Chapter 2
Canonical Coherent States

Abstract This chapter is devoted to a detailed examination of the canonical coherent states (CS), originally introduced by Schrödinger in 1926 and rediscovered in the 1960s for the description of coherent light (lasers). We discuss successively the minimal uncertainty problem, the group-theoretical background of CS, their functional analytic properties and the geometrical context, both in the real and in the complex formulation. We conclude with some simple examples.

This chapter is devoted to a fairly detailed examination of the quintessential example of coherent states—the *canonical coherent states*. It is fair to say that the entire subject of coherent states developed by analogy from this example. As mentioned in the Introduction, this set of states, or rays in the Hilbert space of a quantum mechanical system, was originally discovered by Schrödinger [552] in 1926, as a convenient set of quantum states for studying the transition from quantum to classical mechanics. They are endowed with a remarkable array of interesting properties, some of which we shall survey in this chapter. Apart from initiating the discussion, this will also help us in motivating the various mathematical directions in which one can try to generalize the notion of a CS.

Throughout the book, all Hilbert spaces will be taken to be separable and defined over the complexes. Thus, if $\mathfrak{H}$ is a Hilbert space, its dimension, denoted $\dim\mathfrak{H}$, is (countably) infinite. Also, for any two vectors ϕ, ψ in $\mathfrak{H}$, their scalar product $\langle\phi|\psi\rangle$ will be taken to be antilinear in the first variable ϕ and linear in the second variable ψ (the standard physicist's convention). Unless otherwise stated, we shall use the natural system of units, in which $c = \hbar = 1$.

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2.1 Minimal Uncertainty States

Recall that the quantum kinematics of a free n-particle system is based upon the existence of an irreducible representation of the *canonical commutation relations* (CCR),

$$[Q_i,\, P_j] = iI\delta_{ij}, \quad i,j = 1,2,\ldots,n, \tag{2.1}$$

on a Hilbert space $\mathfrak{H}$. (Here I denotes the identity operator on $\mathfrak{H}$). If n is finite, then according to the well-known uniqueness theorem of von Neumann [vNe55], up to unitary equivalence, there exists only one irreducible representation of (2.1) by self-adjoint operators, on a (separable, complex) Hilbert space (see Sect. 2.2 for a more precise statement). Furthermore, the CCR imply that for any state vector ψ in $\mathfrak{H}$ (note, $\|\psi\| = 1$), the Heisenberg *uncertainty relations* hold:

$$\langle \Delta Q_i\rangle_\psi \, \langle \Delta P_i\rangle_\psi \geq \frac{1}{2}, \quad i = 1,2,\ldots,n, \tag{2.2}$$

where, for an arbitrary operator A on $\mathfrak{H}$,

$$\langle \Delta A\rangle_\psi = [\langle \psi | A^2\psi\rangle - |\langle \psi A\psi\rangle|^2]^{\frac{1}{2}} \tag{2.3}$$

is its standard deviation in the state ψ (to be precise, the vector ψ must belong to the domain of the operator A^2). As already pointed out by Schrödinger (see also [vNe55, Bie81]), there exists an entire family of states, $\eta^{\mathbf{s}}$ in the Hilbert space, labelled by a vector parameter $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$, each one of which saturates the uncertainty relations (2.2):

$$\langle \Delta Q_i\rangle_{\eta^{\mathbf{s}}} \, \langle \Delta P_i\rangle_{\eta^{\mathbf{s}}} = \frac{1}{2}, \quad i = 1,2,\ldots,n. \tag{2.4}$$

We call these vectors *minimal uncertainty states* (MUSTs). In the configuration space, or Schrödinger representation of the CCR, in which

$$\begin{aligned} \mathfrak{H} &= L^2(\mathbb{R}^n, d\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n), \\ (Q_i\psi)(\mathbf{x}) &= x_i\psi(\mathbf{x}), \qquad (P_i\psi)(\mathbf{x}) = -i\frac{\partial}{\partial x_i}\psi(\mathbf{x}), \end{aligned} \tag{2.5}$$

the MUSTs, $\eta^{\mathbf{s}}$, are just the Gaussian wave packets

$$\eta^{\mathbf{s}}(\mathbf{x}) = \prod_{i=1}^{n} (\pi s_i^2)^{-\frac{1}{4}} \, \exp[-\frac{x_i^2}{2s_i^2}]. \tag{2.6}$$

Not surprisingly, quantum systems in these states display behaviour very close to classical systems. More generally, there exists a larger family of states, namely *gaussons* or *gaussian pure states* [557], which exhibits the minimal uncertainty property. These latter states $\eta^{U,V}_{\mathbf{q},\mathbf{p}}$ are parametrized by two vectors, $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ and two real $n \times n$ matrices U and V, of which U is positive definite. In the Schrödinger representation,

$$\eta^{U,V}_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = \pi^{-\frac{n}{4}}[\det U]^{\frac{1}{4}} \exp[i(\mathbf{x} - \frac{\mathbf{q}}{2}) \cdot \mathbf{p}] \times \exp[-\frac{1}{2}(\mathbf{x} - \mathbf{q}) \cdot (U + iV)(\mathbf{x} - \mathbf{q})]. \tag{2.7}$$

In the optical literature, states of the type $\eta^{U,V}_{\mathbf{q},\mathbf{p}}$, for which U is a diagonal matrix but *not* the identity matrix, are called *squeezed states* [150, 598]. Note that when $\mathbf{q} = \mathbf{p} = 0$ and U is diagonal, with eigenvalues $1/s_i^2$, $i = 1, 2, \ldots, n$, the gaussons (2.7) are exactly the MUSTs (2.6). Moreover, if T denotes the orthogonal matrix which diagonalizes U, i.e., $TUT^{-1} = D$, where D is the matrix of eigenvalues of U, then defining the vectors $\mathbf{x}' = T\mathbf{x}$, $\mathbf{q}' = T\mathbf{q}$, $\mathbf{p}' = T\mathbf{p}$, and the matrix $V' = TVT^{-1}$, we may rewrite (2.7) as

$$\eta^{D,V'}_{\mathbf{q}',\mathbf{p}'}(\mathbf{x}') = \pi^{-\frac{n}{4}}[\det D]^{\frac{1}{4}} \exp[i(\mathbf{x}' - \frac{\mathbf{q}'}{2}) \cdot \mathbf{p}'] \times \exp[-\frac{1}{2}(\mathbf{x}' - \mathbf{q}') \cdot (D + iV')(\mathbf{x}' - \mathbf{q}')]. \tag{2.8}$$

It is clear from this relation that, if Q_i', P_i', $i = 1, 2, \ldots, n$, are the components of the rotated vector operators, $\mathbf{Q}' = T^{-1}\mathbf{Q}$, $\mathbf{P}' = T^{-1}\mathbf{P}$, where $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$, $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ are the vector operators of position and momentum, respectively, [see (2.5)] then

$$\langle \Delta Q_i' \rangle_{\eta^{U,V}_{\mathbf{q},\mathbf{p}}} \langle \Delta P_i' \rangle_{\eta^{U,V}_{\mathbf{q},\mathbf{p}}} = \frac{1}{2}, \quad i = 1, 2, \ldots, n. \tag{2.9}$$

To examine some properties of the MUSTs (2.6), take $n = 1$, for simplicity (of notation), and define the *creation* and *annihilation operators*,[1]

$$a^\dagger = \frac{1}{\sqrt{2}}(s^{-1}Q - isP), \qquad a = \frac{1}{\sqrt{2}}(s^{-1}Q + isP), \qquad [a, a^\dagger] = 1. \tag{2.10}$$

[1]For historical reasons, we use here the physicists' notation $a^\dagger$ for the creation operator, but in the rest of the book, we denote by A^* the adjoint of the operator A.

