

Chapter 2

Definitions

In this chapter we describe the conditions and limitations for local thermodynamic equilibrium (LTE) and give the basic definitions for radiation transport and opacities. We also present an outline for the properties of extinction, absorption, and scattering coefficients (as applied in the theory of radiative transfer) and of the various mean and group mean opacities (as used in diffusion approximations).

2.1 Local Thermodynamic Equilibrium (LTE)

Of almost exclusive concern in this book are the conditions appropriate for LTE, which implies that all thermodynamic properties of a small volume of matter are the same as their thermodynamic equilibrium values corresponding to the local electron density and temperature. This means that the electron and ion velocities obey the distribution laws of statistical mechanics and the excitation and ionization equilibria are determined by collisions rather than by radiative rates. Thus, e.g., in the nondegenerate limit the velocity distributions are Maxwellian, the excitation equilibrium is given by the Boltzmann distribution function, and the ionization equilibrium is determined from the Saha equation.

In steady state, from the microscopic point of view, the rate of population change caused by collisional and radiative transitions from state i to all other states j must be balanced by the reverse transitions from all states j back to state i . Expressed mathematically, this is¹

¹Here the factors with p_k are given explicitly, as they are in non-LTE situations where they are unknowns. In LTE situations, as discussed in this book, it is common practice to include them in the effective cross sections.

$$p_i \sum_{j \neq i} (1 - p_j) (C_{ij} + R_{ij}) = (1 - p_i) \sum_{j \neq i} p_j (C_{ji} + R_{ji}), \quad (2.1)$$

where, for a state k , p_k is the ratio of the occupation number (electron population) N_k to the statistical weight g_k

$$p_k \equiv N_k / g_k, \quad (2.2)$$

and C_{kl} and R_{kl} represent the collisional and radiative rates, with the order of the subscripts indicating the state before and after the transition. Transport of particles into and out of the volume of interest has been ignored. Collision rates are proportional to the space density of collision partners and the temperature-dependent rate coefficients while radiative rates are proportional to the spectral radiance [the astronomer's (specific) intensity]. Fermi statistics is assumed so that the availability, $1 - p_k$, of state k to which an electron jumps must be considered explicitly. For continuum processes, such as photoionization or electronic recombination, the summations include integration over the continuum. The condition $C_{kl} \gg R_{kl}$ leads to LTE. In a more restrictive sense it is possible that population ratios satisfy Boltzmann (or Fermi-Dirac) statistics for some high lying levels of an ion, but for tightly bound levels radiative de-excitation can be much faster than collisional excitation. Thus, particularly in heavy elements, the high lying levels may satisfy the LTE requirements, while lower lying levels may not.

A more detailed discussion about LTE, particularly as it applies to nondegenerate conditions and to some limiting cases, is offered by Mihalas (1978) and Mihalas and Mihalas (1984). They also cite some references for collision cross section data.

2.2 The Equation of Radiative Transfer

In general, matter at a temperature T emits, absorbs, and scatters electromagnetic radiation. Chandrasekhar (1960) and Freemann (1965) have formulated the equation of transfer, including the polarization of photons, and have shown that in the nonrelativistic limit polarization effects are steadily damped out. The equation of transfer in dispersive media – i.e., the effect of frequency dependence of the refractive index – has been discussed by Pomraning (1971). Polarization and dispersion will be discussed briefly in Sect. 5.6 and applied in Mie theory.

In the diffusion limit, radiative opacity, i.e., a temperature-dependent weighted mean of absorption, emission, and scattering coefficients, determines the transfer of thermal radiation in matter. However, in the broadest definition, the term “opacity” includes coefficients for electronic and ionic energy conduction.

The spectral volume emission coefficient, $\hat{\epsilon}_\nu$ [power per unit volume per unit frequency interval and per unit solid angle], and the linear coefficients of

absorption,² μ_v^a [per unit length], and scattering, μ_v^s [per unit length], characterize the matter with which radiation of frequency ν and spectral radiance [the astronomer's (specific) intensity], I_ν [power per unit area per unit frequency interval and per unit solid angle], interacts. The variation in time and space of the radiation intensity in this medium is governed by the equation of radiative transfer, as, e.g., described by Chandrasekhar (1960), Kourganoff (1963), Goody (1964), Unsöld (1968), and Penner and Olfe (1968),

$$\begin{aligned} \frac{\partial I_\nu(\mathbf{r}, \hat{\omega}, t)}{c \partial t} + \hat{\omega} \cdot \nabla I_\nu(\mathbf{r}, \hat{\omega}, t) = \tilde{\epsilon}_\nu(\mathbf{r}, t) + \mu_\nu(\rho, T) \left\{ -I_\nu(\mathbf{r}, \hat{\omega}, t) \right. \\ \left. + \mathcal{A}_\nu(\rho, T) \int I_{\nu'}(\mathbf{r}, \hat{\omega}', t) p(\cos \Theta) \frac{d\omega'}{4\pi} \right. \\ \left. + [1 - \mathcal{A}_\nu(\rho, T)] B_\nu(T, \mathbf{r}, t) \right\}. \end{aligned} \quad (2.3)$$

Here, c is the speed of light, t the time, ρ the local density, \mathbf{r} indicates the position coordinates of a point in the medium, $\hat{\omega}$ and $\hat{\omega}'$ are unit vectors indicating the direction of radiation flow leaving and arriving at that point, Θ is the angle between the direction of the incident beam of radiation $\hat{\omega}'$ and the direction of scattered radiation $\hat{\omega}$, $I_{\nu'}$ is the spectral radiance of frequency ν' scattered into a beam with spectral radiance, I_ν , at frequency ν , $\tilde{\epsilon}_\nu$ is the spectral volume emission coefficient at frequency ν per unit solid angle for disequilibrium sources, and T is the temperature of the portion of material that is in thermodynamic equilibrium.

In the most general case the emission, absorption, and scattering coefficients depend also on the radiation field. Thus, radiative transfer is a highly nonlinear process and is not specified until the relationship between the radiation field and ionic, atomic, and molecular level populations (i.e., the equation of state, EOS) is determined.

