2 Constitutive Equations in Hereditary Integral Form

Abstract

Materials respond to external load by deforming and straining, and by developing stresses. The internal stresses corresponding to a given set of strains depend on the constitution of the material itself. For this reason, the rules that permit calculation of internal stresses from known strains, or vice versa, are called constitutive laws or constitutive equations. There are two equivalent ways to describe the mathematical relationships between stresses and strains for viscoelastic materials. One form uses integrals to define the constitutive relations, while the other relates stresses and strains by means of differential equations. Starting from Boltzmann's superposition principle, this chapter develops the integral form of the one-dimensional constitutive equations for linearly viscoelastic materials. This is followed by a discussion of the principle of fading memory, which helps to define the acceptable analytical forms of the material property functions. It is then shown that the closed-cycle condition (i.e., that the steady-state response of a non-aging viscoelastic material to a periodic excitation be periodic) requires that the material property functions depend only on the difference of their arguments. The chapter also examines the relationships between the relaxation modulus and creep compliance functions in the physical time domain as well as in Laplace-transformed space. Various alternative forms of the integral constitutive equations often encountered in practice are discussed as well.

Keywords

Boltzmann · Constitutive · Convolution · Creep · Cycle · Equilibrium · Fading · Glassy · Hereditary · Isothermal · Laplace · Long-term · Matrix · Memory · Operator · Principle · Relaxation · Symbolic

D. Gutierrez-Lemini, *Engineering Viscoelasticity*, DOI: 10.1007/978-1-4614-8139-3_2,

2.1 Introduction

Materials respond to external stimuli by deforming and straining, that is by changing their shape or size, and by developing stresses. The internal stresses corresponding to a given set of strains depend on the constitution of the material itself. For this reason, the rules that permit calculation of internal stresses from known strains, or vice versa, are called constitutive laws, or, constitutive equations—when such relationships are known in analytical form. The terms stress–strain or strain–stress relations or equations, are widely used to emphasize that the first variable is expressed in terms of the second.

There are two equivalent ways to describe the mathematical relationships between stress and strain for linear viscoelastic materials. One way uses integrals to define these relations, while the other relates stresses and strains through linear ordinary differential equations. In this chapter, we develop the integral form of constitutive equations, leaving for Chap. 3 the discussion of their differential counterparts. All the developments are presented in great mathematical detail but to motivate the proofs, some physical insight is also provided. The level of mathematical detail used to present the subject matter and the exercises in this chapter is intended to give the reader the confidence necessary to engage in independent research, irrespective of the field of interest.

For clarity of presentation, only non-aging materials under isothermal conditions are treated in this and subsequent chapters, until Chap. 6, where the dependence of material properties on temperature is examined. All material functions referred to here are thus presumed independent of age and available at the constant temperature implied in the discussions. The dependence of material property functions on temperature will be omitted but assumed understood.[1]

This chapter starts from Boltzmann's superposition principle and develops the integral form of the one-dimensional constitutive equations for a linearly viscoelastic substance. This is followed by a discussion of the principle of fading memory, which helps to define the acceptable forms of relaxation and compliance functions. It is then shown that the closed-cycle condition (that the steady-state response of a non-aging viscoelastic material to a periodic excitation be periodic) requires that the material property functions depend only on the difference of their arguments, and all transients die out. The chapter also examines various relationships between the relaxation modulus and creep compliance functions, both in the time domain and in Laplace-transformed space. Alternative forms of constitutive equations often encountered in practice are also discussed. We conclude the chapter with a discussion of how to evaluate the work done by external agents acting on a linear viscoelastic material. This topic of great practical use, since, as shown in Chap. 1, viscoelastic materials dissipate as heat, some of the energy that is put into them, and hence polymeric materials are often used in industry to dissipate energy.

[1] On this assumption, for instance, $M(t)$ and $C(t)$ will be used for $M(t,T)$ and $C(t,T)$, respectively.

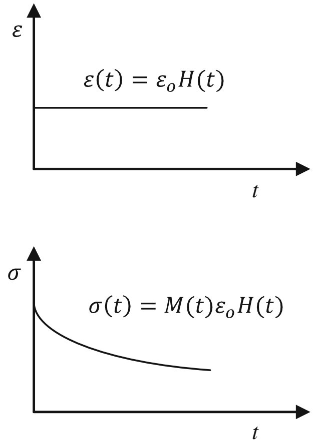

Fig. 2.1 Stress response to a step strain applied at the time the test clock is started

2.2 Boltzmann's Superposition Principle

By definition [c.f. Chap. 1], the tensile relaxation modulus, $M(t,T)$, at any time t, and fixed temperature T describes how the stress varies with time under a step-strain load. To fix ideas, imagine a one-dimensional bar of a linearly viscoelastic material after it is subjected to a strain of magnitude ε_o, suddenly applied at the start of an experiment and held constant thereafter. As seen in (Fig. 2.1), in accordance with Eq. (1.3), the stress response, $\sigma(t)$, of the bar to the applied step strain would be given by:

$$\sigma(t) = \begin{cases} 0, & \text{for } t < 0 \\ M(t)\varepsilon_o, & \text{for } t \geq 0 \end{cases} \tag{a}$$

By the definition of the Heaviside step function H, that: $H(t) = 0$, for negative values of its argument, while $H(t) = 1$, whenever its argument is zero or positive, one can rewrite (a) in the form [c.f. Appendix A]:

$$\sigma(t) = M(t) \cdot H(t)\varepsilon_o \tag{b}$$

Now assume that exactly the same experiment as that described by (a) or (b) were to be carried out using the same material but applying the loading t_1 units of time after "starting the clock." Also assume that all loading[2] and environmental conditions would be the same in both cases. If the material did not age, all its relevant property functions would be exactly the same in both experiments.

[2] The terms "load" and "loading" are used in their broader sense to include tractions, or stresses, as well as displacements, or strains. The exact meaning should be clear from the context in which the term is used.

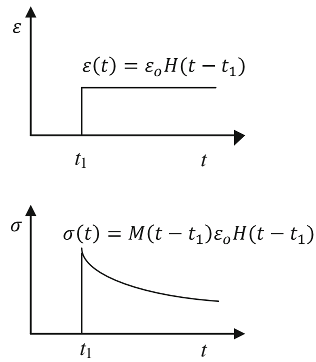

Fig. 2.2 Stress response to a step strain applied t_1 units of time after the test clock is started

Consequently, exactly the same response would be observed in the second experiment as in the first, but with a time delay t_1, as indicated in Fig. 2.2.

Similarly to (a) and (b), the stress response could now be expressed, respectively, as follows:

$$\sigma(t) = \begin{cases} 0, & \text{for} * \tau < \tau_{\varnothing} \\ M(t - t_1)\varepsilon_o, & \text{for} * \tau \geq \tau_{\varnothing} * \end{cases} \tag{c}$$

$$\sigma(t) = M(t - t_1)H(t - t_1)\varepsilon_o \tag{d}$$

It is an easy matter to extend these results to arbitrary load cases. As suggested in Fig. 2.3, any piecewise continuous function of time may be approximated by a series of step functions; with each subsequent step adding an incremental amount to the previous step. Using (c), then, the response to the kth incremental step strain, $\Delta\varepsilon_k$, which is taken to occur at time t_{k+1}, would be:

$$\Delta\sigma_k(t) = M(t - t_k)\Delta\varepsilon_k, \quad t \geq t_k \tag{e}$$

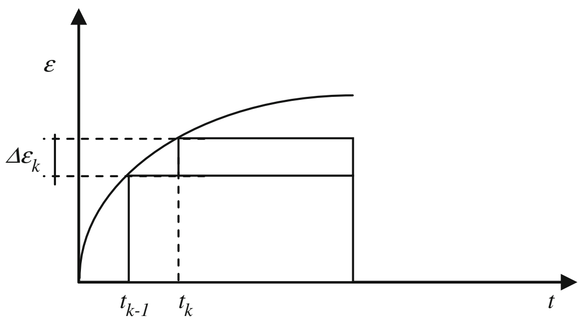

Fig. 2.3 Approximation of a continuous function as a finite series of incremental step functions

According to Boltzmann's principle, the response to each incremental load is independent of those due to the other incremental loads, and the response to the complete load history, as idealized through the series of incremental step-loads, equals the sum of the individual responses:

$$\sigma(t) \approx \sum\nolimits_{k=1}^{N} \Delta\sigma_k(t) = \sum\nolimits_{k=1}^{N} M(t - t_k)\Delta\varepsilon_k, \quad t \geq t_k \tag{f}$$

Dividing and multiplying the right-hand side of (f) by the time interval, $\Delta t_k = t_k - t_{k-1}$, between successive steps, and using the properties of the Heaviside step function, yields:

$$\sigma(t) \approx \sum_{k=1}^{N} \Delta\sigma_k(t) = \sum_{k=1}^{N} M(t - t_k)\frac{\Delta\varepsilon_k}{\Delta t_k}\Delta t_k; \quad t \geq t_k \tag{g}$$

