equilateral triangle. If $\mathbf{F}_{1e}$ is of known magnitude F and is directed along the line from particle 1 to particle 2, while $\mathbf{F}_{2e}$ is of unknown magnitude and its direction is perpendicular to that line, find $\mathbf{F}_{3e}$ (showing its direction on a sketch) so that equilibrium is satisfied.
- 2.1-2.** If in the preceding exercise the force $\mathbf{F}_{2e}$ were directed along the line from particle 2 to particle 3, would the problem have a solution? Explain your answer.
- 2.1-3.** Consider a system of four particles located at the vertices of a rectangle with sides $2a$ and a , as in the figure. Assume that each particle i is subject to a central attractive force $F_{ij} = kd_{ij}$ due to the presence of particle j ($j \neq i$), where d_{ij} is the distance of the two particles and k is a constant. Sketch all forces acting on the four particles and determine the total external force F_{ie} acting on each particle in order for the system to be in equilibrium.



- 2.1-4.** If a system of N particles is subject to central forces $\mathbf{F}_{ij}$ between each pair (i, j) of particles, show that

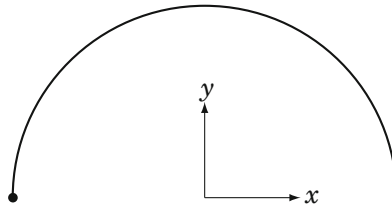
$$\sum_{i=1}^N \sum_{j=1}^{i-1} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \mathbf{0}.$$

- 2.1-5.** Consider two particles i and j separated by distance r and define the function

$$U(r) = 4c \left[\left(\frac{a}{r} \right)^{12} - \left(\frac{a}{r} \right)^6 \right],$$

where a and c are positive constants. Suppose that the equal-and-opposite forces developed between the two particles can be derived from the function $U(r)$, such that $\mathbf{F}_{ij} = \frac{\partial U}{\partial r} \mathbf{e}_{ij}$, where $\mathbf{e}_{ij}$ is a unit vector pointing from i to j . Plot the magnitude F_{ij} of $\mathbf{F}_{ij}$ as a function of r for nominal values $a = c = 1$ and comment on the dependence of F_{ij} on the value of r . Can you attach physical meaning to each of the two power terms in the definition of $U(r)$?

- 2.1-6.** A person weighing 80 kg takes an elevator up 25 stories. If each story is 5 m height, determine the work (in J) done by the gravity force during this ride.
- 2.1-7.** A particle traverses a semicircle of radius 5 ft, starting from the point shown in the figure. Define an angle θ that may be used to define the position of the particle on the semicircle. If there is a constant force $\mathbf{F} = 4\mathbf{i} + 3\mathbf{j}$ acting on the particle throughout its motion, plot the work done by the force as a function θ .

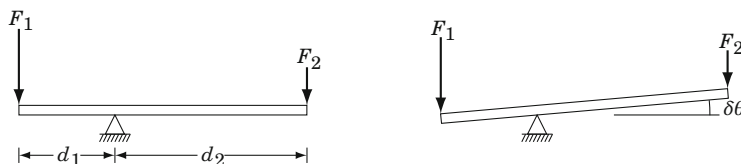


- 2.1-8.** Assume, for simplicity, that the orbit of the earth around the sun is planar and ellipsoidal, with major axes of length 147 million km and 152 million km. Employing *Newton's law of universal gravitation*, the central force acting on the earth by the sun has magnitude

$$F = G \frac{m_s m_e}{d^2},$$

where $G = 6.6710^{-11} \text{ N(m/kg)}^2$ is the gravitational constant, $m_e = 5.9710^{24} \text{ Kg}$ is the mass of the earth, $m_s = 1.9910^{30} \text{ kg}$ is the mass of the sun, and d is the distance between the centers of the two objects. Estimate the work done by the sun's gravity force on the earth during one-quarter of the full trajectory, starting with the earth in the closest distance to the sun. Treat both objects as particles.

- 2.1-9.** Consider the seesaw shown in the figure on the left below, assumed to be in equilibrium in the horizontal position. With the virtual displacement caused by the small rotation about the fulcrum shown in the figure on the right, use the principle of virtual work (treating the angle $\delta\theta$ as infinitesimal) to determine the equilibrium relation between F_1 and F_2 .



2.2 Equilibrium of Rigid Bodies in Two Dimensions

2.2.1 Introduction

When we refer to rigid bodies in solid mechanics, it is with the understanding that no real bodies are perfectly rigid. Indeed, the application of forces invariably causes some deformation, that is, changes in the distances between material points, which lead to changes in the shape and/or volume of the body). The deformation of a body will, in general, be different under various statically equivalent force systems, as exemplified in Fig. 1.25.

In many situations, however, the changes in overall geometry resulting from the deformation can be neglected. The body can then be, for the purposes of statics, idealized as rigid. These are the situations that will be studied in the remainder of this chapter, as well as in the following chapter.

2.2.2 Planar Force Systems

A system of forces $\mathbf{F}_i$ ($i = 1, 2, \dots, N$) is called *planar* if the lines of action of all the forces lie in the same plane. We consider here such a force system acting on a rigid body and we assume, without loss of generality, that the plane of the forces is the xy -plane. In this case, both the force vectors $\mathbf{F}_i$ and the position vectors $\mathbf{r}_i$ drawn from any point O in the xy -plane to any point on the line of action of any of the forces will have nonvanishing components only along the x - and y -axes, namely

$$\mathbf{F}_i = F_{ix}\mathbf{i} + F_{iy}\mathbf{j} \quad , \quad \mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} . \quad (2.33)$$

This means that the moment of the force $\mathbf{F}_i$ about the point O is

$$\mathbf{r}_i \times \mathbf{F}_i = (x_i F_{iy} - y_i F_{ix})\mathbf{k} = M_{iO}\mathbf{k} . \quad (2.34)$$

It is clear from (2.34) that the moment of $\mathbf{F}_i$ may have only one nonvanishing component, namely the one along the z -axis. Consequently, the six component equations constituted by Eqs. (2.18) and (2.19) reduce to only three:

$$\sum_{i=1}^N F_{ix} = 0 \quad , \quad \sum_{i=1}^N F_{iy} = 0 \quad , \quad \sum_{i=1}^N M_{iO} = 0 . \quad (2.35)$$

Note that the force equilibrium equations (2.35)_{1,2} do not need to be written relative to the x - and y -axes, but may instead be written relative to any two axes, say a and b , as in Fig. 2.10a. This is, of course, because the vanishing of the components of $\mathbf{F}_i$ relative to any two axes in the xy -plane is equivalent to the vanishing of the vector $\mathbf{F}_i$ itself. Thus, Eqs. (2.35) may be written equivalently as

$$\sum_{i=1}^N F_{ia} = 0 \quad , \quad \sum_{i=1}^N F_{ib} = 0 \quad , \quad \sum_{i=1}^N M_{iO} = 0 . \quad (2.36)$$

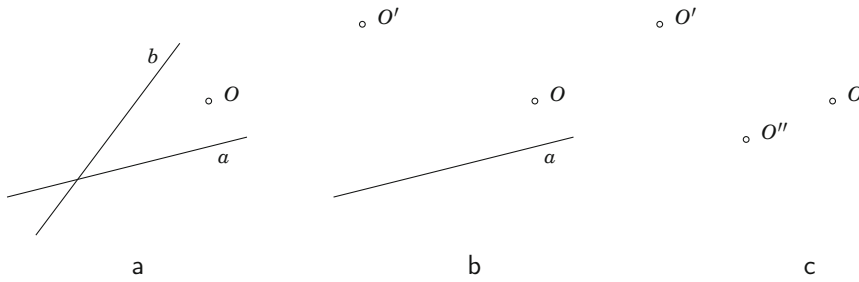


Figure 2.10. Three equivalent ways of writing the equilibrium equations in two dimensions

Another way of expressing planar force equilibrium is by means of the system of equations

$$\sum_{i=1}^N F_{ia} = 0 \quad , \quad \sum_{i=1}^N M_{iO} = 0 \quad , \quad \sum_{i=1}^N M_{iO'} = 0 \quad , \quad (2.37)$$

which means that the component of the resultant force along any axis a is zero and the moments about two points O and O' in the xy -plane are also zero (see Fig. 2.10b). If the line OO' is not perpendicular to the a -axis, then we may establish the equivalence of Eqs. (2.36) and (2.37) by showing that Eq. (2.36) imply Eq. (2.37), and vice versa. To show the first, we assume that Eq. (2.36) hold, implying that the force system is defined by a zero resultant force and zero moment about point O . Such a system clearly produces a zero moment about any other point (such as O') in the xy -plane, so that Eqs. (2.37) are satisfied. Conversely, if Eqs. (2.37) are assumed to hold, then Eq. (2.37)₁ immediately implies that the resultant force (if any) must be perpendicular to the a -axis. Equation (2.37)_{2,3} further imply that the line of action of this resultant must pass through points O and O' . However, since OO' is assumed not to be perpendicular to the a -axis, this is possible only if the resultant force is zero, showing that (2.36) are satisfied. The sets of Eqs. (2.36) and (2.37) are consequently equivalent.

Yet another alternative set of equilibrium equations in two dimensions may be expressed as

$$\sum_{i=1}^N M_{iO} = 0 \quad , \quad \sum_{i=1}^N M_{iO'} = 0 \quad , \quad \sum_{i=1}^N M_{iO''} = 0 \quad , \quad (2.38)$$

where the points O , O' and O'' are not on the same line, as shown in Fig. 2.10c. Again, arguing the equivalence of the sets of equations (2.36) and (2.38) entails establishing that the two sets imply each other. The derivation of (2.38) from (2.36) is trivial, as previously. That Eq. (2.38) imply Eq. (2.36) is established as follows: Eq. (2.38)_{1,2} imply that the line of action of the resultant force is defined by the points O and O' , as argued earlier. However, since O''

is not on the line defined by O and O' , Eq. (2.38)₃ is only possible if the resultant force is equal to zero. Therefore, the equilibrium equations (2.36) and (2.38) are also equivalent.

The equilibrium equations are often written without any explicit reference to the points of application of the forces or moments; thus, for example, we write Eq. (2.35) more succinctly as

$$\sum F_x = 0 \quad , \quad \sum F_y = 0 \quad , \quad \sum M_O = 0 . \quad (2.39)$$

Example 2.2.1 (Square block with four forces):

Consider a unit square block subject to the four forces of equal magnitude F depicted in Fig. 2.11a.

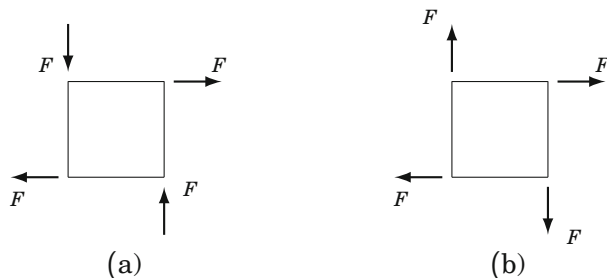


Figure 2.11. Unit square block under the influence of four forces

That this block is in equilibrium may be established by taking the sum of forces in the horizontal and vertical direction, as well as the sum of moments about any of the four vertices, according to (2.35) or (2.36). Alternatively, we may sum the forces in the horizontal direction, and also sum the moments about the top-left and bottom-right points (but not the top-left and top-right or the bottom-left and bottom-right points!), as stipulated by (2.37). Likewise, equilibrium may be established by taking moments about any three of the four vertices of the block, which corresponds to (2.38). Other points than the vertices may be used as well for taking moments. However, given that two of the four forces in this example pass from each of the vertices, the calculation of moments is simplified by taking moments about vertices.

