

Chapter 2

Two Illustrative Examples

In the course of the past 30 years many sequences of Situations have been invented and used and experimented with. The level of student aimed for ranges from pre-school to university, with the highest density at the fourth to eighth grade level. Of these, two are completely or partially available in English. A sequence on introducing statistics, with commentary by Brousseau, was published by Brousseau and Warfield in the *Journal of Mathematical Behavior*.¹ That sequence serves as an introduction to Fundamental Situations, and will be discussed in detail in Chap. 3. The other sequence is far longer and has been taught many times. It provides a complete introduction to rational and decimal numbers as required by “obligatory scholarship” (in the National Programme).² A detailed presentation of that sequence, broken into several sections to keep the size manageable, has been published in the *Journal of Mathematical Behavior*.³ Two particularly intriguing Situations appear early in the sequence and, as with the pots of paint and the Race to Twenty, provide in themselves a nice illustration of the nature and content of a Situation. This chapter contains both, with a sketch of the connecting material between them.

Before presenting the material, I need to give a little explanation of its background and format. In the early 1970s, the Theory of Situations was well developed in Brousseau’s head, but he felt a strong need to prove by experimentation that it could succeed in function as well as in theory. He had the new COREM—the Michelet School—ready and waiting for such an experiment, and the major decision was what concept or sequence of concepts to work with. One that he chose was rational and decimal numbers. He made the choice for both mathematical and technical reasons—the latter being the fact that his presentation could be substituted for part of the Programme without horrendous distortion of the normal sequence. After intensive study, he came up with a sequence of Situations that were then

¹ Brousseau, G., Brousseau, N. and Warfield, 2001.

² *Scolarité obligatoire*.

³ Brousseau, G., Brousseau, N. and Warfield, 2004, 2007, 2008, 2009.

taught by his wife, Nadine, and the other fifth grade teacher at the school. The following year they were taught again with some minor modifications, and thereafter they became part of the standard curriculum at that school. In the course of the next 8 years, Nadine continued to teach the sequence, keeping careful notes as she did so. Eventually the two Brousseaus were persuaded to combine their expertises and write a manual, which was produced in mimeographed form by the IREM of Bordeaux.⁴ It is a remarkable and rich document, in part because of the instructions it presents for carrying out the sequence of Situations, and in part because of the commentary that accompanies the instructions. What the reader needs to realize is that the frequent remarks to the effect that “at this point the students have grasped X, but not Y”, or “some of the students have understood this, but most will need reinforcement” are not theoretical comments based on the image Guy Brousseau had in his head as he invented the Situations, but information backed by 10 years of careful note-taking by Nadine Brousseau as she presented the Situations to a whole succession of fifth grade classes. They need to be read with that understanding.

Part of Brousseau’s preparation for creating the Situations was a study of the historical development of rational and decimal numbers and of the course of development of the teaching of them. Brousseau developed a firm commitment to the idea that a major source of students’ conceptual difficulties is the failure to sort out the different ways in which fractions function, in particular as measures and as linear mappings. Correspondingly, he put together an initial Situation that sets students up to create rational numbers strictly as a method of measurement.

A Didactical Situation: The Thickness of a Sheet of Paper

Preparation of the Materials and the Setting

The teacher has arranged the following:

- On a table in front of the children, five piles of about 200 sheets each of paper of the same size and color but of different thicknesses (for instance, photocopy paper and card stock), placed in a random order. Different piles can have different numbers of sheets. Some of the differences in thickness cannot be perceived merely by touch. The teacher does not need to know the thicknesses in advance; finding the correct measurement is not the issue;
- In the back of the classroom, on another table, five other piles of about 200 sheets of paper of the same types, in a different order, which will be used in Phase 2;
- Ten plastic rulers (two for each group of four or five children);
- A curtain or a screen that can be used to divide the classroom into two sections.

⁴Brousseau, Nadine et Guy, *RATIONNELS et DÉCIMAUX dans la scolarité obligatoire*, Document pour les enseignants et pour les formateurs, I.R.E.M. de Bordeaux, Université de Bordeaux 1, 1987.

First Phase: Search for a Code (About 20–25 min)

The teacher divides the children into groups of four or five.

Presentation of the Situation

Instructions: “Look at the sheets of paper that I have placed in these piles labeled A, B, C, D and E. In each pile, all the sheets have the same thickness, but from one pile to the other, perhaps the thickness is not the same. Can you feel these differences?”