Using these operators and the MUST η^s, for a fixed $s \in \mathbb{R}$, we can generate a very interesting class of other MUSTs (which of course is a subclass of (2.7), and which was already noticed by von Neumann [vNe55]). To do so, define the complex variable

$$z = x + iy = \frac{1}{\sqrt{2}}(s^{-1}q + isp), \qquad (q,p) \in \mathbb{R}^2 \tag{2.11}$$

and write

$$\eta^s = |0\rangle. \tag{2.12}$$

(Note that $a|0\rangle = 0$). Also let $\{|n\rangle\}_{n=0}^{\infty}$ be the normalized eigenstates of the *number operator* $N = a^\dagger a$:

$$N|n\rangle = n|n\rangle, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle, \quad \langle m|n\rangle = \delta_{mn}. \tag{2.13}$$

Then the set of states in $\mathfrak{H}$,

$$\begin{aligned} |z\rangle &= \exp\left(-\frac{|z|^2}{2} + za^\dagger\right)|0\rangle \\ &= \exp\left(-\frac{1}{2}|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \end{aligned} \tag{2.14}$$

for all $z \in \mathbb{C}$, have the eigenvalue property

$$a|z\rangle = z\,|z\rangle. \tag{2.15}$$

It is straightforward to verify that each one of these states $|z\rangle$ is again a MUST satisfying (2.4).

Suppose now that we have a quantized electromagnetic field (in a box), and let $a_k^\dagger, a_k$, $k = 0, \pm 1, \pm 2, \ldots$, be the creation and annihilation operators for the various Fourier modes k. Then in the states

$$|\{z_k\}\rangle = \bigotimes_k |z_k\rangle,$$

the electromagnetic field behaves "classically". More precisely [333, 334], the correlation functions for the field factorize in these states. Thus, let $x = (\mathbf{x}, t)$ be a space-time point and $\mathbf{E}^+(x)$ the positive frequency part of the quantized electric field (Note: $\mathbf{E}^-(x) = \mathbf{E}^+(x)^*$ is the negative frequency part of the field). Then,

$$\mathbf{E}^+(x)\,|\{z_k\}\rangle = \underline{\mathscr{E}}(x)\,|\{z_k\}\rangle,$$

where $\underline{\mathscr{E}}$ is a 3-vector valued function of x, giving the observed field strength at the point x. Let ρ be the density matrix,

$$\rho = |\{z_k\}\rangle\langle\{z_k\}|,$$

and $G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}$ the *correlation functions*,

$$\begin{aligned} G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}(x_1,x_2,\ldots,x_{2n}) &= \\ &= \mathrm{Tr}[\rho E^-_{\mu_1}(x_1)\ldots E^-_{\mu_n}(x_n)E^+_{\mu_{n+1}}(x_{n+1})\ldots E^+_{\mu_{2n}}(x_{2n})], \end{aligned} \tag{2.16}$$

where $E^{\pm}_{\mu_k}$ denotes the μ_k-th component of $\mathbf{E}^{\pm}$. It is then easily verified that

$$G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}(x_1,x_2,\ldots,x_{2n}) = \prod_{k=1}^{n} \overline{\mathscr{E}}_{\mu_k}(x_k) \prod_{\ell=n+1}^{2n} \mathscr{E}_{\mu_\ell}(x_\ell). \tag{2.17}$$

It is because of this factorizability property that the states $|\{z_k\}\rangle$ or the MUSTs $|z\rangle$ were called *coherent states*. However, in the current mathematical literature (though not always in the optical literature), the term coherent state is used to designate an entire array of other mathematically related states, which do not necessarily display either the factorizability property (2.17) or the minimal uncertainty property (2.4). We shall reserve the term *canonical coherent states* for the MUSTs (2.14).

In order to bring out some additional properties of the canonical CS $|z\rangle$, let us use (2.11) to write

$$|z\rangle = \eta^s_{\sigma(q,p)}, \tag{2.18}$$

where z and q, p are related by (2.11). The significance of the σ in this notation will become clear in a while. A short computation shows that

$$\begin{aligned} \langle\eta^s_{\sigma(q,p)}|Q|\eta^s_{\sigma(q,p)}\rangle &= q, \\ \langle\eta^s_{\sigma(q,p)}|P|\eta^s_{\sigma(q,p)}\rangle &= p. \end{aligned} \tag{2.19}$$

In other words, the MUST $\eta^s_{\sigma(q,p)}$ is a translated Gaussian wave packet, centered at the point q in position and p in momentum space. Explicitly, as a vector in $L^2(\mathbb{R}, \mathrm{d}x)$,

$$\eta^s_{\sigma(q,p)}(x) = (\pi s^2)^{-1/4}\, e^{-i\frac{qp}{2}}\, e^{ipx}\, e^{-\frac{(x-q)^2}{2s^2}}, \tag{2.20}$$

which should be compared to (2.7).

2.2 The Group Theoretical Backdrop

A group theoretical property of $|z\rangle$ emerges if we use the Baker–Campbell–Hausdorff identity,

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}\, e^A e^B, \tag{2.21}$$

for two operators A, B, the commutator $[A,B]$ of which commutes with both A and B, and the fact that $a^n|0\rangle = 0,\ n \geq 1$, to write $|z\rangle$ in (2.14) as

$$|z\rangle = \exp(-\frac{1}{2}|z|^2)\, e^{za^\dagger} e^{-\bar{z}a}|0\rangle = e^{za^\dagger - \bar{z}a}|0\rangle, \tag{2.22}$$

where $\bar{z}$ denotes the complex conjugate of z. In terms of (q,p) this is

$$\eta^s_{\sigma(q,p)} = e^{i(pQ-qP)}\eta^s \equiv U(q,p)\eta^s, \tag{2.23}$$

where, $\forall (q,p) \in \mathbb{R}^2$, the operators $U(q,p) = e^{i(pQ-qP)}$ are, of course, unitary. Moreover, we have the integral relation,

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}|\ \mathrm{d}q\,\mathrm{d}p = I. \tag{2.24}$$

The convergence of the above integral is in the *weak sense* (see Sect. 3.1), i.e., for any two vectors ϕ, ψ in the Hilbert space $\mathfrak{H}$,

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \langle\phi|\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}|\psi\rangle\ \mathrm{d}q\,\mathrm{d}p = \langle\phi|\psi\rangle. \tag{2.25}$$

To check the validity of this relation, we use (2.20) to obtain

$$\langle\phi|\eta^s_{\sigma(q,p)}\rangle = (\pi s^2)^{-\frac{1}{4}}\, e^{-\frac{ipq}{2}} \int_{\mathbb{R}} \overline{\phi(x)}\ \exp[ipx]\ \exp[-\frac{(x-q)^2}{2s^2}]\ \mathrm{d}x. \tag{2.26}$$

Hence the left-hand side of (2.25) becomes

$$\begin{aligned}\frac{1}{2\pi\sqrt{\pi}s}\int_{\mathbb{R}^2} \mathrm{d}q\ \mathrm{d}p \int_{\mathbb{R}^2} \mathrm{d}x\ \mathrm{d}x'\ \overline{\phi(x)}\exp[ip(x-x')]\\ \times\exp[-\frac{(x-q)^2}{2s^2} - \frac{(x'-q)^2}{2s^2}]\psi(x').\end{aligned} \tag{2.27}$$

Using the representation

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{ip(x-x')}\ \mathrm{d}p = \delta(x-x'), \tag{2.28}$$

for the δ-distribution, and performing the integration over x', the above integral becomes

$$\begin{aligned}\frac{1}{\sqrt{\pi}s}\int_{\mathbb{R}^2}\overline{\phi(x)}\exp\left[-\frac{(x-q)^2}{s^2}\right]\psi(x)\ \mathrm{d}q\,\mathrm{d}x\\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\exp\left[-q^2\right]\ \mathrm{d}q\int_{\mathbb{R}}\overline{\phi(x)}\psi(x)\ \mathrm{d}x\\ &= \langle\phi|\psi\rangle.\end{aligned} \tag{2.29}$$

The relation (2.24) is called the *resolution of the identity* generated by the canonical CS.

