Chapter 2
Torques

Torque exerted by a force is an important physical quantity in our daily life. It is associated with the rotation of a body to which a force is applied, unlike the force that is related to translation. For a body to be in rotational equilibrium, the sum of all torques exerted on it must be zero.

2.1 Objectives

- To discuss the concept of torque
- To obtain the torque due to more than one force
- To establish the conditions for rotational equilibrium of a rigid body

2.2 Concept of Torque

Torque or moment of a force, $\boldsymbol{M_F}$, is a physical quantity associated with the tendency of a force to produce rotation about any axis.

Torque is a vector quantity, but, in this book, we will use it as a scalar, introducing a sign convention that will allow us to add algebraically several torques due to the forces applied on a body. The sign of torque is taken to be positive (+) if the force tends to produce counterclockwise rotation and negative (−) if the force tends to produce clockwise rotation about an axis.

The effect of rotation depends on the magnitude of the applied force $\boldsymbol{F}$ and on the distance $d_\perp$ (perpendicular) to the axis of rotation. Torque is calculated by the product of the magnitude of force by the distance ($d_\perp$) from the line of action of force $\boldsymbol{F}$ to the axis of rotation. The line of action is the straight line, imaginary, that determines the direction of the force vector. The distance $d_\perp$ is called the moment arm or the lever arm of the force $\boldsymbol{F}$. The segment that defines the lever arm is perpendicular to the line of action of the force and passes through the axis of rotation. The magnitude of the torque, M_F, is defined by (2.1):

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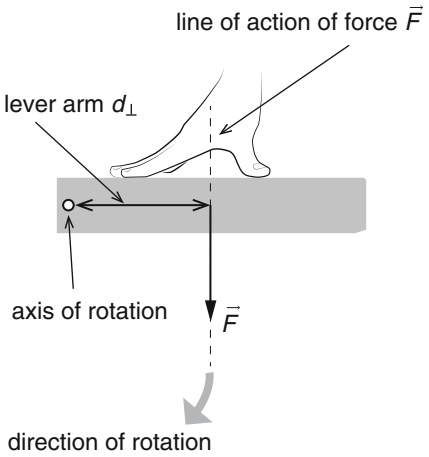


Fig. 2.1 A force $\boldsymbol{F}$ is applied at the center of an object. The line of action of the force $\boldsymbol{F}$, the lever arm $d_\perp$ of the force $\boldsymbol{F}$, and the axis of rotation through O are represented. The *arrow* indicates the direction of rotation

$$M_F = Fd_\perp. \tag{2.1}$$

Its unit in the International System of Units (SI) is N m.

Figure 2.1 shows a force $\boldsymbol{F}$ applied at the center of an object. The rotation axis is an imaginary line, perpendicular to the pivot point O, or fulcrum, of the object. The lever arm $d_\perp$ is perpendicular to the line of action of the force $\boldsymbol{F}$, which tends to rotate the object about the pivot point O clockwise; the magnitude of torque is $M_F = -Fd_\perp$. Now, if the force $\boldsymbol{F}$ with the same magnitude is applied in the opposite direction at the same place, the direction of rotation changes; that is, the body will rotate counterclockwise and the torque is given by $M_F = Fd_\perp$. Note that the axis of rotation is always subjected to the reaction (not shown) to the force $\boldsymbol{F}$. However, this reaction force does not produce torque because its lever arm is equal to zero. You can imagine that Fig. 2.1 represents a top view of a cross section of a door and the axis of rotation is the hinge line and the force is applied to open the door.

If a force $\boldsymbol{F}$ of the same magnitude is now applied at the opposite extremity to the axis, as can be seen in Fig. 2.2, its torque will be twice that of Fig. 2.1 because the lever arm will be $2d_\perp$, i.e., $M_F = -F(2d_\perp)$. In other words, it will now be twice as easy to rotate a bar about its axis of rotation. That is what we usually do instinctively to open a door or when we use a wrench to loosen a nut.

If we now apply a force $\boldsymbol{F}$ at the same place and with the same magnitude as that of Fig. 2.2, but with a different direction, as shown in Fig. 2.3, we see that the torque will be zero because the line of action of force will pass through the axis of rotation and, in this case, the lever arm will be null. If we apply a force to a door in such a way, we will never be able to either open or close it.