Some sheets from each pile are passed around the class—the children touch and compare them. “What do they do in business to compare the different qualities of sheets of paper?” (Weigh them.)

Objective

“You are going to try to invent another way of denoting and recognizing these different types of paper, and of distinguishing them by their thickness alone. You are in competing groups. Each group is going to think of a way of denoting the thickness of the sheets. As soon as you have found a way, you will try it out in a communication game. You can try things out with the sheets and with these rulers.”

Development and Remarks

Nearly all the children try to measure the thickness of a single sheet so as to obtain the required result immediately.

They make remarks like: “It is so thin that one sheet doesn’t have a thickness” or “It is very much smaller than 1 mm” or “You can’t measure one sheet”.

At this stage, there is often a phase of disarray or even discouragement among the students. Then they ask the teacher whether they can take several sheets. Once that idea begins to circulate, they soon try measuring five sheets, ten sheets, or more until they have obtained a thickness sufficiently large to be measured with their rulers. Then they set up systems of designation, such as:

10 sheets 1 mm

60 sheets 7 mm

or

$31 = 2$ mm (The teacher will attempt to get them to realize during the discussion that the latter use of the equality sign is not correct, and suggest an alternative code.)

During this phase, the teacher intervenes as little as she possibly can. She makes remarks only if she sees that in the groups the children are not paying attention to,

or have simply forgotten, the instructions. The children can get up, collect sheets, exchange them, etc.

When the majority of the groups have found a system of designation (and all the group members agree with this system or code), or if time has run out, the teacher passes on to the next phase, the communication game.

Second Phase: Communication Game (10–15 min)

Presentation of the Situation

Instructions: “In order to test the code you have just found, you are going to play a communication game. During this game, you will see whether the thickness-code you have established will let you identify the type of sheet designated.

The children in each group will divide into two sub-groups, one of senders and one of receivers.

- All the sender groups will be located on one side of the screen, all the receiver groups on the other.
- Each sender group will choose one of the types of paper on the first table (A, B, C, D or E), which the receiver group cannot see because of the screen. They will send their receiver group a message which must allow them to determine the type of paper chosen. The receivers will use the piles of paper on the second table at the back of the classroom to find the type of paper chosen by the senders.

When the receivers have succeeded in finding the type of paper, they will become the senders. Points will be assigned to groups whose receivers correctly find the type of paper chosen by the senders.”

Development and Remarks

The teacher’s job is now to

- Pass messages from the senders to the receivers;
- Collect the receivers’ replies,
- After deciding whether the reply agrees with the senders’ choice, notify the whole group of the success or failure of the message.

All messages are written on a single sheet—which we call the “message form” (see Fig. 2.1)—that passes via the teacher between the senders and the receivers of the same group; this sheet bears the number of the group. In addition, so that the teacher can tell whether the reply is right or wrong, the senders write down on another sheet, which they retain—which we could call the “control form”—the type of paper that they choose for each game.

Group number	1		
First game:	S: 10 = 1 mm	1st game	1 D
message sent			
Reply	R: D success	3rd game	3 A
Second game:			Control form
message sent	S: 21 = 1 mm		
Reply	R: B success		
Third game:	S: 8 = 2 mm		
message sent			
Reply	R: A success		
	Message form		

Fig. 2.1 Forms for use in the communication game

Remarks

It is clear that the teacher need not in fact introduce superfluous vocabulary such as “message form” and “control form” nor formal requirements for the presentation of messages—which the children would have to learn to respect. The idea is to give the instructions in such a way that the children grasp the project and make it their own. If this informal presentation results in some group going completely astray from the intended general path, then the teacher intervenes to get them back on track.

If some groups have not been able to construct efficient messages, the teacher has the option of organizing a new phase of discussion, for each group, for the search of a code (with the same instructions as in the first phase). But this situation had never occurred as of the time the manual was written (in 8 identical experiments). The children always succeeded in completing two or three rounds of the game.

- During this game, three different tactics are observed among the children:
- Some choose a number of sheets whose thickness they measure;
 - Some choose one thickness and count the number of sheets;
 - Some look for thickness and number of sheets at random.

Reasonably enough, children tend to choose types of sheet of extreme size, the thinnest or the thickest, so as to facilitate the work of their team mates.