The spectral radiance in equilibrium with the material temperature T , i.e., the LTE source function, is given by the Planck photon frequency distribution

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \left/ \left(e^{\frac{h\nu}{kT}} - 1 \right) \right., \quad (2.4)$$

where h is Planck's constant, and k is Boltzmann's constant. The function B_ν has the same dimensions as I_ν . The linear coefficient for extinction is

$$\mu_\nu = \mu_\nu^{a'} + \mu_\nu^s, \quad (2.5)$$

²The term "absorption coefficient" ("extinction coefficient," etc.) shall always mean the frequency dependent, i.e., monochromatic, absorption coefficient (extinction coefficient, etc.). Contrary to common usage in astrophysics, "opacity" shall indicate the Rosseland mean of the extinction (absorption plus scattering) coefficient. The Planck mean of the absorption coefficient shall specifically be called the "Planck opacity."

and the linear absorption coefficient corrected for stimulated or induced emission³ is

$$\mu_v^{a'} = \mu_v^a \left[1 - \frac{p_u(1 - p_l)}{p_l(1 - p_u)} \right]; \quad (2.6a)$$

μ_v^a is the linear absorption coefficient. The subscripts l and u refer to the lower and upper states involved in the transition, respectively. Again, Fermi statistics is assumed, and the availability $(1 - p_k)$ of the final state k for a transition is considered explicitly. For the case of thermodynamic equilibrium Eq. (2.6a) becomes

$$\mu_v^{a'} = \mu_v^a (1 - e^{-u}), \quad (2.6b)$$

where the reduced (dimensionless) photon energy is

$$u \equiv \frac{h\nu}{kT}. \quad (2.7)$$

The albedo for single scattering is defined as

$$\mathcal{A}_v \equiv \mu_v^s / \mu_v, \quad (2.8)$$

and the scattering phase function $p(\cos \Theta)$ is related to the differential scattering cross section

$$\sigma_v^s(\Theta) d\omega = \sigma_v^s p(\cos \Theta) \frac{d\omega}{4\pi}, \quad (2.9)$$

where σ_v^s is the total scattering cross section and $d\omega$ is the differential of the solid angle.⁴

The relationship between linear coefficient for extinction, mean free path (Λ_v), and cross section is

$$\mu_v = \Lambda_v^{-1} = \sigma_v N_c, \quad (2.10a)$$

where N_c is the number of extinction centers per unit volume. The mass extinction coefficient [area per mass] is

$$\kappa_v = \mu_v / \rho. \quad (2.10b)$$

³Since the induced photons are emitted in the direction of the light beam, they in effect reduce the absorption. It is sometimes argued that the term “negative absorption” is therefore more descriptive. However, in lasers the stimulated emission is much larger than the absorption and the argument is then not appropriate.

⁴In the case of Compton scattering the scattering source term also involves an integration over $d\nu$, where the change in frequency is related to the direction change: $(h\nu)^{-1} - (h\nu_0)^{-1} = (m_0 c^2)^{-1} (1 - \cos \Theta)$ (see Fraser 1966).

Similar relationships exist for absorption and scattering. The spectral mass emission coefficient [power per unit mass per unit frequency interval and per unit solid angle] is

$$j_\nu = \tilde{\epsilon}_\nu / \rho, \quad (2.11)$$

where ρ is the local density of the material. It is evident that of all these radiative properties the cross sections are the more fundamental quantities, nearly independent of atomic mass (isotopic mass) and less dependent on density.⁵ In nonrelativistic cases the induced scattering into and out of the beam cancel, as pointed out independently by Rudkjøbing (1947) and Maria Mayer.⁶

The equation for radiative transfer, Eq. (2.3), is written for the case of radiation in matter at rest with respect to the observer. Since the equation is covariant (Thomas 1930; Fraser 1966; Mihalas 1978), it has the same form in a system moving with constant velocity relative to the rest frame of the observer if all quantities, including absorption and scattering coefficients and source terms, are measured in the moving system. In some cases of interest, however, the fluid is in motion and we want to make use of the cross section data as measured or computed in the laboratory system, i.e., the rest frame of the observer. Fraser (1966) has given the form of the transport equation in the observer's (Eulerian) frame. If scattering is unimportant to first order in v_i/c , where \mathbf{v}_i is the ion drift velocity, the rest frame absorption coefficient corrected for induced emission, $\mu_\nu^{o'} = [1 - e^{-u(T^o)}] \mu_\nu^o$ (Eq. 2.6b), is decreased by $\mathbf{v}_i \cdot \hat{\omega} (\mu_\nu^{o'} + \nu d\mu_\nu^{o'}/d\nu)$ and an additional source term $(\mathbf{v}_i \cdot \hat{\omega}/c) u(T^o) [1 - e^{-u(T^o)}]^{-1} \mu_\nu^{o'} B_\nu(T^o)$ appears on the right side of Eq. (2.3). Physically the new terms account for the Doppler and aberration effects and are particularly important for line absorption and emission. According to Mihalas (1978), it is clear that velocity-field effects on line absorption can lead to very substantial changes in the energy balance of the outer layers of stellar atmospheres. These changes could, in principle, influence the hydrodynamics. Fraser also gives the form of the Compton scattering terms (see also Thomas 1930). Mihalas gives the form of the transport equation for slab and spherical geometries in a co-moving (Lagrangian) frame to order v_i/c for the case where scattering is unimportant (see also Castor 1972). Besides new terms on the left side of Eq. (2.3) with dv_i/dr and $\partial I/\partial \nu$, $(v_i/c) \partial I/\partial r$, $(v_i/c) I/r$, and $[v_i/(cr)] \partial I/\partial \nu$, the rest frame absorption coefficient and source $B_\nu(T^o)$ on the right side of Eq. (2.3) are multiplied by $\nu^0/\nu = 1 - \mathbf{v}_i \cdot \hat{\omega}/c$, where ν^0 is the rest frame frequency.

Techniques have been developed for application to stellar atmospheres to completely linearize the EOS and radiation transfer equations (Mihalas 1978; Mihalas

⁵Density influences the cross section to some extent. Examples are the broadening of spectral lines and the shift of spectral lines and photoelectric edges.

⁶As mentioned in the report by Mayer (1947).

and Mihalas 1984). The method is very effective, but requires enormous computer resources. Reviews edited by Kalkofen (1984, 1987) contain improved methods and applications to stellar modeling.

A large class of problems is concerned with the time independent case and with the matter in LTE, for which the transfer equation reduces to

$$\begin{aligned} \hat{\omega} \cdot \nabla I_v(\mathbf{r}, \hat{\omega}) = \mu_v(\rho, T) \left\{ -I_v(\mathbf{r}, \hat{\omega}) + \mathcal{A}_v(\rho, T) \int I_{v'}(\mathbf{r}, \hat{\omega}') p(\cos \Theta) \frac{d\omega'}{4\pi} \right. \\ \left. + [1 - \mathcal{A}_v(\rho, T)] B_v(T, \mathbf{r}) \right\}. \end{aligned} \quad (2.12)$$

Starting with Eq. (2.12), several subclasses of problems can be defined that are of particular interest from the physical and mathematical point of view. Among them are the angular moments of the spectral radiance. Lenoble (1985) surveys analytical and numerical methods of solution of the transfer equation for steady state and LTE conditions.

