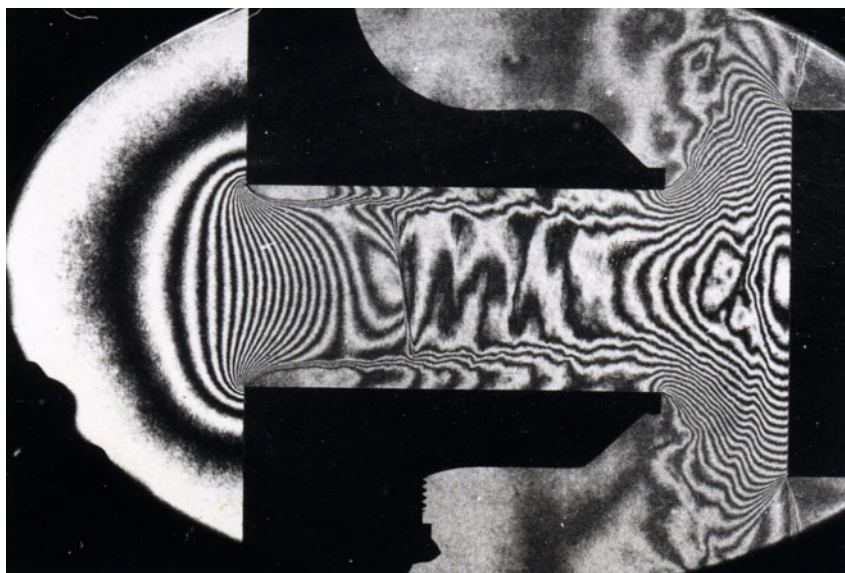


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## TRANSIENT-FLOW EQUATIONS



Isovel lines of measured constant density for the calculated constant Mach Numbers (adiabatic isentropic flow) for a standing pressure wave in a safety valve for pressure ratio of 0.35. Flow pattern is at the limit of becoming unstable. (Courtesy, Föllmer, B. and Zeller, H. [1980].)

## 2-1 Introduction

Equations for the conservation of mass and momentum describe the transient flow in closed conduits. These equations are usually referred to as the continuity and momentum equations. Some authors call a simplified form of the latter, the equation of motion or the dynamic equation. These equations are a set of partial differential equations since the flow velocity and pressure in transient flow are functions of time as well as distance.

In this chapter, the continuity and momentum equations are derived by making a number of simplifying assumptions. A brief introduction to the Reynolds transport theorem, which is used to derive a generalized form of these equations, is first presented. A simplified version of these equations is then derived and various methods for their solution are discussed. Expressions for the wave velocity and a number of models to simulate unsteady friction are presented.

## 2-2 Reynolds Transport Theorem

A number of terms are defined first for the presentation of this theorem which relates the flow variables for a specified quantity of fluid mass, called a *system*, to that of a specified region, called a *control volume* [Roberson and Crowe, 1997]. Everything external to this system is called the *surroundings*, and the system *boundaries* separate the system from its surroundings. The boundary of a control volume is referred to as the *control surface*.

In fluid flow, the shape of a system may change as it travels from one location to another. A control volume usually remains fixed at a location; although in some applications, it may travel and/or deform in shape. For the application of this theorem in this chapter, the shape of the control volume changes with time due to variation in the internal pressure.

The basic conservation laws of mechanics, such as, conservation of mass, momentum and energy are valid for a system. These laws describe the interaction between the system and its surroundings and usually specify the time rate of change of some system property. For example, Newton's second law of motion relates the time rate of change of momentum of a system to the external forces exerted on the system by its surroundings. In the control-volume approach, the boundaries of the system and that of the control volume are the same at the instant a particular conservation law is applied. In other words, all of the system mass is contained in the control volume.

For the analysis of fluid flow, we do not follow the motion of a specified particle or of a specified quantity of mass. Instead, we are interested in the flow through a region. The basic laws, therefore, are written for the flow in a region. The Reynolds transport theorem is useful for this application.

Let  $B$  be an *extensive property* (momentum, energy) of a fluid, and let  $\beta$  be the corresponding intensive property. An *intensive property* is defined as

the amount of  $B$  per unit mass of a system, i.e.,  $\beta = \lim_{\Delta m \rightarrow 0} \Delta B / \Delta m$ . The total amount of  $B$  in a control volume,  $B_{cv}$ , is then

$$B_{cv} = \int_{cv} \beta \rho d\forall \quad (2-1)$$

in which  $m$  = mass,  $\rho$  = mass density, and  $d\forall$  = differential volume of the fluid.

Let us now discuss how the flow variables of a control volume are related to that of a system. To facilitate understanding, our discussion is confined to one-dimensional flow and we assume that the control volume is fixed in space. We are interested in relating the time rate of change of property  $B$  of the system to that of the control volume and the inflow and outflow of  $B$  across the control surface.

Let us consider a system at times  $t$  and  $t + \Delta t$ , as shown in Fig. 2-1. The dashed lines show the control surface, and the solid lines show the boundaries of the system. At time  $t$ , part of the system occupies the control volume while another part is about to move into the control volume. At time  $t + \Delta t$ , part of the system occupies the control volume while another part has moved out. Property  $B$  of the system at times  $t$  and  $t + \Delta t$  may be written as

$$B_{sys}(t) = B_{cv}(t) + \Delta B_{in} \quad (2-2)$$

$$B_{sys}(t + \Delta t) = B_{cv}(t + \Delta t) + \Delta B_{out}$$

where the subscripts “sys” and “cv” refer to the system and the control volume, and the subscripts “in” and “out” refer to the inflow and outflow from the control volume respectively, and  $\Delta B_{in}$  and  $\Delta B_{out}$  are inflow and outflow of property  $B$  into or out of the control volume during time interval  $\Delta t$ .

The time rate of change of property  $B$  of the system is

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{B_{sys}(t + \Delta t) - B_{sys}(t)}{\Delta t} \quad (2-3)$$

By substituting the expressions for  $B_{sys}$  from Eq. 2-2 into Eq. 2-3 and rearranging the terms yield

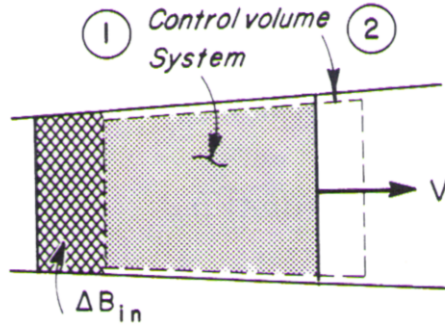
$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{B_{cv}(t + \Delta t) - B_{cv}(t)}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{\Delta B_{out}}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{\Delta B_{in}}{\Delta t} \quad (2-4)$$

Now, as  $\Delta t$  approaches zero in the limit, the first term on the right-hand side of Eq. 2-4 represents the time rate of change of property  $B$  in the control volume, i.e.,