Third Phase: Result of the Games and the Codes (20–25 min)

Presentation of the Situation and Instructions

For this phase, the children come back into their groups of five, as for the first phase of the session. The teacher announces a comparison of the results and prepares on

Type of sheet	Group 1	Group 2	Group 3	Group 4
A	19s, 3mm	10s, 2mm	20s, 4mm	
B	19s, 3mm		4s, 1mm	15s, 2mm
C	19s, 2mm	30s, 2mm	100s 8mm	30s, 3mm 15s, 1mm 20s, 2mm
D	19s, 2mm 12s, 1mm		100s, 9mm	
E			9s, 4mm	13s, 5mm 7s, 3mm

Fig. 2.2 First messages

the blackboard a table showing groups and types of paper, in which she will write the messages exchanged.

Each group in turn provides a “representative”, who reads the messages out loud, explains the chosen code, and indicates the result of the game.

The children compare and discuss the different messages. As they are often very different, in the end the teacher requires them to select amongst the codes.

For instance, after discussing the option

10 = 1 mm

TF (for “très fin”—very thin)

60 sheets 7 mm

the whole class might decide to write

10s; 1 mm

60s; 7 mm

This would result in a table like the following one Fig. 2.2:

When all the messages are written down, the children look at the table and spontaneously make remarks such as “That doesn’t work” or “Here, this is good”, etc.

In general, the resulting remarks can be classified into four categories:

Category 1

If the sheets are of different types, the same number of sheets must correspond to different thicknesses.

Examples from Table 1:

$$\begin{array}{l} 19\text{s}; 3\text{ mm} \rightarrow \text{Type A} \\ 19\text{s}; 3\text{ mm} \rightarrow \text{Type B} \end{array} \left. \vphantom{\begin{array}{l} 19\text{s}; 3\text{ mm} \rightarrow \text{Type A} \\ 19\text{s}; 3\text{ mm} \rightarrow \text{Type B} \end{array}} \right\} \text{“That doesn’t work”}$$

$$\begin{array}{l} 19\text{s}; 2\text{ mm} \rightarrow \text{Type C} \\ 19\text{s}; 2\text{ mm} \rightarrow \text{Type D} \end{array} \left. \vphantom{\begin{array}{l} 19\text{s}; 2\text{ mm} \rightarrow \text{Type C} \\ 19\text{s}; 2\text{ mm} \rightarrow \text{Type D} \end{array}} \right\} \text{“That doesn’t work”}$$

Category 2

For the same type of sheet, the same thickness corresponds to the same number of sheets.

Example from Table 1:

$$\begin{array}{l} 30\text{s}; 2\text{ mm} \rightarrow \text{Type C} \\ 30\text{s}; 3\text{ mm} \rightarrow \text{Type C} \end{array} \left. \vphantom{\begin{array}{l} 30\text{s}; 2\text{ mm} \rightarrow \text{Type C} \\ 30\text{s}; 3\text{ mm} \rightarrow \text{Type C} \end{array}} \right\} \text{“That doesn’t work”}$$

Category 3

If there are twice as many sheets, the thickness is twice as large.

Example from Table 1:

$$\begin{array}{l} 30\text{s}; 3\text{ mm} \rightarrow \text{Type C} \\ 15\text{s}; 1\text{ mm} \rightarrow \text{Type C} \end{array} \left. \vphantom{\begin{array}{l} 30\text{s}; 3\text{ mm} \rightarrow \text{Type C} \\ 15\text{s}; 1\text{ mm} \rightarrow \text{Type C} \end{array}} \right\} \text{“That doesn’t work”}$$

And the children are encouraged to go on, “We should have found 16s; 2 mm, because:

$$\begin{array}{ll} 8\text{s}, & 1\text{ mm} \\ \times 2 \downarrow & \downarrow \times 2 \\ 16\text{s}, & 2\text{ mm} \end{array}$$

Category 4

Differences in the number of sheets must correspond to equal differences in measurement

Example:

$$\begin{array}{l} 19\text{s}; 3\text{ mm} \\ 20\text{s}; 4\text{ mm} \end{array} \left. \vphantom{\begin{array}{l} 19\text{s}; 3\text{ mm} \\ 20\text{s}; 4\text{ mm} \end{array}} \right\} \text{“That doesn’t work because one sheet can’t measure 1 mm”}$$

At the end of the session, the teacher suggests that the children examine the table during the next session, as a class, to verify the measurements by re-measuring and to correct them if necessary.

Remark: The presentation of operations on the numbers using arrows during the search for equivalent pairs is neither formal nor obligatory; it is a familiar manifestation of the use of natural operators which the children have seen in previous classes.