The first angular moment is the net time rate of energy that flows normal (direction $\hat{\mathbf{n}}$) through unit area (inflow = irradiance, outflow = exitance) per unit frequency interval. This normal net flux density \mathbf{F}_v [power per unit area and per unit frequency interval] is, defined⁷ such that

$$\hat{\mathbf{n}} \cdot \mathbf{F}_v(\mathbf{r}) = F_v^+(\mathbf{r}, \hat{\mathbf{n}}) - F_v^-(\mathbf{r}, \hat{\mathbf{n}}) \equiv \pi^{-1} \int \hat{\mathbf{n}} \cdot \hat{\omega} I_v(\mathbf{r}, \hat{\omega}) d\omega, \quad (2.13)$$

where the outflowing component in the $\hat{\mathbf{n}}$ direction is

$$F_v^+(\mathbf{r}, \hat{\mathbf{n}}) = \frac{\int_+ \hat{\mathbf{n}} \cdot \hat{\omega} I_v(\mathbf{r}, \hat{\omega}) d\omega}{\int_+ \hat{\mathbf{n}} \cdot \hat{\omega} d\omega} = \frac{\int_0^{2\pi} \int_0^{\pi/2} I_v(\mathbf{r}, \theta, \phi) \cos \theta \sin \theta d\theta d\phi}{\int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta d\theta d\phi}, \quad (2.14)$$

and the inflowing component is

$$F_v^-(\mathbf{r}, \hat{\mathbf{n}}) = \frac{\int_- \hat{\mathbf{n}} \cdot \hat{\omega} I_v(\mathbf{r}, \hat{\omega}) d\omega}{\int_- \hat{\mathbf{n}} \cdot \hat{\omega} d\omega} = \frac{\int_0^{2\pi} \int_{\pi}^{\pi/2} I_v(\mathbf{r}, \theta, \phi) \cos \theta \sin \theta d\theta d\phi}{\int_{\pi}^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta d\theta d\phi}. \quad (2.15)$$

The + and – signs on the integrals indicate that the integration is to be carried out only on the outflow and inflow side of the surface, respectively. Indicating the space coordinate normal to the surface by r_n , the integral over all angles of Eq. (2.12) can be expressed as

⁷Commonly called flux, $\pi \mathbf{F}_v$, in astrophysics (see, e.g., Unsöld 1968).

$$\begin{aligned}
\pi \nabla \cdot \mathbf{F}_v(\mathbf{r}) &= \pi dF_v(\mathbf{r})/dr_n \\
&= \mu_v(\rho, T) \left\{ - \int I_v(\mathbf{r}, \hat{\omega}) d\omega + \mathcal{A}_v(\rho, T) \int I_{v'}(\mathbf{r}, \hat{\omega}') d\omega' \right. \\
&\quad \left. + 4\pi [1 - \mathcal{A}_v(\rho, T)] B_v(T, \mathbf{r}) \right\},
\end{aligned} \quad (2.16)$$

since

$$\int p(\cos \Theta) \frac{d\omega}{4\pi} = 1. \quad (2.17)$$

Expressing the mean value (the zeroth angular moment) of the radiance by

$$J_v(\mathbf{r}) \equiv \int I_v(\mathbf{r}, \hat{\omega}) \frac{d\omega}{4\pi}, \quad (2.18)$$

Eq. (2.16) becomes

$$\frac{1}{4} \nabla \cdot \mathbf{F}_v(\mathbf{r}) = \frac{1}{4} \frac{dF_v(\mathbf{r})}{dr_n} = \mu_v(\rho, T) [1 - \mathcal{A}_v(\rho, T)] [B_v(T, \mathbf{r}) - J_v(\mathbf{r})]. \quad (2.19)$$

From the definition of the albedo, Eq. (2.8), and with the aid of the definition of the linear coefficient of extinction, Eq. (2.5), this simplifies to

$$\frac{1}{4} \frac{dF_v(\mathbf{r})}{dr_n} = \mu_v^{a'}(\rho, T) [B_v(T, \mathbf{r}) - J_v(\mathbf{r})]. \quad (2.20)$$

Since F_v and J_v come from the homogeneous part of Eq. (2.12) they cannot be integrated over r independently of one another. Equation (2.20) is basic for the definitions of the Planck and Rosseland opacities.

Multiplying Eq. (2.12) by $\hat{\mathbf{n}} \cdot \hat{\omega}$ and integrating over $d\omega$ gives for the second angular moment⁸

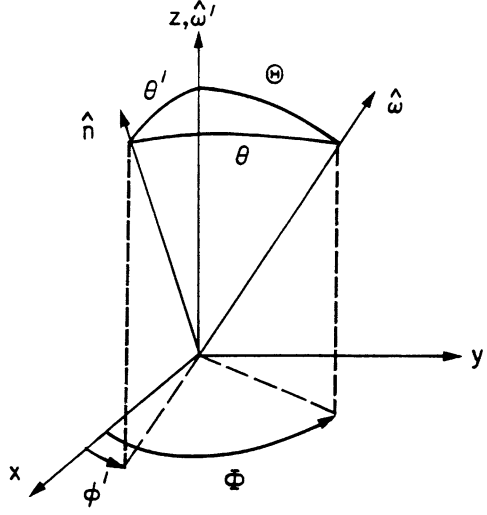
$$\begin{aligned}
\hat{\mathbf{n}} \cdot \nabla K_v(\mathbf{r}) &= \frac{3\mu_v}{4\pi} \left[- \int \hat{\mathbf{n}} \cdot \hat{\omega} I_v(\mathbf{r}, \hat{\omega}) d\omega \right. \\
&\quad \left. + \mathcal{A}_v(\rho, T) \int \int \hat{\mathbf{n}} \cdot \hat{\omega} I_{v'}(\mathbf{r}, \hat{\omega}') p(\cos \Theta) \frac{d\omega'}{4\pi} d\omega \right],
\end{aligned} \quad (2.21)$$

which is related to the radiation pressure (current of momentum). The integral over angles associated with the last term in Eq. (2.12) is

$$\int \hat{\mathbf{n}} \cdot \hat{\omega} B_v(T, \mathbf{r}) d\omega = 0, \quad (2.22)$$

⁸Higher moments correspond to anisotropy in the pressure.

