

Chapter 2

Peridynamic Theory

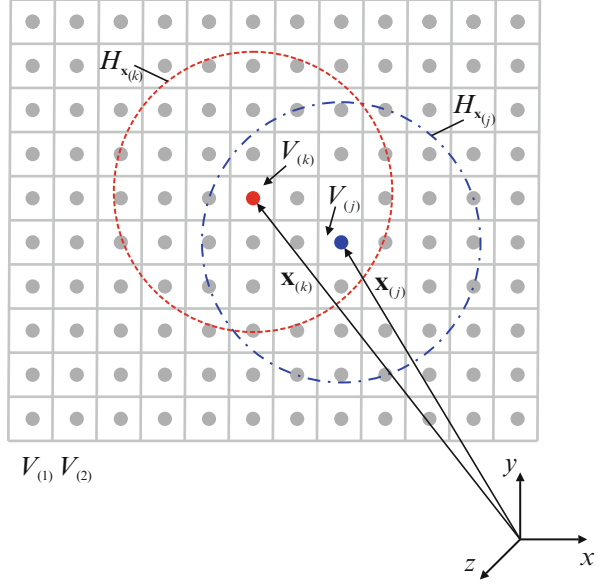
2.1 Basics

At any instant of time, every point in the material denotes the location of a material particle, and these infinitely many material points (particles) constitute the continuum. In an undeformed state of the body, each material point is identified by its coordinates, $\mathbf{x}_{(k)}$ with $(k = 1, 2, \dots, \infty)$, and is associated with an incremental volume, $V_{(k)}$, and a mass density of $\rho(\mathbf{x}_{(k)})$. Each material point can be subjected to prescribed body loads, displacement, or velocity, resulting in motion and deformation. With respect to a Cartesian coordinate system, the material point $\mathbf{x}_{(k)}$ experiences displacement, $\mathbf{u}_{(k)}$, and its location is described by the position vector $\mathbf{y}_{(k)}$ in the deformed state. The displacement and body load vectors at material point $\mathbf{x}_{(k)}$ are represented by $\mathbf{u}_{(k)}(\mathbf{x}_{(k)}, t)$ and $\mathbf{b}_{(k)}(\mathbf{x}_{(k)}, t)$, respectively. The motion of a material point conforms to the Lagrangian description.

According to the peridynamic (PD) theory introduced by Silling (2000), the motion of a body is analyzed by considering the interaction of a material point, $\mathbf{x}_{(k)}$, with the other, possibly infinitely many, material points, $\mathbf{x}_{(j)}$, with $(j = 1, 2, \dots, \infty)$, in the body. Therefore, an infinite number of interactions may exist between the material point at location $\mathbf{x}_{(k)}$ and other material points. However, the influence of the material points interacting with $\mathbf{x}_{(k)}$ is assumed to vanish beyond a local region (horizon), denoted by $H_{\mathbf{x}_{(k)}}$, shown in Fig. 2.1. Similarly, material point $\mathbf{x}_{(j)}$ interacts with material points in its own family, $H_{\mathbf{x}_{(j)}}$.

In other words, the PD theory is concerned with the physics of a material body at a point that interacts with all points within its range, as shown in Fig. 2.1. The range of material point $\mathbf{x}_{(k)}$ is defined by δ , referred to as the “horizon.” Also, the material points within a distance δ of $\mathbf{x}_{(k)}$ are called the family of $\mathbf{x}_{(k)}$, $H_{\mathbf{x}_{(k)}}$. The interaction of material points is prescribed through a micropotential that depends on the deformation and constitutive properties of the material. The locality of interactions depends on the horizon, and the interactions become more local with a decreasing horizon, δ .

Fig. 2.1 Infinitely many PD material points and interaction of points at $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$



Hence, the classical theory of elasticity can be considered a limiting case of the peridynamic theory as the horizon approaches zero (Silling and Lehoucq 2008).

2.2 Deformation

As shown in Fig. 2.2, material point $\mathbf{x}_{(k)}$ interacts with its family of material points, $H_{\mathbf{x}_{(k)}}$, and it is influenced by the collective deformation of all these material points. Similarly, material point $\mathbf{x}_{(j)}$ is influenced by deformation of the material points, $H_{\mathbf{x}_{(j)}}$, in its own family. In the deformed configuration, the material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ experience displacements, $\mathbf{u}_{(k)}$ and $\mathbf{u}_{(j)}$, respectively, as shown in Fig. 2.2. Their initial relative position vector $(\mathbf{x}_{(j)} - \mathbf{x}_{(k)})$ prior to deformation becomes $(\mathbf{y}_{(j)} - \mathbf{y}_{(k)})$ after deformation. The stretch between material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ is defined as

$$s_{(k)(j)} = \frac{(|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}| - |\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|)}{|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|}. \quad (2.1)$$

Associated with material point $\mathbf{x}_{(k)}$, all of the relative position vectors in the deformed configuration, $(\mathbf{y}_{(j)} - \mathbf{y}_{(k)})$ with $(j = 1, 2, \dots, \infty)$, can be stored in an infinite-dimensional array, or a deformation vector state, \mathbf{Y} :

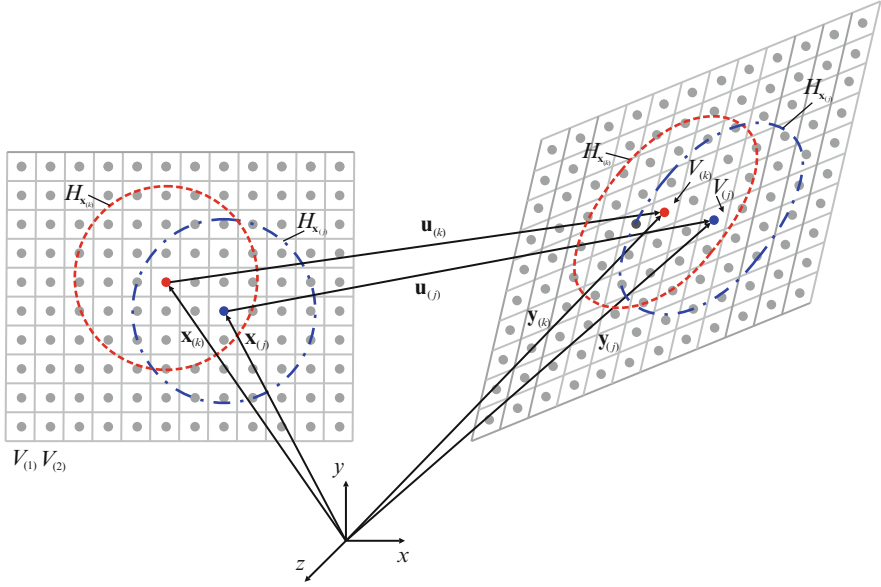


Fig. 2.2 Kinematics of PD material points

$$\underline{\mathbf{Y}}(\mathbf{x}_{(k)}, t) = \begin{Bmatrix} (\mathbf{y}_{(1)} - \mathbf{y}_{(k)}) \\ \vdots \\ (\mathbf{y}_{(\infty)} - \mathbf{y}_{(k)}) \end{Bmatrix}. \quad (2.2)$$

The definitions and mathematical properties of vector states are presented by Silling et al. (2007). Their properties in relation to the derivation of PD equations are summarized in the Appendix.

2.3 Force Density

As illustrated in Fig. 2.3, the material point $\mathbf{x}_{(k)}$ interacts with its family of material points, $H_{\mathbf{x}_{(k)}}$, and it is influenced by the collective deformation of all these material points, thus resulting in a force density vector, $\mathbf{t}_{(k)(j)}$, acting at material point $\mathbf{x}_{(k)}$. It can be viewed as the force exerted by material point $\mathbf{x}_{(j)}$. Similarly, material point $\mathbf{x}_{(j)}$ is influenced by deformation of the material points, $H_{\mathbf{x}_{(j)}}$, in its own family, and the corresponding force density vector is $\mathbf{t}_{(j)(k)}$ at material point $\mathbf{x}_{(j)}$ and is exerted on by material point $\mathbf{x}_{(k)}$. These forces are determined jointly by the *collective* deformation of $H_{\mathbf{x}_{(k)}}$ and $H_{\mathbf{x}_{(j)}}$ through the material model.