The operators $U(q,p)$ in (2.23) arise from a unitary, irreducible representation (UIR) of the *Weyl–Heisenberg group*, $\mathrm{G_{WH}}$, which is a central extension of the group of translations of the two-dimensional Euclidean plane. The UIR in question is the unitary representation of $\mathrm{G_{WH}}$ which integrates the CCR (2.1). An arbitrary element g of $\mathrm{G_{WH}}$ is of the form

$$g=(\theta,q,p),\quad \theta\in\mathbb{R},\quad (q,p)\in\mathbb{R}^2,$$

with multiplication law,

$$g_1g_2=(\theta_1+\theta_2+\xi((q_1,p_1);(q_2,p_2)),\ q_1+q_2,\ p_1+p_2), \tag{2.30}$$

where ξ is the multiplier function

$$\xi((q_1,p_1);(q_2,p_2))=\frac{1}{2}(p_1q_2-p_2q_1). \tag{2.31}$$

Any infinite-dimensional UIR, U^λ, of $\mathrm{G_{WH}}$ is characterized by a real number $\lambda\neq 0$ (in addition, there are also degenerate, one-dimensional, UIRs corresponding to $\lambda=0$, but they are irrelevant here [Per86]) and may be realized on the same Hilbert space $\mathfrak{H}$, as the one carrying an irreducible representation of the CCR:

$$U^\lambda(\theta,q,p)=e^{i\lambda\theta}U^\lambda(q,p):=e^{i\lambda(\theta-\frac{pq}{2})}e^{i\lambda pQ}e^{-i\lambda qP}. \tag{2.32}$$

If $\mathfrak{H}=L^2(\mathbb{R},\mathrm{d}x)$, these operators are defined by the action

$$(U^\lambda(\theta,q,p)\phi)(x)=e^{i\lambda\theta}e^{i\lambda p(x-\frac{q}{2})}\phi(x-q),\qquad \phi\in L^2(\mathbb{R},\mathrm{d}x). \tag{2.33}$$

Thus, the three operators, I,Q,P, appear now as the infinitesimal generators of this representation and are realized as:

$$(Q\phi)(x)=x\phi(x),\quad (P\phi)(x)=-\frac{i}{\lambda}\frac{\partial\phi}{\partial x}(x),\qquad [Q,P]=\frac{i}{\lambda}I. \tag{2.34}$$

For our purposes, we take for λ the specific value, $\lambda=\frac{1}{\hbar}=1$, and simply write U for the corresponding representation.

Denoting the phase subgroup of $\mathrm{G_{WH}}$ (the subgroup of elements $g = (\theta, 0, 0)$, $\theta \in \mathbb{R}$), by Θ, it is easily seen that the left coset space $\mathrm{G_{WH}}/\Theta$ can be identified with $\mathbb{R}^2$ and a general element in it parametrized by (q, p). In terms of this parametrization, $\mathrm{G_{WH}}/\Theta$ carries the *invariant* measure

$$\mathrm{d}\nu(q,p) = \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}. \tag{2.35}$$

The function

$$\sigma : \mathrm{G_{WH}}/\Theta \to \mathrm{G_{WH}}, \quad \sigma(q,p) = (0,q,p), \tag{2.36}$$

then defines a *section* in the group $\mathrm{G_{WH}}$, now viewed as a *fibre bundle*, over the base space $\mathrm{G_{WH}}/\Theta$, having fibres isomorphic to Θ. Thus, the family of canonical CS is the set,

$$\mathfrak{S}_\sigma = \{\eta^s_{\sigma(q,p)} = U(\sigma(q,p))\eta^s \colon (q,p) \in \mathrm{G_{WH}}/\Theta\}, \tag{2.37}$$

and the operator integral in (2.24) becomes

$$\int_{\mathrm{G_{WH}}/\Theta} |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}| \;\mathrm{d}\nu(q,p) = I. \tag{2.38}$$

In other words, the CS $\eta^s_{\sigma(q,p)}$ are labelled by the points (q,p) in the *homogeneous space* $\mathrm{G_{WH}}/\Theta$ of the Weyl–Heisenberg group, and they are obtained by the action of the unitary operators $U(\sigma(q,p))$, of a UIR of $\mathrm{G_{WH}}$, on a fixed vector $\eta^s \in \mathfrak{H}$. The resolution of the identity (2.38) is then a statement of the *square-integrability* of the UIR, U, with respect to the homogeneous space $\mathrm{G_{WH}}/\Theta$. This way of looking at coherent states turns out to be extremely fruitful. Indeed, one could ask if it might not be possible to use this idea to generalize the notion of a CS and to build families of such states, using UIR's of groups other than the Weyl–Heisenberg group, making sure in the process that basic relations of the type (2.36)–(2.38) are still fulfilled. We shall see in Chaps. 7 and 8 that this is indeed possible, and that such an approach yields a powerful generalization of the notion of a coherent state.

Two remarks are in order before proceeding. First and not surprisingly, the same canonical CS may be obtained from the oscillator group H(4), which is the group with the Lie algebra generated by $\{a, a^\dagger, N = a^\dagger a, I\}$. Secondly, it is interesting that the canonical CS are widely used in signal processing, where they generate the so-called *windowed Fourier transform* or *Gabor transform*, since it was introduced in that context by Dennis Gabor in his 1946 landmark paper on communication theory [294]. This is a hint that CS will have an important role in classical physics as well as in quantum physics, and as a matter of fact they may be viewed as a natural bridge between the two. We shall discuss these matters in detail in Chaps. 11 and 12.

2.3 Some Functional Analytic Properties

The resolution of the identity given by the operator integral in (2.38) leads to some interesting functional analytic properties of the CS, $\eta^s_{\sigma(q,p)}$. These properties can be studied in their abstract forms and be used to obtain a generalization of the notion of a CS, but now independently of any group theoretical implications.

Let $\widetilde{\mathfrak{H}} = L^2(\mathrm{G}_{\mathrm{WH}}/\Theta,\ d\nu)$ be the Hilbert space of all complex valued functions on $\mathrm{G}_{\mathrm{WH}}/\Theta$ which are square integrable with respect to $d\nu$. Then (2.38) implies that functions $\Phi : \mathrm{G}_{\mathrm{WH}}/\Theta \to \mathbb{C}$ of the type

$$\Phi(q,p) = \langle \eta^s_{\sigma(q,p)} | \phi \rangle, \tag{2.39}$$

for $\phi \in \mathfrak{H}$, define elements in $\widetilde{\mathfrak{H}}$, and moreover, writing $W : \mathfrak{H} \to \widetilde{\mathfrak{H}}$ for the linear map which associates, via (2.39), an element ϕ in $\mathfrak{H}$ to an element Φ in $\widetilde{\mathfrak{H}}$ (i.e., $W\phi = \Phi$), we see that W is an *isometry* or a norm-preserving linear map:

$$\|W\phi\|^2 = \|\Phi\|^2 = \int_{\mathrm{G}_{\mathrm{WH}}/\Theta} |\Phi(q,p)|^2 \ d\nu(q,p) = \|\phi\|^2. \tag{2.40}$$