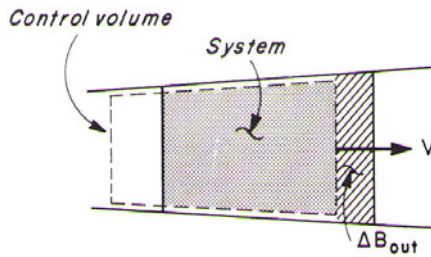
$$\lim_{\Delta t \rightarrow 0} \frac{B_{cv}(t + \Delta t) - B_{cv}(t)}{\Delta t} = \frac{dB_{cv}}{dt} \quad (2-5)$$

By substituting Eq. 2-1 into Eq. 2-5

$$\lim_{\Delta t \rightarrow 0} \frac{B_{cv}(t + \Delta t) - B_{cv}(t)}{\Delta t} = \frac{d}{dt} \int_{cv} \beta \rho d\forall \quad (2-6)$$



(a)



(b)

**Fig. 2-1. System and control volume.**

The second term on the right-hand side of Eq. 2-4 is the rate at which property  $B$  is leaving the control volume. Similarly, the third term of this equation represents the rate at which property  $B$  is entering the control volume. For one-dimensional flow, we may write

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta B_{\text{out}}}{\Delta t} = (\beta \rho A V_s)_{\text{out}} \quad (2-7)$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta B_{\text{in}}}{\Delta t} = (\beta \rho A V_s)_{\text{in}}$$

where  $A$  = cross-sectional area of the conduit and  $V_s$  = flow velocity measured relative to the control surface.

On the basis of Eqs. 2-6 and 2-7, Eq. 2-4 may be written as

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{cv}} \beta \rho dV + (\beta \rho A V_s)_{\text{out}} - (\beta \rho A V_s)_{\text{in}} \quad (2-8)$$

Note that the velocity,  $V$ , is with respect to the control surface, since it accounts for the inflow or outflow from the control volume. For a fixed control

volume,  $V_s$  = fluid flow velocity,  $V$ . However, if the control volume stretches or contracts with respect to time, then the control surface is not fixed and  $V_s$  in Eq. 2-8 is the relative flow velocity, i.e.,  $V_s = (V - W)$ , where  $W$  is the velocity of the control surface at section 1 for inflow and at section 2 for outflow. Both  $V$  and  $W$  are measured with respect to the coordinate axes. Hence, a general form of Eq. 2-8 for an expanding or contracting control volume in a one-dimensional flow may be written as

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{CV}} \beta \rho d\mathcal{V} + [\beta \rho A(V - W)]_{\text{out}} - [\beta \rho A(V - W)]_{\text{in}} \quad (2-9)$$

This is the Reynolds transport theorem relating the properties of the system to those of the control volume.

## 2-3 Continuity Equation

To derive the continuity equation, we apply the law of conservation of mass to a control volume. We consider the flow of a slightly compressible fluid in a conduit having linearly elastic walls. Let the control surface be comprised of sections 1 and 2 and the inside surface of the conduit walls (Fig. 2-2). The control volume may shorten or elongate as pressure changes. Let the velocity (with respect to the coordinate axes) of sections 1 and 2 due to this contraction or expansion be  $W_1$  and  $W_2$ , respectively. Let us assume that the flow is one dimensional and the pressure at the end sections of the control volume is uniform. The radial velocity due to radial expansion and contraction is small and not included in the analysis. However, the effects of radial expansion and contraction are important and are taken into account. The distance  $x$ , flow velocity  $V$ , and discharge  $Q$  are considered positive in the downstream direction.

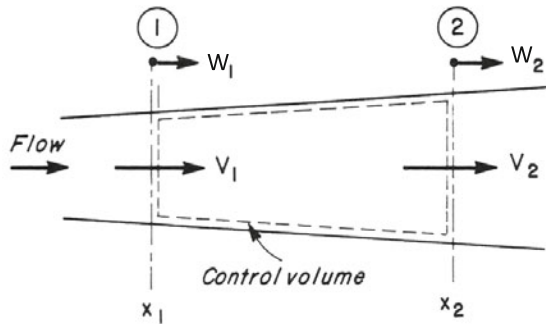


Fig. 2-2. Definition sketch.

To apply the Reynolds transport theorem for the conservation of mass, the intensive property of the fluid is mass/unit mass, i.e.,  $\beta = \lim_{\Delta m \rightarrow 0} \Delta m / \Delta m = 1$ . In addition, since the mass of a system remains constant,  $dM_{\text{sys}}/dt = 0$ . Hence, applying Eq. 2-9 to the control volume shown in Fig. 2-2 and substituting  $\beta = 1$ , we obtain

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho A dx + \rho_2 A_2 (V_2 - W_2) - \rho_1 A_1 (V_1 - W_1) = 0 \quad (2-10)$$

The application of the Leibnitz's rule\* to the first term on the left-hand side gives

$$\begin{aligned} \int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho A) dx + \rho_2 A_2 \frac{dx_2}{dt} - \rho_1 A_1 \frac{dx_1}{dt} \\ + \rho_2 A_2 (V_2 - W_2) - \rho_1 A_1 (V_1 - W_1) = 0 \end{aligned} \quad (2-11)$$

Noting that  $dx_2/dt = W_2$  and  $dx_1/dt = W_1$ , this equation simplifies to

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho A) dx + (\rho AV)_2 - (\rho AV)_1 = 0 \quad (2-12)$$

Based on the mean value theorem†, this equation may be written as

$$\frac{\partial}{\partial t} (\rho A) \Delta x + (\rho AV)_2 - (\rho AV)_1 = 0 \quad (2-13)$$

where  $\Delta x = x_2 - x_1$ . Dividing throughout by  $\Delta x$  and letting  $\Delta x$  approach zero, Eq. 2-13 is simplified as

$$\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho AV) = 0 \quad (2-14)$$

Expansion of the terms inside the parentheses gives

$$A \frac{\partial \rho}{\partial t} + \rho \frac{\partial A}{\partial t} + \rho A \frac{\partial V}{\partial x} + \rho V \frac{\partial A}{\partial x} + AV \frac{\partial \rho}{\partial x} = 0 \quad (2-15)$$

By rearranging terms, using expressions for the total derivatives, and dividing throughout by  $\rho A$ , we obtain

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\*Material presented in Sections 2-3 and 2-4 is based on the collaborative efforts of Professor Clayton Crowe and the author.

According to this rule [Wylie, 1967],

$$\frac{d}{dt} \int_{f_1(t)}^{f_2(t)} F(x, t) dx = \int_{f_1(t)}^{f_2(t)} \frac{\partial}{\partial t} F(x, t) dx + F(f_2(t), t) \frac{df_2}{dt} - F(f_1(t), t) \frac{df_1}{dt}$$

if  $f_1$  and  $f_2$  are differentiable functions of  $t$  and  $F(x, t)$  and  $\partial F / \partial t$  are continuous in  $x$  and  $t$ .

†According to this theorem,  $\int_{x_2}^{x_1} F(x) dx = (x_2 - x_1) F(\xi)$ , where  $x_1 < \xi < x_2$ .

$$\frac{1}{\rho} \frac{d\rho}{dt} + \frac{1}{A} \frac{dA}{dt} + \frac{\partial V}{\partial x} = 0 \quad (2-16)$$

Typically the variables of interest are the pressure intensity  $p$  and the flow velocity  $V$ . To write this equation in terms of these variables, we express the derivatives of  $\rho$  and  $A$  in terms of  $p$  and  $V$  as follows.