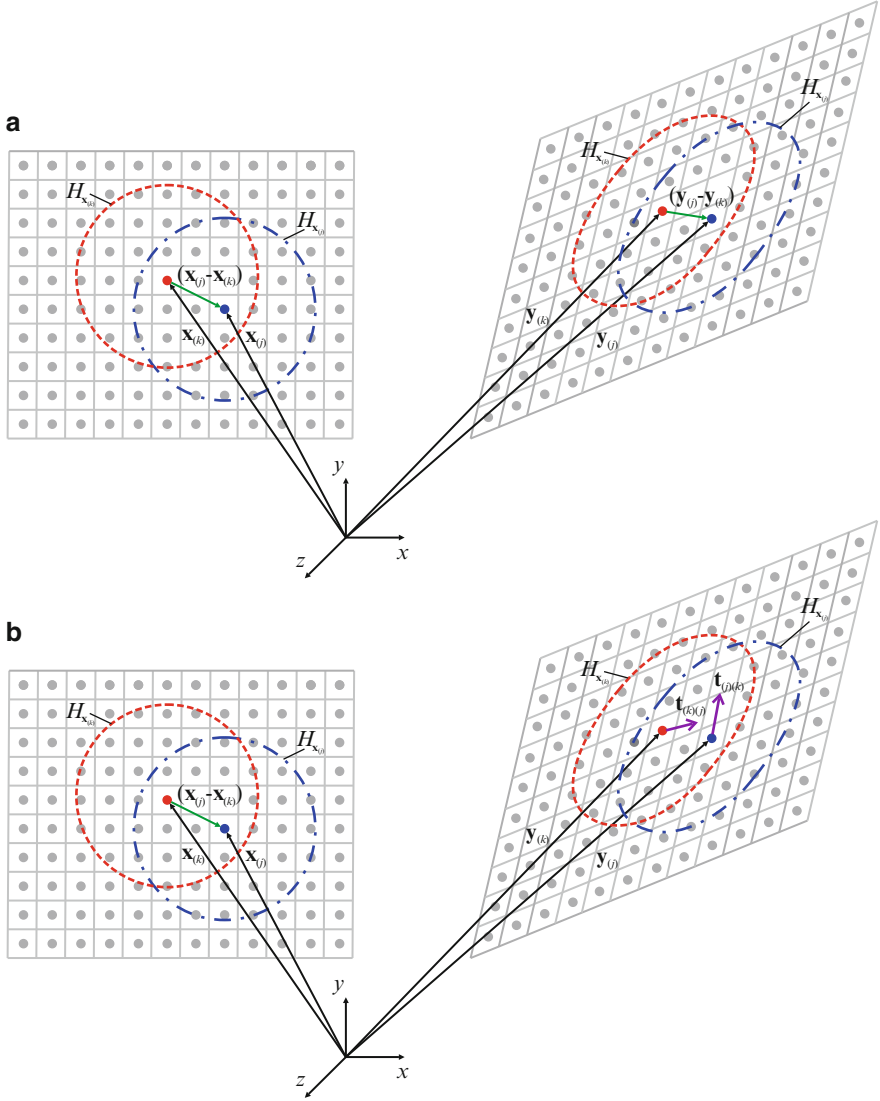


Fig. 2.3 PD vector states: (a) deformation, $\underline{\mathbf{Y}}$, and (b) force, $\underline{\mathbf{T}}$

Associated with material point $\mathbf{x}_{(k)}$, all of the force density vectors, $\mathbf{t}_{(k)(j)}$ with $(j = 1, 2, \dots, \infty)$, can be stored in an infinite-dimensional array, or a force vector state, $\underline{\mathbf{T}}$:

$$\underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) = \left\{ \begin{array}{c} \mathbf{t}_{(k)(1)} \\ \vdots \\ \mathbf{t}_{(k)(\infty)} \end{array} \right\}. \quad (2.3)$$

2.4 Peridynamic States

The PD theory mainly concerns the deformation state, $\underline{\mathbf{Y}}$, and the force state, $\underline{\mathbf{T}}$. As described in Fig. 2.3a, the relative position vector $(\mathbf{y}_{(j)} - \mathbf{y}_{(k)})$ can be obtained by operating the deformation state, $\underline{\mathbf{Y}}$, on the relative position vector $(\mathbf{x}_{(j)} - \mathbf{x}_{(k)})$ as

$$(\mathbf{y}_{(j)} - \mathbf{y}_{(k)}) = \underline{\mathbf{Y}}(\mathbf{x}_{(k)}, t) \langle \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \rangle. \quad (2.4)$$

Similarly, the force density vector, $\mathbf{t}_{(k)(j)}$, shown in Fig. 2.3b, that the material point at location $\mathbf{x}_{(j)}$ exerts on the material point at location $\mathbf{x}_{(k)}$ can be expressed as

$$\mathbf{t}_{(k)(j)}(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) = \underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) \langle \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \rangle. \quad (2.5)$$

The difference between the force state and the deformation state is that the force state is dependent on the deformation state while the deformation state is independent. Therefore, the force state for material point $\mathbf{x}_{(k)}$ depends on the relative displacements between this material point and the other material points within its horizon. Hence, the force state can also be written as

$$\underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) = \underline{\mathbf{T}}(\underline{\mathbf{Y}}(\mathbf{x}_{(k)}, t)). \quad (2.6)$$

2.5 Strain Energy Density

Due to the interaction between material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$, a scalar-valued micropotential, $w_{(k)(j)}$, develops; it depends on the material properties as well as the stretch between point $\mathbf{x}_{(k)}$ and all other material points in its family. Note that the micropotential $w_{(j)(k)} \neq w_{(k)(j)}$, because $w_{(j)(k)}$ depends on the state of material points within the family of material point $\mathbf{x}_{(j)}$. These micropotentials can be expressed as

$$w_{(k)(j)} = w_{(k)(j)}(\mathbf{y}_{(1^k)} - \mathbf{y}_{(k)}, \mathbf{y}_{(2^k)} - \mathbf{y}_{(k)}, \dots) \quad (2.7a)$$

and

$$w_{(j)(k)} = w_{(j)(k)}(\mathbf{y}_{(1^j)} - \mathbf{y}_{(j)}, \mathbf{y}_{(2^j)} - \mathbf{y}_{(j)}, \dots), \quad (2.7b)$$

where $\mathbf{y}_{(k)}$ is the position vector of point $\mathbf{x}_{(k)}$ in the deformed configuration and $\mathbf{y}_{(1^k)}$ is the position vector of the first material point that interacts with point $\mathbf{x}_{(k)}$. Similarly, $\mathbf{y}_{(j)}$ is the position vector of point $\mathbf{x}_{(j)}$ in the deformed configuration and $\mathbf{y}_{(1^j)}$ is the position vector of the first material point that interacts with point $\mathbf{x}_{(j)}$.

The strain energy density, $W_{(k)}$, of material point $\mathbf{x}_{(k)}$ can be expressed as a summation of micropotentials, $w_{(k)(j)}$, arising from the interaction of material point $\mathbf{x}_{(k)}$ and the other material points, $\mathbf{x}_{(j)}$, within its horizon in the form

$$W_{(k)} = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{2} \left(w_{(k)(j)} \left(\mathbf{y}_{(1^k)} - \mathbf{y}_{(k)}, \mathbf{y}_{(2^k)} - \mathbf{y}_{(k)}, \dots \right) + w_{(j)(k)} \left(\mathbf{y}_{(1^j)} - \mathbf{y}_{(j)}, \mathbf{y}_{(2^j)} - \mathbf{y}_{(j)}, \dots \right) \right) V_{(j)}, \quad (2.8)$$

in which $w_{(k)(j)} = 0$ for $k = j$.

