Chapter 2
Basic Tools and Concepts

In this chapter, from the Greeks (through much subsequent development), we derive the tools of spherical astronomy. We will describe the basic theorems of spherical trigonometry and emphasize the usefulness of the sine and cosine laws. We will also describe the ellipse and its properties, in preparation for a subsequent discussion of orbits.

2.1 Circular Arcs and Spherical Astronomy

All astronomical objects outside the solar system are sufficiently far away that their shifts in position due to parallax (caused by periodic motions of the Earth) and proper motion (caused mainly though not exclusively by the objects' own motion) are too small to be discerned—at least by the unaided eye. Historically, this suggested that these objects could be regarded as being fixed to the inner surface of a sphere (or fixed to the outer surface of a transparent sphere), the *celestial sphere*, of some very large radius, centered on the Earth. Objects within the solar system change position with time, e.g., a superior planet's orbital motion causes an eastward motion across our sky relative to the distant stars and the Earth's orbital motion causes the planet to follow a retrograde loop. However, objects within the solar system can be referenced to the celestial sphere at any given instant of time.

We may wish to calculate the distance measured across the sky from one object to another knowing the distance of each of them from a third object, and also knowing an appropriate angle. ("Distance across the sky" is actually an arc length, measured in units of angle such as degrees or radians.) In doing this, we are in essence drawing arcs joining three objects to form a triangle on a spherical surface, so the mathematical relationships involved are those of spherical trigonometry. When we apply them to the sky we are practicing *spherical astronomy*. We note also that the objects do not need to be real; one or more of them can be a reference point, such as the north or south celestial pole.

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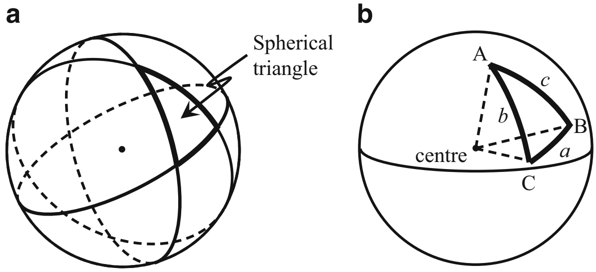
Fig. 2.1 The sphere with a spherical triangle on its surface



A *spherical triangle* is a triangle on the surface of a sphere such that each side of the triangle is part of a *great circle*, which has as its center the center of the sphere (a *small circle* will have its center along a radius of the sphere). For example, the Earth's equator is a great circle (assuming a spherical Earth), and any line of latitude other than the equator is a small circle.

The procedure in spherical astronomy lies primarily in the calculation of one side or angle in a spherical triangle where three other appropriate quantities are known, e.g., we may wish to find the length (in degrees) of one side of a spherical triangle given two other sides and the angle formed between them or find one side given a second side and the angle opposite each side or find one angle given a second angle and the side opposite each angle.

First, we review the basics of spherical trigonometry and then derive the cosine and sine laws, analogous but not identical to the cosine and sine laws of plane geometry. Additional theorems relating the three angles and three sides (involving, for example, haversines[1]) can also be found, but will not be derived here; for such theorems, see Smart's (1977) or Green's (1985) spherical astronomy texts, for example.

Figure 2.1a shows an example of a spherical triangle. The three ellipses are great circles seen in projection, and the spherical triangle (marked by heavy lines) is formed by their intersections. In Fig. 2.1b, we label the three sides of the triangle a, b, and c, and the angles at the three corners A, B, and C. We also draw a radius from the center of the sphere to each corner of the triangle (dashed lines).

Figure 2.2a shows an enlarged view of the spherical triangle and the radii to the center of the sphere. A circle is coplanar with its radii, so the shaded cross-section in this figure is a plane triangle (i.e., with straight sides).

It is important to distinguish between angles A, B, and C, which are the angles seen by an (two-dimensional) observer on the surface of the sphere and angles a, b, and c, which are the angles seen at the center of the sphere (Fig. 2.2b). That is, angle A is the angle (measured in degrees or radians) at the intersection of sides b and c on the surface of the sphere, whereas angle a is the angular separation "across the sky" from point B to point C (also measured in degrees or radians) as viewed from the center of the sphere. The principal equations relating these quantities to each other are the cosine and sine laws of spherical astronomy, which we will now derive.