If we now apply a force $\boldsymbol{F}$ of the same intensity as that of Fig. 2.2, but with a different direction, as shown in Fig. 2.4, we see that the torque, i.e., the turning effect, is decreased, due to the fact that the lever arm which must be perpendicular

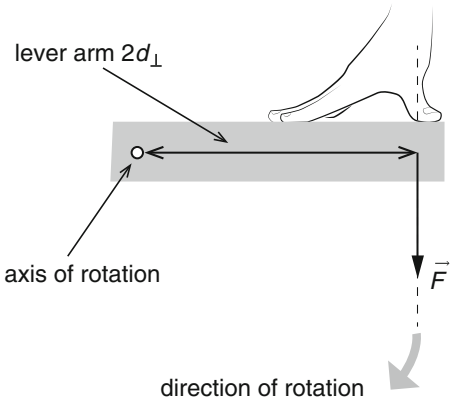


Fig. 2.2 A force $\boldsymbol{F}$ of the same intensity as that of Fig. 2.1 is now applied at the opposite extremity to the axis. The torque will be twice that of Fig. 2.1 because the lever arm was doubled

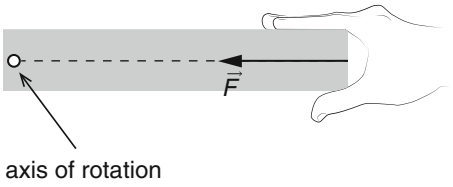


Fig. 2.3 A force $\boldsymbol{F}$ of the same magnitude as that of Fig. 2.2 is now applied at the end of the bar, but perpendicular, and the torque is null because the lever arm is zero, since the line of action, imaginary over the force vector, passes through the axis of rotation

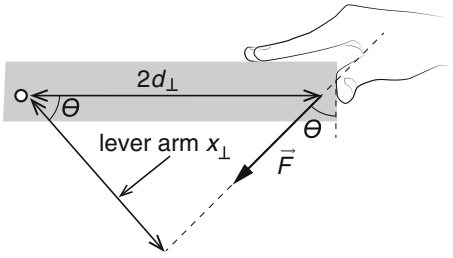


Fig. 2.4 A force $\boldsymbol{F}$ of equal intensity but different direction to that of Fig. 2.2 is applied at the edge of the bar. In this case, the lever arm $x_\perp$, which must be perpendicular to the line of action of force, is shorter than that of Fig. 2.2 and, as a consequence, the magnitude of the torque will be less

to the line of action of force $\boldsymbol{F}$ will be shorter, i.e., $x_\perp < 2d_\perp$. If we want to rotate the bar with the same ease, we have to apply a force with a larger magnitude. In the case of Fig. 2.4, the torque is $M_F = -Fx_\perp$.

We obtain the same result, decomposing the force $\boldsymbol{F}$ in its orthogonal components (Fig. 2.5), as presented in Chap. 1. The intensity of the component $\boldsymbol{F}_y$ can be

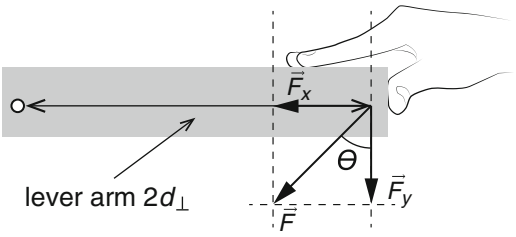


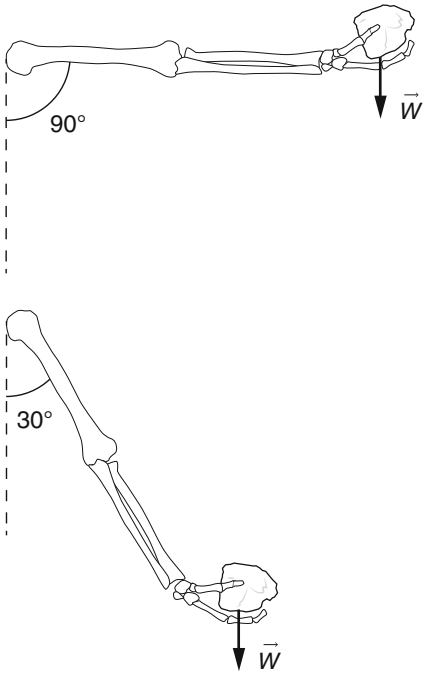
Fig. 2.5 Decomposition of force $\boldsymbol{F}$ in its orthogonal components. The torque due to the component $\boldsymbol{F}_x$ is zero because the lever arm is zero and the torque due to $\boldsymbol{F}_y$ is given by $F_y\,(2d_\perp)$