2.6 Equations of Motion

The PD equations of motion at material point $\mathbf{x}_{(k)}$ can be derived by applying the principle of virtual work, i.e.,

$$\delta \int_{t_0}^{t_1} (T - U) dt = 0, \quad (2.9)$$

where T and U represent the total kinetic and potential energies in the body. This principle is satisfied by solving for the Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{u}}_{(k)}} \right) - \frac{\partial L}{\partial \mathbf{u}_{(k)}} = 0, \quad (2.10)$$

where the Lagrangian L is defined as

$$L = T - U. \quad (2.11)$$

The total kinetic and potential energies in the body can be obtained by summation of kinetic and potential energies of all material points, respectively,

$$T = \sum_{i=1}^{\infty} \frac{1}{2} \rho_{(i)} \dot{\mathbf{u}}_{(i)} \cdot \dot{\mathbf{u}}_{(i)} V_{(i)} \quad (2.12a)$$

and

$$U = \sum_{i=1}^{\infty} W_{(i)} V_{(i)} - \sum_{i=1}^{\infty} (\mathbf{b}_{(i)} \cdot \mathbf{u}_{(i)}) V_{(i)}. \quad (2.12b)$$

Substituting for the strain energy density, $W_{(i)}$, of material point $\mathbf{x}_{(i)}$ from Eq. 2.8, the potential energy can be rewritten as

$$U = \sum_{i=1}^{\infty} \left\{ \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{2} \left[\begin{aligned} &w_{(i)(j)} \left(\mathbf{y}_{(1^i)} - \mathbf{y}_{(i)}, \mathbf{y}_{(2^i)} - \mathbf{y}_{(i)}, \dots \right) \\ &+ w_{(j)(i)} \left(\mathbf{y}_{(1^j)} - \mathbf{y}_{(j)}, \mathbf{y}_{(2^j)} - \mathbf{y}_{(j)}, \dots \right) \end{aligned} \right] V_{(j)} - (\mathbf{b}_{(i)} \cdot \mathbf{u}_{(i)}) \right\} V_{(i)}. \quad (2.13)$$

By using Eq. 2.11, the Lagrangian can be written in an expanded form by showing only the terms associated with the material point $\mathbf{x}_{(k)}$:

$$\begin{aligned} L = & \dots + \frac{1}{2} \rho_{(k)} \dot{\mathbf{u}}_{(k)} \cdot \dot{\mathbf{u}}_{(k)} V_{(k)} + \dots \\ & \dots - \frac{1}{2} \sum_{j=1}^{\infty} \left\{ \frac{1}{2} \left[w_{(k)(j)} \left(\mathbf{y}_{(1^k)} - \mathbf{y}_{(k)}, \mathbf{y}_{(2^k)} - \mathbf{y}_{(k)}, \dots \right) \right. \right. \\ & \quad \left. \left. + w_{(j)(k)} \left(\mathbf{y}_{(1^j)} - \mathbf{y}_{(j)}, \mathbf{y}_{(2^j)} - \mathbf{y}_{(j)}, \dots \right) \right] V_{(j)} \right\} V_{(k)} \dots \\ & \dots - \frac{1}{2} \sum_{i=1}^{\infty} \left\{ \frac{1}{2} \left[w_{(i)(k)} \left(\mathbf{y}_{(1^i)} - \mathbf{y}_{(i)}, \mathbf{y}_{(2^i)} - \mathbf{y}_{(i)}, \dots \right) \right. \right. \\ & \quad \left. \left. + w_{(k)(i)} \left(\mathbf{y}_{(1^k)} - \mathbf{y}_{(k)}, \mathbf{y}_{(2^k)} - \mathbf{y}_{(k)}, \dots \right) \right] V_{(i)} \right\} V_{(k)} \dots \\ & \dots + (\mathbf{b}_{(k)} \cdot \mathbf{u}_{(k)}) V_{(k)} \dots \end{aligned} \quad (2.14a)$$

or

$$\begin{aligned} L = & \dots + \frac{1}{2} \rho_{(k)} \dot{\mathbf{u}}_{(k)} \cdot \dot{\mathbf{u}}_{(k)} V_{(k)} + \dots \\ & \dots - \frac{1}{2} \sum_{j=1}^{\infty} \left\{ w_{(k)(j)} \left(\mathbf{y}_{(1^k)} - \mathbf{y}_{(k)}, \mathbf{y}_{(2^k)} - \mathbf{y}_{(k)}, \dots \right) V_{(j)} V_{(k)} \right\} \dots \\ & \dots - \frac{1}{2} \sum_{j=1}^{\infty} \left\{ w_{(j)(k)} \left(\mathbf{y}_{(1^j)} - \mathbf{y}_{(j)}, \mathbf{y}_{(2^j)} - \mathbf{y}_{(j)}, \dots \right) V_{(j)} V_{(k)} \right\} \dots \\ & \dots + (\mathbf{b}_{(k)} \cdot \mathbf{u}_{(k)}) V_{(k)} \dots \end{aligned} \quad (2.14b)$$

Substituting from Eq. 2.14b into Eq. 2.10 results in the Lagrange's equation of the material point $\mathbf{x}_{(k)}$ as

$$\begin{aligned} \rho_{(k)} \ddot{\mathbf{u}}_{(k)} V_{(k)} + \left(\sum_{j=1}^{\infty} \frac{1}{2} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(k)(j)}}{\partial (\mathbf{y}_{(j)} - \mathbf{y}_{(k)})} V_{(i)} \right) \frac{\partial (\mathbf{y}_{(j)} - \mathbf{y}_{(k)})}{\partial \mathbf{u}_{(k)}} \right. \\ \left. + \sum_{j=1}^{\infty} \frac{1}{2} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(j)(k)}}{\partial (\mathbf{y}_{(k)} - \mathbf{y}_{(j)})} V_{(i)} \right) \frac{\partial (\mathbf{y}_{(k)} - \mathbf{y}_{(j)})}{\partial \mathbf{u}_{(k)}} - \mathbf{b}_{(k)} \right) V_{(k)} = 0 \end{aligned} \quad (2.15a)$$

or

$$\rho_{(k)} \ddot{\mathbf{u}}_{(k)} = \sum_{j=1}^{\infty} \frac{1}{2} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(k)(i)}}{\partial (\mathbf{y}_{(j)} - \mathbf{y}_{(k)})} V_{(i)} \right) - \sum_{j=1}^{\infty} \frac{1}{2} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(i)(k)}}{\partial (\mathbf{y}_{(k)} - \mathbf{y}_{(j)})} V_{(i)} \right) + \mathbf{b}_{(k)}, \quad (2.15b)$$

in which it is assumed that the interactions not involving material point $\mathbf{x}_{(k)}$ do not have any effect on material point $\mathbf{x}_{(k)}$. Based on the dimensional analysis of this equation, it is apparent that $\sum_{i=1}^{\infty} V_{(i)} \partial w_{(k)(i)} / \partial (\mathbf{y}_{(j)} - \mathbf{y}_{(k)})$ represents the force density that material point $\mathbf{x}_{(j)}$ exerts on material point $\mathbf{x}_{(k)}$ and $\sum_{i=1}^{\infty} V_{(i)} \partial w_{(i)(k)} / \partial (\mathbf{y}_{(k)} - \mathbf{y}_{(j)})$ represents the force density that material point $\mathbf{x}_{(k)}$ exerts on material point $\mathbf{x}_{(j)}$. With this interpretation, Eq. 2.15b can be rewritten as

$$\begin{aligned} \rho_{(k)} \ddot{\mathbf{u}}_{(k)} = \sum_{j=1}^{\infty} \left[\mathbf{t}_{(k)(j)} (\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) \right. \\ \left. - \mathbf{t}_{(j)(k)} (\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t) \right] V_{(j)} + \mathbf{b}_{(k)}, \end{aligned} \quad (2.16)$$

where

$$\mathbf{t}_{(k)(j)} (\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) = \frac{1}{2} \frac{1}{V_{(j)}} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(k)(i)}}{\partial (\mathbf{y}_{(j)} - \mathbf{y}_{(k)})} V_{(i)} \right) \quad (2.17a)$$

and

$$\mathbf{t}_{(j)(k)} (\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t) = \frac{1}{2} \frac{1}{V_{(j)}} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(i)(k)}}{\partial (\mathbf{y}_{(k)} - \mathbf{y}_{(j)})} V_{(i)} \right). \quad (2.17b)$$