The range of this isometry, which we denote by $\mathfrak{H}_K$,

$$\mathfrak{H}_K = W\mathfrak{H} \subset \widetilde{\mathfrak{H}}, \tag{2.41}$$

is a closed subspace of $\widetilde{\mathfrak{H}}$ and furthermore, it is a *reproducing kernel Hilbert space*. To understand the meaning of this, consider the function $K(q,p;\, q',p')$ defined on $\mathrm{G}_{\mathrm{WH}}/\Theta \times \mathrm{G}_{\mathrm{WH}}/\Theta$:

$$\begin{aligned} K(q,p;\, q',p') &= \langle \eta^s_{\sigma(q,p)} | \eta^s_{\sigma(q',p')} \rangle \\ &= \exp\big[-\frac{i}{2}(qp' - q'p)\big] \ \exp\big[-\frac{s^2}{4}(p-p')^2\big] \ \exp\big[-\frac{1}{4s^2}(q-q')^2\big] \\ &= \exp\big[z\overline{z}' - \frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2\big] \\ &= \langle \overline{z} | \overline{z}' \rangle = K(z, \overline{z}'), \end{aligned} \tag{2.42}$$

the third and fourth equalities following from (2.18) and (2.20). The function K is a *reproducing kernel*, a name that reflects the reproducing property satisfied by any vector $\Phi \in \mathfrak{H}_K$:

$$\Phi(q,p) = \int_{\mathrm{G}_{\mathrm{WH}}/\Theta} K(q,p;\, q',p')\, \Phi(q',p') \ d\nu(q',p'). \tag{2.43}$$

The function K enjoys the properties:

1. Hermiticity,

$$K(q,p;\, q',p') = \overline{K(q',p';\, q,p)}. \tag{2.44}$$

2. Positivity,

$$K(q,p;\ q,p) > 0. \tag{2.45}$$

3. Idempotence,

$$\int_{G_{WH}/\Theta} K(q,p;\ q'',p'')K(q'',p'';\ q',p')\ d\nu(q'',p'') = K(q,p;\ q',p'). \tag{2.46}$$

The above relations hold for all $(q,p),(q',p') \in G_{WH}/\Theta$. Condition (2.46) is a consequence of (2.38) and is also called the *square integrability property* of K. All three relations are the transcription of the fact that the orthogonal projection operator $\mathbb{P}_K$ of $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}_K$ is an integral operator, with kernel $K(q,p;\ q',p')$.

It is easy to see that the kernel K actually determines the Hilbert space $\mathfrak{H}_K$. Comparing (2.39) and (2.42) we see that

$$(W\eta^s_{\sigma(q',p')})(q,p) = K(q,p;\ q',p'); \tag{2.47}$$

in other words, for fixed (q',p'), the function $(q,p) \mapsto K(q,p;\ q',p')$ is simply the image in $\mathfrak{H}_K$ of the CS $\eta^s_{\sigma(q',p')}$ under the isometry (2.39). Additionally, if Φ is an element of the Hilbert space $\mathfrak{H}_K$, it is necessarily of the form (2.39). Hence, multiplying both sides of that equation by $\eta^s_{\sigma(q,p)}$ and integrating, we get, upon using (2.38),

$$\phi = \int_{G_{WH}/\Theta} \Phi(q,p)\eta^s_{\sigma(q,p)}\ d\nu(q,p). \tag{2.48}$$

This shows that the set of vectors $\eta^s_{\sigma(q,p)}$, $(q,p) \in G_{WH}/\Theta$, is *overcomplete* in $\mathfrak{H}$ and hence, since W is an isometry, the set of vectors

$$\xi_{\sigma(q,p)} = W\eta^s_{\sigma(q,p)}, \qquad \xi_{\sigma(q,p)}(q',p') = K(q',p';q,p), \tag{2.49}$$

(for all $(q,p) \in G_{WH}/\Theta$), is overcomplete in $\mathfrak{H}_K$. (Note that the vectors $\xi_{\sigma(q,p)}$ are the same CS as the $\eta^s_{\sigma(q,p)}$, but now written as vectors in the *Hilbert space of functions* $\mathfrak{H}_K$). The term overcompleteness is to be understood in the following way: Since $\mathfrak{H}_K$ is a separable Hilbert space, it is always possible to choose a countable basis $\{\eta_i\}_{i=1}^{\infty}$ in it, and to express any vector $\phi \in \mathfrak{H}$ as a linear combination of the vectors in this basis. By contrast, the family of CS, $\mathfrak{S}_\sigma$ in (2.37) is labelled by a pair of continuous parameters (q,p), and (2.48), or equivalently (2.38) is the statement of the fact that any vector ϕ can be expressed in terms of the vectors in this family. Clearly, it should be possible to choose a countable set of vectors $\{\eta^s_{\sigma(q_i,p_i)}\}_{i=1}^{\infty}$ from $\mathfrak{S}_\sigma$ and still obtain a basis for $\mathfrak{H}$. This is in fact possible and many different discretizations exist. The most familiar situation is that where the set of points $\{q_i,p_i\}$ is a lattice, such that the area of the unit cell is smaller than a critical value (to be sure, the resulting set of CS is then *overcomplete*). The determination of adequate subsets $\{q_i,p_i\}$ leads to very interesting mathematical problems, for instance in number theory and in the theory of analytic functions [Per86, Sect. 1.4]. These considerations are part of the general problem of CS discretization, that we shall tackle in the last chapter of the book (Chap. 17).

Equation (2.39) also implies a boundedness property for the functions Φ in the reproducing kernel Hilbert space $\mathfrak{H}_K$. Indeed, using the unitarity of $U(\sigma(q,p))$,

$$|\Phi(q,p)| \leq \|\eta\| \, \|\phi\|, \quad \forall (q,p) \in \mathrm{G_{WH}}/\Theta, \tag{2.50}$$

implying that the vectors in $\mathfrak{H}_K$ are all bounded functions. More importantly, this also shows that the linear map

$$E_K(q,p) : \mathfrak{H}_K \to \mathbb{C}, \qquad E_K(q,p)\Phi = \Phi(q,p), \tag{2.51}$$

which simply evaluates each function $\Phi \in \mathfrak{H}_K$ at the point (q,p), and hence called an *evaluation map*, is continuous. As we shall see later, this can in fact be taken to be the defining property of a reproducing kernel Hilbert space and used to arrive at coherent states via a relation of the type (2.47).

The CS $\eta^s_{\sigma(q,p)}$, along with the resolution of the identity relation (2.38), can be used to obtain a useful family of *localization operators* on the *phase space* $\Gamma :=$ $\mathrm{G_{WH}}/\Theta$. Indeed, relations such as (2.19) tend to indicate that the CS $\eta^s_{\sigma(q,p)}$ do in some sense describe the localization properties of the quantum system in the phase space Γ. To pursue this point a little further, denote by Δ an arbitrary Borel set in Γ, considered as a measure space, and let $\mathscr{B}(\Gamma)$ denote the σ-algebra of all Borel sets of Γ. Define the positive, bounded operator

$$a(\Delta) = \int_\Delta |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}| \; \mathrm{d}\nu(q,p). \tag{2.52}$$

This family of operators, as Δ runs through $\mathscr{B}(\Gamma)$, enjoys certain measure theoretical properties:

1. If J is a countable index set and Δ_i, $i \in J$, are mutually disjoint elements of $\mathscr{B}(\Gamma)$, i.e., $\Delta_i \cap \Delta_j = \emptyset$, for $i \neq j$ ($\emptyset$ denoting the empty set), then
$$a(\cup_{i\in J}\Delta_i) = \sum_{i\in J} a(\Delta_i), \tag{2.53}$$
the sum being understood to converge weakly.
2. Normalization:
$$a(\Gamma) = I, \qquad \text{also} \quad a(\emptyset) = 0. \tag{2.54}$$

Such a family of operators $a(\Delta)$ is said to constitute a *normalized, positive operator-valued (POV) measure* on $\mathfrak{H}$. Using the isometry W in (2.39) and the CS $\xi_{\sigma(q,p)}$ in (2.49), we obtain the normalized POV-measure $a_K(\Delta)$ on $\mathfrak{H}_K$:

$$\begin{aligned} a_K(\Delta) &= \int_\Delta |\xi_{\sigma(q,p)}\rangle\langle\xi_{\sigma(q,p)}| \; \mathrm{d}\nu(q,p) \\ &= W a(\Delta) W^*. \end{aligned} \tag{2.55}$$

Note that

$$a_K(\Gamma) = \int_\Gamma |\xi_{\sigma(q,p)}\rangle\langle\xi_{\sigma(q,p)}| \; \mathrm{d}\nu(q,p)$$
$$= \mathbb{P}_K, \tag{2.56}$$

where $\mathbb{P}_K$ is the projection operator, $\mathfrak{H}_K = \mathbb{P}_K\widetilde{\mathfrak{H}}$, which projects onto the reproducing kernel subspace $\mathfrak{H}_K$ of $\widetilde{\mathfrak{H}}$.