Fig. 2.1 Relationship of angles between direction $\hat{\omega}$ and $\hat{\omega}'$ of photons leaving from a point and arriving at that point and the direction \hat{n} normal to the surface on the outflow side of the radiation



because of symmetry. The left side of Eq. (2.21) is based on the definition

$$K_\nu(\mathbf{r}) \equiv \frac{3}{4\pi} \int \hat{n} \cdot \hat{\omega} \hat{n} \cdot \hat{\omega} I_\nu(\mathbf{r}, \hat{\omega}) d\omega, \quad (2.23)$$

similar to Eqs. (2.13) and (2.18). The double integral in Eq. (2.21) can be simplified by expressing $\hat{n} \cdot \hat{\omega}$ in terms of the spherical coordinates associated with direction $\hat{\omega}'$ and \hat{n}

$$\hat{n} \cdot \hat{\omega} = \cos \Theta \cos \theta' + \sin \Theta \sin \theta' \cos(\Phi - \phi'), \quad (2.24)$$

where the polar axis is in direction $\hat{\omega}'$ and Θ is the angle between $\hat{\omega}$ and $\hat{\omega}'$ (see Fig. 2.1). The angle $\Phi - \phi'$ is the difference in azimuth between directions $\hat{\omega}$ and \hat{n} . Since integration over azimuthal angle gives zero, Eq. (2.21) becomes

$$\nabla K_\nu(\mathbf{r}) = -\frac{3\mu_\nu}{4} \left(1 - \mathcal{A}_\nu \overline{\cos \Theta}\right) \mathbf{F}_\nu(\mathbf{r}), \quad (2.25)$$

where

$$\overline{\cos \Theta} \equiv \frac{1}{2} \int \cos \Theta p(\cos \Theta) \sin \Theta d\Theta, \quad (2.26)$$

is the mean value of the cosine of the angle through which the photon scatters. Its effect on the Rosseland opacity is often not fully appreciated. It is discussed in more detail in Sect. 2.4 below. Using Eqs. (2.5) and (2.8), the extinction coefficient modified by the term in parentheses is

$$\mu_v \left(1 - \overline{\mathcal{A}_v \cos \Theta}\right) = \mu_v^{a'} + \mu_v^s \left(1 - \overline{\cos \Theta}\right) = \mu_v^{a'} + \mu_v^{s\mathcal{T}} \equiv \mu_v^{\mathcal{T}}, \quad (2.27)$$

where $\mu_v^{\mathcal{T}}$ is the transport extinction coefficient and $\mu_v^{s\mathcal{T}}$ is the transport scattering coefficient (see, e.g., Frank-Kamenetski 1962). If the phase function $p(\cos \Theta)$ is symmetric in $\cos \Theta$ (e.g., Rayleigh or Thomson scattering) then $\overline{\cos \Theta} = 0$ and the transport scattering coefficient is the same as the ordinary scattering coefficient. The phase function is asymmetric for scattering by anisotropic molecules (Sect. 5.4.3), for scattering in the relativistic domain, and when collective effects are important (Sect. 6.3).

Several cases for which the space derivative of the flux is either near a maximum or near zero, i.e., for which radiation is far from being in equilibrium with matter or nearly in equilibrium with it,⁹ are particularly well illustrated by Eq. (2.20). Suitable mean values of the absorption (or extinction) coefficients can be defined in these cases.

2.3 The Planck or Emission Mean Opacity

Examination of Eq. (2.20) reveals that dF_v/dr_n will be large if

$$\mathcal{A}_v < 1, \quad \text{i.e.,} \quad \mu_v^{a'} \neq 0, \quad (2.28)$$

and

$$J_v \ll B_v(T), \quad (2.29)$$

where $B_v(T)$ represents the equilibrium source of the radiation.¹⁰ In a way similar to the derivation of Eq. (2.20), the radiation outflow from a surface can be obtained

$$\frac{1}{f_S} \frac{dF_v^+(\mathbf{r})}{dr_n} = \mu_v^{a'}(\rho, T) B_v(T, \mathbf{r}), \quad (2.30)$$

where f_S is a factor independent of temperature or material properties and is determined only from the geometry of the isolated surface. Since ρ and T are functions of \mathbf{r} one may express $\mu_v^{a'}$ directly as a function of \mathbf{r} . If the material is nearly isothermal near the surface, then B_v is nearly independent of \mathbf{r} , and integration of Eq. (2.30) gives

⁹Assuming $\mathcal{A}_v \neq 1$. If $\mathcal{A}_v = 1$ for all v then $\mu_v^{a'} = 0$ and $dF_v/dr_n = 0$. This is the case of perfect scattering for which the radiative flux is constant without having to be in equilibrium with matter.

¹⁰This implies optically thin spectral lines (see discussion below).

$$\frac{\Delta F_v^+}{B_v(T)} = f_S \int_0^{\Delta r_n} \mu_v^{a'}(\mathbf{r}) d r_n = f_S \tau_v^{a'}. \quad (2.31)$$

The integral in Eq. (2.31) is referred to as the optical thickness for absorption, $\tau_v^{a'}$. If Δr_n is adjusted to give a value of 1 for the integral, then it defines one mean free path.¹¹

From the definition of emissivity,¹² which is the ratio of the radiant energy at frequency ν emitted from a surface at temperature T , to that of a blackbody at the same temperature

$$\varepsilon_v(\rho, T) \equiv \frac{F_v^+(\rho, T)}{B_v(T)}, \quad (2.32)$$

Eq. (2.31) can be expressed as¹³

$$\Delta \varepsilon_v(\rho, T) = f_S \int_0^{\Delta r_n} \mu_v^{a'}(\mathbf{r}) d r_n. \quad (2.33)$$

From the total flux

$$\pi F^+(\rho, T) = \pi \int_0^\infty F_v^+(\rho, T) d\nu, \quad (2.34)$$

and from Eq. (2.32)

$$F^+(\rho, T) = \int_0^\infty \varepsilon_v(\rho, T) B_v(T) d\nu. \quad (2.35)$$

¹¹A broader definition of monochromatic optical thickness includes scattering:

$$\tau_v \equiv \int_0^{\Delta r_n} \mu_v(\mathbf{r}) d r.$$

¹²Note the difference between the dimensionless emissivity, ε_v , (Eq. 2.32) and the emission coefficients per unit volume, $\tilde{\varepsilon}_v$, and per unit mass, j_v , (Eqs. 2.3 and 2.11).