The bulk modulus of elasticity, of a fluid [Roberson and Crowe, 1997]

$$K = \frac{dp}{d\rho/\rho} \quad (2-17)$$

This equation may be written as

$$\frac{d\rho}{dt} = \frac{\rho}{K} \frac{dp}{dt} \quad (2-18)$$

Now, for a circular conduit having radius  $R$ ,

$$\frac{dA}{dt} = 2\pi R \frac{dR}{dt} \quad (2-19)$$

In terms of the strain,  $\epsilon$ , this equation may be written as

$$\frac{dA}{dt} = 2\pi R^2 \frac{1}{R} \frac{dR}{dt} \quad (2-20)$$

or

$$\frac{1}{A} \frac{dA}{dt} = 2 \frac{d\epsilon}{dt} \quad (2-21)$$

As indicated earlier, we assume that the conduit walls are linearly elastic [Timoshenko, 1941], i.e., stress is proportional to strain. This is true for most common pipe wall materials, e.g., metal, wood, concrete, etc. Then

$$\epsilon = \frac{\sigma_2 - \mu\sigma_1}{E} \quad (2-22)$$

where  $\sigma_2$  = hoop stress,  $\sigma_1$  = axial stress, and  $\mu$  = Poisson ratio. To simplify the derivation, we assume the conduit has expansion joints throughout its length. Therefore, the axial stress,  $\sigma_1 = 0$ . Hence, Eq. 2-22 becomes

$$\epsilon = \frac{\sigma_2}{E} \quad (2-23)$$

Now, the hoop stress in a thin-walled conduit

$$\sigma_2 = \frac{pD}{2e} \quad (2-24)$$

where  $p$  = inside pressure;  $e$  = thickness of the conduit walls and  $D$  = conduit diameter. By taking the time derivative of Eq. 2-24, we obtain

$$\frac{d\sigma_2}{dt} = \frac{p}{2e} \frac{dD}{dt} + \frac{D}{2e} \frac{dp}{dt} \quad (2-25)$$

Based on Eq. 2-23, we may write Eq. 2-25 as

$$E \frac{d\epsilon}{dt} = \frac{p}{2e} \frac{dD}{dt} + \frac{D}{2e} \frac{dp}{dt} \quad (2-26)$$

Using Eqs. 2-19 and 2-21, Eq. 2-26 becomes

$$E \frac{d\epsilon}{dt} = \frac{pD}{2e} \frac{d\epsilon}{dt} + \frac{D}{2e} \frac{dp}{dt} \quad (2-27)$$

which may be simplified as

$$\frac{d\epsilon}{dt} = \frac{\frac{D}{2e} \frac{dp}{dt}}{E - \frac{pD}{2e}} \quad (2-28)$$

It follows from Eqs. 2-21 and 2-28 that

$$\frac{1}{A} \frac{dA}{dt} = \frac{\frac{D}{e} \frac{dp}{dt}}{E - \frac{pD}{2e}} \quad (2-29)$$

Substituting Eqs. 2-18 and 2-29 into Eq. 2-16 and simplifying, the resulting equation becomes

$$\frac{\partial V}{\partial x} + \left( \frac{1}{K} + \frac{1}{\frac{eE}{D} - \frac{p}{2}} \right) \frac{dp}{dt} = 0 \quad (2-30)$$

Since  $p/2 \ll eE/D$  in typical engineering applications, this equation may be written as

$$\frac{\partial V}{\partial x} + \frac{1}{K} \left( 1 + \frac{1}{\frac{eE}{DK}} \right) \frac{dp}{dt} = 0 \quad (2-31)$$

Let us define

$$a^2 = \frac{\frac{K}{\rho}}{1 + \frac{DK}{eE}} \quad (2-32)$$

Note that this expression for the wave velocity is for a conduit with expansion joints. In Chapter 3, we show that  $a$  is the velocity of pressure wave in an elastic conduit filled with a slightly compressible fluid. For other types of support conditions, the expressions for the wave velocity are modified slightly. These expressions are presented in Section 2-6, with their derivation left as an exercise for the reader (Problem 2-6). Substituting Eq. 2-32 and the expression for the total derivative into Eq. 2-31 gives

$$\frac{\partial p}{\partial t} + V \frac{\partial p}{\partial x} + \rho a^2 \frac{\partial V}{\partial x} = 0 \quad (2-33)$$

This equation is called the continuity equation.



## 2-4 Momentum Equation

In this Section, we apply the Reynolds Transport Theorem to derive the momentum equation. The extensive property  $B$  for this application is the momentum of the fluid which is equal to  $mV$ . Therefore, the corresponding intensive property,

$$\beta = \lim_{\Delta m \rightarrow 0} V(\Delta m / \Delta m) = V$$

According to the Newton's second law of motion, the time rate of change of momentum of a system is equal to the resultant of the forces exerted on the system by its surroundings, i.e.,

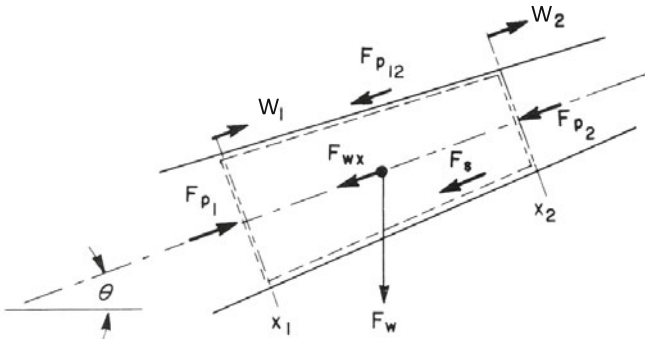
$$\frac{dM_{\text{sys}}}{dt} = \sum F \quad (2-34)$$

By substituting  $\beta = V$  into Eq. 2-9 and using Eq. 2-34, we obtain

$$\frac{d}{dt} \int_{\text{cv}} V \rho d\forall + [\rho A (V - W) V]_2 - [\rho A (V - W) V]_1 = \sum F \quad (2-35)$$

By applying the Leibnitz rule to the first term on the left-hand side of this equation and noting that  $dx_1/dt = W_1$  and  $dx_2/dt = W_2$ , we obtain

$$\begin{aligned} \int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho AV) dx + (\rho AV)_2 W_2 - (\rho AV)_1 W_1 + [\rho A (V - W) V]_2 \\ - [\rho A (V - W) V]_1 = \sum F \end{aligned} \quad (2-36)$$



**Fig. 2-3.** Notation for momentum equation.

By simplifying this equation, applying the mean-value theorem to the first term, and dividing throughout by  $\Delta x$  give

$$\frac{d}{dt}(\rho AV) + \frac{(\rho AV^2)_2 - (\rho AV^2)_1}{\Delta x} = \frac{\sum F}{\Delta x} \quad (2-37)$$

Now let us consider the following forces acting on the control volume (Fig. 2-3):

$$\text{Pressure force at section 1, } F_{p_1} = p_1 A_1 \quad (2-38)$$

where  $p$  = pressure intensity, and the subscript 1 refers to cross section 1.