calculated if we know the angle θ between $\boldsymbol{F}$ and the vertical line, applying the law of trigonometry: $F_y = F\cos\theta$. The resultant torque about θ is then given by

$$M_F = F_x 0 - F_y(2d_\perp) = -F\cos\theta\ \ (2d_\perp) = -Fx_\perp.$$

The torque due to the component $\boldsymbol{F}_x$ is zero and, therefore, can produce no rotation because the line of action of the vector $\boldsymbol{F}_x$ goes through the pivot point O. Figure 2.4 shows that since the angle between $2d_\perp$ and $x_\perp$ is also θ, we can write that $\cos\theta\ (2d_\perp) = x_\perp$.

Example 2.1 In the exercise of lateral lifting of an arm, one object with a 2 kg mass is held by the hand, as can be seen in the figure of Example 2.1. The length of the arm + forearm + center of the hand is 70 cm. The axis of rotation is located at the shoulder. Find the torque exerted by the weight of the object for each situation in which the angle between the arm and the body is (a) 30° downward and (b) 90°.



(a) $M_W = -mgd_\perp = -(2\text{ kg})(10\text{ m/s}^2)d_\perp$; the negative sign indicates that the rotation is clockwise around the shoulder.

$$\text{As } d_\perp = (0.70\text{ m})\sin 30^\circ = (0.70\text{ m})0.5 = 0.35\text{ m},$$

$$M_W = -(2\text{ kg})(10\text{ m/s}^2)(0.35\text{ m}) = -7\text{ N m}.$$

(b) $M_W = -(2\text{ kg})(10\text{ m/s}^2)(0.7\text{ m}) = -14\text{ N m}$.

To maintain the arm outstretched horizontally ($\theta = 90^\circ$) in equilibrium, preventing its fall, it is necessary to apply a muscle force to counteract the torque. This force will be twice as large as the force to maintain the arm at 30°.

Let us see now what happens when the same object is hanging at the elbow with the arm outstretched horizontally. In this case, the torque will decrease to almost half of that with the weight on the hand, because the lever arm has been decreased to almost half. This means also that the muscle has to exert a force around half of the previous example to maintain it outstretched.

Exercise 2.1 Consider that a package of sugar of 1 kg is in the hand of a horizontally outstretched arm. Find the torque due to the weight of the sugar package about the axis of rotation passing through the:

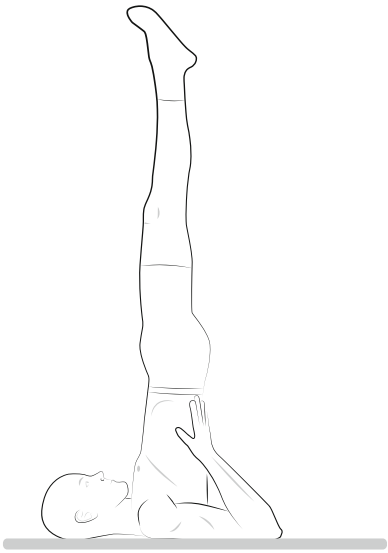
(a) Wrist
(b) Elbow
(c) Shoulder

Consider the following distances: shoulder–elbow = 25 cm, elbow–wrist = 22 cm, and wrist-center of the hand = 6 cm.

(d) Repeat the exercise, considering that the arm is held upward at a 30° angle to the body and then downward at the same angle.

Exercise 2.2 Discuss the reason for progressive difficulty in abdominal exercises, lying flat on the back: (a) with outstretched arms and the hands in the direction of the feet, (b) with the arms crossed over the chest, and (c) with the fingers interlaced under the head.

Exercise 2.3 Consider a person in the yoga posture called sarvanga-asana, as shown in the figure of Exercise 2.3. At the beginning, the person is lying flat on his back, with his legs outstretched. Then, slowly, the legs are lifted until the toes are pointed straight up. To reach the final posture, it is recommended that persons with problems in their spinal column bend their legs before lifting them. Discuss why.