By utilizing the state concept, the force densities $\mathbf{t}_{(k)(j)}$ and $\mathbf{t}_{(j)(k)}$ can be stored in force vector states that belong to material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$, respectively, as

$$\underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) = \left\{ \begin{array}{c} \vdots \\ \mathbf{t}_{(k)(j)} \\ \vdots \end{array} \right\} \text{ and } \underline{\mathbf{T}}(\mathbf{x}_{(j)}, t) = \left\{ \begin{array}{c} \vdots \\ \mathbf{t}_{(j)(k)} \\ \vdots \end{array} \right\}. \quad (2.18a,b)$$

The force densities $\mathbf{t}_{(k)(j)}$ and $\mathbf{t}_{(j)(k)}$ stored in vector states $\underline{\mathbf{T}}(\mathbf{x}_{(k)}, t)$ and $\underline{\mathbf{T}}(\mathbf{x}_{(j)}, t)$ can be extracted again by operating the force states on the corresponding initial relative position vectors

$$\mathbf{t}_{(k)(j)} = \underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) \langle \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \rangle \quad (2.19a)$$

and

$$\mathbf{t}_{(j)(k)} = \underline{\mathbf{T}}(\mathbf{x}_{(j)}, t) \langle \mathbf{x}_{(k)} - \mathbf{x}_{(j)} \rangle. \quad (2.19b)$$

By using Eqs. 2.19a and 2.19b, Lagrange's equation of the material point $\mathbf{x}_{(k)}$ can be recast as

$$\rho_{(k)} \ddot{\mathbf{u}}_{(k)} = \sum_{j=1}^{\infty} (\underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) \langle \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \rangle - \underline{\mathbf{T}}(\mathbf{x}_{(j)}, t) \langle \mathbf{x}_{(k)} - \mathbf{x}_{(j)} \rangle) V_{(j)} + \mathbf{b}_{(k)}. \quad (2.20)$$

Because the volume of each material point $V_{(j)}$ is infinitesimally small, for the limiting case of $V_{(j)} \rightarrow 0$, the infinite summation can be expressed as integration while considering only the material points within the horizon,

$$\sum_{j=1}^{\infty} (\cdot) V_{(j)} \rightarrow \int_V (\cdot) dV' \rightarrow \int_H (\cdot) dH. \quad (2.21)$$

With this replacement, Eq. 2.20 can be written in integral equation form as

$$\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) = \int_H (\underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle) dH + \mathbf{b}(\mathbf{x}, t) \quad (2.22a)$$

or

$$\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) = \int_H (\mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) - \mathbf{t}'(\mathbf{u} - \mathbf{u}', \mathbf{x} - \mathbf{x}', t)) dH + \mathbf{b}(\mathbf{x}, t). \quad (2.22b)$$

2.7 Initial and Constraint Conditions

The resulting PD equation of motion is a nonlinear integro-differential equation in time and space and is free of kinematic linearization, thus it is suitable for geometrically nonlinear analyses. It contains differentiation with respect to time and integration in the spatial domain. It does not contain any spatial derivatives of displacements. Thus, the PD equation of motion is valid everywhere whether or not displacement discontinuities exist in the material. Construction of its solution involves time and spatial integrations while being subject to constraints and/or loading conditions on the boundary, \mathcal{B} , of the material region, \mathcal{R} , and initial conditions on the displacement and velocity fields.

2.7.1 Initial Conditions

Time integration requires the application of initial displacement and velocity values at each material point in \mathcal{R} , and they can be specified as

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}^*(\mathbf{x}) \quad (2.23a)$$

and

$$\dot{\mathbf{u}}(\mathbf{x}, t = 0) = \mathbf{v}^*(\mathbf{x}). \quad (2.23b)$$

In addition to these required initial conditions, the initial conditions may also be necessary on the displacement and velocity gradients, $\mathbf{H}^*(\mathbf{x})$ and $\mathbf{L}^*(\mathbf{x})$, respectively. They can be specified as

$$\mathbf{H}(\mathbf{x}, t = 0) = \mathbf{H}^*(\mathbf{x}) \sim \frac{\partial u_i(x_k, 0)}{\partial x_j}, \quad \text{with } (i, j, k) = 1, 2, 3, \quad (2.24a)$$

and

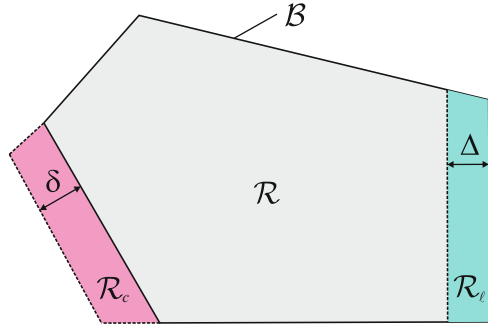
$$\mathbf{L}(\mathbf{x}, t = 0) = \mathbf{L}^*(\mathbf{x}) \sim \frac{\partial \dot{u}_i(x_k, 0)}{\partial x_j}, \quad \text{with } (i, j, k) = 1, 2, 3. \quad (2.24b)$$

The corresponding displacement and velocity fields are superimposed on the initial displacement and velocity fields as

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}^*(\mathbf{x}) + \mathbf{H}^*(\mathbf{x})(\mathbf{x} - \mathbf{x}_{ref}) \quad (2.25a)$$

and

Fig. 2.4 Boundary regions for constraint and external load introduction



$$\dot{\mathbf{u}}(\mathbf{x}, t = 0) = \mathbf{v}^*(\mathbf{x}) + \mathbf{L}^*(\mathbf{x})(\mathbf{x} - \mathbf{x}_{ref}), \quad (2.25b)$$

where \mathbf{x}_{ref} is a reference point (Silling 2004).

2.7.2 Constraint Conditions

The PD equation of motion does not contain any spatial derivatives; therefore, constraint conditions are, in general, not necessary for the solution of an integro-differential equation. However, such conditions can be imposed by prescribing constraints on displacement and velocity fields in a “fictitious material layer” along the boundary of a nonzero volume. Based on numerical experiments, Macek and Silling (2007) suggested that the extent of the fictitious boundary layer be equal to the horizon, δ , in order to ensure that the imposed prescribed constraints are sufficiently reflected on the actual material region. Therefore, a fictitious boundary layer, \mathcal{R}_c , with depth δ , is introduced along the boundary of the actual material region, \mathcal{R} , as shown in Fig. 2.4.

2.7.2.1 Displacement Constraints

The prescribed displacement vector \mathbf{U}_0 can be imposed through the material points in \mathcal{R}_c as

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}_0, \quad \text{for } \mathbf{x} \in \mathcal{R}_c. \quad (2.26)$$

Also, in order to avoid abrupt constraint introduction, it can be applied as

$$\mathbf{u}(\mathbf{x}, t) = \begin{cases} \mathbf{U}_0 \frac{t}{t_0} & \text{for } 0 \leq t \leq t_0 \\ \mathbf{U}_0 & \text{for } t_0 \leq t \end{cases}, \quad (2.27)$$

where t_0 represents the time at which the prescribed displacement is reached. The velocity of each material point, $\dot{\mathbf{u}}(\mathbf{x}, t)$, can be calculated through differentiation.