Similarly,

$$\text{Pressure force at section 2, } F_{p_2} = p_2 A_2 \quad (2-39)$$

Pressure force on the converging sides,

$$F_{p_{12}} = \frac{1}{2} (p_1 + p_2) (A_1 - A_2) \quad (2-40)$$

Component of the weight of fluid along the conduit centerline

$$F_{wx} = \rho g A (x_2 - x_1) \sin \theta \quad (2-41)$$

where  $\theta$  = angle the conduit makes with the horizontal, considered positive for conduit sloping upwards in the downstream direction. Now,

$$\text{Shear force, } F_s = \tau_o \pi D (x_2 - x_1) \quad (2-42)$$

where  $\tau_o$  = shear stress exerted by the conduit walls on the flowing fluid.

Considering the downstream flow direction as positive, it follows from Eqs. 2-38 to 2-42 that

$$\begin{aligned} \sum F &= p_1 A_1 - p_2 A_2 - \frac{1}{2} (p_1 + p_2) (A_1 - A_2) \\ &\quad - \rho g A (x_2 - x_1) \sin \theta - \tau_o \pi D (x_2 - x_1) \\ &= \frac{1}{2} (p_1 - p_2) (A_1 + A_2) - \rho g A (x_2 - x_1) \sin \theta \\ &\quad - \tau_o \pi D (x_2 - x_1) \end{aligned} \quad (2-43)$$

Dividing Eq. 2-43 by  $\Delta x = x_2 - x_1$  gives

$$\frac{\sum F}{\Delta x} = \frac{(p_1 - p_2) (A_1 + A_2)}{2 \Delta x} - \rho g A \sin \theta - \tau_o \pi D \quad (2-44)$$

By substituting Eq. 2-44 into Eq. 2-37 and letting  $\Delta x$  approach zero in the limit, we obtain

$$\frac{\partial}{\partial t}(\rho AV) + \frac{\partial}{\partial x}(\rho AV^2) + A \frac{\partial p}{\partial x} + \rho g A \sin \theta + \tau_o \pi D = 0 \quad (2-45)$$

Let us assume the energy losses for a given flow velocity during the transient state are the same as those for steady flows at that velocity (we will discuss unsteady friction in Section 2-8). If we use the Darcy-Weisbach friction equation for computing the friction losses, then the wall shear stress

$$\tau_o = \frac{1}{8} \rho f V |V| \quad (2-46)$$

where  $f$  = Darcy-Weisbach friction factor. Note that we are writing  $V^2$  as  $V|V|$  to allow for the reverse flow. The substitution of this expression into Eq. 2-45 and the expansion of the terms in parentheses yield

$$\begin{aligned} V \frac{\partial}{\partial t} (\rho A) + \rho A \frac{\partial V}{\partial t} + V \frac{\partial}{\partial x} (\rho AV) + \rho AV \frac{\partial V}{\partial x} \\ + A \frac{\partial p}{\partial x} + \rho g A \sin \theta + \frac{\rho A f V |V|}{2D} = 0 \end{aligned} \quad (2-47)$$

The rearrangement of the terms of this equation gives

$$\begin{aligned} V \left[ \frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho AV) \right] + \rho A \frac{\partial V}{\partial t} + \rho AV \frac{\partial V}{\partial x} \\ + A \frac{\partial p}{\partial x} + \rho g A \sin \theta + \frac{\rho A f V |V|}{2D} = 0 \end{aligned} \quad (2-48)$$

Based on the continuity equation (Eq. 2-14), the sum of the two terms inside the brackets is zero. Hence, dropping the terms inside the brackets and dividing the resulting equation by  $pA$ , we obtain

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} + g \sin \theta + \frac{f V |V|}{2D} = 0 \quad (2-49)$$

This equation is called the momentum equation.

## 2-5 General Remarks

In this section, we discuss various parameters of the governing equations and whether they are hyperbolic, parabolic or elliptic. Each type of these equations describes a particular physical process or phenomenon. For example, wave propagation in a fluid is described by a set of hyperbolic partial differential equations. In addition, once we know the type of the governing equations, suitable numerical methods can be selected for their solution.

The continuity and momentum equations (Eqs. 2-33 and 2-49) describe transient-flows in closed conduits. In these equations, distance  $x$  and time  $t$  are two independent variables and pressure  $p$  and flow velocity  $V$  are two dependent variables. The other variables,  $a$ ,  $\rho$ ,  $f$ , and  $D$ , are the system parameters and usually do not vary with time; these may, however, be functions

of  $x$ . Although wave velocity  $a$  depends on the characteristics of the conduit and on the fluid properties, laboratory tests [Streeter, 1972] show that it is significantly reduced by a reduction of pressure, even when the pressure remains above the liquid vapor pressure. The friction factor  $f$  usually varies with the Reynolds number. However, the effects of such a variation of  $f$  on transient conditions are usually small and may be neglected.

### Classification of Governing Equations

Equations 2-33 and 2-49 are a set of first-order, partial differential equations. We shall now determine the type of these equations, make some qualitative observations for their solution, and discuss methods for numerically integrating them. These equations may be written in the matrix form as

$$\frac{\partial}{\partial t} \begin{pmatrix} p \\ V \end{pmatrix} + \begin{bmatrix} V & \rho a^2 \\ \frac{1}{\rho} & V \end{bmatrix} \frac{\partial}{\partial x} \begin{pmatrix} p \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ -g \sin \theta - \frac{fV|V|}{D} \end{pmatrix} \quad (2-50)$$

or

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial x} = \mathbf{E} \quad (2-51)$$

where

$$\mathbf{U} = \begin{pmatrix} p \\ V \end{pmatrix}; \quad \mathbf{B} = \begin{bmatrix} V & \rho a^2 \\ \frac{1}{\rho} & V \end{bmatrix}; \quad (2-52)$$

$$\mathbf{E} = \begin{pmatrix} 0 \\ -g \sin \theta - \frac{fV|V|}{D} \end{pmatrix}$$

The eigenvalues,  $\lambda$ , of matrix  $\mathbf{B}$  determine the type of the set of partial differential equations. The characteristic equation [Wylie, 1967] of matrix  $\mathbf{B}$  is

$$(V - \lambda)^2 = a^2 \quad (2-53)$$

Hence,

$$\lambda = V \pm a \quad (2-54)$$

Since both eigenvalues are real and distinct, Eqs. 2-33 and 2-49 are a set of hyperbolic partial differential equations. This type of equations describes the propagation of waves in a fluid.

## Initial Conditions

The initial conditions are needed to compute the transient conditions. Mostly the initial conditions correspond to the initial steady-state flows. In this section, we discuss how to specify the initial flow conditions that are compatible with the transient flow equations.