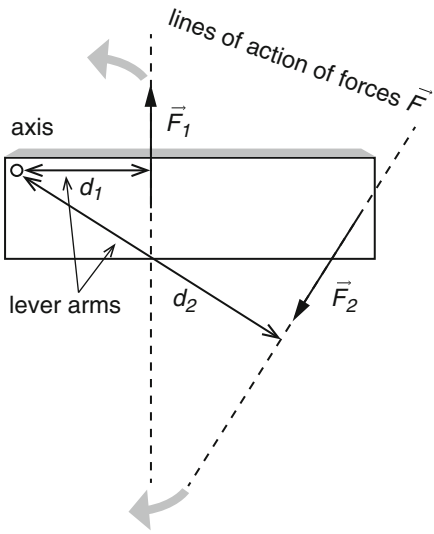


Fig. 2.6 Two forces are applied on the bar. The net torque is the sum of the torques due to each force

When a body is subjected to more than one force, we have to find the torque due to each force and add them to obtain the resultant torque. Figure 2.6 shows two forces acting on a bar. The resultant torque about the axis at pivot point O is given by

$$M = F_1 d_1 - F_2 d_2. \tag{2.2}$$

The direction of rotation is defined by the magnitude of forces and the lever arms involved. If the first (second) term of (2.2) is smaller than the second (first) term, the resultant torque will produce clockwise (counterclockwise) rotation of the bar.

2.3 Binary or Couple

Binary or couple is a system formed by two forces of the same magnitude and opposite direction applied on a body, whose lines of action are separated by a nonzero distance called the binary arm. The forces applied to the tap wrench or to a round doorknob constitute a couple. Figure 2.7 shows a sketch of a couple.

We can find the torque of a couple (2.3) by calculating the torque of each force about the axis of rotation separately and then summing them:

$$M = Fd_{\perp 1} + Fd_{\perp 2} = F(d_{\perp 1} + d_{\perp 2}) = Fx. \tag{2.3}$$

Both forces tend to rotate the bar counterclockwise about an axis through O.

Exercise 2.4 To remove the lug nut that fixes the wheel of a car, a man applies forces with a magnitude of 40 N with each hand on a tire iron, maintaining the hands 50 cm apart. Draw a diagram representing this situation and calculate the torque of the couple of forces exerted by the man.

The axis of rotation through O can be at any place between the couple of forces, as can be seen in Fig. 2.8. Even in such cases, the resultant torque is the same as in (2.3):

$$M = Fd_{\perp 1} + Fd_{\perp 2} = F(d_{\perp 1} + d_{\perp 2}) = Fx.$$

It is important now to analyze the situation in which the forces that constitute the couple have lines of action not perpendicular to the bar, as illustrated in Fig. 2.9. Note that d is the distance between the points of application of the forces

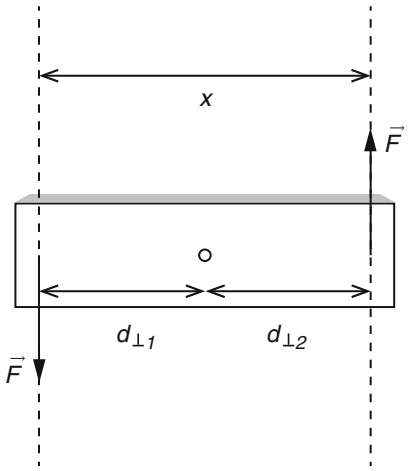


Fig. 2.7 Two forces of equal magnitude but opposite direction are applied at the ends of a bar. The system rotates about the axis through O (perpendicular to this page) at the center of the bar. The binary arm is represented by x

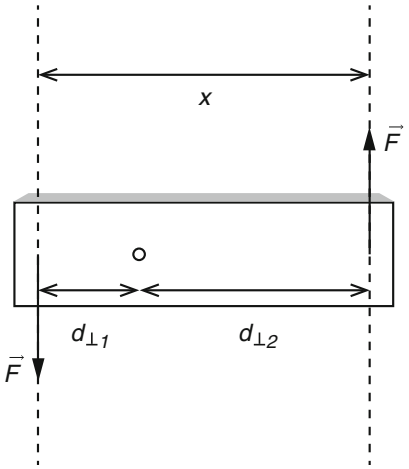


Fig. 2.8 A binary with axis of rotation through O not centered in relation to the forces ***F*** applied at the ends of the bar