[1] hav $\theta = (1/2)[1 - \cos\theta] = \sin^2(\theta/2)$.

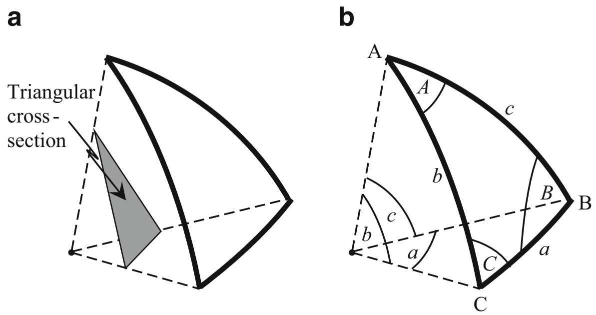


Fig. 2.2 Relating the sides of a spherical triangle to angles *a*, *b*, and *c* at the center of the sphere and angles *A*, *B*, and *C* on the surface of the sphere. The *heavy lines* are arcs of great circles; the *dashed lines* are radii of the sphere

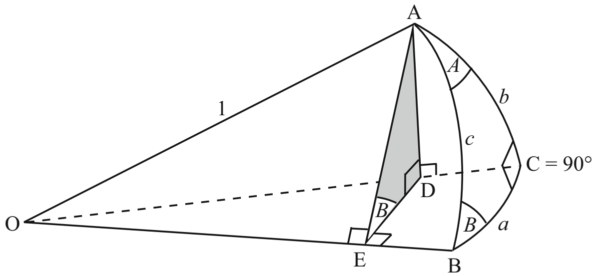


Fig. 2.3 Projection of a right spherical triangle onto a plane perpendicular to the base plane OBC. For convenience, we define points A and B to be located at the apices of angles *A* and *B*, respectively

NB: Small circle arcs have a different relation to interior angles. The laws derived here, therefore, do not apply to them.

2.1.1 The Law of Cosines for a Spherical Triangle

In this section we demonstrate a proof of the cosine law:

$$\cos a = \cos b \, \cos c + \sin b \, \sin c \, \cos A$$

First, we derive some useful results for a right spherical triangle ($\angle C = 90°$ in Fig. 2.3), which will also be useful in proving the sine law.

Take the radius of the sphere to be one unit of distance (OA = 1).

Note that any three points define a flat plane, so planes OAC, OAB, and OBC are each flat, and plane OAC is perpendicular to plane OBC. (To visualize this, it may help to think of arc BC as lying along the equator, in which case arc AC is part of a circle of longitude and arc AB is a "diagonal" great circle arc joining the two. Circles of longitude meet the equator at right angles, and longitudinal planes are perpendicular to the equatorial plane.)

From A, drop a line AD perpendicular to the plane OBC. Plane OAC $\perp$ plane OBC, so point D is on the line OC. From D, draw a line DE $\perp$ OB. (DE is, of course, not $\perp$ OC.) Then:

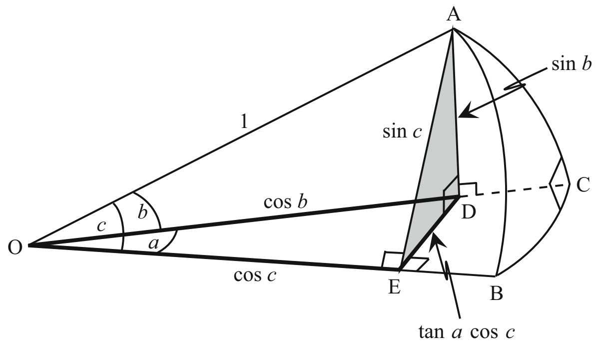


Fig. 2.4 Lengths of relevant sides, taking the radius of the sphere to be one unit of distance (OA = 1)