The block in Fig. 2.11b is in force equilibrium, but not in moment equilibrium. This is easily inferred by noting that the two couples generated by the pairs of horizontal and vertical forces have the same sense. The same conclusion is reached by taking moments about any of the vertices. Interestingly, if the size of the block shrinks to zero (which is tantamount to the block becoming a particle), moment equilibrium plays no role in the equilibrium of the body. This is consistent with the observation in Sect. 2.1.3 about the independence of force and moment equilibrium in continuous bodies but not in particles.

2.2.3 Two-Force and Three-Force Bodies

A simple case of a planar force system occurs when there are only two nonzero forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on a rigid body. Such a *two-force body* is shown in

Fig. 2.12a. For force equilibrium, the two forces must be equal and opposite, and therefore parallel, but for moment equilibrium they must be collinear, since otherwise they would form a couple. In mathematical terms, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are the two forces, then force equilibrium requires that

$$\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0} , \quad (2.40)$$

while moment equilibrium relative to any point O on the plane requires that

$$\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = \mathbf{0} , \quad (2.41)$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ are, respectively, drawn to the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$ from any point. Equations (2.40) and (2.41) guarantee that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal and opposite to each other, as well as collinear. Indeed, (2.40) directly implies that the forces are equal and opposite, therefore (2.41) necessitates that they form a zero force couple, hence they are also collinear.

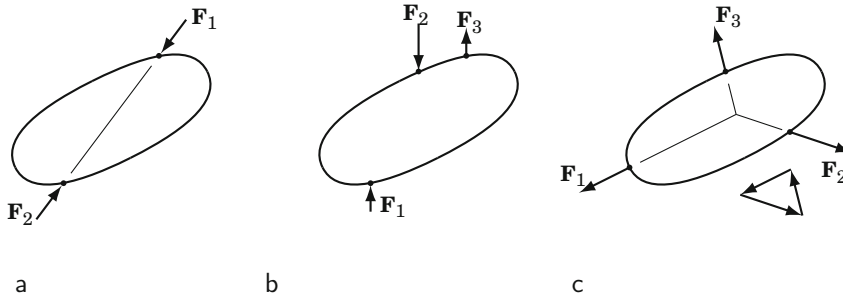


Figure 2.12. Two-force and three-force bodies. (a) Two forces, (b) Three parallel forces, (c) Three concurrent forces

Almost as simple is the case of a *three-force body*, such as those shown in Fig. 2.12bc. Here, each of the three nonzero forces must be equal and opposite to the resultant of the other two, since force equilibrium and property (1.6)(b) (page 15) imply that

$$(\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3 = (\mathbf{F}_2 + \mathbf{F}_3) + \mathbf{F}_1 = (\mathbf{F}_3 + \mathbf{F}_1) + \mathbf{F}_2 = \mathbf{0} . \quad (2.42)$$

Furthermore, each of the forces must be collinear with the resultant of the other two, since (2.42), property (1.10)(c) (page 17) and moment equilibrium about any point implies that

$$\begin{aligned} \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 &= \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times (-\mathbf{F}_1 - \mathbf{F}_2) \\ &= (\mathbf{r}_1 - \mathbf{r}_3) \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}_3) \times \mathbf{F}_2 = \mathbf{0} . \end{aligned} \quad (2.43)$$

This means that the moment of any two of the forces (here, $\mathbf{F}_1$ and $\mathbf{F}_2$) about any point on the line of action of the third (here, $\mathbf{F}_3$) is equal to zero. If two

of the forces are parallel, then the third must also be parallel to them, as shown in Fig. 2.12b. On the other hand, if two of the forces are concurrent, then the third must be concurrent with them (otherwise, it would form a couple with the resultant of the first two), as shown in Fig. 2.12c, with the triangular force polygon shown on the side. The parallel case may, in fact, be regarded as the limit of the concurrent case as the point of concurrency recedes to infinity.

2.2.4 Degrees of Freedom and Constraints

Fixing the position of a rigid body in a plane requires the specification of three independent geometric quantities. These could be, for example, the x - and y -coordinates of one point (say 1) and either the x - or the y -coordinate of another point (say 2), as in Fig. 2.13a. The first two quantities obviously specify the position of point 1. Then, the third quantity fixes the position of point 2, since, due to rigidity, the distance d_{12} between these two points remains constant and equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, so that y_2 can be derived if x_2 is given, and vice versa.

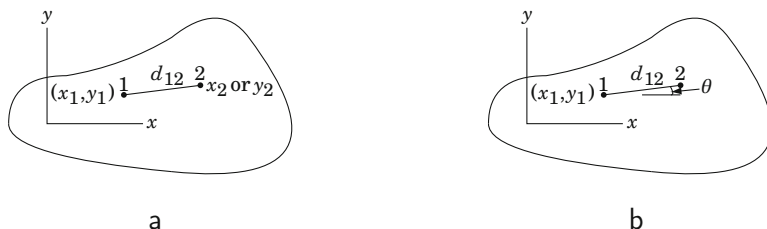


Figure 2.13. Fixing the position of a rigid body in a plane

Alternatively, the third quantity can be the angle subtended by a line between the points, as in Fig. 2.13b. If this angle (with respect to the x -axis) is θ , then $x_2 = x_1 + d_{12} \cos \theta$ and $y_2 = y_1 + d_{12} \sin \theta$. The position of any other point (say 3) is now also fixed due to triangulation, because the constancy of its distances d_{13} and d_{23} from points 1 and 2 is sufficient to fix it in place.

A rigid body confined to planar motion can therefore be said to have three *degrees of freedom* (abbreviated as *dof*), and to completely fix it in space three independent *external constraints* are necessary. That is, the body has to be connected to a fixed frame* by means of external *supports* that prevent changes in three independent kinematic quantities. Supports exert forces or moments (depending on whether what is prevented is a translation or a

*By a *fixed frame* we mean a combination of rigid bodies whose position in space remains unchanged.

rotation) on the body; such forces and moments are called *reactions*. These are conjugate (as defined in Sect. 2.1) to the translations or rotations that they prevent.

One possibility for the rigid body to be fixed is by three external supports, each of which prevents translation in one direction. Examples of such 1-dof supports are shown in Fig. 2.14.

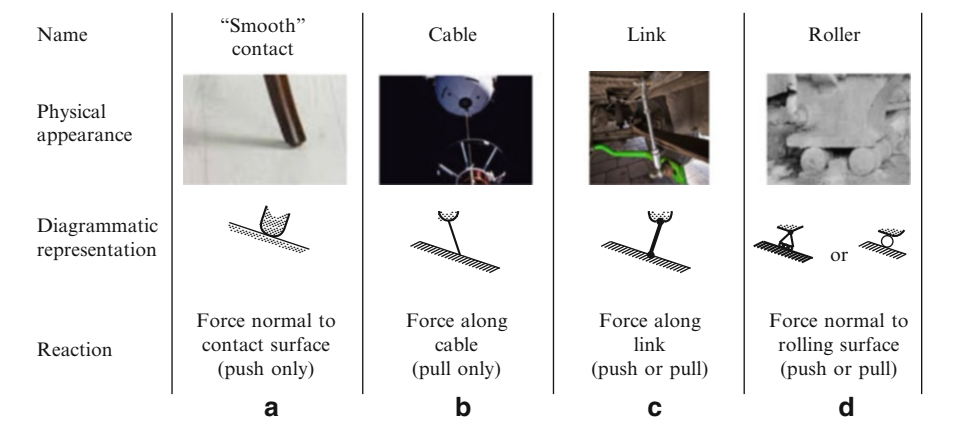


Figure 2.14. 1-dof constraints

Note that the cable and link of Fig. 2.14b,c, respectively, are assumed to be weightless, so that they act as two-force bodies. Note further that the roller support of Fig. 2.14d, while shown as though it rolled on only one surface (and therefore could exert only a push force), is conventionally assumed to act as if it rolled between two parallel surfaces and therefore the reaction force can be a push or a pull.

A set of external supports that prevents all rigid motion in the plane is *proper*, otherwise it is *improper*. Examples of proper and improper supports are shown in Fig. 2.16a–c and d–f, respectively. A set of three 1-dof supports is proper if the lines of action of the reaction forces are not concurrent, as in Fig. 2.16b. If the lines of action are concurrent, then the supports are improper, because, in this case, rotation about the point of concurrency is, at least initially, not prevented (this is the case with the body shown in Fig. 2.16e). The same conclusion applies to the case of the three parallel reaction forces in Fig. 2.16d, since now translation perpendicular to the reactions is not prevented. Similarly, the vertical translation of the body in Fig. 2.16f is unrestrained, as neither of the two supports is resisting such motion.

Another possibility of fixing a rigid body in the plane is a combination of a 1-dof support with a 2-dof support, which may prevent either translation in all directions (such as a “rough”[†] contact support, Fig. 2.15a, or a *pin* or *hinge*

[†]The reason for the quotes around “rough” is explained in Sect. 2.2.5. This constraint is

support, Fig. 2.15b); or else one translation component and rotation (*guided support*, Fig. 2.15d, where the previous remarks on the two-way nature of the roller force apply as well). Figure 2.16b,c illustrates two examples of bodies that are held in place by such a combination of supports.

Name	“Rough” contact	“Smooth” pin or hinge	Fixed (built-in) support	Guided support (fixed connection to roller)
Physical appearance				
Diagrammatic representation				
Reaction	Force in any direction (normal component is push only)	Force in any direction	Force in any direction and moment	Force normal to rolling surface, and moment
	a	b	c	d

Figure 2.15. Two-dimensional 2-dof and 3-dof constraints

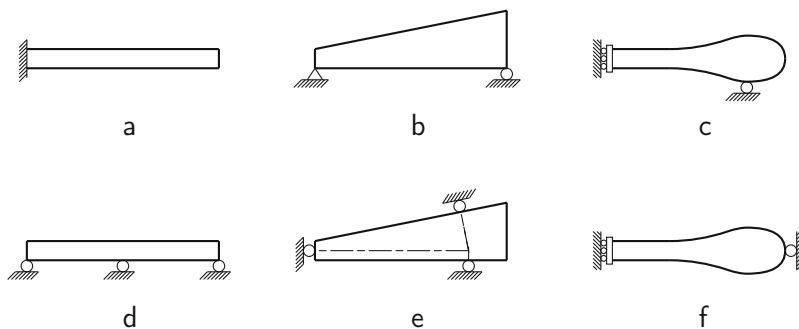


Figure 2.16. Properly (a–c) and improperly (d–f) constrained rigid bodies in two dimensions

Yet another possibility is a single support (called a *fixed* or *built-in* support) that prevents translation (in any direction) and rotation at one point; it is shown in Fig. 2.15c and used in properly supporting the body in Fig. 2.16a.

A body held in place by a fixed support, as in Fig. 2.16a, is said to be *cantilevered*. A body supported by a hinge and a roller, as in Fig. 2.16b, is generally known as *simply supported*. If the roller is moved inward from the

effective only if the normal component of the contact force is a push.

end, then such a body is said to have an *overhang*, as in Fig. 2.17a, but if the overhang composes most of the body's span, then it is also thought of as cantilevered, as in Fig. 2.17b.



Figure 2.17. Simply supported bodies with overhang

A system consisting of a body with the *minimum* number of proper supports is called *statically determinate*, meaning that the three equilibrium equations (2.35) (or any of their equivalents) suffice to find all the unknown reactions. It is easy to see that the systems in Fig. 2.16a–c are statically determinate.