If $\Psi \in \mathfrak{H}_K$ is an arbitrary state vector, and $\Psi = W\psi, \;\; \psi \in \mathfrak{H}$, then by (2.55) and (2.39)

$$\langle\Psi|a_K(\Delta)\Psi\rangle = \langle\psi|a(\Delta)\psi\rangle = \int_\Delta |\Psi(q,p)|^2 \; \mathrm{d}\nu(q,p). \tag{2.57}$$

This means that if $\Psi(q,p)$ is considered as being the phase space wave function of the system, then $a_K(\Delta)$ is the *operator of localization* in the region Δ of phase space. Of course, in order to interpret $|\Psi(q,p)|^2$ as a phase space probability density, an appropriate concept of joint measurement of position and momentum has to be developed. This can in fact be done (see, for example, [13, Bus91, Pru86]). Here, let us just indicate, without proof, an interesting fact which reinforces the interpretation of the $a_K(\Delta)$ as localization operators. On $\mathfrak{H}_K$ define the two unbounded operators Q_K and P_K

$$\langle\Psi|Q_K\Phi\rangle = \int_\Gamma \overline{\Psi(q,p)}\; q\Phi(q,p) \; \mathrm{d}\nu(q,p),$$
$$\langle\Psi|P_K\Phi\rangle = \int_\Gamma \overline{\Psi(q,p)}\; p\Phi(q,p) \; \mathrm{d}\nu(q,p), \tag{2.58}$$

on vectors Ψ, Φ chosen from appropriate dense sets in $\mathfrak{H}_K$. Then it can be shown (see, e.g., [13]) that

$$[Q_K,\, P_K] = iI_K, \quad I_K = \text{ identity operator on } \;\mathfrak{H}_K. \tag{2.59}$$

Thus, multiplication by q and p on Γ, followed by the projection $\mathbb{P}_K$, yields the position and momentum operators on $\mathfrak{H}_K$.

Mathematically, the virtue of this functional analytic description of the coherent states $\eta^s_{\sigma(q,p)}$ is that it points up another possibility of generalization: We would like to associate CS to arbitrary reproducing kernel Hilbert spaces. This will be dealt with in Chap. 5, and it will also take us into the theory of frames, in Chaps. 7 and 17.

2.4 A Complex Analytic Viewpoint

To bring out some complex analytic properties of the canonical CS, let $\phi \in \mathfrak{H}$ be an arbitrary vector. Computing its scalar product with the CS $|\bar{z}\rangle$ using (2.14), we get

$$\begin{aligned}\langle \bar{z}|\phi\rangle &= \exp\left(-\frac{1}{2}|z|^2\right) \sum_{n=0}^{\infty} \frac{\langle n|\phi\rangle}{\sqrt{n!}}\, z^n \\ &= \exp\left(-\frac{1}{2}|z|^2\right) f(z).\end{aligned} \tag{2.60}$$

Here f is an analytic function of the complex variable z. Note that in (2.60) we took the scalar product of $|\phi\rangle$ with $|\bar{z}\rangle$ and not with $|z\rangle$, in order to come out with an analytic function $f(z)$, rather than an anti-analytic function. This was made necessary by our convention that scalar products are linear in the right hand and antilinear in the left hand term. In terms of $z, \bar{z}$ we may write the measure (2.35) as

$$\mathrm{d}\nu(q,p) = \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\pi i}, \tag{2.61}$$

and let us define the new measure

$$\mathrm{d}\mu(z,\bar{z}) = \exp\left(-|z|^2\right) \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\pi i}. \tag{2.62}$$

In this notation, "$\wedge$" denotes the *exterior product* of the two differentials $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ (considered as *one-forms* on the *complex manifold* $\mathbb{C}$). In measure theoretic terms, the quantity $i\,\mathrm{d}z \wedge \mathrm{d}\bar{z}/2$ simply represents the Lebesgue measure $\mathrm{d}x\mathrm{d}y$, $z = x + iy$, on $\mathbb{C}$. Comparing (2.60) and (2.61) with (2.39) and (2.40), we see that $\mathfrak{H}_K$ can be identified with the Hilbert space of all analytic functions in z which are square-integrable with respect to $\mathrm{d}\mu$. Let $\mathfrak{H}_{\mathrm{hol}}$ denote this Hilbert space. Then, the linear map

$$W_{\mathrm{hol}} : \mathfrak{H} \to \mathfrak{H}_{\mathrm{hol}}\,, \qquad (W_{\mathrm{hol}}\phi)(z) = \exp\left(\frac{1}{2}|z|^2\right) \langle \bar{z}|\phi\rangle, \tag{2.63}$$

is an isometry. Using (2.18) and (2.20) we can compute the vectors

$$f_{\sigma(q,p)} = W_{\mathrm{hol}}\, \eta^s_{\sigma(q,p)} = W_{\mathrm{hol}}\, |\bar{z}\rangle = \exp\left(-\frac{1}{2}|z|^2\right) \zeta_{\bar{z}}\,, \tag{2.64}$$

which are the images of the $\eta^s_{\sigma(q,p)}$ in $\mathfrak{H}_{\mathrm{hol}}$. The vectors $\zeta_{\bar{z}} \in \mathfrak{H}_{\mathrm{hol}}$ represent the analytic functions [see (2.42)]:

$$\zeta_{\bar{z}}(z') = e^{z'\bar{z}} = \exp\left[\frac{1}{2}(|z|^2 + |z'|^2)\right] K(z', \bar{z}). \tag{2.65}$$

From this it is clear that the function $K_{\mathrm{hol}} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$,

$$K_{\mathrm{hol}}\,(z',\overline{z}) = \langle \zeta_{\overline{z}'} | \zeta_{\overline{z}} \rangle_{\mathfrak{H}_{\mathrm{hol}}} = e^{z'\overline{z}}, \tag{2.66}$$

is a reproducing kernel for $\mathfrak{H}_{\mathrm{hol}}$. (Here $\langle \cdot | \cdot \rangle_{\mathfrak{H}_{\mathrm{hol}}}$ denotes the scalar product in $\mathfrak{H}_{\mathrm{hol}}$). Indeed, for any $f \in \mathfrak{H}_{\mathrm{hol}}$ and $z \in \mathbb{C}$,

$$\int_{\mathbb{C}} K_{\mathrm{hol}}(z,\overline{z}')f(z')\ \mathrm{d}\mu(z',\overline{z}') = f(z) = \langle \zeta_{\overline{z}} | f \rangle_{\mathfrak{H}_{\mathrm{hol}}}. \tag{2.67}$$

The vectors $\zeta_{\overline{z}}$ satisfy the resolution of the identity relation on $\mathfrak{H}_{\mathrm{hol}}$:

$$\int_{\mathbb{C}} |\zeta_{\overline{z}}\rangle\langle\zeta_{\overline{z}}|\ \mathrm{d}\mu(z,\overline{z}) = I_{\mathfrak{H}_{\mathrm{hol}}}. \tag{2.68}$$