Plane ADE ⊥ plane OBC because line AD ⊥ plane OBC

Plane ADE ⊥ line OB because DE ⊥ OB and AD ⊥ plane OBC

∠ADE = 90° because plane OAC ⊥ plane OBC

∠OED = 90° and ∠OEA = 90° because plane ADE ⊥ line OB

Now define a plane tangent to the sphere (i.e., ⊥ to line OB) at point B; then ∠*B* lies in this tangent plane and, in fact, is equal to the angle of intersection of the two planes OAB and OBC. Plane ADE is parallel to the tangent plane, since both planes are ⊥ line OB, so it follows that:

$$\angle \mathrm{AED} = \angle \mathrm{B}$$

Then

$$\sin c = \frac{\mathrm{AE}}{\mathrm{OA}} = \frac{\mathrm{AE}}{1} = \mathrm{AE}$$
$$\cos c = \frac{\mathrm{OE}}{\mathrm{OA}} = \mathrm{OE}$$
$$\sin b = \frac{\mathrm{AD}}{\mathrm{OA}} = \mathrm{AD}$$
$$\cos b = \frac{\mathrm{OD}}{\mathrm{OA}} = \mathrm{OD}$$

These sides are shown in Fig. 2.4. Then from ΔODE,

$$\cos a = \frac{\mathrm{OE}}{\mathrm{OD}} = \frac{\cos c}{\cos b}$$

or

$$\cos c = \cos a \ \cos b \tag{2.1}$$

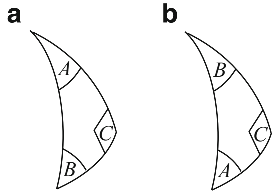


Fig. 2.5 Un-handedness of spherical triangles

Also from ΔODE,

$$\tan a = \frac{\mathrm{DE}}{\mathrm{OE}} = \frac{\mathrm{DE}}{\cos c}$$

or

$$\mathrm{DE} = \tan a \ \cos c$$

Therefore, from ΔADE,

$$\cos B = \frac{\mathrm{DE}}{\mathrm{EA}} = \frac{\tan a \ \cos c}{\sin c} = \tan a \ \cot c$$

Now $\angle C$ has been defined to be a right angle, but there is nothing to distinguish $\angle A$ from $\angle B$. It follows that any rule derived for the left-hand triangle in Fig. 2.5, above, has to be equally true for the right-hand triangle.

Thus, any rule derived for $\angle B$ is equally true for $\angle A$ with suitable relettering:

$$\cos A = \tan b \ \cot c \tag{2.2}$$

(One could derive this directly by redrawing Fig. 2.3 with the plane passing through point B perpendicular to the line OA instead of through point A perpendicular to the line OB.) Another equation can be obtained easily from ΔADE in Fig. 2.4:

$$\sin B = \frac{\mathrm{AD}}{\mathrm{AE}} = \frac{\sin b}{\sin c}$$

$$\therefore \sin b = \sin c \sin B \tag{2.3}$$

With (2.1) to (2.3) in mind, we can now look at the general spherical triangle (no right angles), as shown in Fig. 2.6.

Drop an arc $h \perp$ arc AB from point C to point D in Fig. 2.6. This divides the triangle into two right spherical triangles.

Define arc AD to take up an angle ϕ as seen from the center of the sphere (point O in Fig. 2.4). Then arc DB takes up angle $(c - \phi)$.

Apply (2.1) to each right spherical triangle in Fig. 2.6; then,
ΔADC: side b is opposite to the right angle, so,

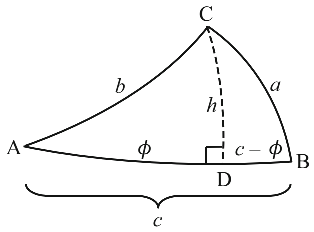


Fig. 2.6 The general spherical triangle as a combination of two right spherical triangles

$$\cos b = \cos h \, \cos \varphi \tag{2.4}$$

ΔBDC: side a is opposite to the right angle, so,

$$\cos a = \cos h \, \cos (c - \phi) \tag{2.5}$$

Divide (2.5) by (2.4) and use the standard trigonometric identity, $\cos(c - \phi) = \cos\, c \cos\, \phi + \sin\, c \sin\, \phi$, to get

$$\begin{aligned} \frac{\cos a}{\cos b} &= \frac{\cos h \, \cos (c-\phi)}{\cos h \, \cos \phi} = \frac{\cos (c-\phi)}{\cos \phi} \\ &= \frac{\cos c \, \cos \phi + \sin c \, \sin \phi}{\cos \phi} = \cos c + \sin c \, \tan \phi \end{aligned}$$

or

$$\cos a = \cos b \, \cos c + \cos b \, \sin c \, \tan \phi \tag{2.6}$$

Now use (2.2) to obtain an equation for $\tan \phi$:

$$\cos A = \tan \phi \, \cot b = \frac{\tan \phi \, \cos b}{\sin b}$$

or

$$\tan \phi = \frac{\cos A \, \sin b}{\cos b} \tag{2.7}$$

Substituting (2.7) into (2.6), we arrive at,

$$\cos a = \cos b \, \cos c + \sin b \, \sin c \, \cos A \tag{2.8}$$

Equation (2.8) is the cosine law for spherical triangles. Because none of the angles in the triangle are right angles, there is nothing to distinguish one angle from another, and the same equation has to apply equally to all three angles:

$$\cos a = \cos b \; \cos c + \sin b \; \sin c \; \cos A$$
$$\cos b = \cos c \; \cos a + \sin c \; \sin a \; \cos B$$
$$\cos c = \cos a \; \cos b + \sin a \; \sin b \; \cos C$$

Note the canonical rotation of angles from one formula to the next.

2.1.2 Law of Sines for a Spherical Triangle

Application of (2.3) to ΔADC and ΔBDC in Fig. 2.6 gives, respectively,

$$\sin h = \sin b \; \sin A \quad \text{and}$$
$$\sin h = \sin a \; \sin B$$

where sides b and a in Fig. 2.6 are opposite the right angles, and so replace side c in (2.3). Therefore,

$$\sin b \quad \sin A = \sin a \; \sin B \quad \text{and}$$
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}$$

Again there is nothing to distinguish one angle from another, so this equation also has to apply equally to all angles:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{2.9}$$

Equation (2.9) is the law of sines for spherical triangles.

2.1.3 Other Laws

Two other formulae which may be useful in particular cases are the *analogue formula* of Smart (1977, p. 10):

$$\sin a \; \cos B = \cos b \sin c - \sin b \; \cos c \; \cos A \tag{2.10}$$

and Smart's (1977, p. 12) *four-parts formula*,

$$\cos a \; \cos C = \sin a \; \cot b - \sin C \; \cot B \tag{2.11}$$

The quantities in each of these may be canonically rotated, as per the cosine and sine laws.

2.1.4 Applications

Uses for spherical trigonometry abound. One example is to use a terrestrial system triangle to find the length of a great circle route for a ship or plane given the initial and final points of the route. The terrestrial coordinate system, (λ, ϕ), consists of latitude, ϕ, longitude, λ, the equator, the poles, and the sense in which latitude and longitude are measured (N or S from the equator and E or W from Greenwich, the "prime meridian," resp.). The units of longitude may be in expressed in units of hours, minutes and seconds of time or in units of degrees, minutes and seconds of arc; latitude is always given in units of arc. The two longitude arcs from the north or south pole to the initial and final points then form two sides of a spherical triangle, and the great circle route forms the third side.

A spherical triangle also can be used to find the arc length between two objects in the sky with a coordinate system appropriate to the sky, or to transform between two coordinate systems. Several different coordinate systems are in use. In the *horizon* or *altazimuth* (A, h) system, the coordinates are *altitude*, h, measured along a vertical circle positive toward the zenith from the horizon, and *azimuth*, A, measured from a fixed point on the horizon, traditionally the North point, CW around toward the East. Both are measured in degrees (and subunits) of arc. For example, an observer can use a theodolite to observe the altitude and azimuth of a star, then use these to find the latitude of the observing site.

For astronomical applications, corrections need to be made for the effect of the Earth's atmosphere on the altitude: refraction by the atmosphere raises the altitude, h, by a value which depends itself on the altitude. At the horizon, the correction is large and typically amounts to ~34$'$. At altitudes above about 40°, and expressed in terms of the zenith distance ($\zeta = 90° - h$), the difference in altitude is ~57.3$''$ tan ζ. A measured altitude must be decreased by this amount to obtain the value of the altitude in the absence of the atmosphere. Another correction must be made for the "dip" of the horizon when the observer is not at ground or sea level (for example, when the observer is on the bridge of a naval vessel).

The *equatorial system* of astronomical coordinates has two variants:

The (H or HA, δ) system, which uses *Hour Angle*, H, and *declination*, δ. The (α, δ) system, which uses *Right Ascension*, α, and declination.