Equations 2-33 and 2-49 describe unsteady, nonuniform flow of a slightly compressible fluid in an elastic conduit. Steady flow may be considered a special case [Stuckenbruck and Wiggert, 1985] in which the time variation of flow velocity,  $\partial V/\partial t$  and of pressure,  $\partial p/\partial t$  are both zero. Hence the governing equations for steady flow may be derived from these two equations by dropping the terms representing the local variation of pressure and flow velocity with respect to time  $t$ ; i.e.,  $\partial p/\partial t$  and  $\partial V/\partial t$  of Eqs. 2-33 and 2-49 are both zero. Therefore, Eqs. 2-33 and 2-49 for *steady flow* become

$$V \frac{dp}{dx} + \rho a^2 \frac{dV}{dx} = 0 \quad (2-55)$$

$$V \frac{dV}{dx} + \frac{1}{\rho} \frac{dp}{dx} + g \sin \theta + \frac{fV|V|}{2D} = 0 \quad (2-56)$$

Note the total derivatives and not the partial derivatives in these equations since both  $p$  and  $V$  are functions of  $x$  only. It follows from Eq. 2-55 that

$$\frac{dV}{dx} = -\frac{V}{\rho a^2} \frac{dp}{dx} \quad (2-57)$$

Substitution of this expression into Eq. 2-56 and simplification of the resulting equation give

$$\frac{dp}{dx} = \frac{\rho [g \sin \theta + fV|V|/(2D)]}{M^2 - 1} \quad (2-58)$$

where  $M = V/a = \text{Mach number}$ . By substituting Eq. 2-58 into Eq. 2-57 and simplifying, we obtain

$$\frac{dV}{dx} = \frac{M^2}{V} \frac{(g \sin \theta + fV|V|/2D)}{1 - M^2} \quad (2-59)$$

For nonzero  $V$ , it is clear from Eq. 2-59 that the velocity gradient  $dV/dx$  is not zero and similarly it is clear from Eq. 2-58 that the pressure gradient  $dp/dx$  is not constant. This is due to the fact that the mass density of the fluid and the flow area of the conduit are functions of  $x$ .

If the initial conditions correspond to steady flow and all the terms of the governing equations have to be included in the analysis, then the initial conditions should be determined from Eqs. 2-58 and 2-59. However, in most of the engineering applications, a number of terms of the governing equations are small as compared to the other terms and may be neglected. This considerably simplifies the analysis without significantly affecting the accuracy of the computed results. These simplified equations are derived in the next section.

### Simplified Equations

In most of the engineering applications, the convective acceleration terms,  $V(\partial p/\partial x)$  and  $V(\partial V/\partial x)$ , are small as compared to the other terms. Similarly, the slope term is usually small and may be neglected. Therefore, dropping these terms from the governing equations, we obtain

$$\begin{aligned}\frac{\partial p}{\partial t} + \rho a^2 \frac{\partial V}{\partial x} &= 0 \\ \frac{\partial V}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{fV|V|}{2D} &= 0\end{aligned}\tag{2-60}$$

It is a common practice in hydraulic engineering to compute pressures in the pipeline in terms of the piezometric head,  $H$ , above a specified datum and use the discharge,  $Q$ , as the second variable instead of the flow velocity  $V$ . Now,  $Q = VA$  and the pressure intensity  $p$  may be written as

$$p = \rho g (H - z)\tag{2-61}$$

in which  $z$  = elevation of the pipe centerline above the specified datum.

We assumed in the derivation of the governing equations (Eqs. 2-33 and 2-49) that the fluid is slightly compressible, and the conduit walls are slightly deformable. Therefore, we may neglect the spatial variation of  $\rho$  and flow area  $A$  due to the variation of the inside pressure with  $x$ . However, the small variation of  $\rho$  and  $A$  is indirectly taken into account by considering the wave velocity  $a$  to have a finite value. Note that if the fluid is considered incompressible and the conduit walls are assumed rigid, then the wave velocity becomes infinite, and a pressure or velocity change is felt instantaneously throughout the system. For a horizontal pipe,  $dz/dx = 0$ . Hence, with these assumptions, it follows from Eq. 2-61 that  $\partial p/\partial t = \rho g (\partial H/\partial t)$  and  $\partial p/\partial x = \rho g (\partial H/\partial x)$ .

By substituting these relationships into Eqs. 2-60 and 2-61, we obtain

$$\frac{\partial H}{\partial t} + \frac{a^2}{gA} \frac{\partial Q}{\partial x} = 0\tag{2-62}$$

$$\frac{\partial Q}{\partial t} + gA \frac{\partial H}{\partial x} + \frac{fQ|Q|}{2DA} = 0\tag{2-63}$$

Steady-state equations corresponding to Eqs. 2-62 and 2-63 may be obtained by substituting  $\partial H/\partial t = 0$  and  $\partial Q/\partial t = 0$ . Hence, it follows from Eq. 2-62 that  $\partial Q/\partial x = 0$ ; i.e.,  $Q$  is constant along the pipe length. Substituting  $\partial Q/\partial t = 0$  into Eq. 2-63, simplifying the resulting equation, and writing it in a finite-difference form, we obtain

$$\Delta H = \frac{f \Delta x Q^2}{2gDA^2}\tag{2-64}$$

where  $\Delta H$  = head loss in pipe length  $\Delta x$  for a flow of  $Q$ . Note that this equation is the same as the Darcy-Weisbach friction equation.

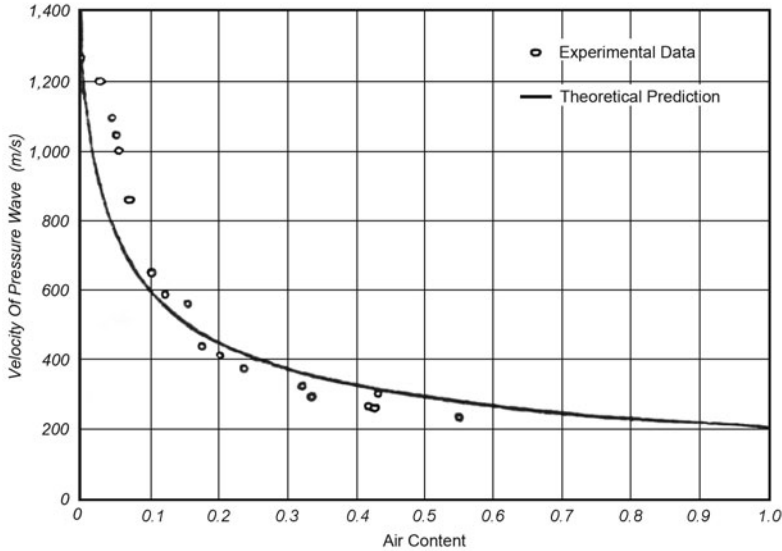
To summarize, steady-state conditions should be computed from Eqs. 2-55 and 2-56 if Eqs. 2-33 and 2-49 are the governing equations. However, if a simplified form of the governing equations (i.e., Eqs. 2-62 and 2-63) is used, then  $Q$  is considered as constant along the pipe, and the piezometric head along the pipe length is computed from Eq. 2-64. However, if complete equations (Eqs. 2-33 and 2-49) are the governing equations, then assuming constant discharge for the initial steady-state conditions and computing the pressure head along the pipe length by using the Darcy-Weisbach equation give erroneous results.