2.7.2.2 Velocity Constraints

The prescribed velocity vector $\mathbf{V}(t)$ can be imposed through the material points in \mathcal{R}_c as

$$\dot{\mathbf{u}}(\mathbf{x}, t) = \mathbf{V}(t), \quad \text{for } \mathbf{x} \in \mathcal{R}_c. \quad (2.28)$$

Their displacement, $\mathbf{u}(\mathbf{x}, t)$, can be obtained from

$$\mathbf{u}(\mathbf{x}, t) = \int_0^t \mathbf{V}(t') dt'. \quad (2.29)$$

If $\mathbf{V}(t) = \mathbf{V}_0 H(t)$, with \mathbf{V}_0 containing constant constraint values, then $\mathbf{u}(\mathbf{x}, t) = \mathbf{V}_0 t$ for all material points in \mathcal{R}_c . The Heaviside step function is represented by $H(t)$. Also, in order to avoid abrupt velocity introduction, it can be applied as

$$\mathbf{V}(t) = \begin{cases} \mathbf{V}_0 \frac{t}{t_0} & \text{for } 0 \leq t \leq t_0 \\ \mathbf{V}_0 & \text{for } t_0 \leq t, \end{cases} \quad (2.30)$$

where t_0 represents the time at which the prescribed velocity is reached.

2.7.3 External Loads

Boundary traction does not directly appear in the PD equation of motion. Therefore, the application of external loads is also different from that of the classical continuum theory. The difference can be illustrated by considering a region, Ω , that is subjected to external loads. If this region is fictitiously divided into two domains, Ω^- and Ω^+ , as shown in Fig. 2.5a, there must be a net force, \mathbf{F}^+ , that is exerted to domain Ω^+ by domain Ω^- so that force equilibrium is satisfied (Kilic 2008).

According to classical continuum mechanics, force \mathbf{F}^+ can be determined by integrating surface tractions over the cross-sectional area, $\partial\Omega$, of domains Ω^- and Ω^+ as

$$\mathbf{F}^+ = \int_{\partial\Omega} \mathbf{T} dA, \quad (2.31)$$

in which \mathbf{T} is the surface tractions (Fig. 2.5b).

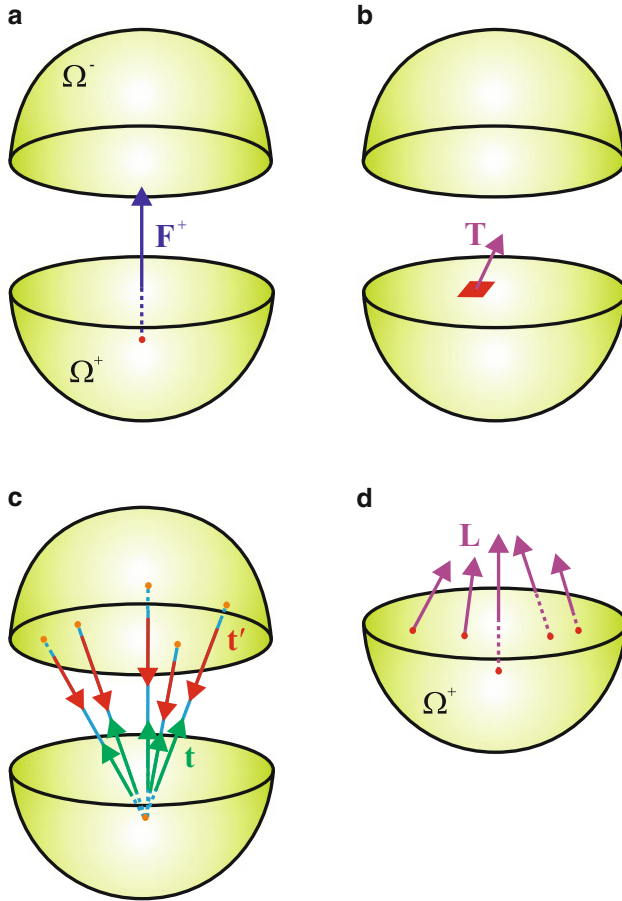


Fig. 2.5 Boundary conditions: (a) domain of interest, (b) tractions in classical continuum mechanics, (c) interaction of a material point in domain Ω^+ with other material points in domain Ω^- , (d) force densities acting on domain Ω^+ due to domain Ω^-

In the case of the PD theory, the material points located in domain Ω^+ interact with the other material points in domain Ω^- (Fig. 2.5c). Thus, the force \mathbf{F}^+ can be computed by volume integration of the force densities (Fig. 2.5d) over domain Ω^+ as

$$\mathbf{F}^+ = \int_{\Omega^+} \mathbf{L}(\mathbf{x}) dV, \quad (2.32a)$$

in which \mathbf{L} , acting on a material point in domain Ω^+ , is determined by

$$\mathbf{L}(\mathbf{x}) = \int_{\Omega^-} [\mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) - \mathbf{t}'(\mathbf{u} - \mathbf{u}', \mathbf{x} - \mathbf{x}', t)] dV. \quad (2.32b)$$

Note that if the volume Ω^- is void, the volume integration in Eq. 2.32b vanishes. Hence, the tractions or point forces cannot be applied as boundary conditions since their volume integrations result in a zero value. Therefore, the external loads can be applied as body force density in a “real material layer” along the boundary of a nonzero volume. Based on numerical experiments, the extent of the boundary layer should be as close to the boundary as possible. Therefore, a boundary layer for external load application, \mathcal{R}_ℓ , with depth Δ , is introduced along the boundary of the material region \mathcal{R} , as shown in Fig. 2.4.

In the case of distributed pressure, $p(\mathbf{x}, t)$, or a point force, $\mathbf{P}(t)$, over the surface \mathcal{S}_ℓ of the boundary layer \mathcal{R}_ℓ , the body force density vector can be expressed as

$$\mathbf{b}(\mathbf{x}, t) = -\frac{1}{\Delta} p(\mathbf{x}, t) \mathbf{n} \quad (2.33a)$$

or

$$\mathbf{b}(\mathbf{x}, t) = \frac{1}{\mathcal{S}_\ell \Delta} \mathbf{P}(t). \quad (2.33b)$$

If $p(\mathbf{x}, t) = p_0(\mathbf{x})H(t)$ and $\mathbf{P}(t) = \mathbf{P}_0H(t)$, with $p_0(\mathbf{x})$ and \mathbf{P}_0 representing the distributed pressure and constant point force, in order to avoid abrupt constraint introduction, they can be applied as

$$\mathbf{b}(\mathbf{x}, t) = -\frac{1}{\Delta} p_0(\mathbf{x}) \mathbf{n} \frac{t}{t_0} \text{ or } \mathbf{b}(\mathbf{x}, t) = \frac{1}{\mathcal{S}_\ell \Delta} \mathbf{P}_0 \frac{t}{t_0} \text{ for } 0 \leq t \leq t_0 \quad (2.34a)$$

and

$$\mathbf{b}(\mathbf{x}, t) = -\frac{1}{\Delta} p_0(\mathbf{x}) \mathbf{n} \text{ or } \mathbf{b}(\mathbf{x}, t) = \frac{1}{\mathcal{S}_\ell \Delta} \mathbf{P}_0, \text{ for } t_0 \leq t, \quad (2.34b)$$

where t_0 represents the time at which the prescribed external load is reached. The displacement and velocity of all points in the boundary layer \mathcal{R}_ℓ are calculated based on the equation of motion.

2.8 Balance Laws

The PD equation of motion must be further governed by the balance of linear momentum, angular momentum, and energy. These balance laws are viewed as having a primitive status in mechanics. The balance of linear momentum and

energy are automatically satisfied, as the principle of virtual work, Eq. 2.9, represents their weak forms. However, the balance of angular momentum must be assured.