Thus, the location of astronomical objects such as the Sun or stars in the sky depends on the observer's latitude, the declination (distance above the celestial equator), δ, of the object, and the time of day.

Declination is measured North (+) or South (–) toward the N or S Celestial Poles from the *celestial equator*, the extension of the Earth's equator into the sky. Both H and α are measured in units of time. H is measured positive westward from the observer's meridian; α is measured eastward from the Vernal Equinox (the ascending node of the *ecliptic*; see below). Note that the (H, δ) system is dependent on the site, because H at any instant depends on the observer's longitude; the (α, δ) system is essentially independent of the observer's location. The latter is of use, for example, for a catalog of stars or other relatively 'fixed' objects.

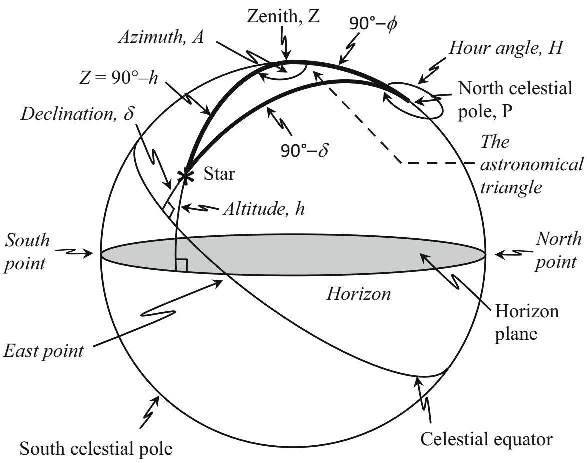


Fig. 2.7 The horizon A, h and equatorial H, δ coordinate systems. The *spherical triangle* whose apices lie at the star, the zenith, and the visible celestial pole is referred to as *the astronomical triangle*. Angle Z is the zenith angle of the star. Note that H, at the north celestial pole, is measured CW from south whereas A, at the zenith, is measured CW from north

At a particular site, H increases with time (the hour angle of the Sun + 12^h defines the apparent solar time) and this causes both altitude and azimuth to change with time also. Thus, the equatorial systems are more fundamental coordinate systems for celestial objects than the altazimuth system. The connection between the two variants of the equatorial system is the sidereal time, Θ:

$$\Theta = H + \alpha \tag{2.12}$$

The origin of the (α, δ) equatorial system is the Vernal Equinox, symbolized by the sign of Aries, γ, so the right ascension of this point is 0. Therefore sidereal time may be defined as,

$$\Theta \equiv H(\gamma) \tag{2.13}$$

(the hour angle of the Vernal Equinox). Recall that the Sun does not move along the celestial equator, but along the ecliptic, causing the different durations of sunshine with season (and latitude). Other consequences of this motion are the *equation of time* and the amplitude (maximum variation of the azimuth of the rising/setting Sun from the East/West points, respectively). See Fig. 2.7 for illustrations of the (A, h) and (H, δ) systems and the quantities needed to compute one set of coordinates from the other. In this figure, ϕ is the observer's latitude, which is equal to the altitude of the celestial pole above the observer's horizon and to the declination of the observer's zenith.

The *ecliptic coordinate system* (λ, β), involves the coordinates *celestial* (or *ecliptic*) *latitude*, β, and *celestial* (or *ecliptic*) *longitude*, λ, analogous to both the terrestrial coordinates (λ, ϕ) and the equatorial system (α, δ). The closer analogy, despite the names, is to the latter because the celestial longitude is measured CCW (viewed from the N) and from the same zero point, the Vernal Equinox, γ. The two

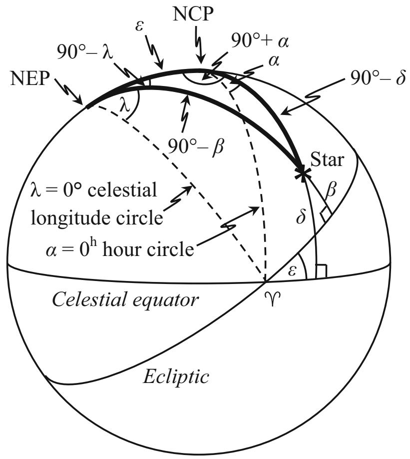


Fig. 2.8 The (λ, β) ecliptic and (α, δ) equatorial coordinate systems. ε is the obliquity of the ecliptic, i.e., the angle between the ecliptic and the celestial equator, and therefore also between the north ecliptic pole (NEP) and the north celestial pole (NCP). The spherical triangle involving the star, the NEP, and the NCP is used to obtain the transformation equations between the systems