In the above derivation, we used the Darcy-Weisbach equation to compute the friction losses. For a general exponential formula for these losses, the last term of Eq. 2-63 may be written as  $kQ|Q|^m/D^b$ , with the values of  $k$ ,  $m$ , and  $b$  depending on the formula employed. For example, for the Hazen-William formula,  $m = 0.85$  and  $b = 4.87$ . With correct values of  $m$  and  $b$ , the results are independent of the formula employed; i.e., the Darcy-Weisbach and the Hazen-William formulas give comparable results [Evangelisti, 1969]. For most of typical engineering applications, the above assumptions are valid and Eqs. 2-62 and 2-63 may be used. However, if any of the above assumptions are not valid, then the analysis should employ complete equations, Eqs. 2-33 and 2-49. From hereon, in our discussion, we will use Eqs. 2-62 and 2-63.

## 2-6 Wave Velocity

An expression for the wave velocity in a slightly compressible fluid confined in a rigid conduit was derived in Section 1-4. However, in addition to the bulk modulus of elasticity and mass density of the fluid, the wave velocity depends upon the elastic properties of the conduit as well as on the external constraints. Elastic properties include the conduit size, wall thickness, and wall material; and the external constraints include the type of supports and the freedom of conduit movement in the longitudinal direction. The bulk modulus of elasticity of a fluid depends upon its temperature, pressure, and the quantity of entrained gases. Pearsall [1965] showed that the wave velocity changes by about 1 percent per  $5^\circ\text{C}$ . The fluid compressibility increases by the presence of free gases, and it has been reported [Pearsall, 1965] that 1 part of air in 10,000 parts of water by volume reduces the wave velocity by about 50 percent. Figure 2-4 shows the variation of wave velocity in an air-water mixture with different air content [Kobori, et al., 1955]. An expression for the wave velocity in a gas-liquid mixture is derived in Section 9-5.

The presence of solids in liquids have less drastic influence on the wave velocity, unless they are compressible. Laboratory studies [Streeter, 1972] and prototype tests [Pearsall, 1965] show that the dissolved gases tend to come out of solution when the pressure is reduced, even when it remains above the liquid vapor pressure. This results in decreasing the wave velocity significantly.



**Fig. 2-4. Variation of wave velocity in an air-water mixture with air content.** (After Kobori, et al. [1955].)

Therefore, the wave velocity of a positive pressure wave may be higher than that of a negative wave. Further prototype tests are needed to quantify the reduction in the wave velocity due to reduction of pressure.

Halliwell [1963] presented the following general expression for the wave velocity

$$a = \sqrt{\frac{K}{\rho[1 + (K/E)\psi]}} \quad (2-65)$$

in which  $\psi$  is a nondimensional parameter that depends on the elastic properties of the conduit;  $E$  = Young's modulus of elasticity of the conduit walls; and  $K$  and  $\rho$  are the bulk modulus of elasticity and density of the fluid, respectively. The moduli of elasticity of materials commonly used for conduit walls and the bulk moduli of elasticity and mass densities of various liquids are listed in [Tables 2-1](#) and [2-2](#).

Expressions for  $\psi$  for various conditions are as follows:

### ***Rigid Conduit***

$$\psi = 0 \quad (2-66)$$

### ***Thick-Walled Elastic Conduit***

Three different cases for the anchoring of the conduit are as follows.



Table 2-1. Young's modulus of elasticity and Poisson's ratio

Material	Modulus of Elasticity, $E^*$ (GPa)	Poisson's Ratio
Aluminum alloys	68-73	0.33
Asbestos cement, transite	24	
Brass	78-110	0.36
Cast iron	80-170	0.25
Concrete	14-30	0.1-0.15
Copper	107-131	0.34
Glass	46-73	0.24
Lead	4.8-17	0.44
Mild steel	200-212	0.27
Plastics		
ABS	1.7	0.33
Nylon	1.4-2.75	
Perspex	6.0	0.33
Polyethylene	0.8	0.46
Polystyrene	5.0	0.4
PVC rigid	2.4-2.75	
Rocks		
Granite	50	0.28
Limestone	55	0.21
Quartzite	24.0-44.8	
Sandstone	2.75-4.8	0.28
Schist	6.5-18.6	

Source: Compiled from Halliwell [1963]; Roark [1965] and Pickford [1969].  
\* To convert  $E$  into  $\text{lb/ft}^2$ , multiply the values in this column by  $145.04 \times 10^3$ .

i. Conduit anchored against longitudinal movement throughout its length

$$\psi = 2(1 + \nu) \left( \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} - \frac{2\nu R_i^2}{R_o^2 - R_i^2} \right) \tag{2-67}$$

in which  $\nu$  = the Poisson ratio and  $R_o$  and  $R_i$  = the external and internal radii of the conduit.

ii. Conduit anchored against longitudinal movement at the upper end

$$\psi = 2 \left[ \frac{R_o^2 + 1.5R_i^2}{R_o^2 - R_i^2} + \frac{\nu(R_o^2 - 3R_i^2)}{R_o^2 - R_i^2} \right] \tag{2-68}$$

iii. Conduit with frequent expansion joints

$$\psi = 2 \left( \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} + \nu \right) \tag{2-69}$$

***Thin-Walled Elastic Conduit***

Three different cases for the anchoring of the conduit against longitudinal movement are:

- i. Conduit anchored against longitudinal movement throughout its length

$$\psi = \frac{D}{e} (1 - \nu^2) \quad (2-70)$$

in which  $D$  = conduit diameter and  $e$  = wall thickness.

- ii. Conduit anchored against longitudinal movement at the upper end [Wylie and Streeter, 1983]

$$\psi = \frac{D}{e} (1 - 0.5\nu) \quad (2-71)$$

- iii. Conduit with frequent expansion joints

$$\psi = \frac{D}{e} \quad (2-72)$$

***Rock Tunnel***

Halliwell [1963] presented long expressions for  $\psi$  for the lined and unlined rock tunnels. Usually the rock characteristics are not known precisely because of nonhomogeneous rock conditions and because of the presence of fissures. Therefore, the following simplified expressions [Parmakian, 1963] may be used instead of Halliwell's expressions.

- i. Unlined tunnel

$$\begin{aligned} \psi &= 1 \\ E &= G \end{aligned} \quad (2-73)$$

in which  $G$  = modulus of rigidity of the rock.

- ii. Steel-lined tunnel

$$\psi = \frac{DE}{GD + Ee} \quad (2-74)$$

in which  $e$  = thickness of the steel-liner and  $E$  = modulus of elasticity of steel.

***Reinforced Concrete Pipe***

The reinforced concrete pipe is replaced by an equivalent steel pipe having equivalent thickness [Parmakian, 1963]

$$e_e = E_r e_c + \frac{A_s}{l_s} \quad (2-75)$$

in which  $e_c$  = thickness of the concrete pipe;  $A_s$  and  $l_s$  are the cross-sectional area and the spacing of steel bars, respectively; and  $E_r$  = ratio of the modulus of elasticity of concrete to that of steel. Usually the value of  $E_r$  varies from 0.06 to 0.1. However, to allow for any cracks in the concrete pipe, a value of 0.05 is suggested [Parmakian, 1963]. The wave velocity may then be determined from Eq. 2-65 for the equivalent thickness  $e_e$  and the modulus of elasticity of steel.