The MUST, $\eta^s = |0\rangle$, is represented in $\mathfrak{H}_{\mathrm{hol}}$ as the constant vector

$$W_{\mathrm{hol}}\ \eta^s = u_0 = K_{\mathrm{hol}}(\,\cdot\,,0), \qquad u_0(z) = 1, \quad \forall z \in \mathbb{C}. \tag{2.69}$$

Since $K_{\mathrm{hol}}(z,\overline{z}) = \|\zeta_{\overline{z}}\|^2$, (2.67) implies that

$$|f(z)| \leq [K_{\mathrm{hol}}(z,\overline{z})]^{\frac{1}{2}}\,\|f\|. \tag{2.70}$$

Hence, in agreement with (2.51), the evaluation map

$$E_{\mathrm{hol}}(z) : \mathfrak{H}_{\mathrm{hol}} \to \mathbb{C}, \qquad E_{\mathrm{hol}}(z)f = f(z), \tag{2.71}$$

is continuous. Actually, the CS $\zeta_{\overline{z}}$ could have been obtained by using this fact alone, i.e., by defining it to be the vector which for arbitrary $f \in \mathfrak{H}_{\mathrm{hol}}$ gives $f(z) = \langle \zeta_{\overline{z}} | f \rangle$. Such a construction would be independent of any group theoretical considerations, and is intrinsic to complex manifolds admitting *Kähler structures* (see Sect. 2.6 below).

The representation $\exp\left(-|z|^2/2\right)\zeta_z$ of the canonical CS, with the ζ_z being vectors in the space of holomorphic functions $\mathfrak{H}_{\mathrm{hol}}$ is known among physicists as the *Fock–Bargmann representation*, and the Hilbert space $\mathfrak{H}_{\mathrm{hol}}$ as the *Bargmann* or *Bargmann–Segal* space, often denoted by $\mathscr{F}^{\mathrm{B}}$ [123,124]. As noted earlier, it consists of entire analytic functions which are square integrable with respect to $\mathrm{d}\mu(z,\overline{z})$,

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \ \text{ the sum converging absolutely for all } z \in \mathbb{C}$$

and

$$\|f\|^2 := \int_{\mathbb{C}} |f(z)|^2\, \mathrm{d}\mu(z,\overline{z}) < \infty. \tag{2.72}$$

The inner product is

$$\langle f\,|\,g\rangle = \int_{\mathbb{C}} \overline{f(z)}\,g(z)\,\mathrm{d}\mu(z,\bar{z}) = \sum_{n=0}^{\infty} n!\,\overline{f_n}\,g_n. \tag{2.73}$$

Going back to the position representation, where the vectors $|n\rangle$ read as $\langle x\,|\,n\rangle = \psi_n(x)$, we see that the operator W_{hol} is an integral operator,

$$f(z) = (W_{\mathrm{hol}}\,\psi)(z) = \int_{\mathbb{R}} \mathscr{K}(z,x)\,\psi(x)\,\mathrm{d}x, \quad \psi \in \mathfrak{H}, \tag{2.74}$$

with kernel

$$\begin{aligned}\mathscr{K}(z,x) &= \pi^{-1/4} e^{-\frac{1}{2}(z^2+x^2)+\sqrt{2}zx} \\ &= \sum_{n=0}^{\infty} \overline{\psi_n(x)}\,\frac{z^n}{\sqrt{n!}}\end{aligned} \tag{2.75}$$

In particular, the basis vectors $|n\rangle \in \mathfrak{H}$, [see (2.13)] are mapped by W_{hol} to the vectors

$$W_{\mathrm{hol}}\,|n\rangle = |u_n\rangle, \qquad u_n(z) = \frac{z^n}{\sqrt{n!}}. \tag{2.76}$$

The inverse transformation is simply

$$\psi(x) = \int_{\mathbb{C}} \overline{\mathscr{K}(z,x)}\,f(z)\,\mathrm{d}\mu(z,\bar{z}), \quad f \in \mathfrak{H}_{\mathrm{hol}}. \tag{2.77}$$

In the Fock–Bargmann representation, the operators $a, a^{\dagger}$ are given by

$$(af)(z) = \frac{\partial f}{\partial z}(z), \qquad (a^{\dagger} f)(z) = z f(z), \qquad f \in \mathfrak{H}_{\mathrm{hol}}. \tag{2.78}$$

Equation (2.66) then implies:

$$K_{\mathrm{hol}}(z',\bar{z}) = \sum_{n=0}^{\infty} u_n(z')\,\overline{u_n(z)}. \tag{2.79}$$

At this point, it is worthwhile to reinterpret the group-theoretical considerations of Sect. 2.2 in the present complex analytic formulation. The starting point is the unitary displacement operator

$$D(z) = e^{(za^{\dagger} - \bar{z}a)}\ (z \in \mathbb{C}), \quad D(-z) = (D(z))^{-1} = D(z)^{\dagger}. \tag{2.80}$$

introduced in Chap. 1, property **P3** [see, also (2.22)]. Using the Baker–Campbell–Hausdorff formula (2.21), we have

$$D(z) = e^{za^\dagger} e^{-\bar{z}a} e^{-\frac{1}{2}|z|^2} = e^{-\bar{z}a} e^{za^\dagger} e^{\frac{1}{2}|z|^2}, \tag{2.81}$$

from which follow the formulas

$$\frac{\partial}{\partial z} D(z) = \left(a^\dagger - \frac{1}{2}\bar{z}\right) D(z) = D(z)\left(a^\dagger + \frac{1}{2}\bar{z}\right),$$

$$\frac{\partial}{\partial \bar{z}} D(z) = -\left(a - \frac{1}{2}z\right) D(z) = -D(z)\left(a + \frac{1}{2}z\right).$$

These operators satisfy the addition formula:

$$D(z)D(z') = e^{\frac{1}{2}z\circ z'} D(z+z'),$$

where $z\circ z'$ is the symplectic product $z\circ z' = z\bar{z}' - \bar{z}z' = 2i\,\mathrm{Im}\, z\bar{z}'$. This in turn entails the covariance formula on a global level:

$$D(z)D(z')D(z)^\dagger = e^{z\circ z'} D(z'),$$

and on a Lie algebra level

$$D(z)\, a\, D(z)^\dagger = a - z, \quad D(z)\, a^\dagger\, D(z)^\dagger = a^\dagger - \bar{z}.$$

The matrix elements of the operator $D(z)$ involve the associated Laguerre polynomials:

$$\langle m|D(z)|n\rangle = D_{mn}(z) = \sqrt{\frac{n!}{m!}}\, e^{-|z|^2/2}\, z^{m-n}\, L_n^{(m-n)}(|z|^2), \quad \text{for } m \geq n, \tag{2.82}$$

with $L_n^{(m-n)}(t) = \frac{m!}{n!}(-t)^{n-m} L_m^{(n-m)}(t)$ for $n \geq m$.