A body may, of course, be held in place by more than the minimum supports necessary for proper constraint, in which case not only are the supports proper, but the body is also *overconstrained*. Here, in general, the three equilibrium equations are not sufficient to determine the (four or more) unknown reactions. In this case, the system is said to be *statically indeterminate*. Examples of overconstrained bodies are shown in Fig. 2.18.

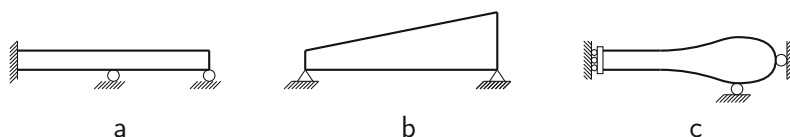


Figure 2.18. Examples of overconstrained bodies

Conversely, a body can be *underconstrained* if it does not have sufficient external supports for proper constraint, or else if, while properly constrained in the exterior, it has one or more *internal degrees of freedom*, such as might be represented by an internal hinge, that allow all or part of it to move, as in Fig. 2.19. In the latter case, the body is referred to as a *mechanism*.

A body that is overconstrained may be transformed into a statically determinate system by means of internal degrees of freedom, as, for example, the *three-hinged* (or *three-pinned*) arch shown in Fig. 2.20a. The analysis of such systems will be undertaken in Sect. 2.4. However, if in the arch of Fig. 2.20a either one of the support hinges were replaced by a roller, the arch would become a mechanism and collapse (say, under its own weight), as in Fig. 2.20b.

It is also possible for a body to be overconstrained with respect to some degree(s) of freedom (and hence statically indeterminate) and underconstrained



Figure 2.19. Examples of mechanisms

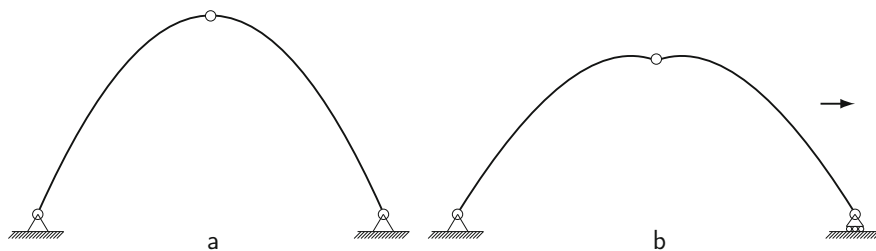


Figure 2.20. Three-hinged (three-pinned) arch: (a) statically determinate (b) collapsing

with respect to others. A simple example is seen in Fig. 2.21, where the body is overconstrained with respect to vertical translation with four reaction forces to be determined by two equilibrium equations, while being clearly underconstrained (actually, unconstrained) with respect to horizontal translation.



Figure 2.21. Example of a body that is both overconstrained and underconstrained

2.2.5 Friction

Contacts or connections are commonly referred to as “smooth” and “rough,” indicating, respectively, the absence and presence of *friction*. Friction is the resistance to motion that is generated when one body slides or tends to slide past another under the influence of some external loading. We consider here only *dry friction*, that is friction between solids, as distinct from *wet* or *fluid friction*, which occurs between layers of fluid or between a solid and a fluid. We put the conventional terms “smooth” and “rough” in quotation marks because, in reality, friction increases both when the surfaces are very rough *and* when they are very smooth. The reason is that, generally, on very rough surfaces the asperities (small bumps in the surface) interfere with the sliding motion and tend to lock the surfaces in place. With very smooth surfaces,

on the other hand, the effective area of contact is larger, tending to increase friction. Very smooth surfaces are also more sensitive to chemical forces that develop between them and may resist the sliding motion (such forces increase as the distance between particles from the two surfaces becomes smaller). But the use of the terms *rough* and *smooth*, while frowned on by tribologists[‡] for the reasons just discussed, is still conventional among engineers.

In the case of contact with friction, as in Fig. 2.15a, the tangential part T of the contact force is the *frictional* force, which resists sliding between the surfaces in contact. Its direction, on each surface, is consequently opposed to the direction of potential sliding motion, as in Fig. 2.22.

In accordance with *Coulomb's law*[§]—also known as the *Amontons–Coulomb law*[¶]—the magnitude of the frictional force at rest cannot exceed that of the normal component (which can only be a push, as shown in Fig. 2.22), denoted N , multiplied by a certain positive number known as the

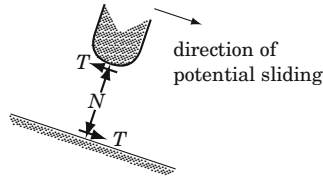


Figure 2.22. Contact forces in Coulomb's law

coefficient of static friction, denoted μ_s . The value of this coefficient depends on the physical characteristics of the two surfaces in contact. Coulomb's law is expressed mathematically as

$$|T| \leq \mu_s N . \quad (2.44)$$

It is important to emphasize here that Coulomb's law specifies only the maximum possible friction force (equal to $\mu_s N$). Frictional forces of smaller magnitude are entirely possible and may apply to the body when it is in equilibrium. In this case, the frictional force cannot be determined from Coulomb's law. Rather, it may be calculated from the equilibrium equations.

Once static friction is overcome and the contacting surfaces are in relative motion, the tangential force resisting the motion is equal to $\mu_k N$, where the positive scalar μ_k is the so-called *coefficient of kinetic friction*, usually smaller than μ_s . If the body is in equilibrium with this force, then the motion will proceed at constant speed. Otherwise acceleration will occur.

[‡]*Tribology* is the discipline concerned with the study of friction, lubrication, and wear.

[§]Charles–Augustin de Coulomb (1736–1806) was a French physicist.

[¶]Guillaume Amontons (1663–1705) was a French scientist who formulated the law of friction well before Coulomb, though the principle was already known to Leonardo da Vinci (1452–1519).

If friction is present in a hinge or pin support, there is resistance to rotation resulting in a moment reaction, and, so long as the friction is not overcome, the support will act like a fixed (built-in) one.

Some representative values of the coefficients of static and kinetic friction are given in Table 2.1. These values depend crucially on the conditions of experiments from which they were estimated and should be considered as coarsely approximate.

Interface	μ_s	μ_k
Aluminum–Steel	0.60	0.45
Copper–Steel	0.55	0.35
Glass–Glass	0.90	0.40
Wood–Wood	0.60	0.50
Rubber–Concrete	1.0	0.80
Ice–Ice	0.10	0.030

Table 2.1. Typical values of the static and kinetic coefficients of dry friction for selected material interfaces

2.2.6 Free-Body Diagrams

Of the forces appearing in the equilibrium equations for a rigid body, some are specified at the outset; these forces (and, possibly, couples) are known as *loads*. The support reactions (also forces and/or couples), on the other hand, are not known to begin with, and it is their determination that constitutes the solution of the equilibrium equations. The remainder of this section will deal with the determination of the reactions in statically determinate rigid bodies under planar force systems.

The procedure consists of four steps. The first is the drawing of a schematic diagram showing a sketch of the body, its supports, and the loads acting on it. In the next step, the supports are replaced by their reactions, resulting in another diagram known as a *free-body diagram*. The drawing of a correct free-body diagram is a crucial step in solving a wide array of problems in solid mechanics. (Free-body diagrams will be discussed in greater generality in Sect. 2.4.)

The next step is the formulation of three independent equilibrium equations, each containing one or more of the reactions. Finally, these equations are solved for the reactions.

In practice, we observe the following conventions:

1. When the number of points at which loads and reactions are present is small, it is more common to designate them by letters $A, B, C, \dots$ than by numbers.

2. The letter designating a point, when equipped with a subscript such as x or y , is also used to designate the component of a force reaction at the point.
3. The assumed sense of a reaction component need not coincide with the positive direction of the corresponding coordinate, especially if intuition indicates that it will be the opposite. If the initial guess is wrong, the value of the component will turn out to be negative, which is entirely acceptable.

Lastly, the equilibrium equations are solved for the unknown forces and/or moments.

The following examples will illustrate these usages.

Example 2.2.2 (Particles connected by a spring.):

We consider, first, the case of an elastic spring connecting two particles, as in Fig. 2.23a. Suppose that the spring is stretched by equal and opposite forces $\mathbf{F}$, as in Fig. 2.23b, and examine the three elements of the system, namely the two particles and the spring itself, assuming that the system is in equilibrium. It is important to note that what constitutes an internal or external force depends crucially on the particular element under consideration. For instance, both forces on each of the two particles are external to them, while only one of them (the applied force $\mathbf{F}$) is external to the full spring-particle system, as shown in Fig. 2.23c. Another critical observation is that drawing the forces on individual elements of a system entails the application of the action-reaction law discussed in Sect. 2.1.

Example 2.2.3 (Flat contact with frictional sliding):

To appreciate the meaning of the inequality (2.44) governing friction, we consider a rigid body of weight W which is in flat contact with a rigid foundation and is pulled by a force F passing through its center of gravity, as in Fig. 2.24a, with the free-body diagram shown in Fig. 2.24b. When the frictional force is below the maximum value $\mu_s N$ stipulated by inequality (2.44), the body remains in equilibrium. In this case, the equilibrium equations (2.35) lead to

$$F = T \quad , \quad W = N \quad , \quad x = \frac{F}{N}h \quad ,$$

that is, the frictional force T is equal and opposite to the pull force F , the weight of the body is balanced by the resultant reaction N normal to the bottom surface of the body, and the reaction N is acting at a point with coordinate x , such that (W, T) form a force couple that balances the force couple of (F, T) .

In the limiting case of impending sliding, the pulling force F becomes equal to the maximum frictional force $T_{max} = \mu_s N$, so that Coulomb's law holds as an equality:

$$F = T_{max} = \mu_s N \quad .$$

Now, the resultant $\mathbf{R}_s$ of the normal and tangential reactions on the frictional interface forms an angle ϕ_s with the normal to the foundation, such that

$$\tan \phi_s = \frac{T}{N} = \frac{\mu_s N}{N} = \mu_s \quad .$$

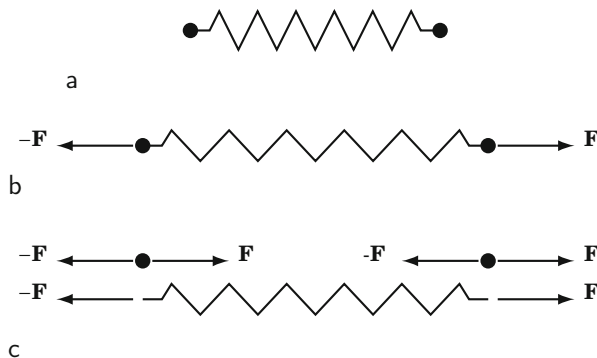


Figure 2.23. Spring connecting two masses. (a) Spring with two masses under no force. (b) Free-body diagram of mass–spring–mass system. (c) Free-body diagrams of the three elements of the system

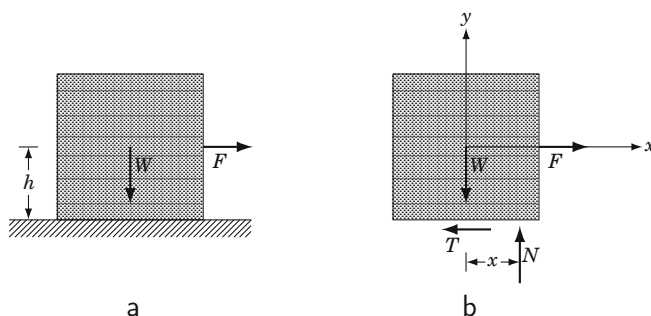


Figure 2.24. Flat contact between a rigid body and a foundation: (a) schematic diagram (b) free-body diagram

The angle $\phi_s = \arctan \mu_s$ is referred to as the *angle of static friction*.