reference circles, the celestial equator and the ecliptic, intersect at the vernal and autumnal equinoxes. The angle between them, known as the *obliquity of the ecliptic*, ε, is about 23.440° at present—it is slowly decreasing with time. The ecliptic system is very important for solar system studies and for celestial mechanics, both of which deal primarily with the solar system, the overlap with stellar kinematics, and dynamics notwithstanding. The relationship between the ecliptic and equatorial systems can be seen in Fig. 2.8, which shows the angles needed to compute one set of coordinates given the other.

The transformation equations between systems are readily obtained by drawing both on a celestial sphere and solving the resulting spherical triangles for the unknown pair of coordinates. Thus to compute the ecliptic longitude and latitude, draw the equatorial (RA) and ecliptic systems on the celestial sphere and use the separation of the poles of the system as one of the triangle legs. For this purpose, the spherical sine and cosine laws are perfectly adequate, even for checking the quadrant of the longitudinal coordinate (which may be in any of the four quadrants).

In Fig. 2.8, note that the great circle arc joining the NEP and the NCP is perpendicular to both the hour circle and the ecliptic/celestial longitude circle through the vernal equinox. The same arc is equal to the obliquity of the ecliptic, ε, the angle at the vernal equinox between the celestial equator and the ecliptic; Challenge [2.8(a)] at the end of this chapter invites you to prove that this is the case.

Example 2.1

An astronomer wants to observe a particular star with a telescope on an altazimuth mount. A star atlas provides the star's α and δ; and the star's hour angle is then given by $H = \Theta - \alpha$ from (2.12), where Θ is the sidereal time. However, because the mount is altazimuth, the coordinates actually needed are the altitude and azimuth. Find the transformation equations to convert the star's coordinates from H and δ to A and h. (Note how H and A are defined, and be careful with signs in equations containing trigonometric functions.) {Hint: The results given in Challenge [2.8] may be helpful.}

Solution to Example 2.1
Figure 2.7 shows the two relevant systems of coordinates. Here, ϕ is the observer's latitude and is equal to the declination of the observer's zenith. The star, the zenith, and the NCP form a spherical triangle referred to as the *astronomical triangle*, with sides $90^\circ - h$ opposite the angle $360^\circ - H$; $90^\circ - \delta$ opposite the azimuth angle, A; and $90^\circ - \phi$ opposite the angle formed at the star. Equations (2.8) and (2.9) then give, respectively,

$$\cos(90^\circ - h) = \cos\left(90^\circ - \phi\right)\ \cos\left(90^\circ - \delta\right) + \sin(90^\circ - \phi)\ \sin\left(90^\circ - \delta\right)\ \cos\left(360^\circ - H\right)$$
$$\sin A = \sin(90^\circ - \delta)\ \sin(360^\circ - H)/\sin(90^\circ - h).$$

The identities,

$$\begin{aligned}\sin(90^\circ - \theta) &= \cos\theta\\ \cos(90^\circ - \theta) &= \sin\theta\\ \sin(360^\circ - \theta) &= -\sin\theta\\ \cos(360^\circ - \theta) &= \cos\theta\end{aligned}$$

then give the transformation equations,

$$\sin h = \sin\phi\ \sin\delta + \cos\phi\ \cos\delta\ \cos H \tag{2.14}$$

$$\sin A = -\cos\phi\ \sin H/\cos h \tag{2.15}$$

Example 2.2
At some time of night, an observer at latitude 30° N sees a star at an altitude of 20° and an azimuth of 150°. Find the star's (H, δ) equatorial coordinates.