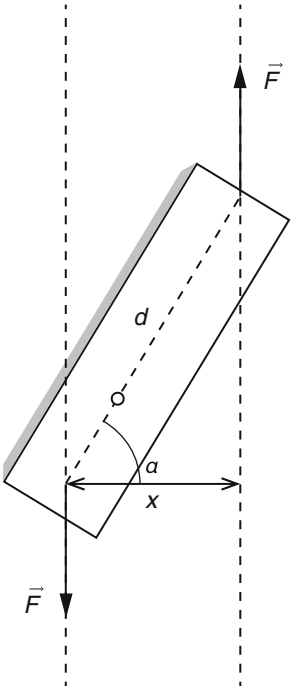


Fig. 2.9 Couple of forces ***F*** not perpendicular to the bar

on the bar. Observe, however, that the arm of the binary is still the distance x. The relation between x and d is given by the following expression:

$$x = d \cos \alpha,$$

and the torque of this couple is $M = Fx$.

2.4 Torque Due to Two or More Nonparallel Forces

2.4.1 Resultant of Two Nonparallel Forces Applied on a Body and Its Line of Action

We have already seen that one of the ways to determine the net torque on a body is by summing the torques due to each force separately. By another method, we use the resultant force and its point of application, or better, its line of action and the correspondent arm.

In the study of torque, it is clear that the point of application of a force is of fundamental importance. Actually, if the line of action of the force is determined, the problem is solved, since the arm of the force corresponds to the perpendicular distance between this line and the axis of rotation.

Observe that the effect of a force on a body does not change if its point of application is changed, since it stays on the line of action of the force. Using this property, the resultant force can be determined, dislocating the vectors on their lines of action to work as shown in Fig. 2.10 for the case of two nonparallel coplanar (which are in the same plane) forces.

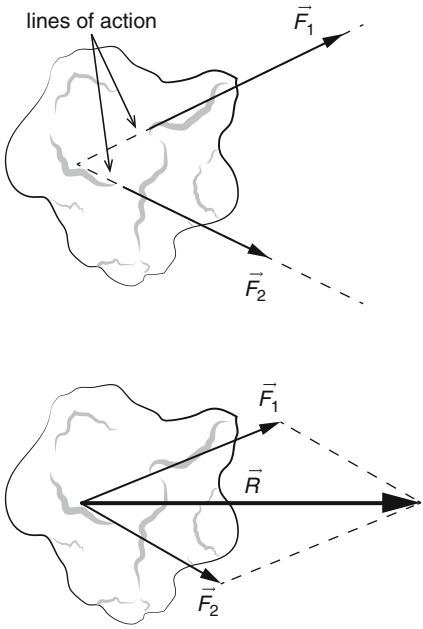


Fig. 2.10 Resultant force ***R*** is obtained by the rule of parallelogram, summing two forces whose points of application on the body do not coincide. Note that the rule is applied, first, by dislocating the points of application of forces to the same point. The line of action of the resultant is also determined by obtaining its torque about any axis

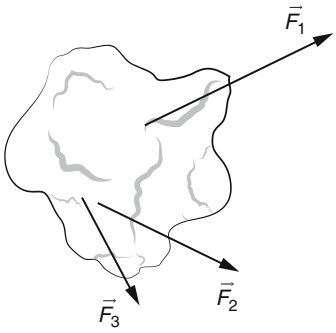
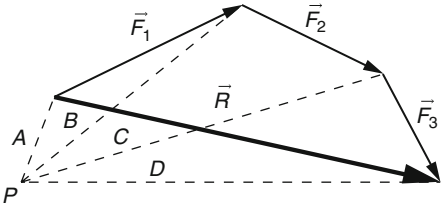


Fig. 2.11 Rule of polygon applied to the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ to obtain the resultant $\boldsymbol{R}$. The polar rays which connect the point P to the head of each vector were drawn in sequence



2.4.2 *Resultant of Two or More Nonparallel Forces Applied on a Body and Its Line of Action: Method of Funicular Polygon*

To find graphically the line of action of the resultant of two or more coplanar forces applied on a body, we construct the funicular polygon. Consider the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ in Fig. 2.11. We want to find the magnitude of the resultant $\boldsymbol{R}$ and its line of action. The magnitude and the direction of the resultant $\boldsymbol{R}$ can be obtained by the rule of polygon. To use this method, we only need to draw straight lines parallel to the lines of action of these forces, which can be done with the help of two squares. The first step consists in the transportation of the vectors to add them by the rule of polygon, as shown in Fig. 2.11. After drawing the resultant, we adopt a point P and connect it with straight lines A, B, C, and D, called polar rays, to the vertices of the polygon.