¹³To obtain Eq. (2.30), Eq. (2.12) has been integrated over angles up to $\theta = \pi/2$ [as measured from the normal to the surface, making use of Eq. (2.24)] for which the mean free path of the photons may be larger than the thickness Δr_n (e.g., $\Delta r_n / \cos \theta$ for a plane surface). Photons leaving the surface at such angles must originate closer to the surface than from a depth Δr_n in order to have not more than one mean free path. Only for a hemisphere with radius Δr_n can one adjust the radius so that photons with frequency ν will have originated within one mean free path when they arrive at the center of the hemisphere. In that case, radiation is independent of $-\pi/2 \leq \theta \leq \pi/2$ and $f_S = 1$; this defines hemispherical emissivity. For further details see Penner and Olfe (1968). For a plane slab $f_S = 2$.

For a blackbody

$$\pi B(T) = \pi \int_0^\infty B_\nu(T) d\nu = \sigma T^4, \quad (2.36)$$

which is the exitance of the body. Here σ is the Stefan-Boltzmann constant

$$\begin{aligned} \sigma &= \frac{2\pi^5 k^4}{15h^3 c^2} = 5.670\,40 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}, \\ &= 3.539\,17 \times 10^{11} \text{ eV m}^{-2} \text{ K}^{-4} \text{ s}^{-1}, \end{aligned} \quad (2.37a)$$

or

$$\begin{aligned} \sigma k^{-4} &= 1.560\,55 \times 10^{84} \text{ J}^{-3} \text{ m}^{-2} \text{ s}^{-1}, \\ &= 6.418\,15 \times 10^{27} \text{ eV}^{-3} \text{ m}^{-2} \text{ s}^{-1}. \end{aligned} \quad (2.37b)$$

It is customary to define the total emissivity

$$\varepsilon(\rho, T) \equiv \frac{F^+(\rho, T)}{B(T)}, \quad (2.38)$$

which, with the aid of Eq. (2.35), leads to the relationship between total and spectral emissivity

$$\varepsilon(\rho, T) = \frac{\int_0^\infty \varepsilon_\nu(\rho, T) B_\nu(T) d\nu}{B(T)}. \quad (2.39)$$

The relationship between spectral emissivity and absorption coefficient, as given by Eq. (2.33), makes it desirable to define a mean absorption coefficient that has become known as the Planck mean opacity or “emission mean opacity”¹⁴

$$\mu_P(\rho, T) = \frac{\int_0^\infty \mu_\nu^{a'}(\rho, T) B_\nu(T) d\nu}{B(T)}, \quad (2.40)$$

and therefore,

$$d\varepsilon(\rho, T) = f_S \mu_P(\rho, T) dr_n. \quad (2.41)$$

It should be noted that the Planck opacity is a valid approximation under conditions of LTE *only* in the regions where the dimensions are small in comparison to the radiation mean free path and that it is a mean of only the absorption

¹⁴In an optically thin medium induced emission cannot occur; $\mu_\nu^{a'}$ should then be replaced by μ_ν^a .

coefficient corrected for induced emission,¹⁴ i.e., scattering is not included. If the material exhibits very strong spectral lines, the optical depth varies rapidly with frequency and it becomes meaningless to apply the Planck mean opacity unless the medium is optically thin at *all* frequencies in the important region under the Planck function.

2.4 The Rosseland Mean Opacity

Much more frequently it occurs that dF_v/dr_n is small in applications of Eq. (2.20). Using the same conditions for the inequalities (2.28), Eq. (2.20) requires then that

$$J_v(\mathbf{r}) \simeq B_v(T, \mathbf{r}), \quad (2.42)$$

which indicates that the radiation is nearly in equilibrium with the matter and has a nearly isotropic distribution. An expansion of I_v about $B_v(T)$ suggests itself as a solution

$$I_v(\mathbf{r}, \hat{\omega}) = B_v(T, \mathbf{r}) + \frac{\hat{\omega} \cdot \nabla I_v^{(1)}(\mathbf{r})}{\mu_v(\rho, T)} + \dots, \quad (2.43)$$

where $I_v^{(1)}$ is independent of direction. From this one obtains with the aid of equations related to the first and second angular moments, Eqs. (2.13) and (2.22),

$$\pi \hat{\mathbf{n}} \cdot \mathbf{F}_v(\mathbf{r}) = \mu_v^{-1} \int \hat{\mathbf{n}} \cdot \hat{\omega} \hat{\omega} \cdot \nabla I_v^{(1)}(\mathbf{r}) d\omega. \quad (2.44)$$

Substituting the expansion presented by Eq. (2.43) into the transfer equation, Eq. (2.12), multiplying by $\hat{\mathbf{n}} \cdot \hat{\omega}$ and integrating over $d\omega$ yields for the component of the flux in the $\hat{\mathbf{n}}$ direction with the aid of Eqs. (2.24), (2.26), and (2.44)

$$\pi \hat{\mathbf{n}} \cdot \mathbf{F}_v(\mathbf{r}) \left[1 - \mathcal{A}_v(\rho, T) \overline{\cos \Theta} \right] \mu_v(\rho, T) = -\frac{4\pi}{3} \hat{\omega} \cdot \nabla B_v(T, \mathbf{r}), \quad (2.45a)$$

or using the definition of the transport extinction coefficient, Eq. (2.27),

$$\pi \hat{\mathbf{n}} \cdot \mathbf{F}_v(\mathbf{r}) = -\frac{4\pi}{3\mu_v^T(\rho, T)} \hat{\omega} \cdot \nabla B_v(T, \mathbf{r}). \quad (2.45b)$$

Since B_v depends only indirectly on \mathbf{r} , through the dependence of T on \mathbf{r} , the frequency integrated flux can be expressed as

$$\pi \mathbf{F}(\mathbf{r}) = -\frac{4\pi}{3} \nabla T(\mathbf{r}) \int_0^\infty \frac{\partial B_v(T)/\partial T}{\mu_v^T(\rho, T)} dv. \quad (2.46)$$