The linear momentum, \mathbf{L} , and angular momentum (about the coordinate origin), \mathbf{H}_0 , of a fixed set of particles at time t in volume V are given by

$$\mathbf{L} = \int_V \rho(\mathbf{x}) \dot{\mathbf{u}}(\mathbf{x}, t) dV \quad (2.35a)$$

and

$$\mathbf{H}_0 = \int_V \mathbf{y}(\mathbf{x}, t) \times \rho(\mathbf{x}) \dot{\mathbf{u}}(\mathbf{x}, t) dV, \quad (2.35b)$$

while the total force, \mathbf{F} , and torque, $\mathbf{\Pi}_0$, about the origin are given by

$$\mathbf{F} = \int_V \mathbf{b}(\mathbf{x}, t) dV + \int_V \int_H \mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle dH dV - \int_V \int_H \mathbf{T}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle dH dV \quad (2.35c)$$

and

$$\begin{aligned} \mathbf{\Pi}_0 = & \int_V \mathbf{y}(\mathbf{x}, t) \times \mathbf{b}(\mathbf{x}, t) dV + \int_V \int_H \mathbf{y}(\mathbf{x}, t) \times \mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle dH dV \\ & - \int_V \int_H \mathbf{y}(\mathbf{x}, t) \times \mathbf{T}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle dH dV . \end{aligned} \quad (2.35d)$$

Thus, the balance of linear momentum, $\dot{\mathbf{L}} = \mathbf{F}$, and angular momentum, $\dot{\mathbf{H}}_0 = \mathbf{\Pi}_0$, results in

$$\begin{aligned} \int_V \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) dV = & \int_V \mathbf{b}(\mathbf{x}, t) dV \\ & + \int_V \int_H \mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle dH dV \\ & - \int_V \int_H \mathbf{T}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle dH dV \end{aligned} \quad (2.36a)$$

and

$$\begin{aligned}
\int_V \mathbf{y}(\mathbf{x}, t) \times \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) dV &= \int_V \mathbf{y}(\mathbf{x}, t) \times \mathbf{b}(\mathbf{x}, t) dV \\
&+ \int_V \int_H \mathbf{y}(\mathbf{x}, t) \times \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle dH dV \\
&- \int_V \int_H \mathbf{y}(\mathbf{x}, t) \times \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle dH dV .
\end{aligned} \tag{2.36b}$$

Because $\underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle = \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle = \mathbf{0}$ for $\mathbf{x}' \notin H$, these equations can be rewritten to include all of the material points in volume V as

$$\begin{aligned}
\int_V \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) dV &= \int_V \mathbf{b}(\mathbf{x}, t) dV \\
&+ \int_V \int_V \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle dV' dV \\
&- \int_V \int_V \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle dV' dV
\end{aligned} \tag{2.37a}$$

and

$$\begin{aligned}
\int_V (\mathbf{y}(\mathbf{x}, t) \times \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t)) dV &= \int_V \mathbf{y}(\mathbf{x}, t) \times \mathbf{b}(\mathbf{x}, t) dV \\
&+ \int_V \int_V \mathbf{y}(\mathbf{x}, t) \times \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle dV' dV \\
&- \int_V \int_V \mathbf{y}(\mathbf{x}, t) \times \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle dV' dV .
\end{aligned} \tag{2.37b}$$

If the parameters \mathbf{x} and \mathbf{x}' in the third integrals on the right-hand side of Eqs. 2.37a, b are exchanged, the third integrals become

$$\int_V \int_V \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle dV' dV = \int_V \int_V \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle dV dV' \tag{2.38a}$$

and

$$\begin{aligned}
& \int_V \int_V (\mathbf{y}(\mathbf{x}, t) \times \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle) dV' dV \\
&= \int_V \int_V (\mathbf{y}(\mathbf{x}', t) \times \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dV dV' .
\end{aligned} \tag{2.38b}$$

Therefore, Eqs. 2.37a, b can be rewritten as

$$\int_V (\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) - \mathbf{b}(\mathbf{x}, t)) dV = 0 \tag{2.39a}$$

and

$$\begin{aligned}
\int_V (\mathbf{y}(\mathbf{x}, t) \times \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t)) dV &= \int_V \mathbf{y}(\mathbf{x}, t) \times \mathbf{b}(\mathbf{x}, t) dV \\
&\quad - \int_V \int_V ((\mathbf{y}(\mathbf{x}', t) - \mathbf{y}(\mathbf{x}, t)) \times \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dV' dV .
\end{aligned} \tag{2.39b}$$

Hence, the balance of linear momentum, Eq. 2.39a, is automatically satisfied for arbitrary force density vectors $\underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle$ and $\underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle$.

The difference between the locations of material points at \mathbf{x} and \mathbf{x}' in the deformed configuration can be written by using the state notation as

$$\mathbf{y}(\mathbf{x}', t) - \mathbf{y}(\mathbf{x}, t) = (\mathbf{y}' - \mathbf{y}) = \underline{\mathbf{Y}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle, \tag{2.40}$$

where $\mathbf{y}' = \mathbf{y}(\mathbf{x}', t) = \mathbf{x}' + \mathbf{u}'$ and $\mathbf{y} = \mathbf{y}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}$. Considering only the material points within the horizon, substituting from Eq. 2.40 into Eq. 2.39b results in

$$\begin{aligned}
& \int_V \mathbf{y}(\mathbf{x}, t) \times (\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) - \mathbf{b}(\mathbf{x}, t)) dV \\
&= - \int_V \int_H (\underline{\mathbf{Y}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle \times \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dH dV .
\end{aligned} \tag{2.41}$$

While invoking the requirement of a balance of linear momentum, Eq. 2.39a, in order to satisfy the balance of angular momentum, the integral on the right-hand side of Eq. 2.41 must be forced to vanish, i.e.,

$$\int_H (\underline{\mathbf{Y}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle \times \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dH = 0 \tag{2.42a}$$

or

$$\int_H ((\mathbf{y}' - \mathbf{y}) \times \mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dH = 0. \quad (2.42b)$$

It is apparent that this requirement is automatically satisfied if the force vectors $\mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) = \mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle$ and $\mathbf{t}'(\mathbf{u} - \mathbf{u}', \mathbf{x} - \mathbf{x}', t) = \mathbf{T}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle$ are aligned with the relative position vector of the material points in the deformed state, $(\mathbf{y}' - \mathbf{y})$. However, their general form that satisfies the requirement of Eq. 2.42b can also be derived in terms of the deformation gradient and stress tensors of classical continuum mechanics.

2.9 Bond-Based Peridynamics

As a special case, the force density vectors can also be equal in magnitude as well as being parallel to the relative position vector in the deformed state, shown in Fig. 2.6, in order to satisfy the requirement for balance of angular momentum. Thus, they can be expressed in the form

$$\begin{aligned} \mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) &= \mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle = \frac{1}{2} C \frac{\mathbf{y}' - \mathbf{y}}{|\mathbf{y}' - \mathbf{y}|} \\ &= \frac{1}{2} \mathbf{f}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) \end{aligned} \quad (2.43a)$$

and

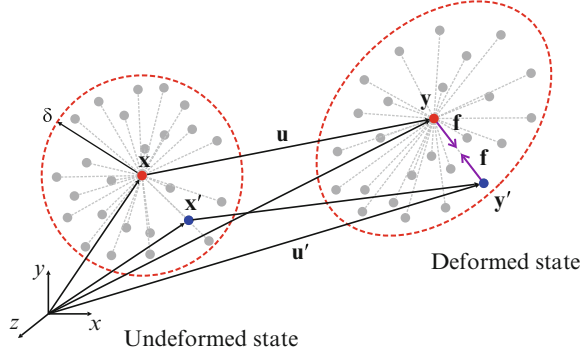
$$\begin{aligned} \mathbf{t}'(\mathbf{u} - \mathbf{u}', \mathbf{x} - \mathbf{x}', t) &= \mathbf{T}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle \\ &= -\frac{1}{2} C \frac{\mathbf{y}' - \mathbf{y}}{|\mathbf{y}' - \mathbf{y}|} = -\frac{1}{2} \mathbf{f}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t), \end{aligned} \quad (2.43b)$$

where C is an unknown auxiliary parameter that depends on the engineering material constants, pairwise stretch between \mathbf{x}' and \mathbf{x} , and the horizon. This particular form of the force vectors is referred to as “bond-based” peridynamics, as introduced by Silling (2000). As shown in Fig. 2.6, the bond-based peridynamic theory is concerned with pairwise interactions of material points.