Coming back to the Weyl–Heisenberg group $\mathrm{G_{WH}} = \{(\theta, z)\,,\, \theta \in \mathbb{R}, z \in \mathbb{C}\}$, we see that the group law (2.30) becomes

$$(\theta_1, z_1)(\theta_2, z_2) = (\theta_1 + \theta_2 + \mathrm{Im}\, z_1\bar{z}_2, z_1 + z_2), \quad (\theta, z)^{-1} = (-\theta, -z),$$

so that the unitary representation (2.32) reads in $\mathfrak{H}_{\mathrm{hol}}$ as

$$(\theta, z) \mapsto e^{i\lambda\theta} D(z), \tag{2.83}$$

$$(\theta_1, z_1)(\theta_2, z_2) \mapsto e^{i\lambda\theta_1} D(z_1) e^{i\lambda\theta_2} D(z_2) = e^{i(\theta_1+\theta_2+\mathrm{Im}\, z_1\bar{z}_2)} D(z_1+z_2). \tag{2.84}$$

To complete the picture we need to define also a discrete symmetry, the *parity* P, acting on $\mathfrak{H}$ as a linear operator through

$$\mathsf{P}|n\rangle = (-1)^n |n\rangle, \quad \text{or} \quad \mathsf{P} = e^{i\pi a^\dagger a}. \tag{2.85}$$

It satisfies the following identities:

$$\begin{aligned} \mathsf{P}^2 &= 1, \\ \mathsf{P}\, a\, \mathsf{P} = -a\,;& \mathsf{P}\, a^\dagger\, \mathsf{P} = -a^\dagger, \\ \mathsf{P} D(z) \mathsf{P} &= D(-z)\,. \end{aligned} \tag{2.86}$$

The displacement operators $D(z)$ obey some interesting integral formulas. First, by an explicit calculation using associated Laguerre polynomials, (2.82), one has,

$$\int_{\mathbb{C}} \langle m\,|\,D(z)\,|\,n\rangle\, \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{\pi i} = \delta_{mn}\, 2(-1)^m, \tag{2.87}$$

which in turn implies

$$\int_{\mathbb{C}} D(z)\, \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{\pi i} = 2\,\mathsf{P}\,. \tag{2.88}$$

Moreover, using the orthogonality of the associated Laguerre polynomials, one obtains from (2.82) the ground state projector $P_0 := |0\rangle\langle 0|$ as the Gaussian average of $D(z)$:

$$\int_{\mathbb{C}} e^{-\frac{1}{2}|z|^2} D(z)\, \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{\pi i} = |0\rangle\langle 0|\,. \tag{2.89}$$

More generally, for $\mathrm{R}e(s) < 1$,

$$\int_{\mathbb{C}} e^{\frac{s}{2}|z|^2} D(z)\, \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{\pi i} = \frac{2}{1-s} \exp(\ln \frac{s+1}{s-1} a^\dagger a)\,, \tag{2.90}$$

where the convergence holds in norm for $\mathrm{R}e(s) < 0$ and weakly for $0 \leq \mathrm{R}e(s) < 1$.

Finally, combining (2.88) with the third relation from (2.86), one gets a resolution of the identity

$$\int_{\mathbb{C}} D(z)\, 2\mathsf{P}\, D(-z)\, \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{\pi i} = 1\,. \tag{2.91}$$

2.5 An Alternative Representation and Squeezed States

There is an interesting alternative representation of the canonical CS (2.14) in terms of vectors in the Bargmann space. Indeed, denoting them again by $|z\rangle$, in terms of the Bargmann space vectors ζ_z, we have $|z\rangle = \exp[-|z|^2/2]\, |\zeta_z\rangle$. Thus we may write, in view of our earlier discussion,

$$|z\rangle = \exp[-|z|^2/2] \sum_{n=0}^{\infty} \overline{u_n(\bar{z})}\; |u_n\rangle = \exp[-|z|^2/2] \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |u_n\rangle\,. \tag{2.92}$$

A second representation can be obtained obtained using the *squeezed basis*, which is the basis associated to the squeezed states of quantum optics (see, e.g., [577, 598, 610, 613]). The point of departure here is the expression (2.79) for the reproducing kernel. As will be shown in Chap. 5, the functional form of the reproducing kernel is independent of the orthonormal basis used to express it. Thus, in (2.79) we may replace the basis $\{u_n\}_{n=0}^{\infty}$, also called the *Fock–Bargmann basis*, by any other orthonormal basis of $\mathfrak{H}_{\mathrm{hol}}$. To this end we choose the squeezed basis

$$u_n^{\xi} = S(\xi)u_n, \qquad n = 0, 1, 2, \ldots, \tag{2.93}$$

where $S(\xi)$ is the (unitary) *squeeze operator*

$$S(\xi) = e^{\xi K_+ - \overline{\xi} K_-}, \qquad \xi \in \mathbb{C}\,, \tag{2.94}$$

with

$$K_+ = \frac{1}{2}z^2, \qquad K_- = \frac{1}{2}\partial_z^2, \qquad K_0 = \frac{1}{2}\left[z\partial_z + \frac{1}{2}\right] \tag{2.95}$$

being the well-known generators of the Lie algebra of the SU(1,1) group on $\mathfrak{H}_{\mathrm{hol}}$. The vectors u_n^{ξ} have the explicit forms

$$u_n^{\xi}(z) = \frac{1}{\sqrt{n!}}(1-|\zeta|^2)^{\frac{1}{4}}\left[\frac{\overline{\zeta}}{2}\right]^{\frac{n}{2}} e^{\frac{\zeta}{2}z^2} H_n\left(\left[\frac{1-|\zeta|^2}{2\overline{\zeta}}\right]^{\frac{1}{2}} z\right), \quad \zeta = \frac{\tanh(|\xi|)}{|\xi|}\xi\,, \tag{2.96}$$

where the H_n are the complex Hermite polynomials (i.e., the usual Hermite polynomials, written in terms of the complex variable z),

$$H_n(z) = n!\sum_{m=0}^{[n/2]} \frac{(-1)^m(2z)^{n-2m}}{m!\,(n-2m)!}\,. \tag{2.97}$$

They satisfy the orthogonality relations [39, 40, 215, 269, 318],

$$\iint_{\mathbb{C}} \overline{H_m(z)}H_n(z)\, e^{-2\varepsilon[\frac{x^2}{1+\varepsilon}+\frac{y^2}{1-\varepsilon}]}\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi\sqrt{1-\varepsilon^2}}{2\varepsilon}\,\left[\frac{2}{\varepsilon}\right]^n n!\,\delta_{mn}\,,\ z = x+iy, \tag{2.98}$$

for any nonzero $\varepsilon < 1$. Going back to (2.79) we see that we may also write

$$K_{\mathrm{hol}}(z', \overline{z}) = e^{z'\overline{z}} = \sum_{n=0}^{\infty} u_n^{\xi}(z')\overline{u_n^{\xi}(z)}, \tag{2.99}$$

form which it follows that

$$\begin{aligned}|z\rangle &= \exp[-\frac{|z|^2}{2}] \sum_{n=0}^{\infty} \overline{u_n^{\xi}(\bar{z})}\, u_n^{\xi} \\ &= \exp[-\frac{|z|^2}{2}] \sum_{n=0}^{\infty} \frac{(1-|\zeta|^2)^{\frac{1}{4}}}{\sqrt{n!}} \left[\frac{\zeta}{2}\right]^{\frac{n}{2}} e^{\frac{\bar{\zeta}}{2}z^2} H_n\left(\left[\frac{1-|\zeta|^2}{2\zeta}\right]^{\frac{1}{2}} z\right) u_n^{\xi},\end{aligned} \tag{2.100}$$

and this holds for any complex ξ. This expression should be compared to (2.92).

More generally, since any two orthonormal bases of a Hilbert can be mapped unitarily into one another, taking an arbitrary orthonormal basis $\{v_n\}_{n=0}^{\infty}$ of $\mathfrak{H}_{\text{hol}}$ we may write

$$|z\rangle = \exp[-|z|^2/2] \sum_{n=0}^{\infty} \overline{v_n(\bar{z})}\, v_n\,. \tag{2.101}$$

Finally, let us note that the squeezed states of quantum optics, which we already encountered in Sect. 2.1, are themselves defined to be the vectors,

$$|z,\xi\rangle := S(\xi)|z\rangle = \exp[-\frac{|z|^2}{2}] \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} u_n^{\xi}\,, \tag{2.102}$$

in the squeezed basis, so that alternatively, we may also write them as

$$|z,\xi\rangle = \exp[-\frac{|z|^2}{2}] \sum_{n=0}^{\infty} \overline{u_n^{-\xi}(\bar{z})}\, u_n\,, \tag{2.103}$$

in the Fock–Bargmann basis.