After that, we transport these polar rays, using the same direction as the original situation. For this, the straight line A is drawn in a way to intercept the line of action of vector $\boldsymbol{F}_1$ at any point. From this point of intersection, we draw the line B until it cuts the line of action of $\boldsymbol{F}_2$. From this intersection, we draw line C until it meets the line of action of $\boldsymbol{F}_3$. Finally, from this point, we draw line D. At the intersection of lines A and D, the line of action of the resultant $\boldsymbol{R}$ should pass and, furthermore, should be parallel to the vector $\boldsymbol{R}$, obtained by the rule of parallelogram (Fig. 2.11). We have to be careful to draw always parallel lines, when transporting vectors, and polar rays from one figure to another. Note that sometimes the line of action of the resultant can pass outside the object subjected to the forces. The resultant of three forces applied on a body by the method of funicular polygon is shown in Fig. 2.12.

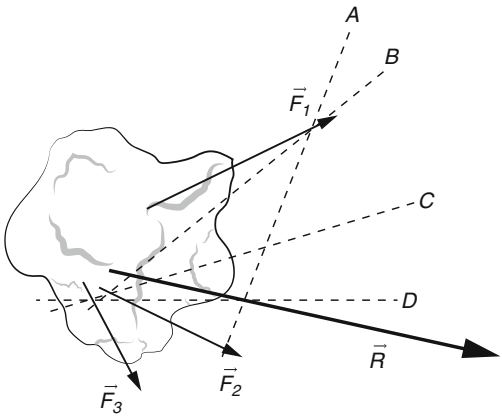
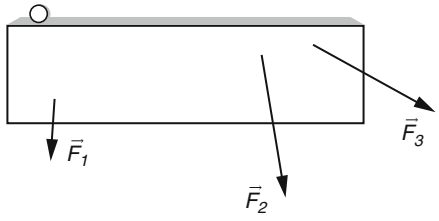
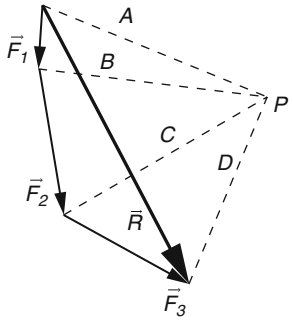


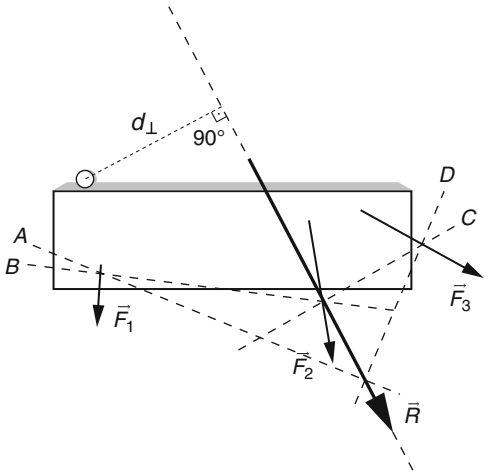
Fig. 2.12 Construction of funicular polygon to determine the line of action of the resultant of the three coplanar forces

Once the magnitude and direction of the resultant force and its line of action have been determined, the torque due to this resultant can be calculated about any axis of rotation. This can be done after obtaining the arm $d_\perp$ and using (2.1) and specifying the direction of rotation.

Example 2.2 Consider a bar of 38 cm in length subjected to three coplanar forces with magnitude $F_1 = 5.1$ N, $F_2 = 12.6$ N, and $F_3 = 11.8$ N, as illustrated in the figure of Example 2.2. Determine the resultant force vector by the method of polygon and then apply the method of funicular polygon to obtain the line of action of the resultant force. Then, find the torque of the resultant force about the axis of rotation through O.







The first thing to do is to transport the force vectors to do a vector addition. We obtain for the magnitude of the resultant the value $R = 26.3$ N.