Once sliding begins, the frictional force becomes equal to $\mu_k N$ and the resultant reaction forms an *angle of kinetic friction* equal to $\phi_k = \arctan \mu_k$.

Example 2.2.4 (Ladder):

Another example involving the just-discussed concept of friction is that of a person standing on a ladder, assumed to be in “smooth” contact with the wall and in “rough” contact with the ground, as shown schematically in Fig. 2.25a.

In the free-body diagram of Fig. 2.25b, W is the resultant of the weights of the person and the ladder. (The diagram also shows the concurrency of this resultant with those of the two contact forces.) It can be readily seen that the force system acting on the body consists of two equal and opposite vertical forces, $N_B = W$, and two equal and opposite horizontal forces, $N_A = T_B$. For moment equilibrium, then,

$$W c \tan \alpha = T_B h ,$$

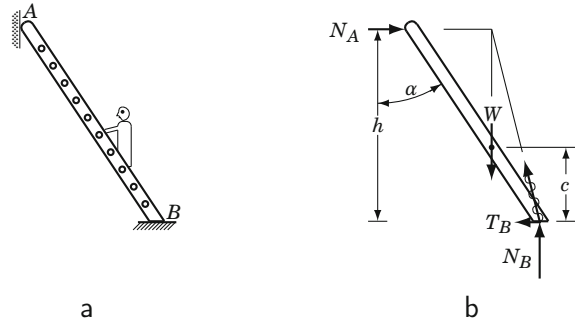


Figure 2.25. Person on a ladder: (a) schematic diagram (b) free-body diagram

and therefore

$$T_B = W \frac{c}{h} \tan \alpha .$$

Since, however, $T_B \leq \mu_s N_B = \mu_s W$, it follows that, for stability,

$$\frac{c}{h} \tan \alpha \leq \mu_s .$$

Example 2.2.5 (Lever):

Let us consider next the hinged rod shown in Fig. 2.26. Here, a force F is applied

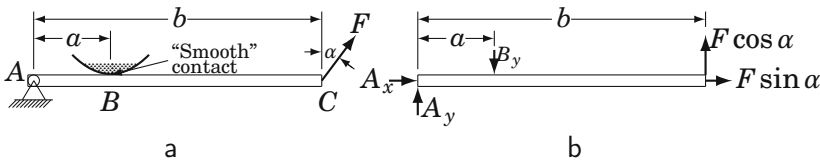


Figure 2.26. Hinged lever: (a) schematic diagram (b) free-body diagram

at one end C of a rigid bar that is hinged at the other end A , with the intention of transmitting a larger force to a rigid body (potentially so as to lift it) in “smooth” contact with the bar at point B .

The equilibrium equations (2.35) applied to the forces shown in the free-body diagram (note that B_y is assumed to be downward, as expected on physical grounds, and not necessarily in accord with the usual convention for positive and negative forces) are

$$A_x + F \sin \alpha = 0 \quad , \quad A_y - B_y + F \cos \alpha = 0 \quad , \quad -aB_y + bF \cos \alpha = 0 \quad ,$$

where the third equation represents moment equilibrium about point A . Note that this equation can be solved directly for B_y (while the first equation yields A_x); once B_y is determined it can be inserted into the second equation to determine A_y . On the other hand, the second and third equations can be combined linearly (by multiplying the former by a and subtracting it from the latter) to yield

$$-aA_y + (b-a)F \cos \alpha = 0 \quad ,$$

but this is none other than moment equilibrium about B , illustrating the replacement of one of the force equilibrium equations (here, the force equilibrium equation in the y -direction) by another moment equilibrium equation, as previously discussed in Sect. 2.2.2. We now have three equations, each of which contains one and only one unknown reaction.



Figure 2.27. Practical lever problem

In practice the problem may be framed differently: the weight of the body to be lifted may be specified (say W) and what is to be determined is the magnitude of the lifting force F , which may be applied over a pulley, as shown in Fig. 2.27. It will be shown later that a force applied by means of a weightless cable over a frictionless pulley remains constant in magnitude.

Example 2.2.6 (Lever-based exercise machine):

In the exercise machine shown schematically in Fig. 2.28, the lever, balanced on the fulcrum A , is pressed down by the force F in order to raise the roller B supporting the weight stack which offers the resistance R . Since B can move only vertically, the moment arm of the resisting force R remains constant, while that of the applied motive force F (necessarily downward) decreases, so that this force increases progressively through the movement.

Example 2.2.7 (Pulley-based exercise machine):

In the machine shown schematically in Fig. 2.29a, used for leg extension and leg curl, the motive force F is applied normally (by means of a roller, or possibly a low-friction pad) to a crank attached rigidly (though with variable attachment points) to a wheel. As the wheel turns, it pulls a cable that is strung over some pulleys and raises the weight stack. In this machine, the moment arm of both the applied force F and the resisting force R remains constant through the movement.

In the machine shown in Fig. 2.29b, a (possibly bent) hinged bar takes the place of the crank and wheel. Here, the moment arm of the resisting force is variable.

Example 2.2.8 (Simply supported beam):

As was noted above, a body is called simply supported if the supports consist of one pin or hinge and one roller or “smooth” contact (the orientation of the contact surface being such that at equilibrium the force is a push). A straight bar that is intended to carry primarily transverse forces is called a *beam*. A simply supported beam, where the load F is the resultant of any loads that may

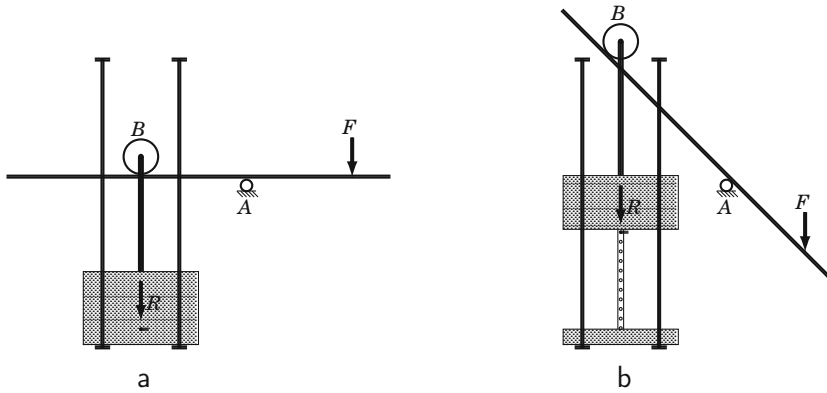


Figure 2.28. Lever-based exercise machine: (a) initial position (b) position with weights raised

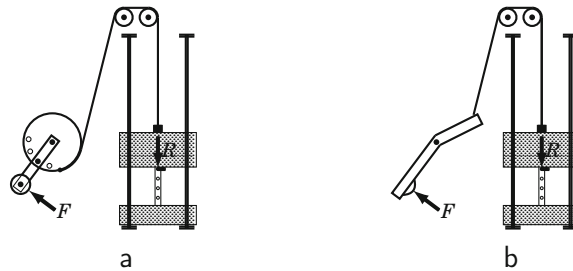


Figure 2.29. Pulley-based exercise machine

be acting on the beam, is shown in Fig. 2.30a, with the free-body diagram shown in Fig. 2.30b.

Equilibrium of forces in the x -direction immediately leads to $A_x = F \sin \alpha$. A_y and C_y can be solved from uncoupled equations if these represent moment equilibrium about C and A , respectively:

$$\sum M_C = bF \cos \alpha - LA_y = 0 \Rightarrow A_y = \frac{b}{L} F \cos \alpha$$

and

$$\sum M_A = -aF \cos \alpha + LC_y = 0 \Rightarrow C_y = \frac{a}{L} F \cos \alpha .$$

Equilibrium of forces in the y -direction can now be used as a check on the results:

$$A_y + C_y = \frac{a+b}{L} F \cos \alpha = F \cos \alpha ,$$

since $a + b = L$.

It is important to point out that the use of the resultant in place of the actual loads acting on the beam is sufficient for the purpose of calculating support reactions, but not if one desires more information, as will be discussed in Sect. 2.4.

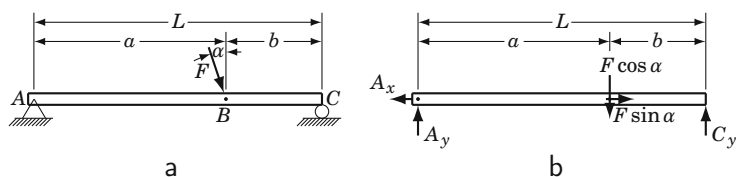


Figure 2.30. Simply supported beam: (a) schematic diagram (b) free-body diagram

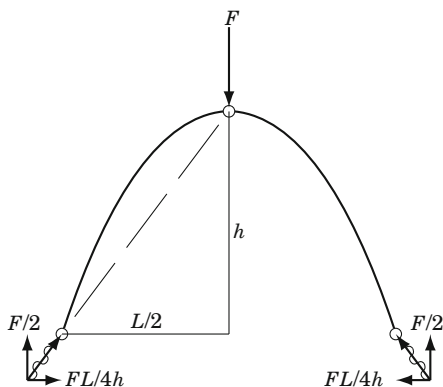


Figure 2.31. Support reactions on an apex-loaded three-hinged arch

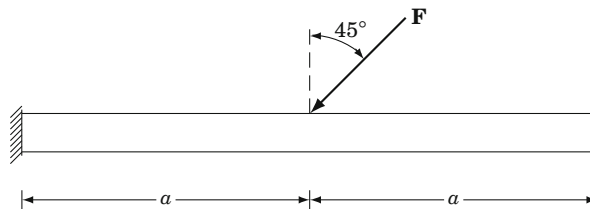
Example 2.2.9 (Three-hinged arch):

For the three-hinged arch of Fig. 2.20a, however it may be loaded, each of the two hinge supports has two reactions and therefore the three global equilibrium equations are not sufficient to determine them. But since the pin joint at the apex is frictionless, only a force—and no moment—can be transmitted there, so that if the arch is sectioned at the pin (this is an example of the *method of sections*, which will be discussed in Sect. 2.4), the resultant moment about the pin on each member must be zero. This requirement provides an additional equilibrium equation.

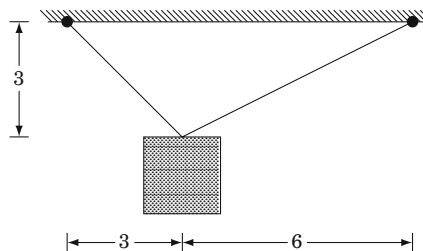
In the simplest case where the arch is symmetric and the only load is a downward force F acting at the pin, this equilibrium is equivalent to each member being a two-force body, so that the support reactions must be directed toward the pin. By symmetry, the vertical components of the reactions are $F/2$, and therefore the (equal and opposite) horizontal components must, by similar triangles as seen in Fig. 2.31, have the value $FL/4h$ if L is the span and h is the rise of the arch.