Passing to the limit as N increases without bound and the size of successive intervals is made vanishingly small:

$$\sigma(t) = \lim_{\lim N \to \infty} \sum_{k=1}^{N} \Delta\sigma_k(t) = \lim_{\substack{N \to \infty \\ t_k \to \tau}} \sum_{k=1}^{N} M(t - t_k)\frac{\Delta\varepsilon_k}{\Delta t_k}\Delta t_k; \quad t \geq t_k \tag{h}$$

Since this process turns the discrete set t_k into a continuous spectrum, we use the letter τ to denote it and arrive at[3] (see, for instance, [1]):

$$\sigma(t) = \int_{0^+}^{t} d\sigma(t) = \int_{0^+}^{t} M(t - \tau)\frac{d}{\partial\tau}\varepsilon(\tau)d\tau \tag{i}$$

To allow the strain to have a step discontinuity at time $t = 0^+$, we add (a) and (i) and write:

$$\sigma(t) = M(t)\varepsilon(0^+) + \int_{0^+}^{t} M(t - \tau)\frac{d}{\partial\tau}\varepsilon(\tau)d\tau \tag{2.1a}$$

The term $M(t)\ \varepsilon(0^+)$ may be taken inside the integral, using that $\varepsilon(0^-) \equiv 0$, because:

$$\int_{0^-}^{0^+} M(t - \tau)\frac{d}{\partial\tau}\varepsilon(\tau)d\tau = M(t)\int_{0^-}^{0^+} \frac{d}{d\tau}\varepsilon(\tau)d\tau \equiv M(t)\varepsilon(0^+)$$

[3] The notation x^+ is used to signify a value of x that is just larger than x. Similarly, x^- means a value of x just less than x.

Hence, (2.1a) may be alternatively expressed as:

$$\sigma(t) = \int_{0^-}^{t} M(t-\tau)\frac{d}{\partial\tau}\varepsilon(\tau)d\tau \tag{2.1b}$$

Had we chosen the applied action to be a stress instead of strain history, entirely similar arguments would have led to the strain–stress forms:

$$\varepsilon(t) = C(t)\sigma(0^+) + \int_{0^+}^{t} C(t-\tau)\frac{d}{d\tau}\sigma(\tau)d\tau \tag{2.2a}$$

$$\varepsilon(t) = \int_{0^-}^{t} C(t-\tau)\frac{d}{d\tau}\sigma(\tau)d\tau \tag{2.2b}$$

Equations in (2.1a, b) and (2.2a, b) show that the response of a viscoelastic substance at any point in time depends not only on the value of the action at that instant, but also on the integrated effect, or complete history of all past actions. In other words, the response at the present instant inherits the effects of all past actions. For this reason, viscoelastic materials are also frequently called hereditary materials; and viscoelasticity, hereditary elasticity.

Example 2.1 The (one-dimensional) viscoelastic response to a constant strain-rate loading, $\varepsilon\ (t) = R{\cdot}t$, may be expressed in the elastic form: $\sigma(t) = E_{eff}(t){\cdot}\varepsilon\ (t)$. Derive an expression for $E_{eff}(t)$, the constant-rate effective modulus, for a viscoelastic substance.

Solution:

Assume the relaxation modulus of the viscoelastic material to be $M(t)$ and compute its stress response with (2.1a), using that $d\varepsilon\ (s)/ds = d(Rs)/ds = R$, and introducing the change of variables $t - \tau = u$, to arrive at:

$$\sigma(t) = M(t)\varepsilon(0^+) + R\int_{0^+}^{t} M(t-\tau)d\tau = R\int_{0}^{t} M(u)du$$

Multiplying and dividing this expression by t, recalling that $\varepsilon(t) = R \cdot t$, and re-ordering:

$$\sigma(t) = Rt\frac{1}{t}\int_0^t M(u)du \equiv \left[\frac{1}{t}\int_0^t M(u)du\right]\varepsilon(t) \equiv E_{eff}(t)\varepsilon(t)$$

With the obvious definition of the constant-rate effective modulus, E_{eff}:

$$E_{eff}(t) \equiv \frac{1}{t}\int_0^t M(u)du \tag{2.3}$$

This expression can be used to evaluate the stress response of a viscoelastic material to constant strain-rate loading, by means of the elastic-like expression: $\sigma(t) = E_{eff}(t) \cdot \varepsilon(t)$.

Had the roles of strain and stress been reversed, we would have employed (2.2a) to derive the following definition of the constant-rate effective compliance:

$$D_{eff}(t) \equiv \frac{1}{t}\int_0^t C(u)du \tag{2.4}$$

As before, this can be used to determine the strain at any specified time, of a viscoelastic material subjected to constant-rate stress, using the elastic-like form: $\varepsilon(t) = D_{eff}(t) \cdot \sigma(t)$.

Example 2.2 Obtain the instantaneous response of a viscoelastic material with relaxation modulus, $M(t)$, to a general strain history $\varepsilon(t)$.

Solution:

We evaluate the stress response using expression (2.1a) at $t = 0$, to get:

$$\sigma(t) = M(0)\varepsilon(0^+) \equiv M_g\varepsilon(0^+) \tag{2.5}$$

In similar fashion, (2.1b) would yield the instantaneous strain response to an arbitrary stress history σ(t), as:

$$\varepsilon(t) = C(0)\sigma(0^+) \equiv C_g\sigma(0^+) \tag{2.6}$$

This example indicates that the instantaneous, impact, or glassy response of a non-aging viscoelastic material is elastic, with operating properties equal to its glassy modulus, or its glassy compliance, depending on whether strain or stress, respectively, is the controlled variable.

Example 2.3 Obtain the equilibrium response of a viscoelastic substance with relaxation modulus, $M(t)$, to a general strain history $\varepsilon(t)$.

Solution:

We evaluate the stress response using expression (2.1a) as $t \to \infty$:

$$\sigma(\infty) = \lim_{t\to\infty} \int_{0^-}^{t} M(t-\tau)\frac{d\varepsilon}{d\tau}d\tau = \lim_{t\to\infty}\left[\int_{0^-}^{0^+} M(t-\tau)\frac{d\varepsilon}{d\tau}d\tau + \int_{0^+}^{t} M(t-\tau)\frac{d\varepsilon}{d\tau}d\tau\right]$$

Noting that $\varepsilon(t) \equiv 0$, $t < 0$:

$$\sigma(\infty) = M(\infty)\varepsilon(0^+) + M(\infty)\lim_{t\to\infty}\int_{0^+}^{t}\frac{d\varepsilon}{d\tau}d\tau$$

Or, after canceling like terms, since the integral evaluates to: $\varepsilon(\infty) - \varepsilon(0)$, and $M(\infty)$ is the equilibrium modulus M_e:

$$\sigma(\infty) = M_e\varepsilon(\infty) \tag{2.7}$$

By the same procedure, starting with (2.2a), it is found that the long-term strain response to an arbitrary stress history, σ(t), is given as:

$$\varepsilon(\infty) = C_e\sigma(\infty) \tag{2.8}$$

This example indicates that the long-term response of a non-aging viscoelastic material is elastic, with operating properties equal to either its long-term or equilibrium modulus, or its long-term or equilibrium compliance, depending on whether strain or stress is the controlled variable.

2.3 Principle of Fading Memory

Loosely speaking, we say that a material has fading memory if the influence of an action on its response becomes less important as time goes by. Accordingly, the mathematical implications of the fading memory hypothesis—often called principle—can be established by loading and unloading a viscoelastic system, and monitoring its response after the load is removed. Before establishing the consequences of the principle of fading memory on a rigorous basis, we develop them by examining the response of a viscoelastic material to the relaxation and creep experiments; with which we are already familiar. The results of these experiments are the relaxation modulus and the creep compliance. As discussed in Chap. 1, the general shapes of these functions are as shown in Fig. 2.4.

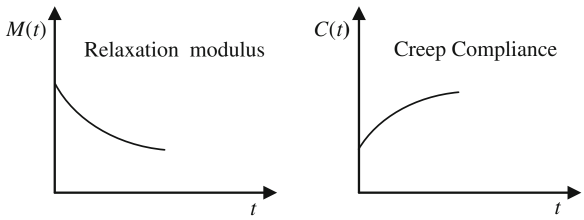

Fig. 2.4 Functional forms of the stress *relaxation modulus* and *creep compliance* of a viscoelastic material used to explain the fading memory hypothesis

The functional forms shown in the figure indicate that the fading memory hypothesis should require that the relaxation modulus be a monotonically decreasing function of time, with monotonically decreasing slope. In similar fashion, the creep compliance should be a monotonically increasing function of time, with monotonically decreasing slope. We now proceed with the rigorous proofs of these statements. To do that, we will take the applied action to be a step strain of magnitude ε_o, applied to a one-dimensional viscoelastic system starting at time $t = 0$ and ending at time $t = t^*$: $\varepsilon(t) = \varepsilon_o[\mathrm{H}(t) - H(t - t^*)]$.