Results (from the data collected in the years of using this material before the manual was written): At the end of this session all the children know how to measure the thickness of a certain number of sheets of paper, to write the corresponding pair, and to reject a type of paper that does not correspond to a pair that they are given. The majority are therefore able to set up a comparison strategy and use it to accept a type of paper as corresponding to a measurement. Some of them have formulated this strategy. Most of the children can analyze a table of measurements and point out incompatibilities by implicit use of the linear model.

Comparison of Thicknesses and Equivalent Pairs

Preparation of Materials and Scene

The material is the same as for Session 1: piles of sheets set out in the same way, rulers.

First Phase (25–30 min)

Presentation of the Situation: Instructions

The teacher asks the children to return to the examination of the table drawn up during the first session.

This observation is first done individually so that the children can spot the most obvious incompatibilities among the measurements. Then the teacher suggests that they fix up the errors that they have seen, line by line (for each type of paper).

Development and Observations

After observation, and on request, the children come up one at a time to the board to indicate the “messages” which seem to them to be incorrect, and possibly to propose a correction.

These corrections are discussed by the whole group of children. If they all agree, the correction is made, otherwise, they suggest concrete verification: they count out the number of sheets indicated by the message and measure them. After collective verification, the new message is adopted and written into the table. (This manipulation is made by groups: two groups verifying one message, two others verifying another message, and so on.) It has often happened that when the same type of paper is measured by different groups of children, these groups do not obtain compatible measurements. This is due to reading errors or to the fact that the children have compressed the sheets to a greater or lesser extent. They are swift to notice this and point it out.

Likewise, for types of paper close in thickness it has happened that the measurements did not permit the recognition of the type of paper in question. It is during this phase that the children are likely to notice that they have more chance of distinguishing between papers of similar thickness by taking a larger number of sheets. Thus, fifteen or twenty sheets of paper of almost the same thickness have measurements so close that the children cannot distinguish between them with precision. At this stage they often suggest measuring the thickness of fifty, eighty, or a hundred sheets.

The following types observations can be made from Fig. 2.2, and have been made in classrooms using this Situation:

For Type A : we have 10s; 2 mm and 20s; 4 mm. That's fine.

For Type C:

15 s; 1 mm

$\times 2 \downarrow \quad \downarrow \times 2$

30s; 2 mm, so Group 4's other measurements are probably off.

The four measurements: 20s; 4 mm, 10s; 2 m, and 15s; 1 mm, 30s; 2 mm, would thus be conserved.

On the other hand, in the Type B row, the measurements 4s; 1 mm and 15s; 2 mm would be contested and rejected by the children, with the suggestion that they be re-done.

A different collection of measurements might produce the following line of reasoning:

Second Phase: Completion of Table; Search for Missing Values (20–25 min)

Presentation of the Situation: Instructions

In general, certain types of paper were not chosen during the communication game (the extremes being easier to communicate) At this point, students tend to notice that, and that they have no measurements on the table.

The teacher then suggests that the children complete the table by measuring the missing types of paper.

Development and Remarks

The children carry out the work in groups of five which are no longer competitive but co-operative and not restricted to the same groupings as in the previous session. (Several groups take the same type of paper and later verify the compatibility of their measurements.)

When all the children agree, the new measurements are written into the table. At the end of this phase, the table is therefore entirely correct and complete. There are many compatible measurements for each type of paper.

Third Phase: Communication Game (15 min)

The teacher suggests that the children re-play the communication game from the first activity, taking into account all the observations and corrections that they have made (large number of sheets, compression of the paper, etc.). The game is thus replayed to the great satisfaction of the children, who succeed in their task even if the teacher adds one or two new types of paper.

Results observed: These children know how to adapt the number of sheets chosen to the necessities for discrimination of their thickness (to augment the number of sheets when the thicknesses are very close). They know how to calculate pairs corresponding to the same type of paper. Everyone now knows how to use the linear model to analyze a table. A small group of the children is able to use relationships of closeness between pairs. A large number of the children have learned to judge declarations and to argue their own points of view.

Summary of the Rest of the Sequence (Session 3)

After verifying that the children know how to recognize sheets designated by their thicknesses, as for example (48 s; 9 mm), the teacher asks them to find other ways of writing each thickness, and then to rank the sheets from the thinnest to the thickest. Then she gives them (25 s; 7 mm) to rank with the others.