Thus, several mean free paths inside of any system boundary the flux is given by the diffusion approximation

$$\pi \mathbf{F}(\mathbf{r}) = -D(\rho, T) \nabla u_{\text{rad}}(T, \mathbf{r}), \quad (2.47)$$

where, with the aid of Eq. (2.36), the radiation energy density [energy per unit volume] is

$$u_{\text{rad}}(T, \mathbf{r}) = \frac{4\pi}{c} B(T, \mathbf{r}) = a_S T^4(\mathbf{r}), \quad (2.48)$$

where $a_S = 4\sigma/c = 7.56577 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4} = 4.72218 \times 10^3 \text{ eV m}^{-3} \text{ K}^{-4}$ is Stefan's constant. Combining equations for the flux, Eqs. (2.46) and (2.47), with the equation for radiation energy density, Eq. (2.48), led Rosseland (1924) to establish the relationship between the radiative energy flux and the photon diffusion coefficient [area per unit time]

$$D(\rho, T) = \frac{c}{3\mu_R(\rho, T)}, \quad (2.49)$$

where

$$\frac{1}{\mu_R(\rho, T)} = \int_0^\infty \frac{\partial B_\nu(T)/\partial T}{\mu_\nu^T(\rho, T)} d\nu \bigg/ \frac{\partial B(T)}{\partial T}, \quad (2.50)$$

with μ_R the Rosseland mean extinction coefficient or opacity.¹⁵ The flux, Eq. (2.46), is also expressed in terms of the energy density, Eq. (2.48) or the corresponding temperature, using the expression for the Rosseland mean opacity, Eq. (2.50), as

$$\pi \mathbf{F}(\mathbf{r}) = -\frac{c}{3\mu_R} \nabla u_{\text{rad}}(T, \mathbf{r}) = -\frac{4a_S c}{3\mu_R} T^3 \nabla T(\mathbf{r}). \quad (2.47a)$$

If the scattering phase function $p(\cos \Theta)$ is symmetric in $\cos \Theta$ then the transport extinction coefficient is equal to the ordinary extinction coefficient. An example where the phase function is asymmetric is given in Sect. 6.3.

¹⁵For the case of radiation transfer in a moving fluid, Fraser (1966) obtained expressions in the diffusion approximation for radiation energy density and flux in the observer's (Eulerian) frame:

$$u_{\text{rad}} = (4\sigma/c)[T^4 - (c\mu_R^a)^{-1}DT^4/Dt],$$

$$\pi \mathbf{F} = (4\sigma/3)(4T^4 \mathbf{v}_i/c - \mu_R^{-1} \nabla T^4),$$

where μ_R^a is the Rosseland mean opacity without scattering, μ_R is the usual Rosseland mean opacity with Compton scattering, \mathbf{v}_i is the ion drift velocity, and $D/Dt \equiv \partial/\partial t + \mathbf{v}_i \cdot \nabla$.

2.5 Other Mean Opacities

Other mean absorption coefficients (opacities), such as the flux weighted mean (also known as the radiation pressure mean)¹⁶

$$\mu_f = \frac{\int_0^\infty \mu_v^{a'} F_v dv}{F}, \quad (2.51)$$

or the absorption mean

$$\mu_a = \frac{\int_0^\infty \mu_v^{a'} J_v dv}{J}, \quad (2.52)$$

are used if it is demanded that the opacity yield the correct amount of flux or radiation energy absorbed, respectively. However, μ_f and μ_a cannot be calculated directly since F_v and J_v are not known a priori. A modification, approximating the monochromatic flux, converts the radiation pressure mean opacity into the Chandrasekhar mean opacity. Applications with these mean opacities are rare.

2.6 Differences Between the Various Mean Opacities

There are five important differences between the mean values: (1) In the Planck opacity the large values of $\mu_v^{a'}$ (i.e., the peaks of absorption lines) contribute most heavily to the mean value, while in the Rosseland opacity the small values of μ_v (i.e., the continuum extinction in the windows between the absorption lines) contribute most heavily to the integral and hence, to the mean value. (2) The individual processes contributing to the Planck, flux, or absorption mean opacities are additive, it is not possible to add them in the Rosseland mean. (3) The weighting functions used in the Planck and Rosseland opacities peak at different photon frequencies. (4) The Rosseland opacity includes scattering effects and the Planck opacity does not. (5) For the Rosseland opacity to be meaningful and nonzero the extinction coefficient must possess the property $\mu_v \sim \lim_{v \rightarrow 0} v^k$ where $k < 3$ (see, e.g., Huebner and Fullerton 1974), or must have a low frequency cut-off such as the plasma frequency; Rayleigh scattering, if considered alone, would be unsatisfactory.

In order to clarify the meaning of these differences it is instructive to examine the relationships between the various mean opacities. If the flux has a known frequency distribution, e.g., the solar spectrum, which remains unchanged in the traversal through a thin medium, then the flux mean opacity, Eq. (2.51), is applicable. If in

¹⁶Biermann (1933) pointed out that if μ_v^T is used in place of μ_v^a then Eq. (2.51) gives again the Rosseland opacity, but F_v must be known.

addition F_ν is the blackbody spectrum defined by a temperature T at position \mathbf{r} in the medium at which the absorption coefficient μ_ν is determined then the Planck mean, Eq. (2.40), is equivalent to the flux mean opacity. Analogously the Rosseland mean opacity, although apparently of entirely different character than the Planck or flux mean, is the same as the flux mean opacity through the assumption that the local flux depends only on its local gradient and on the local value of the absorption coefficient of the medium. This can be shown by substituting $\partial B_\nu(T)/\partial T$, as obtained from the differential form of Eq. (2.46), into the definition of the Rosseland opacity, Eq. (2.50). With the aid of Eq. (2.34) the flux mean opacity is again obtained. For a gray medium the frequency distribution of the absorption coefficient is a constant and the Planck, Rosseland, and flux mean opacities are identical to within the above-mentioned restriction about the inclusion of the scattering term. Further discussions and a comparison of the flux and absorption mean opacities with the more widely used Rosseland and Planck mean opacities can be found in papers by Przybylski (1960) and Carbon (1974).