Exercises

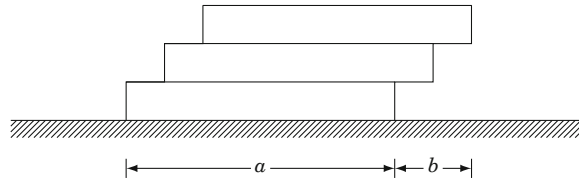
- 2.2-1.** Let forces $\mathbf{F}_1 = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{F}_2 = -\mathbf{i} + 3\mathbf{j}$ act on a rigid body at points with coordinates $(2, 0)$ and $(1, 1)$, respectively. Find a third force $\mathbf{F}_3$ and its line of action such that the three-force system be in equilibrium.
- 2.2-2.** Let forces $\mathbf{F}_1 = 2\mathbf{j}$ and $\mathbf{F}_2 = -5\mathbf{j}$ act on a rigid body at points with coordinates $(0, 0)$ and $(1, -1)$. What is the line of action of a third force $\mathbf{F}_3$, such that the three forces be in equilibrium? How would your answer change if the rigid body is additionally subject to a moment $\mathbf{M} = 5\mathbf{k}$?
- 2.2-3.** Consider a two-dimensional rigid body in the shape of a unit square whose vertices A , B , C , and D have coordinates $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. Suppose that there are forces acting only at the four vertices of the body, such that $\mathbf{F}_A = \mathbf{i} + \mathbf{j}$, $\mathbf{F}_B = \mathbf{i} + 4\mathbf{j}$, $\mathbf{F}_C = -2\mathbf{i} + F_{Cy}\mathbf{j}$, and $\mathbf{F}_D = F_{Dx}\mathbf{i} + F_{Dy}\mathbf{j}$. Determine the components F_{Cy} , F_{Dx} , and F_{Dy} , such that the body remain in equilibrium. Obtain the solution three times using a suitable set of equilibrium equations in the form (2.35), (2.37), and (2.38).
- 2.2-4.** Draw the free-body diagram for the cantilevered body and use the equilibrium equations to find the reactions.



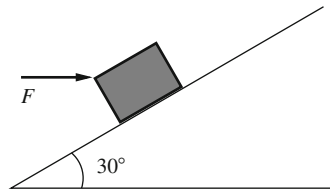
- 2.2-5.** Draw the free-body diagram for the system shown in the figure, as well as for each its three constituent parts (two cables and the box). Determine the forces on each cable assuming that the box weighs 600 kips.



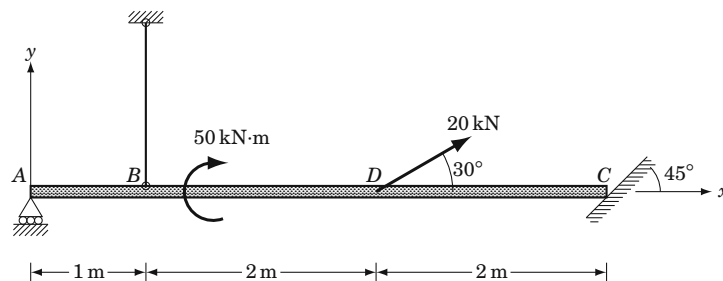
- 2.2-6.** Let three rectangular objects of uniform density have width a and be stacked on top of each other as in the figure. If each object has weight W , find the maximum allowable offset b of the top object relative to the bottom object before the system is unable to remain in an equilibrium state.
- 2.2-7.** Find the solution to the problem in Exercise 2.2-6 for the case of n objects, where n is any positive integer.



- 2.2-8.** A rectangular block has mass of 100 kg is resting on an inclined surface, as in the figure. If the static coefficient of friction between the block and the surface is $\mu_s = 0.25$, find the magnitude F of the horizontal force needed to prevent its downward sliding. Also, find the magnitude F of the horizontal force needed to initiate an upward sliding of the block. If the kinetic coefficient of friction is $\mu_k = 0.2$, find the magnitude of the force F needed to sustain the upward sliding motion of the block under constant velocity.

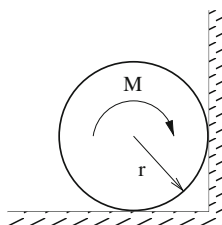


- 2.2-9.** The bar shown in the figure below is supported by a roller at point A and a rigid link at point B and rests on a frictional surface at point C . The bar is subjected to the external force and moment shown in the figure, as well as to its own weight $W = 10$ kN.



- (a) Draw the free-body diagram of the bar showing all external forces.
- (b) If friction is neglected, determine the magnitude of the force exerted on the bar by the rigid link element.
- (c) If friction is included and sliding at point C is imminent, determine again the force exerted on the bar by the truss element. Assume that the static friction coefficient is $\mu_s = 0.5$.

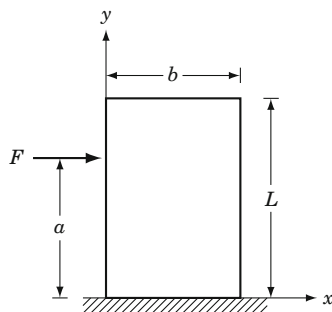
2.2-10. Find the magnitude M of the maximum moment that may be applied on the cylinder of mass m and radius r shown in the figure without inducing sliding. Let the static coefficient of friction be μ_s on both frictional interfaces.



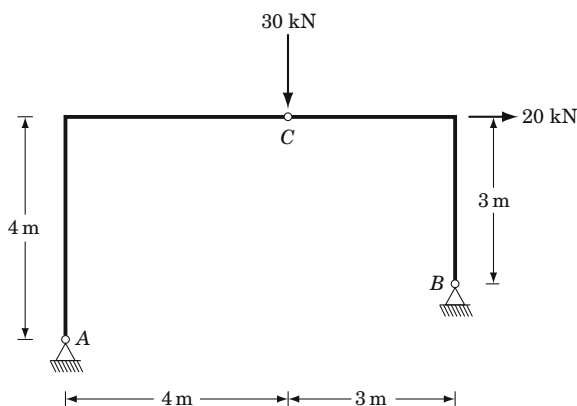
How would the answer to this problem change if the sense of the moment is reversed?

2.2-11. A homogeneous two-dimensional box of weight W is subject to a horizontal force F , as shown in the figure. The box is supported by a frictional surface with which the static coefficient of friction is $\mu_s = 0.25$.

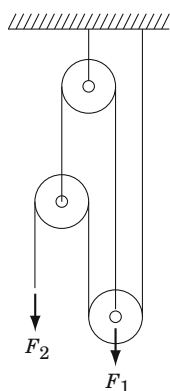
- (a) If $F = 0.2W$, draw the free-body diagram of the box showing all external forces and reactions. Does the body slide for this value of F ?
- (b) Find the critical value F_c of the force F for which the box is at the onset of sliding.
- (c) For $F = F_c$, find the horizontal coordinate x_c of the point at which the vertical reaction of the surface acts on the box to maintain equilibrium.
- (d) Use the result of part (c) to determine a condition satisfied by the dimensions a and b , such that sliding of the box commences before tipping.
- (e) What is the smallest value $F_{\min}$ of the horizontal force F that will tip the box, assuming that F may act at any point on the left edge of the box? What is the point at which the force acts in this case?



2.2-12. Find the reactions at points A and B in the three-hinged arch shown in the figure.



2.2-13. Find the force F_2 required to keep the system of pulleys in equilibrium under the influence of the force F_1 .



2.2-14. Show the degree of freedom allowed by the improper supports of Fig. 2.16d and e.

2.2-15. Explain why the support of Fig. 2.16f is improper.

2.3 Equilibrium of Rigid Bodies in Three Dimensions

2.3.1 Three-Dimensional Force Systems

The equilibrium equations for a rigid body in three dimensions are given by Eqs. (2.18) and (2.19). Resolving all the vectors that appear in these equations with respect to a right-handed Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ leads to

$$\sum_{i=1}^N F_{ix} \mathbf{i} + \sum_{i=1}^N F_{iy} \mathbf{j} + \sum_{i=1}^N F_{iz} \mathbf{k} = \mathbf{0} \quad (2.45)$$

and

$$\sum_{i=1}^N M_{Oix} \mathbf{i} + \sum_{i=1}^N M_{Oiy} \mathbf{j} + \sum_{i=1}^N M_{Oiz} \mathbf{k} = \mathbf{0}, \quad (2.46)$$

where

$$M_{Oix} = F_{iy}z_i - F_{iz}y_i, \quad M_{Oiy} = F_{iz}x_i - F_{ix}z_i, \quad M_{Oiz} = F_{ix}y_i - F_{iy}x_i \quad (2.47)$$

are the components of the moment $\mathbf{M}_i$ about a point O of the external force $\mathbf{F}_i$ acting at a point whose position vector relative to O has Cartesian coordinates (x_i, y_i, z_i) . It follows from (2.45) and (2.46) that the six equilibrium equations may be expressed in component form as

$$\sum_{i=1}^N F_{ix} = 0, \quad \sum_{i=1}^N F_{iy} = 0, \quad \sum_{i=1}^N F_{iz} = 0 \quad (2.48)$$

and

$$\sum_{i=1}^N M_{Oix} = 0, \quad \sum_{i=1}^N M_{Oiy} = 0, \quad \sum_{i=1}^N M_{Oiz} = 0. \quad (2.49)$$

As in the two-dimensional case, there exist many equivalent sets of equilibrium equations alternative to the canonical equations (2.48) and (2.49). To derive one such set, we start with the five Eqs. (2.48)_{1,2} and (2.49) and note that their enforcement reduces the statically equivalent force system to a force passing through point O and directed along the z -axis. Now, instead of considering the canonical sixth Eq. (2.48)₃, we may take either

$$\sum_{i=1}^N M_{Aix} = 0 \quad (2.50)$$

or

$$\sum_{i=1}^N M_{Aiy} = 0 \quad (2.51)$$

about *any* point A which does not lie on the z -axis.

Example 2.3.1 (Box subject to four forces):

Consider a rectangular box with sides of length a , b , and c along the x -, y -, and z -axes, respectively, and let it be subject to the four forces and an unspecified moment, shown in Fig. 2.32. Force equilibrium is readily established using Eq. (2.48), since there are no forces acting along the x - or z -axes, while the forces along the y -axis trivially sum to zero. For moment equilibrium to hold, the x -, y -, and z -moments about the origin should vanish, that is

$$-Fc - Fc + Fc + M_x = 0$$

$$M_y = 0$$

$$3Fa - Fa - Fa + M_z = 0.$$

This implies that the box is in equilibrium if the applied moment is equal to $Fc\mathbf{i} - Fa\mathbf{k}$.

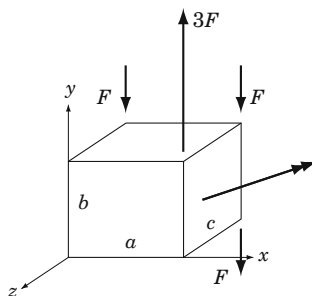


Figure 2.32. A box in equilibrium

2.3.2 Constraints in Three Dimensions

The analysis of the equilibrium of rigid bodies in three dimensions follows the same principles as in two dimensions. However, a three-dimensional rigid body has six degrees of freedom. This may be easily understood by fixing the position of any one particle in the body (three constraints) and noting that the body still possesses the freedom to rotate about any three mutually perpendicular axes, as shown in Fig. 2.33 for the case of a cube; eliminating this degree of freedom requires the specification of three additional constraints. Given the six degrees of freedom, there is a much greater variety of possible support constraints in three-dimensional rigid bodies.

Since a built-in support prevents all rigid-body motion, in three dimensions it may be regarded as a 6-dof constraint. At the other extreme, the 1-dof constraints of Fig. 2.14 function as they do in two dimensions: they prevent translation in one direction only.