By the end of the session, the teacher tells the children a method of writing the thickness of a sheet: (50s; 4 mm) designates a pile of 50 sheets which is 4 mm thick; the thickness of one of these sheets is written $\frac{4}{50}$, read as “four fiftieths of a millimeter”. To verify that they are able to use this new notation, she gives them some exercises, such as

- Write the thickness of papers A,B,C and D as fractions,
- Write the fraction designating a stack of 20 sheets whose thickness is 3 mm.
- Write the fraction for a sheet of paper if it takes 30 to “make” 5 mm.
- Can there be than one fraction designating the same thickness? Find some.

The follow up to part d is to point out that to say that $2/9$ mm and $4/18$ mm are the same thickness, we write $2/9 = 4/18$.

Results observed: Within the context of the Situation, the children know how to find equivalent pairs. They know how to compare the thickness of sheets of paper (many of them using two methods, either specifying the number of sheets or specifying the thickness). Using these comparisons, they have a strategy for ranking pairs. They know how to designate the thickness of a sheet of paper by means of a fraction and how to find equivalent fractions.

Remark: This knowledge occurs within the Situation. It is unlikely that it is yet possible to take a question out of context and ask it independently. The results can not yet be depended upon as “acquired” knowledge, nor do the children identify them as such.

This is followed by a sequence of lessons in which the students figure out how to carry out the basic operations (addition, subtraction, multiplication and division by a whole number) in the context of the sheets of paper, then convert to the more standard notation and become accustomed to it. The learning is reinforced by some exercises involving the same kind of measuring of other objects, for instance different types of small nail, each individual nail being lighter than the available scale can measure.⁵ With that well in hand, the students are introduced to decimal numbers as a type of rational number which is convenient for making approximations and has a useful notation. The basic mechanism is a Situation in which students need to take one fraction as a target and find other fractions that get closer and closer to it. They discover that this works out much the most easily if all fractions involved have denominators that are powers of ten. Once they have mastered the use of these fractions, they are given the decimal notation for them.

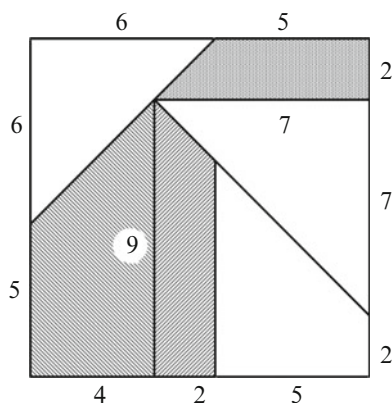
The next stage is to progress to fractions as linear mappings (that is, taking $5/9$ of something rather than simply using $5/9$ to designate a quantity). This is introduced with the Situation of the Puzzle. This Situation has been used a number of times outside of the sequence described in the manual, so that the descriptions of student responses are based on an even larger collection of observations that are those in the Stacks of Paper Situation.

Another Situation: The Puzzle

The Problem-Situation

The first Situation put to students for study of fractions as linear mappings is the following.

⁵For details, see Brousseau, Brousseau and Warfield (2004) pp 17–20.

Fig. 2.3 The Puzzle

Instructions

“Here are some puzzles (Example: Fig. 2.3 above) like the ones we have been making patterns with for the past few days. You remember the nice patterns we have been making using a few pieces at a time. We want the younger class to be able to play with those puzzles, but the ones we have been using are too small for their fingers. So you are going to make some similar puzzles, larger than the models, according to the following rule: the segment that measures 4 cm on the model will measure 7 cm on your reproduction. I shall give a puzzle to each group of four or five students, but every student will do at least one piece or a group of two will do two. When you have finished, you must be able to reconstruct figures that fit together exactly the same way as the model.”

Development

After a brief planning phase in each group, the students separate. The teacher puts an enlarged representation of the complete puzzle on the chalkboard.

Most of the students think that the thing to do is to add 3 cm to every dimension. Even if a few doubt this model, their conviction is not as strong as that of their classmates who do, and they seldom succeed in convincing them. The result, obviously, is that the pieces are not compatible. Discussions, diagnostics and rather heated discussions ensue. The blame first falls on the execution of the plan. Verification is attempted—some students re-do all the pieces. In the end they have to succumb to the evidence, which is not easy for them to do! The teacher intervenes only to give encouragement and to verify facts, without making particular requirements. By means of successive rearrangements, some students produce a puzzle that roughly reproduces the form of the model. The teacher, invited along with the other groups of students to confirm success, in this case suggests that the competitors use the model to form a figure with the original piece (such as Fig. 2.4) that cannot be