Their substitution into Eq. 2.22b results in the bond-based PD equation of motion of the material point \mathbf{x}

$$\rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) = \int_H \mathbf{f}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) dH + \mathbf{b}(\mathbf{x}, t), \quad (2.44)$$

Fig. 2.6 Deformation of PD material points \mathbf{x} and \mathbf{x}' , and developing equal and opposite pairwise force densities



in which the force density vector, $\mathbf{f}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x})$ is referred to as the pairwise response function by Silling and Askari (2005). It is defined as the force vector per unit volume squared that the material point at \mathbf{x}' exerts on the material point at \mathbf{x} . The force density vector can be assumed linearly dependent on the stretch between these material points in the form

$$\mathbf{f}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}) = [c_1 s(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}) - c_2 T] \frac{\mathbf{y}' - \mathbf{y}}{|\mathbf{y}' - \mathbf{y}|}, \quad (2.45)$$

where the mean value of the temperatures at material points \mathbf{x}' and \mathbf{x} relative to the ambient temperature is denoted by T . The stretch $s(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x})$ can be interpreted as the strain in the classical continuum theory, and it is defined as

$$s(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}) = \frac{|\mathbf{y}' - \mathbf{y}| - |\mathbf{x}' - \mathbf{x}|}{|\mathbf{x}' - \mathbf{x}|}. \quad (2.46)$$

For an isotropic material, the peridynamic material parameters c_1 and c_2 in Eq. 2.45 can be determined by considering an infinite homogeneous body under isotropic expansion, as suggested by Silling and Askari (2005). The body is also subjected to uniform temperature change, T . Equating the energy densities of peridynamic and classical continuum theory leads to the determination of c_1 and c_2 as

$$c_1 = c = \frac{18\kappa}{\pi\delta^4} \quad \text{and} \quad c_2 = c\alpha, \quad (2.47a, b)$$

in which κ is the bulk modulus and α is the coefficient of thermal expansion of the material. The PD material parameter c is referred to as the bond-constant. In this case, the PD theory limits the number of independent material constants to one for isotropic materials with a constraint on the Poisson's ratio. It permits only total deformation without distinguishing the distortional and volumetric deformations. Furthermore, it does not allow plastic incompressibility.

2.10 Ordinary State-Based Peridynamics

As shown in Fig. 2.7, the force density vectors having unequal magnitudes while being parallel to the relative position vector in the deformed state also satisfy the requirement for balance of angular momentum, Eq. 2.42b. Thus, they can be defined in the form

$$\mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) = \underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle = \frac{1}{2} A \frac{\mathbf{y}' - \mathbf{y}}{|\mathbf{y}' - \mathbf{y}|} \quad (2.48a)$$

and

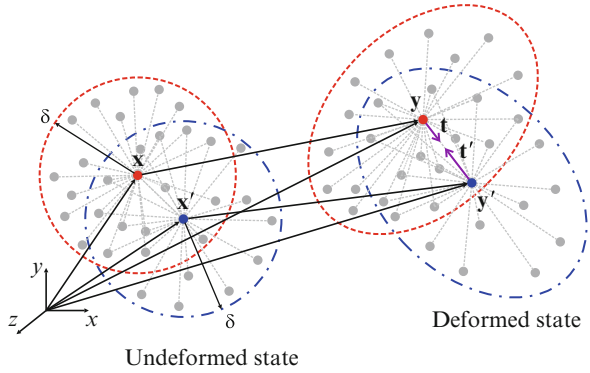
$$\mathbf{t}'(\mathbf{u} - \mathbf{u}', \mathbf{x} - \mathbf{x}', t) = \underline{\mathbf{T}}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle = -\frac{1}{2} B \frac{\mathbf{y}' - \mathbf{y}}{|\mathbf{y}' - \mathbf{y}|}, \quad (2.48b)$$

where A and B are auxiliary parameters that are dependent on engineering material constants, deformation field, and the horizon. As coined by Silling et al. (2007), the choice of the force density vectors in this form is referred to as “ordinary state-based” peridynamics. It permits decoupled distortional and volumetric deformations. Also, it enables the enforcement of plastic incompressibility.

In light of the definition of the strain energy density function, Eq. 2.8, and the expressions for force density vectors in terms of micropotentials, Eqs. 2.17a, b, while considering the requirement on their direction, Eqs. 2.48a, b, the force density vectors can be related to the strain energy density function, W , as

$$\mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) \sim \frac{\partial W(\mathbf{x})}{\partial(|\mathbf{y}' - \mathbf{y}|)} \frac{\mathbf{y}' - \mathbf{y}}{|\mathbf{y}' - \mathbf{y}|}, \quad (2.49a)$$

Fig. 2.7 Deformation of PD material points \mathbf{x} and \mathbf{x}' , and developing unequal pairwise force densities



or

$$\mathbf{t}'(\mathbf{u} - \mathbf{u}', \mathbf{x} - \mathbf{x}', t) \sim \frac{\partial W(\mathbf{x}')}{\partial(|\mathbf{y} - \mathbf{y}'|)} \frac{\mathbf{y}' - \mathbf{y}}{|\mathbf{y}' - \mathbf{y}|}. \quad (2.49b)$$

These relations permit the determination of the auxiliary parameters A and B in Eq. 2.48, and thus the peridynamic constitutive parameters that describe the material behavior. The explicit forms of the expressions for these parameters are derived in Chap. 4 for isotropic and in Chap. 5 for fiber-reinforced composite materials.

2.11 Nonordinary State-Based Peridynamics

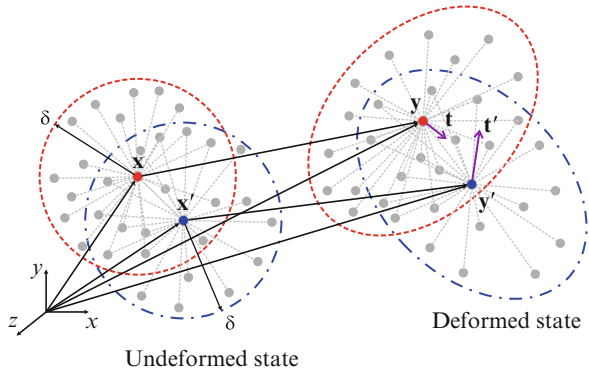
As shown in Fig. 2.8, a general form of a force density vector that satisfies the requirement of Eq. 2.42b necessary for balance of angular momentum can be derived by applying the principle of virtual displacements to Eq. 2.22a as

$$\begin{aligned} \rho(\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) \cdot \Delta \mathbf{u} = & \int_H (\mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle \\ & - \mathbf{T}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle) \cdot \Delta \mathbf{u} dH + \mathbf{b}(\mathbf{x}, t) \cdot \Delta \mathbf{u}, \end{aligned} \quad (2.50)$$

where $\Delta \mathbf{u}$ represents the virtual displacement vector applied to the PD material point at \mathbf{x} . This equation can also be written in matrix notation as

$$\begin{aligned} \rho(\mathbf{x}) \ddot{\mathbf{u}}^T(\mathbf{x}, t) \Delta \mathbf{u} = & \int_H (\mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle \\ & - \mathbf{T}(\mathbf{x}', t) \langle \mathbf{x} - \mathbf{x}' \rangle)^T \Delta \mathbf{u} dH + \mathbf{b}^T(\mathbf{x}, t) \Delta \mathbf{u}. \end{aligned} \quad (2.51)$$

Fig. 2.8 Deformation of PD material points \mathbf{x} and \mathbf{x}' , and developing force densities in arbitrary directions



Noting that $\underline{\mathbf{T}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle = \underline{\mathbf{T}}(\mathbf{x}', t)\langle \mathbf{x} - \mathbf{x}' \rangle = \mathbf{0}$ for $\mathbf{x}' \notin H$ and integrating Eq. 2.51 throughout the body result in

$$\begin{aligned} \int_V (\rho(\mathbf{x})\ddot{\mathbf{u}}^T(\mathbf{x}, t) - \mathbf{b}^T(\mathbf{x}, t))\Delta\mathbf{u} dV &= \int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle)^T \Delta\mathbf{u} dV' dV \\ &\quad - \int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}', t)\langle \mathbf{x} - \mathbf{x}' \rangle)^T \Delta\mathbf{u} dV' dV. \end{aligned} \quad (2.52)$$