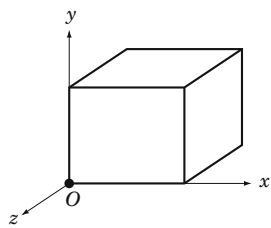


Figure 2.33. Cube with fixed vertex point O that may freely rotate about the x -, y -, and z -axes

Name	"Smooth" contact	Cable	Link	Roller
Physical appearance				
Diagrammatic representation				
Reaction	Force normal to contact surface (push only)	Force along cable (pull only)	Force along link (push or pull)	Force normal to rolling surface (push or pull)
	a	b	c	d

Figure 2.14. (reproduced)

An exception is the roller pictured in Fig. 2.14d. If the circle on the right represents a ball, or if the casters shown on the left can swivel, then the support does, in fact, provide a 1-dof constraint. But if the circle represents a cylinder or if the casters are somehow fixed to roll in one direction, and the surface is “rough,” then the support (shown more explicitly in Fig. 2.35a) is a 2-dof constraint in that it also prevents translation parallel to the axis of the cylinder or caster. In that way it is equivalent to a support provided by two non-collinear links, as shown in Fig. 2.34a. (In Fig. 2.34a,b, any link can be replaced by a cable or a “smooth” contact, with the limitation that the force must be a pull or a push, respectively.)

A 3-dof constraint preventing all translation at a point (but allowing all rotation about it) is provided by three non-coplanar links as shown in Fig. 2.34b or, more simply, by the *ball-and-socket joint* of Fig. 2.34c. As can be seen, a perspective depiction is necessary in order to bring out the three-dimensional nature of the constraints.

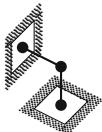
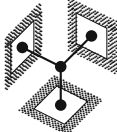
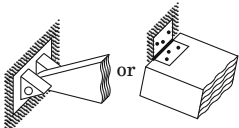
Name	Double link	Triple link	Ball-and-socket joint	Pin or hinge
Illustration				
Reactions	Force along each link	Force along each link	Force in any direction	Force in any direction; moment about any axis perpendicular to pin
	a	b	c	d

Figure 2.34. Some three-dimensional multi-dof constraints

The fixed collar-rod connection of Fig. 2.15c (page 68) is a 4-dof constraint, allowing translation along the rod and rotation about it. It can, however, be made into a 3-dof constraint by allowing rotation at the connection to the collar.

A pin joint or hinge in three dimensions, shown in Fig. 2.34d, is a 5-dof constraint: only rotation about the pin axis is allowed.

Some additional three-dimensional supports are shown in Fig. 2.35.

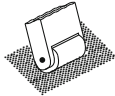
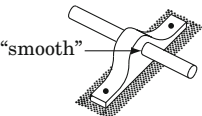
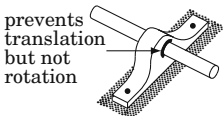
Name	Roller on a “rough” surface	Journal bearing	Thrust bearing
Illustration			
Reactions	Force perpendicular to surface and parallel to pin axis	Force and moment in any direction (about any axis) perpendicular to shaft	Force in any direction; moment about any axis perpendicular to shaft
	a	b	c

Figure 2.35. Additional three-dimensional multi-dof constraints

When combining supports to provide six constraints, care must be taken to make sure that they are proper. A pin can be combined with a single link (or the like) provided the reaction force there is not aligned with the pin axis.

By analogy with the two-dimensional case, if the supports exert forces only, then for static determinacy there must be six force reactions. These can be provided by combinations ranging from a triple link (or, *equivalently*, a ball-and-socket joint), a double link, and a single link (or a similar 1-dof constraint) to six separate 1-dof constraints. For the supports to be proper,

the following conditions are necessary:

- (a) The reaction forces must not be parallel to one plane, otherwise nothing prevents translation perpendicular to that plane.
- (b) No more than three of the reaction forces can be concurrent. By contradiction, suppose that four of them are concurrent, say at point O , and the other two pass through points A and B , whose position vectors relative to O are $\mathbf{r}_A$ and $\mathbf{r}_B$, with the lines of action of the forces given by the unit vectors $\mathbf{n}_A$ and $\mathbf{n}_B$, respectively, as in Fig. 2.36. If the scalar values of the reaction forces at A and B are R_A and R_B , then the moment about O is $\mathbf{r}_A \times R_A \mathbf{n}_A + \mathbf{r}_B \times R_B \mathbf{n}_B$. Consider, now, an axis through point O that is parallel to the vector $(\mathbf{r}_A \times \mathbf{n}_A) \times (\mathbf{r}_B \times \mathbf{n}_B)$; the moment component about that axis is identically zero, and therefore the body is free to rotate about it.

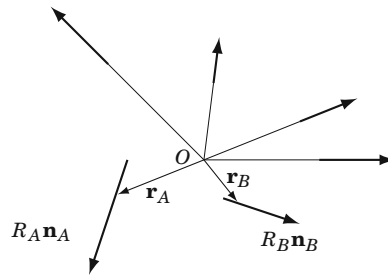


Figure 2.36. System of 6 reaction forces of which 4 are concurrent

- (c) Since concurrent vectors become parallel as the point of concurrency recedes to infinity, it follows that, for the system to be statically determinate, no more than three of the reactions can be parallel. If, for example, a body rests on four supports all of which exert vertical forces, then the remaining two horizontal forces cannot prevent it from rotating about a vertical axis.

Examples of improperly constrained three-dimensional bodies are shown in Fig. 2.37.

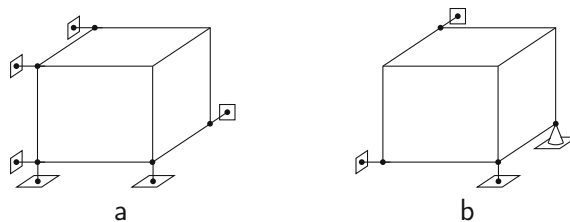


Figure 2.37. Improper combinations of single, double, and triple links

Example 2.3.2 (Box with improper supports):

There are two ways to establish that a set of supports on a body is improper. The first is to visually identify a rigid deformation (translation or rotation) that is not restrained by the supports. In the case of the box in Fig. 2.37a (reproduced again for clarity together with its free-body diagram in Fig. 2.38), we may note, by inspection, that a rotation about the x -axis is completely uninhibited by the supports, which is sufficient to render the supports improper.

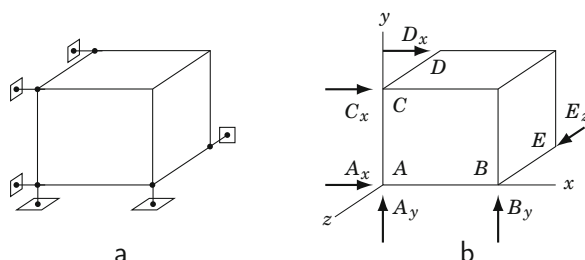


Figure 2.38. An analysis of a three-dimensional rigid body with improper supports

The second way to establish that the set of supports is improper is to draw a free-body diagram (although there are no external loads!) and examine the reaction forces. With reference to Fig. 2.38b, it is clear that there is at least one reaction in each of the three directions x , y , and z , which implies that all three translations are restrained. Likewise, at least one reaction contributes to the moment about the y - and z -axes (as well as all axes parallel to them). This, in turn, means that rotations about any axes parallel to the y - and z -axes are also restrained. However, this is not the case when it comes to moments about the x -axis. Indeed, no reactions contribute to this moment, since the A_x , A_y , B_y and E_z reactions intersect the x -axis and the C_x and D_x reactions are parallel to this axis. Therefore, any external load that results in a nonzero moment about the x -axis cannot be reacted upon by any of the supports and would result in an uninhibited rotation of the body about the x -axis.

The following example illustrates the procedure followed in the solution of three-dimensional equilibrium problems.

Example 2.3.3 (Cantilever assemblage of rigidly connected bars):

Let us consider the cantilevered assemblage of rigidly connected bars shown in Fig. 2.39. Let the respective mean lengths of bars AB , BC , and CD be a , b , and c , so that the coordinates of points A , B , C , and D are $(0,0,c)$, $(a,0,c)$, (a,b,c) , and $(a,b,0)$, respectively. In drawing a free-body diagrams, we simply replace the applied force F by its components and the fixed support D by the reactions acting there, as in Fig. 2.40.

The equations of force equilibrium yield $F_{Dx} = 0$, $F_{Dy} = F \cos \alpha$, and $F_{Dz} = -F \sin \alpha$. Equilibrium of moments about D can be expressed in vector form as

$$\mathbf{i}M_{Dx} + \mathbf{j}M_{Dy} + \mathbf{k}M_{Dz} + (-\mathbf{i}a - \mathbf{j}b + \mathbf{k}c) \times (-\mathbf{j}F \cos \alpha + \mathbf{k}F \sin \alpha) = 0, \quad (2.52)$$

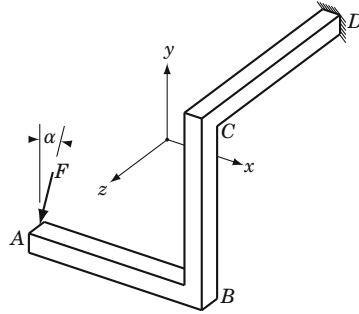


Figure 2.39. Three-dimensional bar assemblage

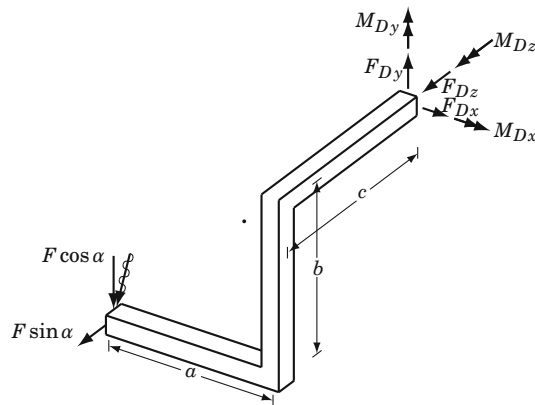


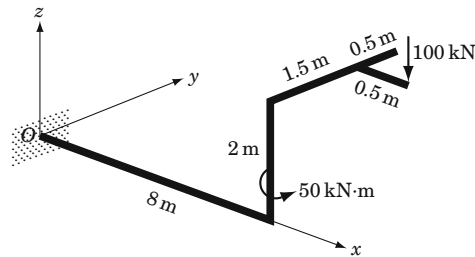
Figure 2.40. Free-body diagram for assemblage of Fig. 2.39

leading to

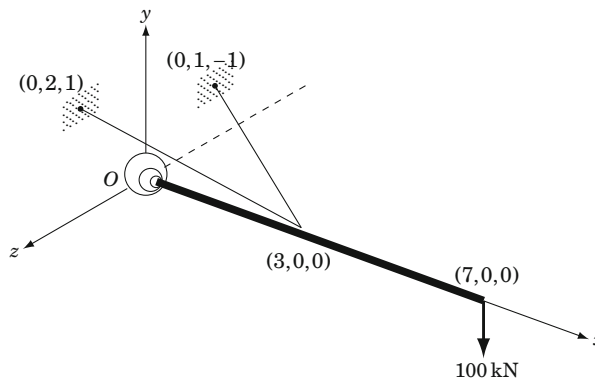
$$M_{Dx} = F(c \cos \alpha - b \sin \alpha), \quad M_{Dy} = Fa \sin \alpha, \quad M_{Dz} = Fa \cos \alpha. \quad (2.53)$$

Exercises

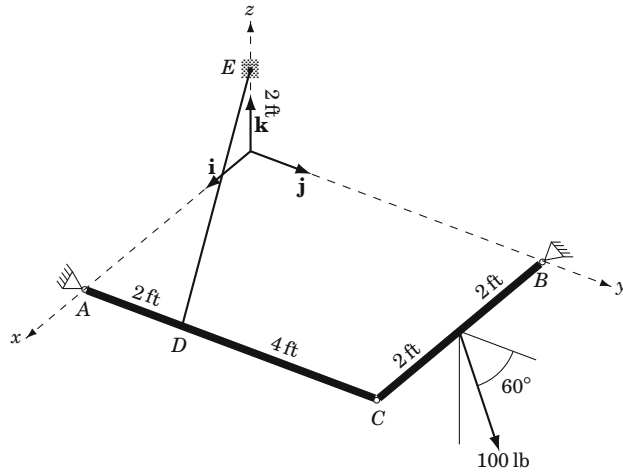
- 2.3-1.** Identify an equivalent system of equilibrium equations to (2.48) and (2.49) that include a single force equilibrium equation, say, $(2.48)_1$ and five moment equilibrium equations. Make sure to justify the equivalence of this system to (2.48) and (2.49).
- 2.3-2.** For the improper combinations of constraints of Fig. 2.37b, find the degree(s) of freedom that are allowed using both geometric and mathematical arguments.
- 2.3-3.** Determine the reactions at the built-in support O of the three-dimensional rigid body depicted in the figure.