Fig. 2.4 A figure made with the original pieces

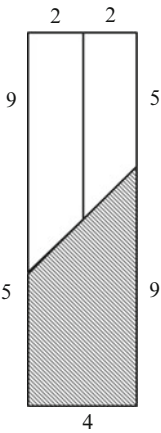
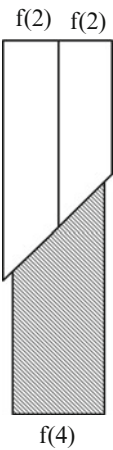


Fig. 2.5 An attempt to reproduce the same figure with fudged enlarged pieces



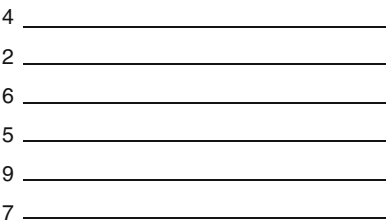
reproduced with the pieces they have made (Fig. 2.5). It is generally quite easy to find three sides, a , b , and c , such that $a + b = c$ and $f(a) + f(b) \neq f(c)$.

For a variety of social and intellectual reasons, there is a general resistance to the idea of reconsidering the initial model of the process. The result is generally a small dose of pandemonium.

When at last the children accept that there must be another law and set about searching for it, things move along more quickly, especially if one of them, or the teacher, displays the lengths on the blackboard (Fig. 2.6).

They might first find the image of 2: if $4 \rightarrow 7$ then $2 \rightarrow 3.5$. In general, at this stage, the idea is not contested, as if, as soon as the other model is rejected, this one replaces it. “We would need the image of 1.” “Yes, then we could find all the others.” “For that, 4 will have to be divided into four parts, and 7 will have to be divided into

Fig. 2.6 Side lengths in the original puzzle



four parts as well.” The model of commensuration that they have been taught would allow them to write directly: 4 times the image of 1 measures 7; the image of 1 is therefore $\frac{7}{4}$. Before they get to it, though, they usually calculate using more elaborate procedures that they can construct because of their experience working with fractions as measures. These calculations now allow success duly verified by construction.

Summary: Analysis of the Two Situations

I shall finish this chapter with a brief analysis of the Situations it presents, so as to point out how they fit into the general structure. I need to point out, though, one danger to making such an analysis. Situations of Action, Communication, etc. are by no means cut and dried. As Brousseau himself remarks, there is frequently some overlap and some mixture of the ordering. The specific labeling could often be argued, if the labeling were important in itself. Since labeling serves more the function of a lens than a road map, such an argument would be of dubious worth.

To start with the sheets of paper, the phase when the students are moving about, attempting to measure single sheets of paper, trying out various sizes of stacks and checking if the ruler can handle them is unambiguously a Situation of Action. When they settle down to work out a system whereby the transmitting team can describe the paper in such a way that the receiving team can recognize it, they are in a classic Situation of Communication, where they not only must formulate what they need to communicate, but invent for themselves a code for doing so. The game of messages remains part of the Situation of Communication, but when the messages are assembled and the discussion of the resulting table begins, a Situation of Validation has been reached. Their discussion covers with many iterations the issue of why two codes are inconsistent, or a particular code is not self-consistent. After it they return briefly to the Situation of Action, folding back in the Pirie/Kieran sense⁶ to the original activity, but now with new layers of understanding of it.

⁶Pirie and Kieran describe the learning process by positing a sequence of concentric circles of increasing levels of abstraction. The student’s path of learning progresses through the sequence, but not uniformly outward. From time to time there is a return to an inner circle, which they characterize as “folding back”, because at each return the knowledge is in some sense thicker. See, for instance Pirie, S.E.B., and Kieran, T.E. (1994).

The Situation of the Puzzle again starts with an unambiguous Situation of Action, and very lively action it is. Didacticians refer to this as a particularly robust Situation, meaning that even when presented out of its original context and to quite different collections of students, it can be depended on to engage the students and to maintain a high interest level that will bring them to invent the concept of multiplying by a fraction. The Situation of Action lasts as long as the students are simply cutting out pieces and pushing them together. Formulation begins when they finally accept that making repeated minor modifications of the additive model just isn't going to work for them, and begin to discuss alternative ways to proceed. Since this procedure is self-checking, this Situation does not include a Situation of Validation. That aspect of the learning of this particular concept is taken up in the Situations that follow it.



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