Expression (2.1a) will be used to establish the corresponding response. Before we proceed, we put (2.1a) in a form more suitable to our purposes, integrating it by parts and writing the resulting derivative of the modulus in terms of the time difference, $t - \tau$; thus:

$$\sigma(t) = M(0)\varepsilon(t) + \int_0^t \frac{\partial M(t-\tau)}{\partial(t-\tau)}\varepsilon(\tau)d\tau \tag{2.9}$$

Inserting the step-strain load into this expression leads to the response after the load is removed ($t > t^*$):

$$\sigma(t) = M(0)\varepsilon_o[H(t) - H(t-t^*)] + \int_0^{t^*} \frac{\partial M(t-\tau)}{\partial(t-\tau)}[H(\tau) - H(\tau - t^*)]\varepsilon_o d\tau; \quad t > t^* \tag{j}$$

By the definition of the Heaviside unit step function, the term inside the first set of brackets is zero. The other term in the expression may be evaluated using the mean-value theorem of integral calculus[4] [c.f. Appendix A]:

$$\sigma(t) = t^* \cdot \{\frac{\partial M(t-\lambda t^*)}{\partial(t-\lambda t^*)}\}[H(\lambda t^*) - H(\lambda t^* - t^*)]\varepsilon_o; \quad t > t^*; \quad 0 < \lambda < 1 \tag{k}$$

[4] The mean value theorem of integral calculus states that $\int_a^b f(x)dx = (b-a)f[a+\lambda(b-a)]; \quad 0<\lambda<1.$

Since $\lambda t^* < t^*$, the second Heaviside step function inside the brackets vanishes, so that:

$$\sigma(t) = t^* \cdot \{\frac{\partial M(t-\lambda t^*)}{\partial(t-\lambda t^*)}\}\varepsilon_o; \quad t > t^*; \quad 0<\lambda<1 \tag{l}$$

For the influence of an action removed at $t = t^*$ to eventually disappear, so that $\sigma \to 0$, it is necessary that:

$$\lim_{t\to\infty}\{\frac{\partial M(t-\lambda t^*)}{\partial(t-\lambda t^*)}\} = 0; \quad \forall t^* < \infty; \quad 0<\lambda<1 \tag{m}$$

Or, equivalently:

$$\lim_{t\to\infty}\{\frac{\partial}{\partial t}M(t)\} = 0 \tag{2.10}$$

Otherwise, the material would retain permanent memory of the effect of the applied load, and the process would induce irreversible changes.

As may be seen from (2.9), the derivative, $\partial M(s)/\partial s$, of the relaxation function with respect to its argument acts as a weighting factor on the applied action, ε. For the effect of the action to be less and less pronounced with the passage of time, it is necessary that the weighting factor be a monotonically decreasing function of its argument. That is,

$$\left|\frac{\partial}{\partial t}M(t)\right|_{t=t_2} \le \left|\frac{\partial}{\partial t}M(t)\right|_{t=t_1}; \quad t_2 > t_1 \tag{2.11}$$

Also, as experimental evidence shows [c.f. Chap. 1]:

$$|M(t)|_{t_2} \le |M(t)|_{t_1}; \quad t_2 > t_1 \tag{2.12}$$

In similar fashion, repeating the previous arguments with a step stress applied at $t = 0$ and removed at $t = t^*$, leads to the following requirements for the creep compliance function:

$$\lim_{t\to\infty}\{\frac{\partial}{\partial t}C(t)\} = 0 \tag{2.13}$$

$$\left|\frac{\partial}{\partial t}C(t)\right|_{t=t_2} \le \left|\frac{\partial}{\partial t}C(t)\right|_{t=t_1}; \quad t_2 > t_1 \tag{2.14}$$

$$|C(t)|_{t=t_2} \ge |C(t)|_{t=t_1}; \quad t_2 > t_1 \tag{2.15}$$

Geometrically, then, the fading memory hypothesis simply requires that the relaxation modulus and creep compliance be monotonically decreasing and increasing functions of their arguments, respectively, and also that the absolute values of their slopes decrease monotonically. In addition, as indicated in Chap. 1, experimental observations indicate that:

- The relaxation modulus decreases with observation time and is bounded by the glassy modulus for fast processes and by the equilibrium modulus for very slow processes.
- The creep compliance increases with observation time and is bounded by the glassy and equilibrium compliances for very fast and slow processes, respectively.

The fading memory principle embodied in (2.10)–(2.15), together with the experimental observations, requires that the general forms of the relaxation and creep compliance functions be as shown in Fig. 2.4.

Example 2.4 As an application of the fading memory principle, we evaluate the stress responses of a viscoelastic material to two arbitrary loading programs, $\varepsilon_1(t)$ and $\varepsilon_2(t)$, which reach the same constant value, ε^*, at time t^* and remain at that level from that point on, as indicated in Fig. 2.5.

Solution:

Use (2.1a) to evaluate the response as $t \to \infty$, splitting the integration interval from 0^+ to t^*, and t^* to ∞; and note that the derivatives of the strain histories $\varepsilon_1(t)$ and $\varepsilon_1(t)$ vanish after $t = t^*$ to write:

$$\sigma_1(\infty) = M(\infty)\varepsilon_1(0^+) + \lim_{t\to\infty}\int_{0^+}^{t^*} M(t-\tau)\frac{d\varepsilon_1}{d\tau}d\tau = M(\infty)\varepsilon^*$$

$$\sigma_2(\infty) = M(\infty)\varepsilon_2(0^+) + \lim_{t\to\infty}\int_{0^+}^{t^*} M(t-\tau)\frac{d\varepsilon_2}{d\tau}d\tau = M(\infty)\varepsilon^*$$

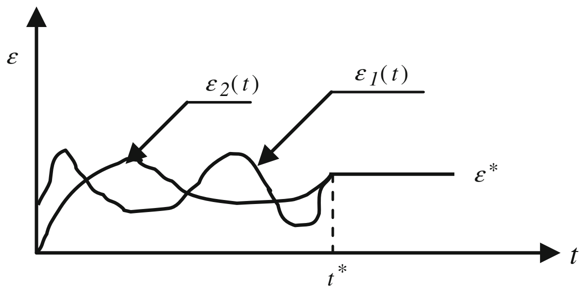

Fig. 2.5 Example 2.4: Two arbitrary loading histories which become identical and constant after a finite time

Since the integrals evaluate to $M(\infty)[\varepsilon_1(t^*) - \varepsilon_1(0^+)]$ and $M(\infty)[\varepsilon_2(t^*) - \varepsilon_2(0^+)]$, and also, $\varepsilon_1(t^*) = \varepsilon_2(t^*) = \varepsilon^*$, it follows that: $\sigma_1(\infty) = \sigma_2(\infty) = M(\infty) \cdot \varepsilon^*$. Or, using the alternate notations for the long-term or equilibrium modulus $M(\infty) \equiv M_\infty \equiv M_e$:

$$\sigma(\infty) = M_\infty \varepsilon^* \equiv M_e \varepsilon^* \tag{2.16}$$

Proceeding in an entirely similar fashion, but using (2.2a), one would find that the long-term, equilibrium strain response to an arbitrary stress history, $\sigma(t)$ would be given by:

$$\varepsilon(\infty) = C_\infty \sigma^* \equiv C_e \sigma^* \tag{2.17}$$

These expressions clearly show that a viscoelastic material would "remember" only that the loading got to ε^*—or σ^*, for that matter—but not how it got there. That is, after sufficiently long, a viscoelastic material will have effectively forgotten the details of the loading history; in agreement with the principle of fading memory.

2.4 Closed-Cycle Condition

This section examines the mathematical consequences of the physical expectation stated in Chap. 1, that the response of a linear viscoelastic material to harmonic loading ought to be harmonic, of the same frequency as the excitation, but out of phase with it. This so-called closed-cycle condition, that: "the steady-state response to harmonic loading also be harmonic," is satisfied by materials that do not age. That is, by materials whose property functions depend only on one timescale: the time measured from when the load was first applied, irrespective of the time elapsed since their manufacturing.