- 2.3-4.** A rigid boom is kept in place by a ball-and-socket support at point O and by two rigid links, as in the figure. Determine the reactions at the ball-and-socket as well as the forces in the rigid links due to the force applied at the tip of the boom. For the given applied force, could the rigid links be replaced by cables?

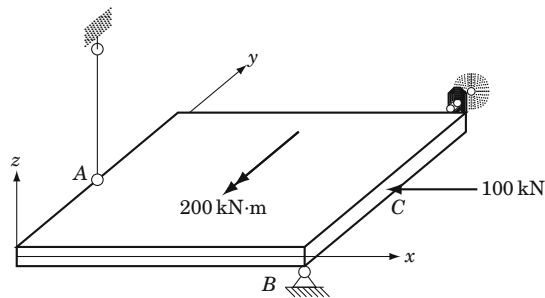


- 2.3-5.** The two rods AC and BC are hinged together at C and are supported by the cable DE and the ball-and-socket joints at A and B . Rod BC is subjected to a force on the plane normal to the x -axis.



- (a) Draw the free-body diagram of the system of the two rods.
- (b) Using the free-body diagram of part (a), determine the tension T in the cable.
Hint: This can be accomplished by writing a single equilibrium equation.

2.3-6. The 1×1 m homogeneous square plate shown in the figure weighs 500 kN and is supported by a rigid link at A (midway on the edge of the plate), a ball-and-socket joint at B and a single journal bearing at D (assume that the bearing is equivalent to a double link along the x - and z -axis, and therefore generates no reaction moments). In addition, the plate is subject to a 100 kN force acting midway on the side BD and a moment of 200 kN·m, as shown in the figure.



- (a) Draw the free-body diagram of the plate.
- (b) Using the free-body diagram of part (a), determine all reaction forces acting on the plate.
- (c) For the given loading, is it possible to replace the rigid link of the system with an inextensible rope without upsetting equilibrium?

- 2.3-7.** Stand against a “smooth” wall with your back straight, your legs together and your heels touching the wall. Now, keeping your back straight, attempt to lift one of your legs. What do you observe? Explain your observation using a sketch of your body with all resultant forces and reactions.
- 2.3-8.** Stand against a “smooth” wall with your back straight, your legs together and one of your shoulders touching the wall so that your body is perpendicular to the wall. Now, keeping your back straight, attempt to lift forward your leg closest to the wall. What do you observe? Returning to the original position, attempt next to lift forward your leg furthest to the wall. What do you observe now? Explain your observations using a sketch of your body with all resultant forces and reactions for both cases.

2.4 Method of Sections

2.4.1 Introduction

The calculation of the support reactions on a rigid body in equilibrium is only one of the purposes of solving the equilibrium equations. In fact, it is not even the most important one. Far more important is the determination of whether the body can carry the loads imposed on it without failing in some way, and that knowledge can be obtained only if the *internal reactions*, that is, the forces and moments that one part of the body exerts on another, are known. If, for example, parts of the body are glued together, the forces transmitted by the glue must be known in order to determine whether it will hold. The same is true if the body is held together with fasteners (nails, bolts, and the like) or through the cohesion of the material itself.

It is important to keep in mind is that the definition of any particular “body” is arbitrary: any portion of matter occupying a certain region in space may also be regarded as a body, and therefore any part of what had previously been defined as a body is in turn another body (which can be called a *subbody*). Furthermore (as was already discussed in Sect. 2.1 in reference to particles), a body is in equilibrium if, and only if, all of its subbodies are in equilibrium.

If the subbodies are imagined as parts into which the original body is divided, then the internal reaction components (force and moment) are conjugate to the imagined relative displacements (translation and rotation) between the parts, as illustrated in Fig. 2.41.

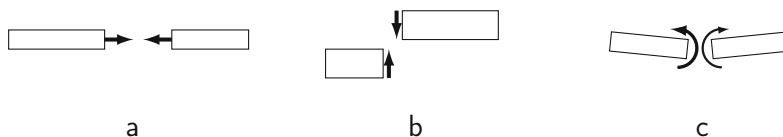


Figure 2.41. Internal reactions and imagined relative displacements of the subbodies

The determination of internal reactions by means of solving the equilibrium equations for subbodies (into which the body is divided by means of a surface drawn through it) is known as the *method of sections*.^{*} As with the determination of support reactions, the first step in the method is the drawing of free-body diagrams. And, of course, no more unknown force or moment components can be determined by statics than the number of independent equilibrium equations. This means that, however, the cohesive forces are distributed over the dividing surface, only their resultants can be found in

^{*}This is the general definition, but “method of sections” is sometimes used in a restricted sense for a specific application of the method that will be discussed in Sect. 3.2.

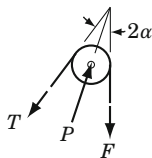


Figure 2.42. Free-body diagram of a pulley

this way. In order to determine such distributions, the deformability of the body must be taken into account, and this will be undertaken starting with Chap. 6.

Example 2.4.1 (Forces in a pulley):

As our first case, we will analyze the forces on the pulley of Fig. 2.27 (page 76). The cut will be made through the cable on both sides of the pulley, and between the pulley and the pin, resulting in the free-body diagram shown in Fig. 2.42. Here T is the tension in the cable to the left of the pulley while P is the push exerted on the pulley by the frictionless pin. Since the free body is a three-force body, the line of action of this push must meet those of the cable tensions and, since it acts through the center of the pulley, it bisects the angle between them. If this angle is 2α , then equilibrium of forces perpendicular to the line of action of P requires that $F \sin \alpha = T \sin \alpha$ and therefore $T = F$, proving that in a weightless cable going around a frictionless pulley the tension is the same on both sides of the pulley. The remaining force equilibrium equation leads to

$$P = 2F \cos \alpha .$$

2.4.2 Slender Bodies

In the case of long slender bodies (usually, but not necessarily, straight) with a well-defined axis, it is generally convenient to define the components of the internal reaction in relation to the axis, as shown in Fig. 2.43, where the body axis is chosen to coincide with the x -axis. The component of the force along the axis is called, naturally enough, the *axial force* and will be denoted P , which stand for “push” or “pull.” The axial component of the moment, which tends to twist the body about the axis, is called the *torsional moment*, *twisting moment* or *torque*, and will be denoted T .[†] The force component transverse to the axis is called the *shear force* and usually denoted V , while the transverse moment is *bending moment*, denoted M . Both the shear force and the bending moment can be further decomposed into components along the two axes perpendicular to the body axis. Note that, while we have here defined $V_y = F_y$, so that its positive direction is the positive y -direction when it is acting on a cut facing

[†]The symbol T is also used, depending on the context, to denote the tension in a cable or the tangential force at a frictional interface.

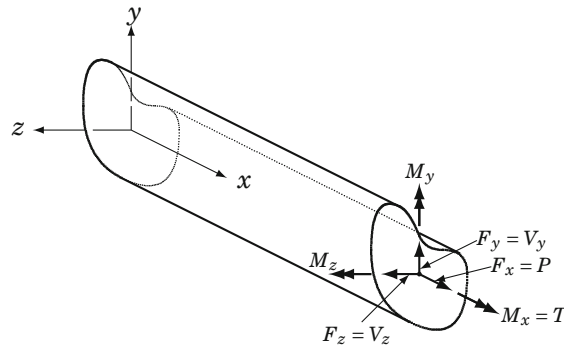


Figure 2.43. Internal reactions in a slender body

the positive x -axis (and vice versa), as shown in the figure, there is another convention, common among structural engineers, with the opposite definition, namely $V_y = -F_y$.

For the other internal reactions, the common sign conventions, regardless of any choice of coordinate system, are as follows (shown in Fig. 2.44):

- (a) Axial force: positive when in tension (each subbody pulls on the other), negative when in compression.
- (b) Torque: positive according to the right-hand rule around an axis pointing outward from the cut.
- (c) Bending moment: positive when causing bending that is convex upward.

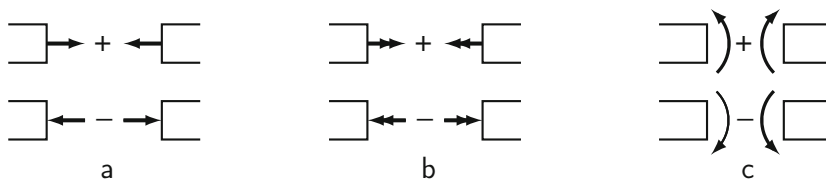


Figure 2.44. Sign conventions for (a) axial force, (b) torque, (c) bending moment

If the loading on a slender body includes forces or moments that are transverse (perpendicular to the axis), resulting in bending moments and shear forces, then, as we already mentioned in Sect. 2.2 (page 76) in a two-dimensional context, it is common to refer to the member as a *beam*. If the body carries only a torsional moment, then it is usually called a *shaft*. Columns, posts, and struts are designed primarily to carry compressive axial

force. A slender body carrying an axial force that is either compressive or tensile is generally known simply as a *bar*.

If the loading on a slender body is planar (say in the xy -plane), then the only internal reactions are $F_x = P$, $F_y = V$, and $M_z = M$. The bending moment is usually depicted by means of a curved arrow having the sense of the moment, as shown in Fig. 2.45.

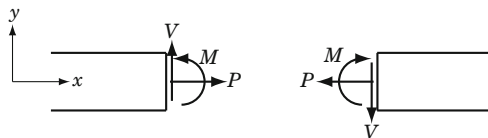


Figure 2.45. Internal reactions in a slender body under planar loading

Example 2.4.2 (Internal reactions in a beam):

We consider, as an example, the beam loaded as shown in Fig. 2.30 (page 78). We divide it into two sections (subbodies) by means of a transverse cut at a distance x from the left end. The corresponding pairs of free-body diagrams are shown in Fig. 2.46. Note that the internal reactions P , V , and M are shown as functions of

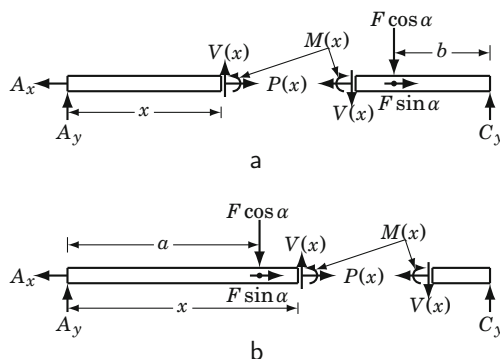


Figure 2.46. Free-body diagrams of sections of the beam of Fig. 2.30. (a) $x < a$, (b) $x > a$

x , since their values depend on the location of the cut. For each of the two cases, $x < a$ and $x > a$, these values can be found by solving the equilibrium equations for either of the two subbodies. Furthermore, the support reactions appearing in the subbody chosen must be determined beforehand from the equilibrium of the whole body. This is not always the case: if the beam is cantilevered, then a subbody that includes the free end has no unknowns other than the internal reactions.