Expressing weighting functions of Eqs. (2.40) and (2.50) specifically in terms of u as given by Eq. (2.7) and dropping the notation for the functional dependence on ρ and T [and the independent variable u in $\mu(u)$] Planck¹⁷ and Rosseland opacities can be written as

$$\mu_P = \frac{15}{\pi^4} \int_0^\infty \frac{\mu_\nu^a u^3}{e^u(1 - e^{-u})} du, \quad (2.53a)$$

$$= \frac{15}{\pi^4} \int_0^\infty \mu^a u^3 e^{-u} du, \quad (2.53b)$$

and

$$\frac{1}{\mu_R} = \frac{15}{4\pi^4} \int_0^\infty (\mu^T)^{-1} \frac{u^4}{e^u(1 - e^{-u})^2} du, \quad (2.54a)$$

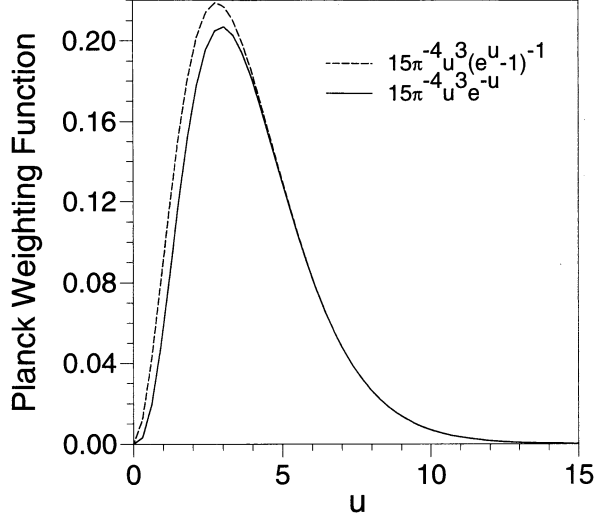
$$= \frac{15}{4\pi^4} \int_0^\infty \left[\mu^a + \frac{\mu^{sT}}{(1 - e^{-u})} \right]^{-1} \frac{u^4}{e^u(1 - e^{-u})^3} du. \quad (2.54b)$$

Here μ and μ^T are the same as given in Eqs. (2.5) and (2.27) except expressed as a function of u instead of ν .

Since the bound-free and the free-free absorption coefficients vary approximately as u^{-3} (exactly as u^{-3} in the Kramers semiclassical approximation), it has become customary to introduce the (dimensionless) reduced extinction coefficient, $D(u)$,

¹⁷In an optically thin case replace μ^a by μ^a in Eq. (2.53a) and replace $D^a(u)$ by $D^a(u)$ in Eq. (2.55a); do not use Eqs. (2.53b) or (2.55b). See footnote 14 by Eq. (2.40).

Fig. 2.2 Weighting function for the Planck mean opacity corresponding to Eqs. (2.53a), dashed curve, and (2.53b), solid curve



$$D^{\mathcal{T}}(u) = D^{a'}(u) + D^{s\mathcal{T}}(u) = Au^3 \frac{\mu^{\mathcal{T}}}{1 - e^{-u}} = Au^3 \frac{\mu^{a'} + \mu^{s\mathcal{T}}}{1 - e^{-u}}, \quad (2.55a)$$

$$= Au^3 \left[\mu^a + \frac{\mu^{s\mathcal{T}}}{1 - e^{-u}} \right]. \quad (2.55b)$$

Here the superscripts a and s stand for absorption and scattering, \mathcal{T} stands for transport [see Eq. (2.27)], and

$$\begin{aligned} A &= \frac{3^{3/2}}{16} \frac{4\pi\epsilon_0 mc}{he^2} kT \frac{10^{-3}M}{\rho N_0}, \\ &= 1.33011 \times 10^{-11} TM/\rho, \text{ [m]}, \text{ } (T \text{ in K}), \\ &= 1.54352 \times 10^{-7} kTM/\rho, \text{ [m]}, \text{ } (kT \text{ in eV}), \\ &= 2.10007 \times 10^{-6} kTM/\rho, \text{ [m]}, \text{ } (kT \text{ in Rydberg units}). \end{aligned} \quad (2.56)$$

In this equation ϵ_0 is the permittivity of vacuum, m the rest mass of the electron, e its charge, M the atomic (or molecular) mass number,¹⁸ and $N_0 = 6.02214 \times 10^{23} \text{ g-mol}^{-1}$ the Avogadro constant.

Using the reduced absorption (extinction) coefficients, the Planck mean opacity, including stimulated emission, Eqs. (2.53a) or (2.53b)¹⁹ and the Rosseland mean

¹⁸ $10^{-3}M$ is the atomic (or molecular) mass in kg/g-mol.

¹⁹See footnote by Eq. (2.53a).

opacity, Eqs. (2.54a) or (2.54b)²⁰ become

$$\kappa_P = \frac{\mu_P}{\rho} = \frac{1}{A\rho} \int_0^\infty W_P(u) D^{a'}(u) du, \quad (2.57)$$

$$\frac{1}{\kappa_R} = \frac{\rho}{\mu_R} = A\rho \int_0^\infty \frac{W_R(u)}{D^T(u)} du. \quad (2.58)$$

Here the Planck weighting function is

$$W_P(u) = \frac{15}{\pi^4} e^{-u}, \quad (2.59)$$

and the Rosseland weighting function is

$$W_R(u) = \frac{15}{4\pi^4} \frac{u^7}{e^u (1 - e^{-u})^3}. \quad (2.60)$$

The maxima of the weighting functions, which exclude the u dependence of the various μ -coefficients, are at $u \simeq 2.8$ and $u \simeq 3.0$, for Eqs. (2.53a) and (2.53b), respectively, and at $u \simeq 3.8$ for both of Eqs. (2.54a) and (2.54b). The weighting function W_P has its maximum at $u = 0$ and W_R has its maximum at $u \simeq 7.0$. The weighting functions for the Planck mean, i.e., Eqs. (2.53a) and (2.53b), are presented in Fig. 2.2 and those for the Rosseland mean, i.e., Eqs. (2.54a) and (2.54b), in Fig. 2.3. The Planck and Rosseland weighting functions, Eqs. (2.59) and (2.60), for the various reduced μ -coefficients are presented in Fig. 2.4.

A quantity frequently used in radiation transport is the (dimensionless) transmissivity

$$T_\nu(\Delta r) \equiv e^{-\tau_\nu}, \quad (2.61)$$

where τ_ν depends on Δr , the thickness of the layer traversed by the radiation [see, Eq. (2.31)]. The contribution to the radiation from sources in the thickness Δr must be negligible. Since τ_ν varies strongly with ν in the neighborhood of a spectral line, the transmissivity will not decrease exponentially with optical thickness if it is averaged over a spectral band $\Delta\nu$.

Even though it may include scattering (see footnote 11), the quantity

$$\alpha_\nu = 1 - e^{-\tau_\nu}, \quad (2.62)$$

²⁰Below the plasma cut-off frequency $\mu^a \rightarrow \infty$. An equivalent way of expressing this is to set the lower limit on the integrations equal to u_p , i.e., the value of u corresponding to the plasma cut-off frequency.