As will be shown in what follows, the closed-cycle condition requires that the kernels of the constitutive integrals, M and C, depend only on the difference of their arguments, and also, that all transients die out. In other words, the closed-cycle condition requires that $M(t, \tau) = M(t - \tau)$, and $C(t, \tau) = C(t - \tau)$, as has been assumed without proof in our derivations, so far. A physical proof of this implication of the closed-cycle condition can be constructed rewriting (2.1a), say, using $M(t, \tau)$, in place of $M(t - \tau)$, in order to remove the assumption made so far in our derivations that the kernel of the constitutive equation depends only on the difference of its arguments:

$$\sigma(t) = M(0)\varepsilon(t) - \int_0^t \frac{\partial}{\partial \tau}\{M(t, \tau)\}\varepsilon(\tau)d\tau \tag{a}$$

The first term in this expression is simply the instantaneous value of the stress response. The second term, the hereditary component, is calculated as follows. In the time interval between τ and $\tau + \mathrm{d}\tau$ of the past, the strain was $\varepsilon(\tau)$. Since the material is assumed to be linear, its memory of this past action should be proportional to the product $\varepsilon(\tau)$ and the duration of the action; that is: $\varepsilon(\tau)\ \mathrm{d}\tau$; producing the stress: $\frac{\partial}{\partial\tau}M(t,\tau)\cdot\varepsilon(\tau)d\tau$. If the material does not age, its properties must be independent of the time when the experiment starts. For this to be the case, the kernel $M(t,\ \tau)$ can only be a function of the difference $t - \tau$. Clearly, the same is true of the creep compliance. In particular, and for this reason, such kernels are called difference kernels.

Proceeding now with the mathematical proof, we evaluate the stress response to a periodic strain of period $p : \varepsilon(t+p) = \varepsilon(t)$, using (a):

$$\sigma(t+p) = M(0)\varepsilon(t+p) - \int_{0}^{t+p} \frac{\partial}{\partial\tau}M(t+p,\tau)\varepsilon(\tau)d\tau \tag{b}$$

Next, introduce the change of variable $\tau = \tau' + p$ and use the stated periodicity of the applied strain, $p : \varepsilon(t+p) = \varepsilon(t)$, to write:

$$\sigma(t+p) = M(0)\varepsilon(t) - \int_{-p}^{t} \frac{\partial}{\partial\tau'}M(t+p,\tau'+p)\varepsilon(\tau')d\tau'$$

Splitting the interval of integration from $-p$ to 0, and from 0 to t; and afterward replacing the new variable of integration, τ' with the original symbol τ, for simplicity, get:

$$\sigma(t+p) = M(0)\varepsilon(t) - \int_{-p}^{0} \frac{\partial}{\partial\tau}M(t+p,\tau+p)\varepsilon(\tau)d\tau - \int_{0}^{t} \frac{\partial}{\partial\tau}M(t+p,\tau+p)\varepsilon(\tau)d\tau \tag{c}$$

Now, use that: $\sigma(t) = M(0)\varepsilon(t) - \int_0^t \frac{\partial}{\partial\tau}M(t,\tau)\varepsilon(\tau)d\tau$, to cast (c) in the form:

$$\begin{aligned}\sigma(t+p) = \sigma(t) + \int_{0}^{t} \left[\frac{\partial}{\partial\tau}M(t,\tau) - \frac{\partial}{\partial\tau}M(t+p,\tau+p)\right]\varepsilon(\tau)d\tau \\ - \int_{-p}^{0} \frac{\partial}{\partial\tau}M(t+p,\tau+p)\varepsilon(\tau)d\tau\end{aligned} \tag{d}$$

It then follows that, for the response to be periodic, that is, for $\sigma(t+p)=\sigma(t)$:

$$\int_0^t \left[\frac{\partial}{\partial\tau}M(t,\tau)-\frac{\partial}{\partial\tau}M(t+p,\tau+p)\right]\varepsilon(\tau)d\tau=0;\quad \forall\varepsilon(t) \tag{e}$$

Together with:

$$\int_{-p}^{0}\frac{\partial}{\partial\tau}M(t+p,\tau+p)\varepsilon(\tau)d\tau=0 \tag{f}$$

Condition (e) implies that:

$$M(t,\tau)-M(t+p,\tau+p)=0 \tag{g}$$

Differentiating this expression with respect to p, and setting $p=0$, afterward, leads to[5]:

$$-\frac{\partial}{\partial(t+p)}M(t+p,\tau+p)\bigg|_{p=0}-\frac{\partial}{\partial(\tau+p)}M(t+p,\tau+p)\bigg|_{p=0}=0 \tag{h}$$

The general solution of this equation is an arbitrary function of $t-\tau$, as is easily verified by direct substitution. Consequently:

$$M(t,\tau)=M(t-\tau) \tag{2.18}$$

According to the fading memory principle, condition (e) is met for arbitrary excitations only in the limit as $t\to\infty$, if the kernel $|\partial M/\partial t\,|$ of the integral is bounded, as indicated by relation (2.10). This means, additionally, that the lower limit in the integral in (b) must be taken as $-\infty$; and that the approximation:

$$\int_0^t \frac{\partial}{\partial\tau}M(t+p,\tau+p)\varepsilon(\tau)d\tau\approx\int_{-\infty}^{t}\frac{\partial}{\partial\tau}M(t+p,\tau+p)\varepsilon(\tau)d\tau \tag{2.19}$$

holds only for sufficiently long times. Otherwise, the response to a periodic excitation, even of a non-aging material, will be non-periodic.

[5] Here, use is made of the total derivative: $\frac{d}{dp}f(x,y)=\frac{\partial}{\partial x}f(x,y)\frac{dx}{dp}+\frac{\partial}{\partial y}f(x,y)\frac{dy}{dp}$.

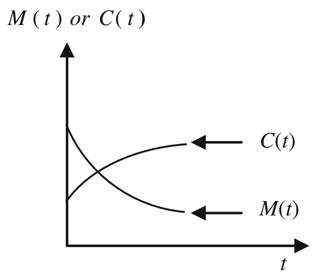

Fig. 2.6 Side-by-side comparison of relaxation modulus and creep compliance

Summarizing: for the response of a viscoelastic material to a cyclic excitation to also be periodic, its material property functions must depend on the difference between current time and loading time.[6] That is, the closed-cycle condition (that the response to periodic excitation be periodic) can only be satisfied by non-aging materials.

2.5 Relationship Between Modulus and Compliance

Expressions (2.1a, b) and (2.2a, b) relate stresses to strains, through the corresponding relaxation modulus and creep compliance. This suggests that the two expressions may be combined in some form to obtain the relationship between the two property functions. Before we go into the mathematical details of this, we use what we have learned already about these two material functions and compare their forms side-by-side in Fig. 2.6.

As suggested by the figure, it is reasonable to expect that the values of C and M at $t = 0$, as well as at sufficiently long times, might be reciprocals of each other. Although one could argue that $C(t) \cdot M(t) \approx 1$ elsewhere, the figure shows that, in general, the creep compliance and the relaxation modulus are not reciprocals of each other. As will be shown subsequently, the values of the relaxation modulus and its creep compliance, for the extreme cases of glassy and equilibrium response are indeed reciprocals of each other, as they would be for elastic solids. However, unlike for elastic materials, the relaxation modulus is not the reciprocal of the creep compliance.

A relationship between relaxation modulus and creep compliance may be derived using (2.1b) to evaluate the stress response to a step-strain history $\varepsilon(t) = \varepsilon_o H(t)$, together with the fact that $dH(t)/dt = \delta(t)$ [c.f. Appendix A]; thus:

$$\sigma(t) = \int_{0^-}^{t} M(t-\tau)\varepsilon_o \delta(\tau) d\tau \equiv M(t)\varepsilon_o \tag{a}$$

[6] Material property functions which depend on the difference between current and loading time are known as “difference” kernels.

Putting this result into (2.2b), taking ε_o outside the integral, and using that $\varepsilon(t) = \varepsilon_o H(t)$:

$$\varepsilon(t) = \left[\int_{0^-}^{t} C(t-\tau)\frac{d}{d\tau}M(\tau)d\tau\right]\varepsilon_o = \varepsilon_o H(t) \tag{b}$$

Canceling out ε_o produces the first form of the relationship between a relaxation modulus and its corresponding creep compliance:

$$\int_{0^-}^{t} C(t-\tau)\frac{d}{d\tau}M(\tau)d\tau = H(t) \tag{2.20}$$

Proceeding in the reverse order, applying a step stress $\sigma(t) = \sigma_o H(t)$, and then calculating the corresponding strain response, the result would be:

$$\int_{0^-}^{t} M(t-\tau)\frac{d}{d\tau}C(\tau)d\tau = H(t) \tag{2.21}$$

As stated earlier, (2.20) and (2.21) show that, in general, the relaxation modulus and creep compliance are not reciprocals of each other. Additional, practical information can be gained by examining the behavior of these expressions as time approaches 0 and ∞; as well as by invoking the consequences of the fading memory principle.