### ***Wood-Stave Pipe***

The thickness of a uniform steel pipe equivalent to the wood-stave pipe is determined [Parmakian, 1963] from Eq. 2-75 by using  $E_r = \frac{1}{60}$ ,  $e_c$  = thickness of wood staves, and  $A_s$  and  $l_s$  are the cross-sectional area and the spacing of the steel bands, respectively. The wave velocity is then computed from Eq. 2-65.

### ***Polyvinyl Chloride (PVC) and Reinforced Plastic Pipes***

Watters et al. [1976] show that Eq. 2-65 may be used to determine the wave velocity in the polyvinyl chloride (PVC) and in the reinforced plastic pipes, provided a proper value of the modulus of elasticity for the wall material is used.

### ***Noncircular Conduits***

The following expression for  $\psi$  is obtained from the equation for the wave velocity in the thin-walled rectangular conduits presented by Jenkner [1971] by using the steady-state bending theory and by allowing the corners of the conduit to rotate:

$$\psi = \frac{\beta b^4}{15e^3d} \quad (2-76)$$

in which  $\beta = 0.5(6 - 5\alpha) + 0.5(d/b)^3[6 - 5(b/d)^2]$ ,  $\alpha = [1 + (d/b)^3]/[1 + (d/b)]$ ,  $b$  = width of the conduit (longer side), and  $d$  = depth of the conduit (shorter side).

Thorley and Guymer [1976] included the influence of the shear force on the bending deflection of the thick-walled ( $l/e < 20$ ) rectangular conduits while deriving the equations for the wave velocity. From these equations, the following expression is obtained for a *thick-walled conduit* having a square cross section:

$$\psi = \frac{1}{15} \left( \frac{l}{e} \right)^3 + \frac{l}{e} \left( 1 + \frac{e}{2G} \right) \quad (2-77)$$

in which  $e$  = wall thickness,  $(l - e)$  = inside dimension of the conduit, and  $G$  = shear modulus of the wall material. Based on the equations presented by

Thorley and Twyman [1977], the following expression is obtained for  $\psi$  for a *thin-walled hexagonal conduit*:

$$\psi = 0.0385 \left(\frac{l}{e}\right)^3 \tag{2-78}$$

in which  $l$  = mean width of one of the flat sides of the hexagonal section.

**Table 2-2.    Bulk modulus of elasticity and density of common liquids at atmospheric pressure**

Liquid	Temperature (°C)	Density $\rho^*$ (kg/m <sup>3</sup> )	Bulk Modulus of Elasticity, $K^{**}$ (GPa)
Benzene	15	880	1.05
Ethyl alcohol	0	790	1.32
Glycerin	15	1,260	4.43
Kerosene	20	804	1.32
Mercury	20	13,570	26.2
Oil	15	900	1.5
Water, fresh	20	999	2.19
Water, sea	15	1,025	2.27

Source: Compiled from Pearsall [1965]; Baumeister [1967] and Pickford [1969].

\* To convert the specific weight of the liquid into lb/ft<sup>3</sup>, multiply the values of this column by  $62.43 \times 10^{-3}$ .

\*\* To convert  $K$  into lb/in<sup>2</sup>, multiply the values of this column by  $145.04 \times 10^3$ .

### 2-7 Solution of Governing Equations

As demonstrated previously, the momentum and continuity equations are quasi-linear, hyperbolic, partial differential equations. A closed-form solution of these equations is not available. However, by neglecting or by linearizing the nonlinear terms, various graphical [Parmakian, 1963; Bergeron, 1961] and analytical [Rich, 1963; Wood, 1937] methods have been developed. These methods are approximate and cannot be used to analyze large systems or systems with complex boundary conditions.

The following methods, suitable for computer analyses, are available for numerically integrating the nonlinear, hyperbolic partial differential equations:

- Method of characteristics;
- Finite-difference methods;
- Finite-element method;

Spectral method, and  
Boundary-integral method.

The method of characteristics has become popular and is extensively used for the solution of one-dimensional, hydraulic transient problems, especially if the wave velocity is constant. This method has proven to be superior to other methods in several aspects, such as correct simulation of steep wave fronts, illustration of wave propagation, ease of programming, and efficiency of computations [Evangelisti, 1969; Wylie and Streeter, 1983; Lister, 1960; Abbott, 1966; Streeter and Lai, 1962]. Details of this method are presented in the next chapter; and its use and necessary boundary conditions are developed in Chapters 4 through 10.

The finite-difference methods [Perkins et al., 1964; Smith, 1978; Chaudhry and Yevjevich, 1981; Chaudhry, 1983; Chaudhry and Hussaini, 1983] may be classified into two categories: explicit and implicit. Both of these categories have several schemes. Implicit methods usually have the advantage that they allow larger time steps. However, if too large a time step is used, then the accuracy of the scheme is adversely affected and numerical oscillations may be produced in some cases that may yield totally incorrect results [Holloway and Chaudhry, 1985]. Both of these methods are briefly discussed in Chapter 3. The finite-element method [Chung, 1978; Baker, 1983] does not offer any significant advantage for the solution of one-dimensional problems. The spectral method [Gottlieb and Orszag, 1976-1977] is not suitable for nonperiodic boundary conditions and the boundary-integral method [Liggett, 1984] does not efficiently handle the time-dependent problems as compared to the other available methods, especially if shocks or bores are formed. Neither of these methods are discussed further herein.

## 2-8 Unsteady Friction

In the derivation of the governing equations in Sections 2-3 and 2-4, we assumed that the steady friction formulas may be used to compute the transient-state head losses. Although this approximation yields satisfactory results for computing the first peak of transient pressures, the computed pressure oscillations show very slow dissipation as compared to that measured in the laboratory experiments or that measured during field tests on actual projects. This does not pose serious limitations for determining the maximum or minimum pressures in a typical installations or typical operations. However, the computed results are not reliable for multiple operations, such as starting the pumps following a power failure, load acceptance on turbines following load rejection, or for sequential starting or stopping of turbo-machinery, etc.

Several methods have been proposed to account for the unsteady friction effects in transient flow computations. These methods may be classified into

three categories: Quasi-two-dimensional, convolution integral, and instantaneous, acceleration-based methods. Brief descriptions of the first two methods and details of the third method are presented in the following paragraphs. This discussion is based on the paper by Reddy, Silva, and Chaudhry [2012]. Literature review by Ghidaoui [2001] is a good source on the topic.

### Quasi-two-dimensional models

These models provide accurate simulation of the phenomenon [Vardy and Hwang 1991; Brunone et al. 1995; Silva-Araya and Chaudhry 1997; Pezzinga 1999; Zhao and Ghidaoui 2004]. However, they are computationally intensive, and thus have been used primarily for simple piping systems.

### Convolution integral methods

Zielke [1968] introduced these methods by developing an exact solution for the laminar unsteady friction. These methods, suitable for one-dimensional models, use past local accelerations and weighting functions. These solutions are time consuming and require large computer memory. Trikha [1975] proposed a less demanding version of Zielke's method, but with reduced accuracy. Similar versions were proposed by Kagawa et al. [1983], Suzuki et al. [1991], and Schohl [1993]. The convolution integral method was extended to turbulent flow by Vardy and Brown [1995, 2003, 2004] for smooth and for rough pipes. These solutions provide acceptable results at the expense of numerical accuracy, because of the approximation of the convolution integral by a limited number of weighted coefficients [Vitkovsky et al. 2006b].