Solution to Example 2.2
Now we need the equations for the inverse of the transformation in Example 2.1, i.e., from the horizon to the equatorial system. Using Fig. 2.7 and the procedure in Example 2.1 again these are, from the cosine law,

$$\sin\delta = \sin\phi\ \sin h + \cos\phi\ \cos h\ \cos A \tag{2.16}$$

and, from the sine law,

$$\sin H = -\cos h\ \sin A/\cos\delta \tag{2.17}$$

Then with the values $\phi = 30^\circ$, $h = 20^\circ$, and $A = 150^\circ$, (2.16) yields

$$\sin\delta = -0.53376, \text{so that}\, \delta = -32.^\circ260$$

and from (2.17), $\sin(H) = -0.55562$, so

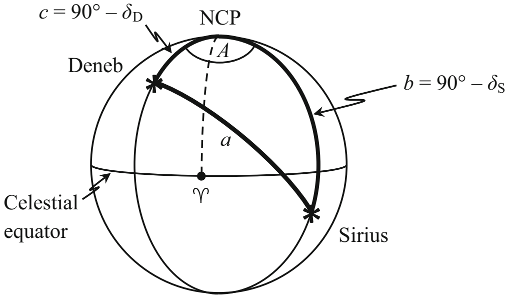


Fig. 2.9 The (α, δ) equatorial coordinate system, and the spherical triangle relating Deneb, Sirius, and the NCP. The RA arcs of the two stars are shown; angle A is the difference in right ascension between these two arcs

$$H = -33^\circ\!.753 \text{ or } -33^\circ\!.753/(15^\circ/\mathrm{h}) \cong -02^\mathrm{h}15^\mathrm{m} = 02^\mathrm{h}15^\mathrm{m}\mathrm{East}$$

Example 2.3
What is the angular distance across the sky from Deneb (α Cygni) at $\alpha = 20^\mathrm{h}\ 40^\mathrm{m}\ 24^\mathrm{s}$, $\delta = +45^\circ\ 10'$ to Sirius (α Canis Majoris) at $\alpha = 6^\mathrm{h}\ 43^\mathrm{m}\ 48^\mathrm{s}$, $\delta = -16^\circ\ 41'$?

Solution to Example 2.3
Figure 2.9 illustrates the (α, δ) equatorial coordinate system, the relevant spherical triangle, and the angles involved. The $\alpha = 0$ line is also shown for reference, as a dashed line from the NCP to the vernal equinox. Subscripts D and S signify Deneb and Sirius, respectively. We want to find the great circle arc length of side a.

The arc lengths of sides b and c are

$$b = 90^\circ - \delta_\mathrm{S} = 90^\circ - \left(-16^\circ\ 41'\right) = 106^\circ\ 41' = 10^\circ\!.668$$
$$c = 90^\circ - \delta_\mathrm{D} = 90^\circ - 45^\circ\ 10' = 44^\circ\ 50' = 44^\circ\!.83$$

There are several ways of expressing the angle A, all of which are equivalent by the fact that $\cos\theta = \cos(-\theta) = \cos(360^\circ - \theta)$. Here we take the smallest positive value of A, but it would be equally correct and perhaps simpler to take $A = \alpha_\mathrm{S} - \alpha_\mathrm{D}$ to obtain a negative angle or $A = \alpha_\mathrm{D} - \alpha_\mathrm{S}$ to obtain a positive angle $>180^\circ$.

$$c = 90^\circ - \delta_\mathrm{D}$$

From Fig. 2.9,

$$\begin{aligned} A &= \left(24^\mathrm{h} - \alpha_\mathrm{D}\right) + \alpha_\mathrm{S} = \left(24^\mathrm{h} - 20^\mathrm{h}40^\mathrm{m}24^\mathrm{s}\right) + 6^\mathrm{h}43^\mathrm{m}48^\mathrm{s} \\ &= 10^\mathrm{h}03^\mathrm{m}24^\mathrm{s} = \left(10^\mathrm{h} \times 15^\circ/\mathrm{h}\right) + \left(3^\mathrm{m} \times 1/60\,\mathrm{h/m} \times 15^\circ/\mathrm{h}\right) \\ &\quad + \left(24^\mathrm{s} \times 1/3600\,\mathrm{h/s} \times 15^\circ/\mathrm{h}\right) = 150^\circ\!.85 \end{aligned}$$

Then, using the cosine law to find side a,

$$\begin{aligned}\cos a &= \cos b \ \cos c + \sin b \ \sin c \ \cos A \\ &= \cos 106.68^\circ \ \cos 44.83^\circ + \sin 106.68^\circ \ \sin 44.83^\circ \ \cos 150.85^\circ \\ &= -0.7934.\end{aligned}$$

The inverse cosine is double-valued, so, within round-off error,

$$a = \cos^{-1}(-0.7934) = 142.6^\circ \ \text{or} \ 217.4^\circ.$$

The shortest angular distance between any two objects on a sphere is always $\leq 180^\circ$, so the distance across the sky from Deneb to Sirius is $142^\circ\!.6$.