Looking at the left-hand section of Fig. 2.46a, we find that, for force equilibrium, $P(x) = A_x$ and $V(x) = -A_y$, while moment equilibrium about the point $(x, 0)$ yields $M(x) = A_y x$. Inserting the previously derived values of A_x and A_y , we find that, for $x < a$,

$$P(x) = F \sin \alpha, \quad V(x) = -\frac{b}{L} F \cos \alpha, \quad M(x) = \frac{bx}{L} F \cos \alpha \quad (x < a),$$

and the same result will be found for the right-hand section. Similarly, for $x > a$, we find from either section of Fig. 2.46b that

$$P(x) = 0, \quad V(x) = \frac{a}{L}F \cos \alpha, \quad M(x) = \frac{a(L-x)}{L}F \cos \alpha \quad (x > a).$$

In later chapters, we will introduce the common practice of plotting the values of P , V , and M against x , producing what are known, respectively, as the *axial-force*, *shear-force* (or simply *shear*), and *bending-moment* (or simply *moment*) *diagrams*.

As was pointed out in the discussion of Sect. 2.2, any other loading of the beam that is statically equivalent to that of Fig. 2.30 will produce the same support reactions. This is not true, however, of the internal reactions.

Example 2.4.3 (Internal reactions in a beam under a uniformly distributed load):

We now suppose that the loading is transverse only (i.e., $\alpha = 0$, so that $P(x) = 0$) but that, rather than being concentrated at $x = a$, it is uniformly distributed over the beam, with an intensity (force per unit length) of F/L . The equivalent

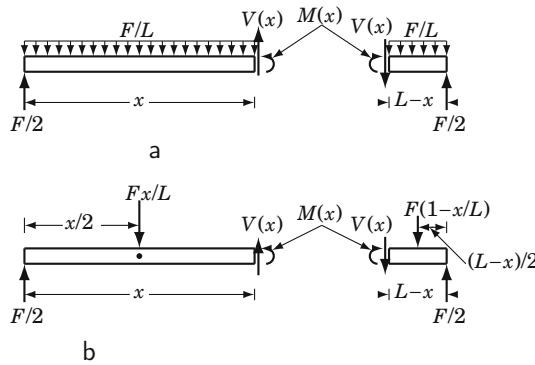


Figure 2.47. Free-body diagrams for a uniformly loaded simply supported beam. (a) Original loading, (b) Statically equivalent loading

concentrated force would then be at $x = L/2 = a = b$, and the end reactions (vertical only) are $A_y = C_y = F/2$. If we now cut the beam at x , we obtain the free-body diagrams shown in Fig. 2.47a, which can be replaced for the purpose of equilibrium analysis of the sections by those of Fig. 2.47b.

Once again, the equilibrium of either the right-hand or the left-hand portion can be used to determine the internal reactions, namely,

$$V(x) = F \left(-\frac{1}{2} + \frac{x}{L} \right), \quad M(x) = \frac{F}{2} x(L-x).$$

We note, in particular, that the maximum bending moment (which occurs at $x = L/2$) is $FL/8$. For the corresponding problem with the load F concentrated at $x = L/2$ (the special case of Fig. 2.30 with $a = 0$ and $a = b = L/2$) the maximum bending moment is $FL/4$.

As we can see from the examples, in straight slender bodies the internal axial force is determined only by the axial loads, while the transverse loading determines the shear force and bending moment. Similarly, the internal torque is determined only by the torque loads. The three types of slender bodies—bars carrying axial loads, shafts carrying torque, and beams carrying transverse loads—will be studied separately, and the corresponding internal force and moment diagrams will be studied in greater depth in conjunction with the deformation of such bodies, once the relations between force and deformation are introduced in Chap. 6. Special consideration must be given to bars subject to an axial force that is compressive: if such bars are slender enough, then they are liable to buckle when the load reaches a critical value. The phenomenon of buckling will be studied on its own in Chap. 10.

This simple categorization of slender bodies does not work if they are curved or bent, like the members of the three-pinned arch of Fig. 2.20 (page 70) or of the three-dimensional assemblage of Fig. 2.39 (page 89).

Example 2.4.4 (Internal reactions in a three-pinned arch):

To determine the internal reactions in the arch, we make a cut through the left-hand member at a point whose coordinates (with the left support as origin) are x, y and where the slope is $dy/dx = \tan \theta$, as shown in Fig. 2.48. We can

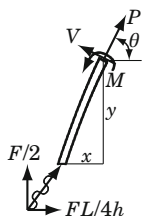


Figure 2.48. Free-body diagram of a portion of an apex-loaded three-hinged arch

solve directly for the internal forces (the axial force P and the shear force V) by considering force equilibrium not in the x - and y -directions but in the tangential and normal directions at x, y , yielding

$$P = -\frac{F}{2} \left(\sin \theta + \frac{L}{2h} \cos \theta \right) \quad , \quad V = -\frac{F}{2} \left(\cos \theta - \frac{L}{2h} \sin \theta \right).$$

Note that the axial force is entirely compressive, a characteristic of arch action. (Of course, if the load were upward, the axial force would be tensile.)

Moment equilibrium leads to

$$M = \frac{F}{2} \left(x - \frac{Ly}{2h} \right).$$

At the apex, where $x = L/2$ and $y = h$, the moment vanishes as it should.

In the special case where the arch is a triangle so that the members are straight, we have $y/x = 2h/L = \tan \theta$. Consequently, $V = M = 0$, and $P = -(F/2)\sqrt{1 + (L/2h)^2}$.

Example 2.4.5 (Internal reactions in a three-dimensional assemblage):

For the internal reactions in the assemblage of Fig. 2.39 (page 89), we perform cuts in each of its three members. Since the assemblage is cantilevered, if we analyze the equilibrium of subbodies that include the free end, then there is no need to know the reactions shown in Fig. 2.40. The corresponding free-body diagrams are shown in Fig. 2.49. Note that the diagrams omit, for the sake of clarity, those internal reaction components that are zero by inspection, namely, P and T in bar AB , and V_x in the other two bars. Note also that the assumed positive directions of the internal reactions in bar CD point in the negative x -, y -, and z -directions, since the cut faces the negative z -direction.

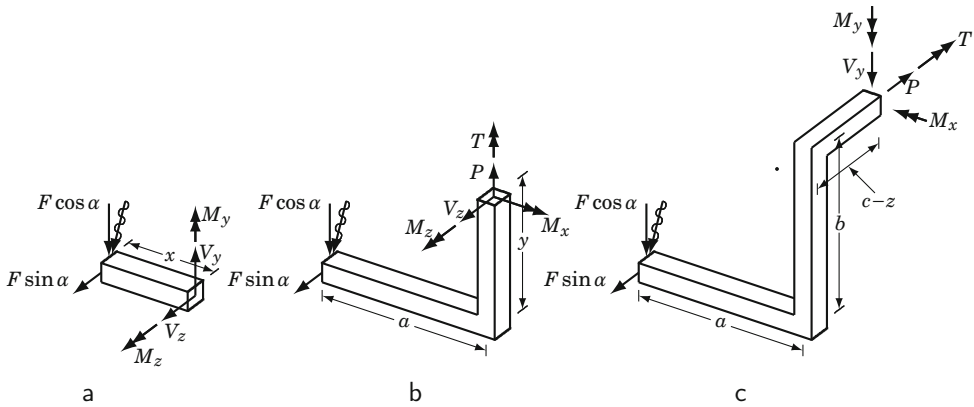


Figure 2.49. Free-body diagrams for the bars of Fig. 2.39. (a) Bar AB , (b) Bar BC , (c) Bar CD

In bar AB , the nontrivial equilibrium equations are four in number and yield the following results:

$$\begin{aligned}\sum F_y &= 0 \Rightarrow V_y = F \cos \alpha \\ \sum F_z &= 0 \Rightarrow V_z = -F \sin \alpha \\ \sum M_y &= 0 \Rightarrow M_y = -(F \sin \alpha)x \\ \sum M_z &= 0 \Rightarrow M_z = -(F \cos \alpha)x.\end{aligned}$$

In bar BC , we find

$$\begin{aligned}\sum F_y &= 0 \Rightarrow P = F \cos \alpha \\ \sum F_z &= 0 \Rightarrow V_z = -F \sin \alpha \\ \sum M_x &= 0 \Rightarrow M_x = (F \sin \alpha)y \\ \sum M_y &= 0 \Rightarrow T = -(F \sin \alpha)a \\ \sum M_z &= 0 \Rightarrow M_z = -(F \cos \alpha)a.\end{aligned}$$

Note that as absolute forces and moments, these reactions are continuous across the joint B , except that their internal functions may change: the shear force V_y in AB becomes the axial force P in BC , while the bending moment M_y becomes the torque T .

Finally, in bar CD

$$\sum F_y = 0 \Rightarrow V_y = -F \cos \alpha$$

$$\sum F_z = 0 \Rightarrow P = F \sin \alpha$$

$$\sum M_x = 0 \Rightarrow M_x = -(F \sin \alpha)b + (F \cos \alpha)(c - z)$$

$$\sum M_y = 0 \Rightarrow M_y = (F \sin \alpha)a$$

$$\sum M_z = 0 \Rightarrow T = -(F \cos \alpha)a$$

Exercises

- 2.4-1.** In the ladder of Example 2.2.4 (page 74), find the internal reactions at a section of the ladder halfway between A and the rung on which the person is standing.
- 2.4-2.** Find the internal reactions on the transverse section located at $a/2$ from the built-in end of the body in Exercise 2.2-4 (page 79).
- 2.4-3.** Find the internal reactions at the hinge C in Exercise 2.3-5 (page 90).
- 2.4-4.** Find the internal reactions on the transverse section located midway between points C and D of the bar in Exercise 2.2-9 (page 80).
- 2.4-5.** Find the internal reactions at the point with coordinates $(5,0,0)$ in Exercise 2.3-4 (page 90).
- 2.4-6.** In the pulley of Example 2.4.1 (page 94), suppose that there is friction between the pulley and the pin, the kinetic coefficient of friction being μ_k . If the radii of the pulley and the pin are R and r , respectively, find the relation between the cable tension T and the force F representing a weight that is (a) raised, (b) lowered.
- 2.4-7.** In the assemblage of Fig. 2.39 (page 89), suppose that the force F applied at A is not as shown but is instead an axial pull on bar AB . Find the internal reactions in the bars.
- 2.4-8.** Suppose that the three-hinged arch of Example 2.2.9 (page 78) has the shape of a parabola given by $y = h[1 - (2x/L)^2]$, the origin being halfway between the supports. Find the internal reactions at the sections where (a) $x = L/4$ and (b) $y = h/2$.

Introduction to Solid Mechanics

An Integrated Approach

Lubliner, J.; Papadopoulos, P.

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