The unitary squeeze operator $S(\xi)$ in (2.94) can be written as

$$S(\xi) = \sum_{n=0}^{\infty} |u_n^{\xi}\rangle\langle u_n|\,, \tag{2.104}$$

and thus on the Bargmann space of analytic functions $\mathfrak{H}_{\text{hol}}$ it has the integral kernel,

$$\begin{aligned}S^{\xi}(z,\bar{w}) &= \sum_{n=0}^{\infty} u_n^{\xi}(z)\overline{u_n(w)} \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}(1-|\zeta|^2)^{\frac{1}{4}} \left[\frac{\bar{\zeta}}{2}\right]^{\frac{n}{2}} e^{\frac{\zeta}{2}z^2}\, H_n\left(\left[\frac{1-|\zeta|^2}{2\bar{\zeta}}\right]^{\frac{1}{2}} z\right) \frac{\bar{w}^n}{\sqrt{n!}}\,.\end{aligned}$$

After some manipulation this can be brought into the form,

$$S^{\xi}(z,\bar{w}) = (1-|\zeta|^2)^{\frac{1}{4}}\, e^{\frac{1}{2}(\zeta z^2 - \bar{\zeta}\bar{w}^2)}\, e^{\sqrt{1-|\zeta|^2} z\bar{w}}\,. \tag{2.105}$$

From (2.102) it follows that the function,

$$z \mapsto e^{-\frac{|w|^2}{2}} S^{\xi}(z,\overline{w}) = (1-|\zeta|^2)^{\frac{1}{4}}\, e^{-\frac{|w|^2}{2}}\, e^{\frac{1}{2}(\zeta z^2 - \overline{\zeta}\overline{w}^2)}\, e^{\sqrt{1-|\zeta|^2} z\overline{w}}. \tag{2.106}$$

is just the Bargmann space function representing the squeezed coherent state vector $|\overline{w},\xi\rangle$. Furthermore, taking the limit $\xi \to 0 \Longrightarrow |\zeta| \to 0$, in the above equation, yields the Bargmann space function for the canonical coherent state $|\overline{w}\rangle$ [see (2.99)], as it should.

2.6 Some Geometrical Considerations

As already pointed out, the existence of the CS $\zeta_{\overline{z}}$ can be traced back to certain intrinsic geometrical properties of $\mathbb{C}$, considered as a one-dimensional, complex *Kähler manifold*. While we do not intend, at this point, to discuss this notion in any depth, it is still possible to get a general idea of what is involved. To begin with, $\mathbb{C}$ may be thought of as being either a one-dimensional complex manifold or a two-dimensional real manifold $\mathbb{R}^2$, equipped with a complex structure. In the first case, one works with the *holomorphic coordinate* z (or the *antiholomorphic coordinate* $\overline{z}$). In the second case, one uses the real coordinates q, p. Considered as a real manifold, $\mathbb{R}^2$ is *symplectic*, i.e., it comes equipped with a closed, non-degenerate *two-form* [compare with (2.35) and (2.61)]

$$\Omega = \mathrm{d}q \wedge \mathrm{d}p = \frac{1}{i}\,\mathrm{d}z \wedge \mathrm{d}\overline{z}, \tag{2.107}$$

while considered as a complex manifold, $\mathbb{C}$ admits the *Kähler potential* function:

$$\Phi(z',\overline{z}) = z'\overline{z}, \tag{2.108}$$

from which the two-form emerges upon differentiation:

$$\Omega = \frac{1}{i}\frac{\partial^2 \Phi(z,\overline{z})}{\partial z \partial \overline{z}}\; \mathrm{d}z \wedge \mathrm{d}\overline{z}. \tag{2.109}$$

Similarly, the Kähler potential also determines the reproducing kernel:

$$K_{\mathrm{hol}}(z',\overline{z}) = \exp\left[\Phi(z',\overline{z})\right], \tag{2.110}$$

while the measure $\mathrm{d}\mu$ [see (2.62)], defining the Hilbert space $\mathfrak{H}_{\mathrm{hol}}$ of holomorphic functions, is given in terms of it by

$$\mathrm{d}\mu(z,\overline{z}) = \exp\left[-\Phi(z,\overline{z})\right]\,\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{2\pi i}. \tag{2.111}$$

Continuing, if we define the complex *one-form*

$$\Theta = -i\partial_{\bar{z}}\Phi(z,\bar{z}) = -iz\,\mathrm{d}\bar{z}, \tag{2.112}$$

we get

$$\Omega = \partial_z\Theta, \tag{2.113}$$

where $\partial_z, \partial_{\bar{z}}$ denote (exterior) differentiation with respect to z and $\bar{z}$, respectively. It appears therefore, that it is the Kähler structure of $\mathbb{C}$, (or the fact that it comes equipped with the Kähler potential Φ) which leads to the existence of the Hilbert space $\mathfrak{H}_{\mathrm{hol}}$ of holomorphic functions and consequently, the CS $\zeta_{\bar{z}}$ [the appearance of these latter being a consequence of the continuity of the evaluation map (2.71)]. Once again, this situation is generic to all Kähler manifolds.

Let $\mathbb{P}(z)$ be the one dimensional projection operator onto the vector subspace of $\mathfrak{H}_{\mathrm{hol}}$ generated by the vector $\zeta_{\bar{z}}$, and denote this subspace by $\mathfrak{H}_{\mathrm{hol}}(z)$. The collection of all these one-dimensional subspaces, as z ranges over $\mathbb{C}$, defines a (holomorphic) *line bundle* over the manifold $\mathbb{C}$ — a structure which is intimately related to the existence of a *geometric prequantization* of $\mathbb{C}$. We hasten to add, however, that while a complex Kähler structure is in some sense ideally suited to the existence of a geometric prequantization, a family of CS may define a geometric prequantization even in the absence of such a structure and, in fact, it provides a complete quantization, as we shall see in Chap. 11.

2.7 Outlook

We have quickly gleaned through a number of illustrative properties of the canonical coherent states. As mentioned earlier, each one of these properties can be taken as the starting point for a generalization of the notion of a CS. From a purely physical point of view, for example, it is useful to look for generalizations which preserve the minimal uncertainty property. In doing so, it is useful to exploit some of the group theoretical properties as well. Mathematical generalizations could be based on group theoretical, analytic or related geometrical properties. We shall attempt to describe a bit of all of these various possibilities and along the way, we shall be naturally led to some powerful applications of the theory of CS to wavelets and signal analysis. But before digging further into the mathematics, we shall exhibit some examples of physical applications illustrating the use of the canonical coherent states. For instance, one could mention

1. *Quantization theory*: this is one of the most obvious applications of canonical CS — which goes back to the early days of quantum mechanics. The idea is to establish a correspondence between classical observables, that is, real valued functions on phase space, and quantum observables, represented by self-adjoint

operators on a Hilbert space, in such a way that Poisson brackets correspond to commutators. Note, however, that such a goal is impossible to reach in most cases. Details on this quantization technique may be found in earlier works [17, 33, 135, 136, 488, 511]. More important, we will devote to the topic of quantization the full Chap. 11.

2. *Atomic physics*: as another illustration of the efficiency of CS methods, one may consider the example of a system of N two-level atoms interacting with a radiation field, for instance, the Dicke model [251, Scu97]. An extensive description of such applications of CS to atomic physics may be found in the textbook [Gaz09].
3. *Quantum measurements and quantum information theory*: we have here the most recent application of canonical CS, in the domain which has witnessed in the last years a spectacular renewal, thanks to the prospect of using the phenomenon of entanglement for designing a quantum computer. For details we refer to the books by Helstrom [Hel76], Holevo [H0l01] or Busch et al. [Bus91]. Another direction to quote is the rôle of CS in the description of the phenomenon of decoherence [619, 620].

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