Next, from the point P, we draw the polar rays and then transfer them to the diagram of forces. The point of intersection between the first and the last polar rays defines the point where the line of action of the resultant force must pass. Then, the arm of the resultant can be obtained using the scale:

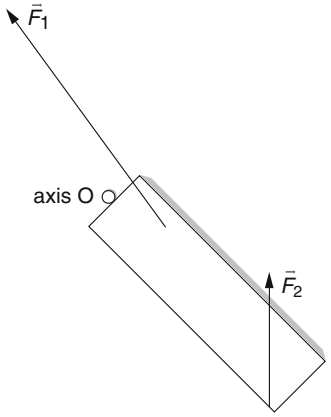
$$d_\perp = 16.5 \text{ cm} = 0.165 \text{ m}.$$

Hence, the net torque due to the resultant force about the axis at O is

$$M_R = -Rd_\perp = -(26.3 \text{ N})(0.165 \text{ m}) = -4.3 \text{ N m}.$$

Direction of rotation: clockwise

Exercise 2.5 Determine the resultant of the coplanar forces and its torque about the axis through O of the figure of Exercise 2.5 by the method of funicular polygon. Utilize the following scales:

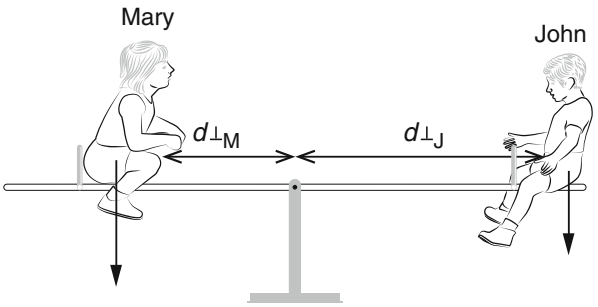


1 cm in the drawing = 10 N for the force.
1 cm in the drawing = 20 cm for the distance.

2.5 Rotational Equilibrium

For a body to be in rotational equilibrium, the sum of the torques produced by all the forces acting on the body must be zero. This is the condition used by the two-pan balance. The torque of the weight of an object in one of the pans of the balance makes it fall (rotate), if there is nothing in the other pan. As the calibrated mass is placed on the other pan, the torque due to its weight begins to equilibrate the balance. When the weights on both pans are the same, the magnitude of both torques will be the same and the net torque will be zero, due to opposite signs, and the balance will be in equilibrium.

Example 2.3 John and Mary are playing on a seesaw, as can be seen in the figure of Example 2.3. John, with a mass of 20 kg, is seated 2 m from the pivot point, called the fulcrum. At what distance from the fulcrum Mary, with a mass of 30 kg, has to sit so that the board will be in horizontal equilibrium?



The sketch shows the situation:

To be in equilibrium, the sum of torques of weights of Mary and John must be zero, that is,

$$M_M + M_J = 0,$$

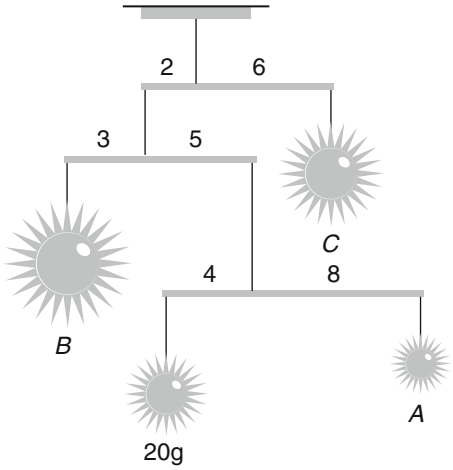
$$M_J = -(\text{Johns mass})gd_{\perp J} = -(20\,\text{kg})\left(10\,\text{m/s}^2\right)(2\,\text{m}) = -400\,\text{N m}.$$

Hence, in equilibrium, $M_M = +\ 400$ N m $= (30$ kg$)(10$ m/s$^2)d_{\perp M}$.