Exchanging the parameters \mathbf{x} and \mathbf{x}' in the second integral on the right-hand side of Eq. 2.52 leads to

$$\int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}', t)\langle \mathbf{x} - \mathbf{x}' \rangle)^T \Delta\mathbf{u} dV' dV = \int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle)^T \Delta\mathbf{u}' dV' dV. \quad (2.53)$$

This relationship permits the right-hand side of Eq. 2.52 to be rewritten as

$$\begin{aligned} &\int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle)^T \Delta\mathbf{u} dV' dV - \int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}', t)\langle \mathbf{x} - \mathbf{x}' \rangle)^T \Delta\mathbf{u} dV' dV \\ &= \int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle)^T (\Delta\mathbf{u} - \Delta\mathbf{u}') dV' dV. \end{aligned} \quad (2.54)$$

The difference in virtual displacements of material points at locations \mathbf{x} and \mathbf{x}' can be written in state form as

$$\Delta\mathbf{u}' - \Delta\mathbf{u} = \Delta\underline{\mathbf{Y}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle. \quad (2.55)$$

Therefore, Eq. 2.54 can be rewritten as

$$\begin{aligned} &\int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle)^T (\Delta\mathbf{u} - \Delta\mathbf{u}') dV' dV \\ &= - \int_V \int_V (\underline{\mathbf{T}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle)^T (\Delta\underline{\mathbf{Y}}(\mathbf{x}, t)\langle \mathbf{x}' - \mathbf{x} \rangle) dV' dV. \end{aligned} \quad (2.56)$$

With this equation, Eq. 2.52 can be written in the form

$$\int_V (\rho(\mathbf{x})\ddot{\mathbf{u}}^T(\mathbf{x}, t) - \mathbf{b}^T(\mathbf{x}, t))\Delta\mathbf{u} dV = - \int_V \Delta W_I dV, \quad (2.57)$$

where ΔW_I corresponds to the virtual work of the internal forces at location \mathbf{x} due to its interactions with all other material points:

$$\Delta W_I = \int_V (\mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle)^T (\Delta \mathbf{Y}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dV'. \quad (2.58)$$

Considering only the material points within the horizon, Eq. 2.58 can be rewritten as

$$\Delta W_I = \int_H (\mathbf{T}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle)^T (\Delta \mathbf{Y}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dH. \quad (2.59)$$

The corresponding internal virtual work at location \mathbf{x} in classical continuum mechanics can be expressed as

$$\Delta \hat{W}_I = \text{tr}(\mathbf{S}^T \Delta \mathbf{E}) \quad (2.60)$$

where $\mathbf{S} = \mathbf{S}^T$ is the second Piola-Kirchhoff (Kirchhoff) stress tensor, and the Green-Lagrange strain tensor, $\mathbf{E} = \mathbf{E}^T$, can be related to the deformation gradient tensor, \mathbf{F} ,

$$\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}). \quad (2.61)$$

Using Eq. 2.61, the virtual form of the Green-Lagrange strain tensor can be written as

$$\Delta \mathbf{E} = \frac{1}{2} (\Delta \mathbf{F}^T \mathbf{F} + \mathbf{F}^T \Delta \mathbf{F}). \quad (2.62)$$

After substituting from Eq. 2.62 into Eq. 2.60, the internal virtual work expression in classical continuum mechanics takes the form

$$\Delta \hat{W}_I = \text{tr}(\mathbf{S}^T \mathbf{F}^T \Delta \mathbf{F}) = \text{tr}(\mathbf{P} \Delta \mathbf{F}), \quad (2.63)$$

where $\mathbf{P} = (\mathbf{S}^T \mathbf{F}^T)$ is the first Piola-Kirchhoff (Lagrangian) stress tensor.

By using the vector state reduction to a second-order tensor, given in Eq. A.8, the deformation gradient tensor, which corresponds to the deformation state in PD theory, can be obtained as

$$\mathbf{F} = (\mathbf{Y} * \mathbf{X}) \mathbf{K}^{-1}, \quad (2.64)$$

whose virtual form can be written as

$$\Delta \mathbf{F} = (\Delta \mathbf{Y} * \mathbf{X}) \mathbf{K}^{-1}, \quad (2.65)$$

in which the explicit form of the shape tensor, \mathbf{K} , serving as a volume-averaging quantity, is derived in the Appendix; it is symmetric and diagonal. The symbol $*$ denotes the convolution of vector states, also defined in the Appendix.

Substituting from Eq. 2.65 into the internal virtual work expression of classical continuum mechanics, Eq. 2.63, in conjunction with Eq. A.7, results in

$$\Delta \hat{W}_I = \text{tr} \left(\mathbf{P} \left(\int_H \underline{w} \langle \mathbf{x}' - \mathbf{x} \rangle \Delta \underline{\mathbf{Y}} \langle \mathbf{x}' - \mathbf{x} \rangle \otimes \underline{\mathbf{X}} \langle \mathbf{x}' - \mathbf{x} \rangle dH \right) \mathbf{K}^{-1} \right), \quad (2.66)$$

where the influence (weight) function, \underline{w} , is a scalar state, and \otimes denotes the dyadic product of two vectors, i.e., $\mathbf{C} = \mathbf{a} \otimes \mathbf{b}$ or $C_{ij} = a_i b_j$. The scalar state influence function provides a means to control the influence of PD points away from the current point.

Using Eqs. A.4 and 2.55, this equation can be expressed in indicial form as

$$\Delta \hat{W}_I = P_{ij} \left(\int_H \underline{w} \langle \mathbf{x}' - \mathbf{x} \rangle (\Delta u'_i - \Delta u_i) (x'_k - x_k) dH \right) K_{kj}^{-1}, \text{ with } (i, j, k) = 1, 2, 3 \quad (2.67)$$

Because the shape tensor is symmetric, this equation can be rearranged in the form

$$\Delta \hat{W}_I = \int_H \underline{w} \langle \mathbf{x}' - \mathbf{x} \rangle P_{ij} K_{jk}^{-1} (x'_k - x_k) (\Delta u'_i - \Delta u_i) dH, \text{ with } (i, j, k) = 1, 2, 3 \quad (2.68a)$$

or, in matrix form,

$$\Delta \hat{W}_I = \int_H (\underline{w} \langle \mathbf{x}' - \mathbf{x} \rangle \mathbf{P} \mathbf{K}^{-1} (\mathbf{x}' - \mathbf{x}))^T (\Delta \mathbf{u}' - \Delta \mathbf{u}) dH. \quad (2.68b)$$

After invoking Eq. 2.55 into Eq. 2.68b, equating the virtual work expressions from the PD theory, Eq. 2.59, and classical continuum mechanics, Eq. 2.68b, results in

$$\begin{aligned} & \int_H (\underline{\mathbf{T}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle)^T (\Delta \underline{\mathbf{Y}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dH \\ & \equiv \int_H (\underline{w} \langle \mathbf{x}' - \mathbf{x} \rangle \mathbf{P} \mathbf{K}^{-1} (\mathbf{x}' - \mathbf{x}))^T (\Delta \underline{\mathbf{Y}}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle) dH. \end{aligned} \quad (2.69)$$

This requirement leads to the relation between the force vector state and the deformation gradient and stress tensors of classical continuum mechanics as

$$\mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) = \mathbf{I}(\mathbf{x}, t) \langle \mathbf{x}' - \mathbf{x} \rangle \equiv \underline{w} \langle \mathbf{x}' - \mathbf{x} \rangle \mathbf{P} \mathbf{K}^{-1} (\mathbf{x}' - \mathbf{x}) \quad (2.70)$$

Although this expression for the force density vector, Eq. 2.70, is identical to that derived by Silling et al. (2007), this derivation based on the principle of virtual displacements proves that the force density vector is valid for any material model provided that the Piola-Kirchhoff stress tensor can be obtained directly or by using incremental procedures. Therefore, this equation also forms the basis for implementing any material behavior in the PD theory.

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