### Instantaneous acceleration-based (IAB) methods

These models are based on the assumption that the damping attributable to unsteady friction is caused by instantaneous local and convective accelerations. The accelerations are computed from the average cross-sectional values without taking into consideration the velocity distribution at a cross section. Carsten and Roller [1959] introduced this concept. Since then several different formulations have been proposed [Brunone and Golia, 1990; Vitkovsky et al. 2006a, Brunone et al. 1991b; Bergant et al. 2001; Bughazem and Anderson, 2000; Vardy and Brown, 1995, and 2003; Ramos et al. 2004]. Of these formulations, one- and two-coefficient models appear to give satisfactory results and are presented herein.

The friction term in the momentum equation may be divided into steady and unsteady parts as

$$\frac{\partial H}{\partial x} + \frac{1}{g} \frac{\partial V}{\partial t} + J_s + J_u = 0 \quad (2-79)$$

in which  $J_s$  and  $J_u$  are the steady and unsteady friction terms, respectively. The steady friction may be expressed by the Darcy-Weisbach relation as

$$J_s = \frac{fV|V|}{2gD} \quad (2-80)$$

An expression for  $J_u$  for a one-coefficient model may be written as

$$J_u = \frac{k}{g} \left[ \frac{\partial V}{\partial t} + \text{Sign}(V)a \left| \frac{\partial V}{\partial x} \right| \right] \quad (2-81)$$

The expression for a two-coefficient model used by Lourerio and Ramos [2003], Ramos [2004] and Vitkovsky et al. [2000] are similar and are of the form

$$J_u = \frac{1}{g} \left[ K_{ut} \frac{\partial V}{\partial t} + K_{ux} \text{Sign}(V)a \left| \frac{\partial V}{\partial x} \right| \right] \quad (2-82)$$

in which  $K_{ut}$  and  $K_{ux}$  are two decay coefficients related to the local and convective accelerations, respectively. It has been shown numerically that the term  $K_{ut}\partial V/\partial t$  affects the phase shift of transient pressure waves and  $K_{ux}\partial V/\partial x$  affects the rate of damping [Ramos et al. 2004].

Reddy, Silva and Chaudhry [2012] presented an equation for the estimation of decay coefficients for IAB models. To develop this equation, a genetic algorithm (GA) was used to reproduce time-history of pressure oscillations recorded in 14 experiments, conducted in laboratories all over the world. The pipe material for these experiments includes steel, copper, and PVC, pipe diameter ranges from 0.012 m to 0.4 m and pipe length from 14 m to 160 m. Transients were produced by valve closure at the upstream or downstream ends of the piping systems.

The decay coefficients for one- and two-coefficient IAB models were determined for both methods of characteristics and finite-difference methods. The values that reproduced the time history of the experimental pressure oscillations range from 0.015 to 0.060 for  $K$  in the one-coefficient model and from 0.025 to 0.053 for  $K_{ux}$  and from 0.006 to 0.057 for  $K_{ut}$  in the two-coefficient model.

## Example

*Compute the wave velocity in the steel penstock of the Kootenay Canal hydroelectric power plant, BC, Canada. The data for different segments of the penstock are listed in Table 2-3. The values of  $E$  for steel,  $G$  for concrete, and  $K$  and  $\rho$  for water are 207 GPa, 20.7 GPa, 2.19 GPa, and 999 kg/m<sup>3</sup>, respectively.*

## Solution

For transient analysis, the wave velocity in each segment of the penstock may be determined as follows.

**Table 2-3. Data for penstock**

Pipe	Length (m)	Diameter (m)	Wall Thickness, (mm)	Remarks
1	244.	6.771	19	Expansion coupling at one end
2	36.5	5.55	22	Encased in concrete

**Pipe 1**

$$\frac{D}{e} = \frac{6.71}{0.019} = 353$$

As the pipe is anchored at one end,

$$\begin{aligned}\psi &= \frac{D}{e} (1 - 0.50\nu) & (\text{Eq. 2-71}) \\ &= 353 (1 - 0.15) \\ &= 300.05\end{aligned}$$

$$a = \sqrt{\frac{K}{\rho [1 + (K/E) \psi]}} \quad (\text{Eq. 2-65})$$

$$\frac{K}{E} = \frac{2.19}{207} = 0.0106$$

$$\begin{aligned}a &= \sqrt{\frac{2.19 \times 10^9}{999 (1 + 0.0106 \times 300.05)}} \\ &= 724 \text{ m/s}\end{aligned}$$

**Pipe 2**

Equations for a steel-lined tunnel (Eq. 2-74) may be used to compute the wave velocity in pipe 2.

$$\begin{aligned}\psi &= \frac{DE}{GD + Ee} \\ &= \frac{5.55 \times 207 \times 10^9}{20.7 \times 10^9 \times 5.55 + 207 \times 10^9 \times .022} \\ &= 9.62 & (\text{Eq. 2-74})\end{aligned}$$

$$\begin{aligned}a &= \sqrt{\frac{2.19 \times 10^9}{999 (1 + 0.0106 \times 9.62)}} \\ &= 1410 \text{ m/s}\end{aligned}$$



## 2-9 Summary

In this chapter, the momentum and continuity equations describing the transient flows in closed conduits are derived and the assumptions made in the derivations are discussed. It is demonstrated that these equations are quasi-linear, hyperbolic, partial differential equations. Various numerical methods available for their solution are discussed and a number of models to simulate unsteady friction and expressions for the wave velocity in the closed conduits are presented.

## Problems

**2-1** Derive the momentum equation considering the conduit walls are rigid and the fluid is compressible.

**2-2** Compute the wave velocity in a 3.05-m-diameter steel penstock having a wall thickness of 25 mm if it:

- i. is embedded in a concrete dam;
- ii. is anchored at the upstream end; and
- iii. has expansion joints throughout its length.

**2-3** Determine the wave velocity in a reinforced concrete pipe having 1.25-m diameter, 0.15-m wall thickness, and carrying water. The 20-mm reinforcing bars have a spacing of 0.5 m, and the pipe has expansion joints throughout its length.

**2-4** A 0.2-m-diameter copper pipe having a wall thickness of 25 mm is conveying kerosene oil at 20°C from a container to a valve. If the valve is closed instantly, at what velocity would the pressure waves propagate in the pipe? Assume the pipe is anchored at the upper end.

**2-5** Figure 5-13 shows the power conduits of an underground hydroelectric power station. Compute the wave velocity in each segment of the conduit. Assume modulus of rigidity of rock is 5.24 GPa.

**2-6** Derive the continuity equation if the conduit is:

- i. anchored against longitudinal movement throughout its length; and
- ii. anchored against longitudinal movement at the upper end.

**Answers****2-2**

- i. 1413 m/s
- ii. 992 m/s
- iii. 978 m/s

**2-3** 913 m/s**2-4** 1232 m/s**References**

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