Before proceeding, we note that the integrals in (2.20) and (2.21) correspond to a special class of integrals known as Stieltjes convolutions. Convolution integrals are presented in the next section in the context of viscoelasticity and are fully discussed in Appendix A. As shown in the Appendix, by the commutative property of convolution integrals, Eqs. (2.20) and (2.21) are mathematically equivalent and either one could have been derived from the other.

2.5.1 Elastic Relationships

The relationships between a relaxation modulus and its creep compliance, corresponding to short and long term are obtained by taking the limit of either (2.20) or (2.21), as $t \to 0^+$ and $t \to \infty$, respectively. To do that, (2.20) is rewritten by splitting its integration interval into two intervals going from 0^- to 0^+, and 0^+ to t:

$$\int_{0^-}^{0^+} C(t-\tau)\frac{d}{d\tau}M(\tau)d\tau + \int_{0^+}^{t} C(t-\tau)\frac{d}{d\tau}M(\tau)d\tau = H(t) \tag{c}$$

The relationship between the short-term property functions is developed by letting t → 0; noting that the first integral evaluates to $M(0^+)C(0^+)$ and that the second integral vanishes. Proceeding thus, and using the notation $M(0^+) = M_g$, and $C(0^+) = C_g$, to denote glassy quantities, leads to:

$$M_g = {}^{1}\!/\!{}_{C_g} \tag{2.22}$$

In similar fashion, taking the limit of either (2.20) or (2.21) as t → ∞, and using the notation $M(\infty) = M_{\mathrm{e}}$, and $C(\infty) = C_{\mathrm{e}}$, to denote equilibrium properties:

$$M_e = {}^{1}\!/\!{}_{C_e} \tag{2.23}$$

It is left as an exercise for the reader to derive (2.23).

The last two expressions show that, as pointed out at the beginning of the section, in the extreme cases of short-term (or glassy) and long-term (or equilibrium) response, a relaxation function and its compliance counterpart are indeed reciprocals of each other, just as for elastic materials.

The monotonic nature of the modulus and compliance functions, stated in (2.12) and (2.15), can be used to establish a relationship between them which also shows that modulus and compliance are not, in general, simple inverses of each other [2].

Indeed, using (2.21), say, with the facts that $M(t)$ is a monotonically decreasing function of its argument, so that $M(t-\tau) \geq M(t)$, for all $\tau \geq 0$; and $M(t) = \mathrm{C}(t) \equiv 0$, for $t < 0$, there results:

$$H(t) = \int_{0^-}^{t} M(t-\tau)\frac{\partial}{\partial\tau}C(\tau)d\tau \geq M(t)\int_{0^-}^{t}\frac{\partial}{\partial\tau}C(\tau) = M(t)C(t) \tag{d}$$

That is:

$$M(t)C(t) \leq 1 \tag{2.24}$$

Which, as stated before, shows that in general: $M(t) \neq 1/C(t)$.

2.5.2 Convolution Integral Relationships

The mathematical relationships listed in (2.20) and (2.21) are known as Stieltjes convolution integrals [c.f. Appendix A]. Formally, the Stieltjes integral of two functions, ϕ and ψ, is defined as [3]:

$$\varphi(t-\tau) * d\psi(\tau) \equiv \int\limits_{-\infty}^{t} \varphi(t-\tau)\frac{d}{d\tau}\psi(\tau)d\tau \equiv \varphi * d\psi \tag{e}$$

In which $\varphi(t)$ is assumed continuous in $[0,\infty)$; $\psi(t)$, vanishes at $-\infty$; and the form on the far right is used when the argument, t, is understood.

In line with the mathematical structure of relaxation and compliance functions, the further assumption is made that φ and ψ vanish for all negative arguments, which allows splitting the interval of integration from $-\infty$ to 0^-, and from 0^- to t, to write, more simply:

$$\varphi(t-\tau) * d\psi(\tau) \equiv \int\limits_{0^-}^{t} \varphi(t-\tau)\frac{d}{d\tau}\psi(\tau)d\tau \tag{f}$$

Alternatively, integrating by parts:

$$\varphi(t-\tau) * d\psi(\tau) \equiv \varphi(t)\psi(0^+) + \int\limits_{0^+}^{t} \varphi(t-\tau)\frac{d}{d\tau}\psi(\tau)d\tau \tag{g}$$

As shown in Appendix A, under the stated restrictions on the functions involved, the convolution integral is commutative, associative and distributive. Thus, for any three well-behaved functions, f, g, and h:

$$\begin{gathered} f * g = g * f \\ f * (g * h) = (f * g) * h = f * h * h \\ f * (g + h) = f * g + f * h \end{gathered} \tag{h}$$

Based on their definition, the convolution integral allows writing viscoelastic constitutive equations in elastic-like fashion. Corresponding to (2.1a, b) or (2.2a, b), for instance, we write:

$$\sigma(t) = M(t-\tau) * d\varepsilon(\tau) \equiv M * d\varepsilon \tag{2.25}$$

$$\varepsilon(t) = C(t-\tau) * d\sigma(\tau) \equiv C * d\sigma \tag{2.26}$$

Additionally, corresponding to (2.20) and (2.21), above:

$$C(t-\tau) * dM(\tau) \equiv C * dM = H(t) \tag{2.27}$$

$$M(t-\tau) * dC(\tau) \equiv M * dC = H(t) \tag{2.28}$$

These expressions clearly show that the relaxation modulus and creep compliance are, in general, not mere inverses, but convolution inverses of each other. In addition, the viscoelastic relations in (2.25) and (2.26) look exactly like elastic constitutive equations, if the operation of multiplication is replaced by that of convolution. Using this fact, it is straightforward to write down the viscoelastic constitutive counterparts of any given elastic constitutive equations. This is done by simply replacing the elastic property of interest (modulus or compliance) with the corresponding viscoelastic property, and ordinary multiplication with the convolution operation between the material property function and the applied action (strain or stress).

Example 2.5 Write the viscoelastic version of the three-dimensional constitutive equations of a linear isotropic elastic solid which has its stress–strain equations split into a spherical and a deviatoric part as follows[7]: $\sigma_S = 3K\varepsilon_S; \quad \sigma_{Dij} = 2G\varepsilon_{Dij}; \; i, j = 1, 3$

Solution:

Although three-dimensional constitutive equations will be discussed at length in Chap. 8, this exercise is meant to get the reader comfortable with writing the viscoelastic counterparts of elastic constitutive equations. So, whatever the meaning of the symbols involved, replace the elastic products with convolutions to write the results directly: $\sigma_S(t) = 3K(t-\tau) * d\varepsilon_S(\tau); \quad \sigma_{Dij}(t) = 2G(t-\tau) * d\varepsilon_{Dij}(\tau); \; i, j = 1, 3.$

2.5.3 Laplace-Transformed Relationships

Since linear viscoelastic constitutive equations correspond to convolution integrals, one may apply the Laplace transform to convert them into algebraic equations. As explained in Appendix A, any piecewise continuous function, $f(t)$, of exponential order—that is, bounded by a finite exponential function—has a Laplace transform, $\bar{f}(s)$, defined as:

$$L\{f(t)\} \equiv \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt \tag{i}$$

[7] The 3 × 3 stress and strain matrices—indeed any square matrix of any order—may be split into a spherical and a deviatoric part. The spherical part is a diagonal matrix with each of its three non-zero entries equal to the average of the diagonal elements of the original matrix. Therefore, any one of its non-zero entries may be used to represent it. The deviatoric part of the matrix is, by definition, the matrix that is left over from such decomposition. This decomposition is discussed fully in Appendix B.

Some properties of the Laplace transform are presented in Appendix A. We list the following two and use them to transform convolution integrals in the time domain, t, into algebraic expressions in the transform variable, s.

$$\text{Transform of first derivative:} \quad L\{\frac{d}{dt}f\} = s\bar{f}(s) - f(0) \tag{j}$$

$$\text{Transform of the convolution:} \quad L\{f * g\} = \bar{f}(s)\bar{g}(s) \tag{k}$$

Indeed, applying these expressions to the convolution forms (2.25) and (2.26), respectively, results in the following algebraic form of the constitutive equations:

$$\overline{\sigma}(s) = s\overline{M}(s)\overline{\varepsilon}(s) \tag{2.29}$$

$$\overline{\varepsilon}(s) = s\overline{C}(s)\overline{\sigma}(s) \tag{2.30}$$

The same results would have been obtained if the Laplace transform had been applied to the original stress–strain and strain–stress equations, (2.1a, b) and (2.2a, b). For example, if the Laplace transform is applied to both sides of (2.1a), the relationship in (2.29) would be obtained, after collecting terms as follows:

$$\overline{\sigma}(s) = \overline{M}(s)\varepsilon(0^+) + \overline{M}(s) \cdot [s\varepsilon(s) - \varepsilon(0^+)] = s\overline{M}(s) \cdot \overline{\varepsilon}(s) \tag{l}$$

The advantage of taking the Laplace transform of viscoelastic constitutive equations is that the transformed expressions involve only products of the transform of the material property function of interest (modulus or compliance) and the Laplace transform of the input function—strain or stress, just like elastic constitutive equations do. In other words, the Laplace transform converts a viscoelastic constitutive equation into an elastic-like expression between transformed variables. Conversely, if each material property in an elastic constitutive relation is replaced by its Carson[8] transform and each input variable in it is replaced by its Laplace transform, the resulting expression must stand for the Laplace transform of the corresponding viscoelastic constitutive equation. Thus, as in the case of the convolution notation, this equivalence between elastic constitutive relations and the Laplace transform of viscoelastic equations allows one to write down the transformed viscoelastic constitutive equations directly from the elastic ones. This equivalence forms the basis of a so-called elastic–viscoelastic correspondence principle, which is presented in Chap. 9.