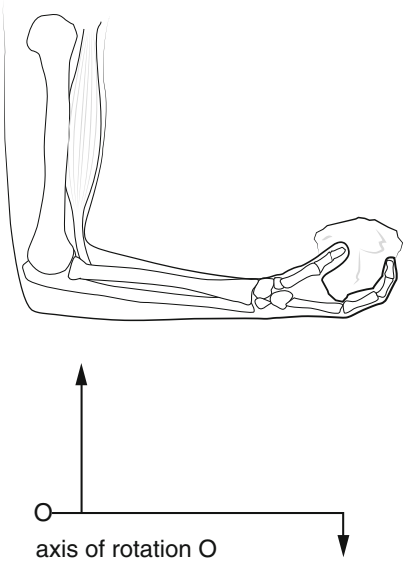
Then, $d_{\perp\,M} = (400$ N m$)/(300$ kg m/s$^2) = 1.33$ m.

As Mary weighs more than John, she has to sit closer to the fulcrum of the seesaw than John for the seesaw to be in horizontal equilibrium, that is, at 1.33 m from the pivot point O.

Exercise 2.6 You want to construct a mobile with four ornaments and three light rods with negligible mass as shown in the figure of Exercise 2.6. The distances in centimeters and the mass of one of the ornaments in grams are indicated in the figure. Determine the masses of ornaments A, B, and C, considering that the mobile should be in equilibrium.



Example 2.4 The forearm of the figure of Example 2.4 is maintained at 90° to the upper arm. An object weighing 15 N is held in the hand. The distance between the object and the axis of rotation through O, located at the elbow joint between the ulna and the humerus, is 30 cm. The biceps muscle is attached at 3.5 cm from the joint O.



(a) Find the torque produced by the weight of the object.
(b) Find the upward force that the biceps muscle exerts on the forearm (made up of the ulna and the radius). Consider only these two forces and that the forearm is in equilibrium.

(a) $M_W = -(15\text{ N})(0.30\text{ m}) = -4.5\text{ N m}$.
(b) If the forearm is in rotational equilibrium, the magnitude of the torque due to the muscle force must be equal to that calculated in (a) with the opposite sign, that is, $M_b = 4.5\text{ N m} = F_b(0.035\text{ m})$.

Hence, $F_b = (4.5\text{ N m})/(0.035\text{ m}) = 128.6\text{ N}$. Note that the muscle force is 8.6 times larger than the weight of the object.

Exercise 2.7 In Example 2.4, the weight of the forearm and hand was neglected. Consider now that the weight of the forearm plus the hand of 20 N is applied at 15 cm from O. Find:

(a) The torque due to the weight of the forearm-hand
(b) The muscle force exerted by the biceps muscle

2.6 Answers to Exercises

Exercise 2.1 (a) $M = 0.60\text{ N m}$; (b) $M = 2.80\text{ N m}$; (c) $M = 5.30\text{ N m}$; the same result is obtained when the angle between the outstretched arm and the body is 30°, either upward or downward since $d_\perp$ is always equal to $d\sin 30°$: (d) $M = 0.30\text{ N m}$; (e) $M = 1.40\text{ N m}$; (f) $M = 2.65\text{ N m}$. The direction of rotation depends on the arm, either the left or the right arm.

Exercise 2.2 The progressive difficulty is due to the increase in the magnitude of torque which is a consequence of the increase in the distance of application of total weight (head-neck + torso + arms-forearms-hands and a part of the abdomen) to the axis of rotation at the lower abdominal region. As the arms-forearms-hands are brought closer to the head, the torque becomes larger. To overcome it, a larger torque with the opposite direction of rotation should be exerted, which demands larger muscle force.

Exercise 2.3 As the angle of the set thigh-leg-foot with the ground (horizontal) becomes smaller, the torque becomes larger, because the distance $d_\perp$ is larger, and the weight of the set is applied a little above the knee. As the leg is elevated, the torque will be less because the distance $d_\perp$ will be less. If the torque is larger, this must be equilibrated by another torque of opposite sign produced by muscle force. Therefore, the larger the torque, the larger the muscle force should be in order to maintain equilibrium. Bending the legs, the torque due to the weight will be much less, because the distance $d_\perp$ will shorten and the muscle needs to exert a much smaller force to reach the final posture of sarvanga-asana.

Exercise 2.4 $M = 20\text{ N m}$.

Exercise 2.5 $m_A = 10\text{ g}$; $m_B = 50\text{ g}$; $m_C = 26.7\text{ g}$.

Exercise 2.6 (a) $M = -3.0\text{ N m}$; (b) the torque of the force exerted by the biceps muscle must equilibrate the sum of torques due to the weight of the object plus the weight of the forearm-hand. So, $F_b = 214.3\text{ N}$.

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