Fig. 2.3 Weighting function for the Rosseland mean opacity corresponding to Eqs. (2.54a), dashed curve, and (2.54b), solid curve

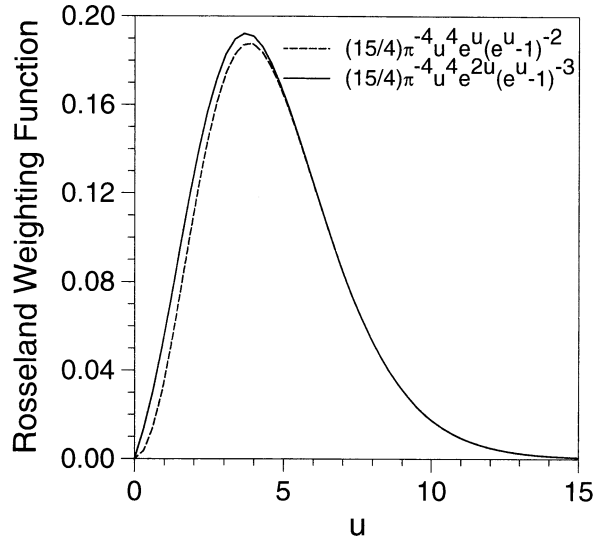
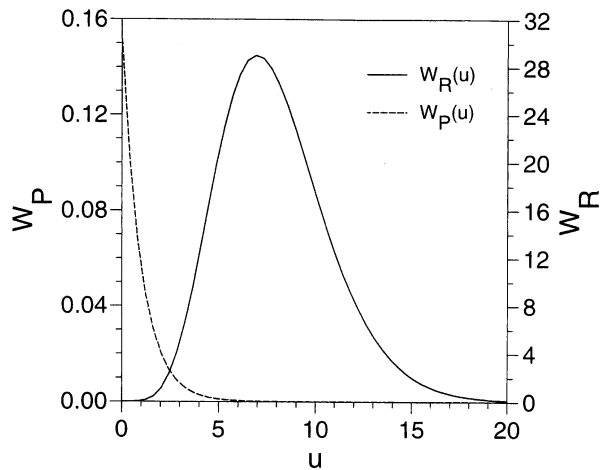


Fig. 2.4 Planck (left scale) and Rosseland (right scale) weighting functions corresponding to Eqs. (2.59) and (2.60)



is called absorptivity. Under conditions of thermal equilibrium, absorptivity and emissivity are equal (Kirchhoff's law). The width in wave numbers of an imaginary all-absorbing line with a rectangular profile that is equivalent to the integrated absorption of a real line or molecular band is

$$w = \int_0^\infty \alpha_{\tilde{\nu}} d\tilde{\nu} = \int_0^\infty [1 - \exp(-\tau_{\tilde{\nu}})] d\tilde{\nu}. \quad (2.63)$$

This is known as the “equivalent width.” The curve defined by w as a function of path length or optical depth is known as the “curve of growth.” It is useful for comparing

theoretical results with measurements and for deriving frequency-averaged absorption coefficients from transmissivity measurements (see section “Random Line Models” in Chap. 7 and in Sects. 11.2.2 and 13.2.1).

In regions near media boundaries, where the diffusion approximation based on Planck or Rosseland mean opacities is poor, group mean absorption (extinction) coefficients can be used. The Planck group mean opacity for the interval Δu_i centered at u_i is

$$\kappa_P(u_i, \Delta u_i) \equiv \frac{\pi^4}{15A\rho} \int_{u_i-\Delta u_i/2}^{u_i+\Delta u_i/2} W_P(u) D^3(u) du \bigg/ \int_{u_i-\Delta u_i/2}^{u_i+\Delta u_i/2} \frac{u^3}{e^u(1-e^{-u})} du, \quad (2.64)$$

and the Rosseland group mean opacity is

$$\kappa_R(u_i, \Delta u_i) \equiv \frac{15}{4\pi^4 A\rho} \int_{u_i-\Delta u_i/2}^{u_i+\Delta u_i/2} \frac{u^4}{e^u(1-e^{-u})^2} du \bigg/ \int_{u_i-\Delta u_i/2}^{u_i+\Delta u_i/2} \frac{W_R(u)}{D^T(u)} du. \quad (2.65)$$

Similar group mean opacities can be defined for the flux and the absorption means (see Eqs. 2.51 and 2.52). For the transport group mean opacity

$$\kappa_T(u_i, \Delta u_i) \equiv \frac{1}{\rho\Delta r} \int_{u_i-\Delta u_i/2}^{u_i+\Delta u_i/2} e^{-\mu_v^T \Delta r} du \bigg/ \int_{u_i-\Delta u_i/2}^{u_i+\Delta u_i/2} du, \quad (2.66)$$

a meaningful thickness, Δr , of the medium must be specified. It is not possible to assign any significance to the transport mean opacity if the limits on the integrals in Eq. (2.66) are 0 and ∞ .

2.7 Summary

In this chapter we have defined local thermodynamic equilibrium (LTE) and distinguished it from the steady state. We have briefly discussed the equation of radiative transfer. Finally we introduced the coefficients for absorption and scattering, and their sum, the extinction coefficient. These coefficients have been related to various mean values of opacity.

Excepting the few cases where numerical values for dimensioned quantities were introduced, the equations in this chapter were presented in such a way that any system of units can be used. Expressions for the opacities were given in various but equivalent forms, see Eqs. (2.53), (2.54), (2.57), and (2.58). Their application must be considered with care. Only in the deep interior of a medium in LTE, at distances from an interface much *larger* than the *maximum* mean free path (for photons with energies *between* spectral lines in the region where the Rosseland weighting function is large), is the Rosseland mean opacity appropriate for use in the diffusion approximation, which, by combining Eqs. (2.47a) and (2.58) can be written as

$$\pi \mathbf{F}(\mathbf{r}) = -\frac{4a_{\text{Sc}}}{3\kappa_{\text{R}}\rho} T^3 \nabla T(\mathbf{r}). \quad (2.67)$$

At points in a medium with distances to an interface *smaller* than the *minimum* mean free path (for photons with energies coinciding with the strongest absorption lines in the region where the Planck weighting function is large) the Planck mean opacity may be used. At points between the minimum and maximum mean free paths from an interface only the frequency dependent equation for radiative transfer can describe the transport of radiative energy correctly (see Chap. 14). However, the group mean opacities described at the end of this chapter can give a very good approximate solution, particularly if an interpolation scheme is used for which the group Rosseland and group Planck mean opacities are the limiting values for the appropriate optical depths.

Opacity

Huebner, W.F.; Barfield, W.D.

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