Example 2.6 Use the elastic–viscoelastic correspondence to write down the viscoelastic version of the three-dimensional constitutive equations of the linear isotropic elastic solid of Example 2.5.

[8] The s-multiplied Laplace transform of a function is simply called the Carson transform of the function.

Solution:

Using the elastic–viscoelastic correspondence, write the Laplace transform of the elastic expressions as: $\bar{\sigma}_S = 3s\bar{K}\bar{\varepsilon}_S; \quad \bar{\sigma}\mathrm{D}_{ij} = 2s\bar{G}\bar{\varepsilon}\mathrm{D}_{ij}; \quad i, j = 1, 3$. The viscoelastic constitutive equations are obtained taking the inverse Laplace transform of the forms given. Thus, $\sigma_S = 3K * d\varepsilon_S; \quad \sigma\mathrm{D}_{ij} = 2G * d\varepsilon\mathrm{D}_{ij}; \quad i, j = 1, 3$.

The relationship between the relaxation modulus, M, and the creep compliance, C, in the transformed plane, can be obtained either by applying the Laplace transform to (2.27) or (2.28), or by combining the algebraic expressions (2.29) and (2.30). In either case, there results:

$$\overline{M}(s)\overline{C}(s) = \frac{1}{s^2} \tag{2.31}$$

Example 2.7 The relaxation modulus of a viscoelastic solid is given by $M(t) = M_e + M_1 e^{-\alpha t}$. Use expression (2.31) and Laplace transform inversion to obtain its creep compliance, assuming the latter is a function of the form: $C(t) = C_e - C_1 e^{-\beta t}$.

Solution:

According to (2.31), the creep compliance function would be given by the inverse Laplace transform of the function $1/s^2\bar{M}(s)$. Hence, we first evaluate this function, then invert it, and equate it to the Laplace transform $\bar{C}(s) = C_e/s - C_1/(s+\beta)$, of the desired creep compliance. Proceeding thus, using the table of transforms included in Appendix A, and simplifying, there results: $s^2\bar{M}(s) = \dfrac{s+\alpha}{s[M_e\alpha + (M_e + M_1)s]}$. Expanding this rational function into its partial fractions, as explained in Appendix A; using the notation $M_e + M_1 = M_g$, simplifying and equating the result to the Laplace transform of $C(t)$, there results: $\frac{C_e}{s} - \dfrac{C_1}{(s+\beta)} = \dfrac{1/M_e}{s} + \dfrac{(M_e - M_g)/M_e M_g}{(s + \alpha M_e/M_g)}$. Equating coefficients of the corresponding powers of s yields: $C_e = 1/M_e$, $C_1 = (M_g - M_e)/(M_e M_g)$, $\beta = M_e\,\alpha/M_g$.

More general methods of approximate and exact inversion of material property functions given as sums of exponential functions are presented in Chap. 7.

2.6 Alternate Integral Forms

Depending on preference, and the application at hand, the integral constitutive equations for viscoelastic substances may be written in several different ways. The mathematical operations that are used to transform one constitutive form into another—most typically, integration by parts—require that the material property functions involved, and their time derivatives, be bounded. On occasion, the transformations also assume that the material property functions vanish identically for all negative time.

For ease of reference, we list Boltzmann's equation (2.1b), where it was noted that $\varepsilon\,(t)\;\equiv\;0$, for $t < 0$, allowed us to write [4]:

$$\sigma(t) = \int\limits_{0^-}^{t} M(t-\tau)\frac{d\varepsilon}{d\tau}d\tau \tag{2.32a}$$

On the physical expectation that $M(t)$ be bounded for all values of time, and requiring that $\varepsilon\,(t) \to 0$, as $t \to -\infty$, which is satisfied, since both $\varepsilon\,(t) \to 0$, and $d\varepsilon/dt\;\equiv\;0$, for $t < 0$, one may extend the lower limit of integration to $-\infty$, in (2.32a), without altering its value. Thus,

$$\sigma(t) = \int\limits_{-\infty}^{t} M(t-\tau)\frac{d\varepsilon}{d\tau}d\tau \tag{2.33a}$$

Another useful form is obtained integrating (2.32a) by parts and simplifying:

$$\sigma(t) = M(0)\varepsilon(t) - \int\limits_{0}^{t} \frac{\partial}{\partial\tau}M(t-\tau)\varepsilon(\tau)d\tau \tag{2.34a}$$

Using the notation $M\;(0)\;\equiv\;M_g$, and introducing the normalized function $m(t)\;\equiv\;M(t)/M_g$:

$$\sigma(t) = M_g\left\{\varepsilon(t) - \int\limits_{0}^{t} \frac{\partial}{\partial\tau}m(t-\tau)\varepsilon(\tau)d\tau\right\} \tag{2.35a}$$

Using the notation $\sigma_g\,(t)\;\equiv\;M_g\cdot\varepsilon\,(t)$, and taking M_g inside the integral, produces the form:

$$\sigma(t) = \sigma_g(t) - \int\limits_{0}^{t} \frac{\partial}{\partial\tau}m(t-\tau)\sigma_g(\tau)d\tau \tag{2.36}$$

An important application of this is in the derivation of constitutive equations for materials that are termed hyper-viscoelastic. Equations for hyper-viscoelastic materials are derived from those of hyper-elastic materials. A material is termed hyper-elastic, if there exists a potential function of the strains, say, W, such that each individual stress component in such a material may be computed as the derivative of W with respect to the corresponding strain [5]. Since both the glassy or short-term and the equilibrium or long-term responses of a viscoelastic solid are elastic, either can be used to define the potential function. We proceed by using the glassy response; thus

$$\sigma_g(t) = \frac{\partial}{\partial \varepsilon(t)} W_g(\varepsilon(t)) \tag{2.37}$$

The stress–strain law in (2.36) would then take the equivalent form:

$$\sigma(t) = \frac{\partial}{\partial \varepsilon(t)} W_g(\varepsilon(t)) - \int_0^t \frac{\partial}{\partial \tau} m(t-\tau) \frac{\partial}{\partial \varepsilon(\tau)} W_g(\varepsilon(\tau)) d\tau \tag{2.38}$$

Another form, which allows a generalization to non-linear viscoelasticity, is derived by introducing the strain relative to the configuration at time t: $\varepsilon_{rel}(t, \tau) = \varepsilon(t) - \varepsilon(\tau)$. Using that $\varepsilon_{rel}(t, 0) = \varepsilon(t)$ in (2.35a) yields:

$$\sigma(t) = M_g \left\{ \varepsilon_{rel}(t) + \int_0^t \frac{\partial}{\partial \tau} m(t-\tau) \varepsilon_{rel}(\tau) d\tau \right\} \tag{2.39}$$

Constitutive Eqs. (2.34a, b), (2.35a, b), (2.36) and (2.39) are also frequently written in terms of integral operators, using convolution integral notation, but the exact form of the kernel (i.e., the derivative of the relaxation function) is not disclosed. With the obvious definitions, those equations would read:

$$\sigma(t) = M_g\{\varepsilon(t) - \Gamma(\mathrm{t-s}) * \varepsilon(s)\} \tag{2.40a}$$

$$\sigma(t) = M_g\{1 - \Gamma(\mathrm{t-s}) *\}\varepsilon(t) \tag{2.40b}$$

$$\sigma(t) = \sigma_g(t) - \Gamma(\mathrm{t-s}) * \sigma_g(s) \tag{2.41a}$$

$$\sigma(t) = \{1 - \Gamma(\mathrm{t-s}) *\}\sigma_g(t) \tag{2.41b}$$

$$\sigma(t) = M_g \varepsilon_{rel}(t, 0) + M_g \Gamma(\mathrm{t-s}) * \varepsilon_{rel}(t, s) \tag{2.42}$$

Example 2.8 Use (2.37) and (2.38) to develop the stress–strain law for a hyper-viscoelastic material having normalized relaxation function, $m = a + (1-a)e^{-t/\eta}$, if it is known that its glassy response can be established from the potential function of the strains $W_g = ½E\varepsilon^2(t)$.

Solution:

The hyper-viscoelastic form is derived by putting the given functions into (2.37) and (2.38) directly. Evaluating (2.37) first: $\sigma_g(t) = \frac{\partial}{\partial \varepsilon} W_g(\varepsilon(t)) = E \cdot \varepsilon(t)$.

Taking this result and m into (2.38) and re-arranging: $\sigma(t) = E \cdot \varepsilon(t) - \int\limits_0^t \left[\frac{\partial}{\partial \tau}\{E[a+ (1-a)]e^{-(t-\tau)/\eta}\} \cdot \varepsilon(\tau)\right] d\tau$.

Comparing this with (2.34a) shows that hyper-viscoelastic material in question is linearly viscoelastic with relaxation modulus: $M(t) = E[a + (1-a)e^{-t/\eta}]$.

We end this section by presenting some of the constitutive equations in strain–stress form which are the exact counterparts of the foregoing expressions. These strain–stress forms are derived by reversing the roles of stress and strain in the arguments that led to the previous forms. For instance, the strain–stress equations analogous to (2.32a)–(2.35a) are:

$$\varepsilon(t) = \int\limits_{0^-}^{t} C(t-\tau)\frac{d\sigma}{d\tau}d\tau \tag{2.32b}$$

$$\varepsilon(t) = \int\limits_{-\infty}^{t} C(t-\tau)\frac{d\sigma}{d\tau}d\tau \tag{2.33b}$$

$$\varepsilon(t) = C(0)\varepsilon(t) - \int\limits_{0}^{t} \frac{\partial}{\partial \tau} C(t-\tau)\sigma(\tau)d\tau \tag{2.34b}$$

$$\varepsilon(t) = C_g\left\{\sigma(t) - \int\limits_{0}^{t} \frac{\partial}{\partial \tau} c(t-\tau)\tau(\tau)d\tau\right\} \tag{2.35b}$$

2.7 Work and Energy

Under the action of external agents, be these loads or displacements, a deformable body will change its configuration and, if not properly restrained, undergo large-scale motion. At any rate, as the points of application of the external agents move, work—defined as the product of force and displacement—is performed on the body. To develop an expression for the work performed on a body by the external agents, we use a uniaxial specimen of constant cross-sectional area, A, initial length, l, and volume, V, that is loaded at its ends by either a displacement u or a force F. With this, the rate of work of the external forces—that is, force times displacement rate—can then be expressed as:

$$\frac{dW}{dt} \equiv F\frac{du}{dt} \tag{2.43}$$

Multiplying and dividing by the specimen's volume $V = A{\cdot}l$, and using that the strain is given $\varepsilon = u/l$, we cast the previous expression in the form:

$$\frac{dW}{dt} \equiv F\frac{du}{dt} = \frac{F}{A}A \cdot l \cdot \frac{d(u/l)}{dt} \equiv V \cdot \sigma \cdot \frac{d\varepsilon}{dt} \tag{2.44}$$

The total work performed during a time interval (0, t) is given by the integral:

$$W|_0^t = \int_0^t \frac{dW}{ds}ds \equiv \int_0^t F(s)\frac{du}{ds}ds \tag{2.45}$$

Combining (2.44) and (2.45) and dividing the result by the specimen's volume, V, produces the work per unit volume, W_V, that is input into the system:

$$W_V|_0^t = (1/V)W|_0^t = \int_0^t \sigma(s)\frac{d\varepsilon(s)}{ds}ds \tag{2.46}$$

In practical applications, we insert an appropriate form or another of the constitutive equation, such as (2.1b) and write (2.46) as:

$$W_V|_0^t = \int_{s=0}^{t}\int_{\tau=0}^{s} M(s-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau\frac{d\varepsilon(s)}{ds}ds \tag{2.47}$$

Suitable functions (bounded and piecewise continuous) allow interchanging the order of integration. Before doing this, we note that the relaxation modulus, $M(t)$, is defined only for positive values of time. One can continue it to negative values of its argument in an arbitrary manner. In particular, it is sometimes convenient to assume $M(t)$ either as an even or an odd function of time, that is,

$$M(t) = M(-t) \tag{2.48}$$

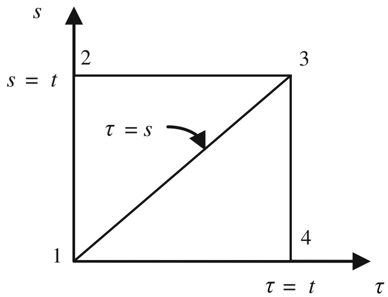

Fig. 2.7 Region of integration used for change of variables in work expression

$$M(t) = -M(-t) \tag{2.49}$$

Here, we take M to be an even function of time. With this, the integral in (2.47) in the τ-s plane is taken over the area of a right triangle with base of length t, whose hypotenuse is the ray $\tau = s$, as indicated in Fig. 2.7.

Using (2.48) and noting that the square $\overline{1234}$ in the figure is made up of two triangles of equal area, and that on account of (2.48), the value of M is the same on points that are symmetrically located about the diagonal of the square, there results:

$$W_V|_0^t = \frac{1}{2}\int\limits_{\tau=0}^{t}\left[\int\limits_{s=0}^{t} M(s-\tau)\frac{d\varepsilon(s)}{ds}ds\right]\frac{d\varepsilon(\tau)}{d\tau}d\tau \tag{2.50}$$

The expressions derived here apply equally to any other one-dimensional pair or work-conjugate quantities, such as shearing force and deflection, torque and twist angle, or bending moment and rotation.

2.8 Problems

P.2.1 Determine the constant-rate effective modulus, $E_{eff}(t)$, of a one-dimensional solid made of a viscoelastic material whose relaxation modulus is: $M(t) = E_e + E_1 e^{-t/\tau_1}$.

$$\text{Answer}: \; E_{eff}(t) = E_e + \frac{E_1}{(1/\tau_1)}\left[1 - e^{-t/\tau_1}\right]$$

Hint: Use $M(t)$ with the defining expression derived in Example 2.1 and carry out the indicated integration.

P.2.2 Use convolution notation to derive the relationship between relaxation modulus and creep compliance.

$$\text{Answer}: \; M(t-\tau) * dC(\tau) = H(t)$$

Hint: Combine (2.25) and (2.26) to get $\sigma(t) = M(t-\tau) * d\varepsilon(\tau) \equiv M(t-\tau) * d\{C(\tau - s) * d\sigma(s)\}$; then, use that: $\sigma(t) = \sigma_o H(t)$ and thus $d\sigma(t) = \sigma_o \delta(t)$ to evaluate the convolution integral inside the braces, and obtain: $\sigma_o = M(t-\tau) * dC(\tau)\sigma_o$, from which the desired result follows.

P.2.3 As presented in Chap. 7, a popular analytical form used to represent relaxation functions consists of a finite sum of decaying exponentials, which in the literature is usually referred to as a Dirichlet–Prony series or, more simply, Prony series:

$$M(t) = M_e + \sum_{i=1}^{N} M_i e^{-t/\tau_i}; \quad M_e \geq 0; \quad and: \ M_i, \quad \tau_i > 0, \ \forall i$$

In this expression, M_e represents the equilibrium modulus, which is zero for a viscoelastic liquid [c.f. Chap. 1]. Also, although the τ_i's represent relaxation times of the material and are thus material properties, in practice, they, as well as M_e and the coefficients M_i, are all established by fitting the Prony series to experimental data. Prove that such forms satisfy the requirements of fading memory.

Hint:

(a) Evaluate the derivative of the series as $t \to \infty$ to show it satisfies (2.10).

$$\lim_{t\to\infty}\left\{\frac{\partial}{\partial t}M(t)\right\} \equiv \lim_{t\to\infty}\left\{-\sum_{1}^{N}\frac{M_i}{\tau_i}e^{-t/\tau_i}\right\} = 0$$

(b) Compare the values of the function at $t_2 > t_1$, to prove that the Prony series is a monotonically decreasing function, in accordance with (2.12).

$$M(t_2) = \sum_{1}^{N} M_i e^{-t_2/\tau_i} \leq \sum_{1}^{N} M_i e^{-t_1/\tau_i} = M(t_1); \quad \forall\, t_2 > t_1$$

(c) Evaluate the derivative of the series at $t_2 > t_1$ and prove that the absolute value of its derivative is also monotonically decreasing, satisfying (2.11).

$$\left|\frac{\partial}{\partial t}M(t)\right|_{t=t_2} \equiv \left|-\sum_{1}^{N}\frac{M_i}{\tau_i}e^{-t_2/\tau_i}\right| \leq \left|-\sum_{1}^{N}\frac{M_i}{\tau_i}e^{-t_1/\tau_i}\right| \equiv \left|\frac{\partial}{\partial t}M(t)\right|_{t=t_1}; \quad \forall\, t_2 > t_1$$

P.2.4 As discussed in Chap. 7, the power-law form: $M(t) = M_e + M_t(1 + \frac{t}{\alpha})^{-p}$ is also used to represent the relaxation function of viscoelastic solids. Show that this form satisfies the requirements of fading memory. In this expression, M_e, M_t, α and p, are all positive.

Hint:

(a) Proceed as in P.2.3 and evaluate the derivative of the given power-law form as $t \to \infty$ to show it satisfies (2.10).

$$\lim_{t\to\infty}\left\{\frac{\partial}{\partial t}M(t)\right\} \equiv \lim_{t\to\infty}\left\{-\frac{pM_t}{\alpha}\frac{1}{(1+t/\alpha)^{(1+p)}}\right\} = 0$$

(b) Compare the values of the given function at $t_2 > t_1$, to prove that this power-law form is a monotonically decreasing function, in accordance with (2.12).

$$M(t_2) \equiv M_e + M_t(1 + \frac{t_2}{\alpha})^{-p} \leq M_e + M_t(1 + \frac{t_1}{\alpha})^{-p}; \quad t_2 > t_1$$

(c) Evaluate the derivative of the function at $t_2 > t_1$ and prove that the absolute value of its derivative is also monotonically decreasing, satisfying (2.11).

$$\left|\frac{\partial}{\partial t}M(t)\right|_{t_2} \equiv \left|-\frac{pM_t}{\alpha}(1+\frac{t_2}{\alpha})^{-(1+p)}\right| \leq \left|-\frac{pM_t}{\alpha}(1+\frac{t_1}{\alpha})^{-(1+p)}\right| \equiv \left|\frac{\partial}{\partial t}M(t)\right|_{t_1};$$
$$t_2 > t_1$$

P.2.5 Compute the steady-state response of the one-dimensional solid of P.2.1 if it is subjected to the cyclic strain history $\varepsilon(t) = \varepsilon_o \cos(\omega t)$.

$$\text{Answer}: \ \sigma(t) = \left[E_e + \frac{E_1(\omega\tau)^2}{1+(\omega\tau)^2}\right]\varepsilon_o \cos(\omega t) - \frac{E_1(\omega\tau)}{1+(\omega\tau)^2}\varepsilon_o cos(\omega t)$$

Hint: Take the strain history into (2.1b); use integration-by-parts twice; simplify, and discard the transient term: $E_1\varepsilon_o\frac{(\omega\tau)}{1+(\omega\tau)^2}e^{-t/\tau}$ to obtain the desired result.

P.2.6 Repeat problem P.2.5 if the cyclic strain history is $\varepsilon(t) = \varepsilon_o \sin(\omega t)$.

$$\text{Answer}: \ \sigma(t) = \left[E_e + \frac{E_1(\omega\tau)^2}{1+(\omega\tau)^2}\right]\varepsilon_o \sin(\omega t) + \frac{E_1(\omega\tau)}{1+(\omega\tau)^2}\varepsilon_o cos(\omega t)$$

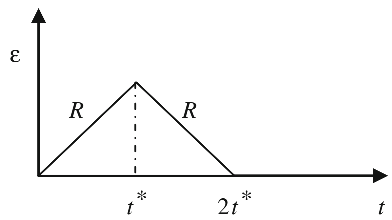

Fig. 2.8 Problem 2.7

Hint: Take the strain history into (2.1b); use integration-by-parts twice; simplify, and discard the transient term $E_1\varepsilon_o\frac{(\omega\tau)}{1+(\omega\tau)^2}e^{-t/\tau}$ to obtain the desired result.

P.2.7 A uniaxial bar of a viscoelastic solid with relaxation modulus $M(t)$ is subjected to a constant-rate load–unload strain history, as shown in Fig. 2.8. Prove that a non-zero stress will exist in the bar at the time when the strain reaches zero at the end of the load-unload cycle.

Hint:

Using that $\varepsilon(t) = \begin{cases} Rt, & t \le t^* \\ Rt^* + R(t - t^*), & t \ge t^* \end{cases}$, evaluate (2.1b) at $t = 2t^*$. Split the integration interval into two parts: from 0 to t^* and t^* to $2t^*$ and introduce a change of variables to arrive at the result: $\sigma(2t^*) = -R\int_0^{t^*} E(s)ds + R\int_{t^*}^{2t^*} E(s)ds$. Proceed as in Example 2.1 and multiply and divide this expression by t^* to cast the result into the form: $\sigma(2t^*) = Rt^*\left[-\frac{1}{t^*}\int_0^{t^*} E(s)ds + \frac{1}{t^*}\int_{t^*}^{2t^*} E(s)ds\right]$. The quantities inside the brackets are the average values of the relaxation function in the respective intervals of integration. Because the relaxation modulus is a monotonically decreasing function of time, the first integral inside the brackets is numerically larger than the second. This proves that, while $\varepsilon(2t^*)$ is zero, $\sigma(2t^*)$ is negative.

P.2.8 In Chap. 1, it was pointed out that in an elastic solid the stress corresponding to a given strain will always be the same, irrespective of the time it takes to apply the strain, and that contrary to this, the stress in a viscoelastic material will depend on the rate of straining, and hence, on the time it takes the strain to reach a specified value. Considering two constant strain-rate histories, $\varepsilon_1(t) = R_1 \cdot t$ and $\varepsilon_2(t) = R_2 \cdot t$, derive an expression for the duration, t_2, at which a viscoelastic material subjected to a strain history $\varepsilon_2(t) = R_2 \cdot t$ would develop the same stress response as it would after t_1 units of time under the strain history $\varepsilon_1(t) = R_1 \cdot t_1$.

$$\text{Answer}: \; t_2 = \frac{E_{eff}(t_1)R_1}{E_{eff}(t_2)R_2}t_1$$

Hint: Proceeding as in Example 2.1, evaluate (2.1b) for each constant strain-rate load and arrive at $\sigma(t_1) = E_{eff}(t_1)R_1t_1$ and $\sigma(t_2) = E_{eff}(t_2)R_2t_2$; where the average or effective modulus, $E_{eff}(t)$, is given by (2.3). The result follows from these relations.

P.2.9 The work per unit volume, $W_V(t)$, performed by external agents acting for t units of time on a uniaxial bar of a viscoelastic material is given by: $W_V(t) = \int_0^t \sigma(s)d\varepsilon(s)$. Evaluate the work per unit volume, done in a complete cycle, on a bar of a viscoelastic materials with relaxation modulus $M(t)$, if the applied excitation is $\varepsilon(t) = \varepsilon_o sin(\omega t)$.

$$\text{Answer}: \; W_V = \pi\sigma_o\varepsilon_o sin\delta$$

Hint: Using that the response to a periodic excitation will be periodic and of the same frequency as the excitation, but out of phase with it, let δ be the phase angle, and take the response to be $\sigma(t) = \sigma_o sin(\omega t + \delta)$. Insert the stress and strain into the expression for the work per unit volume, and write: $W_V = \int_t^{t+p} \sigma_0 \sin(\omega s + \delta)\varepsilon_o \omega \cos(\omega s)ds$; where $p = 2\pi/\omega$ is the period. Now use trigonometric identities to expand the circular function $\sin(\omega s + \delta) = sin(\omega s)\cos(\delta) + cos(\omega s)\sin(\delta)$, perform the integration, using the periodicity of the circular functions, and simplify to arrive at the result. As will be explained in Chap. 4, the phase angle, δ, is a characteristic of the material's relaxation modulus.

P.2.10 Repeat Problem P2.9 using the periodic strain history $\varepsilon(t) = \varepsilon_o cos(\omega t)$.

Answer: The result is the same as for a sine function history: $W_V = \pi\sigma_o\varepsilon_o sin\delta$

Hint: Proceed as in Problem 2.9.

References

1. A.D. Drosdov, Finite elasticity and viscoelasticity; a course in the nonlinear mechanics of solids, World Scientific, pp. 267–271, 279–283 (1996)
2. A.C. Pipkin, Lectures on viscoelasticity theory (Springer, New York, 1986), pp. 14–16
3. H.-P. Hsu, Fourier analysis (Simon and Schuster, New York, 1970) pp. 88–92
4. R.M. Christensen, Theory of viscoelasticity, 2nd edn. (Dover, New York, 2003), pp. 3–9
5. L.E. Malvern, Introduction to the mechanics of a continuous medium (Prentice-Hall, Englewood, 1963) pp. 278–290, 282–285

http://www.springer.com/978-1-4614-8138-6

Engineering Viscoelasticity
Gutierrez-Lemini, D.
2014, XV, 353 p. 130 illus., Hardcover
ISBN: 978-1-4614-8138-6