See Schlosser et al. (1991/1994) or Kelley and Milone (2011) for more details on and worked examples of transformations.

2.2 Properties of Ellipses

Spherical astronomy is a very important tool in solar system astronomy, as well as other areas, but it is not, of course, the only one. We now turn from a consideration of the circular and spherical to review basic properties of ellipses. This is useful for understanding the orbits of solar system objects, considered in Chap. 3, as well as extra-solar planets (Milone and Wilson 2014, Chap. 16).

An ellipse is the locus of points $P(x, y)$, the sum of whose distances from two fixed points is constant.

That is, in Fig. 2.10,

$$\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 = \text{constant} \tag{2.18}$$

The two fixed points are the *foci* of the ellipse (singular: *focus*). Relevant geometric definitions of the quantities in Figs. 2.10 and 2.11 are:

a = semi-major axis (therefore the length of the major axis is $2a$);
b = semi-minor axis (therefore the length of the minor axis is $2b$);
f = distance of each focus from the center of the ellipse;
r = distance from one focus to a point on the ellipse (e.g., point P);
$r_{\min}$ = distance from either focus to the nearest point on the ellipse;
$r_{\max}$ = distance from either focus to the farthest point on the ellipse.

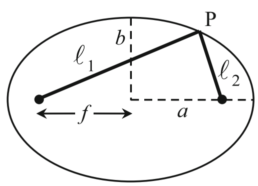


Fig. 2.10 The ellipse definition illustrated

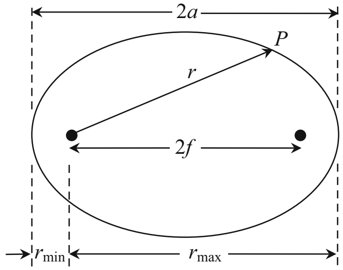


Fig. 2.11 Major axis: $2a$

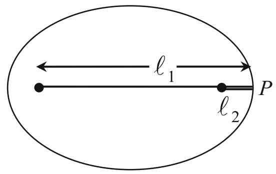


Fig. 2.12 Illustration of $\ell_1 + \ell_2 = 2a$

We can now evaluate the constant in (2.18), above. We do this by noting that (2.18) is true for every point on the ellipse. It is therefore also true if we move point P to the right end of the ellipse (Fig. 2.12) so that ℓ_1 and ℓ_2 lie along the major axis. The foci are symmetrically placed, so ℓ_2 equals the distance from the left focus to the left end of the major axis and ℓ_1 and ℓ_2 add up to $2a$ for this point.

Because $\ell_1 + \ell_2 =$ constant, independently of our choice of point P, we have

$$\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 = 2a \tag{2.19}$$

Some other relationships which follow from Figs. 2.10 and 2.11 are

$$\begin{aligned} r_{\min} &= a - f \\ r_{\max} &= a + f \end{aligned} \tag{2.20}$$

$$r_{\max} + r_{\min} = (a + f) + (a - f) = 2a \tag{2.21}$$

$$r_{\max} - r_{\min} = (a + f) - (a - f) = 2f \tag{2.22}$$

If we place point P at the end of the minor axis (Fig. 2.13), then $\ell_1 = \ell_2$ and it follows from $\ell_1 + \ell_2 = 2a$ that the distance from either focus to the end of the minor axis is equal to the length of the semi-major axis, a.

Then, using Pythagoras' theorem in Fig. 2.13, we have

$$b^2 = a^2 - f^2 = a^2\left(1 - \frac{f^2}{a^2}\right) \tag{2.23}$$

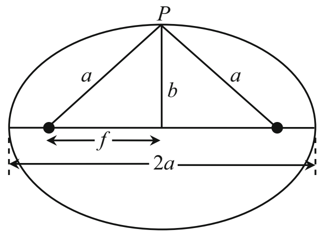


Fig. 2.13 The distance from either focus to one end of the semi-minor axis is equal to the length of the semi-major axis

Re-arranging (2.23) gives

$$f^2 = a^2 - b^2 \tag{2.24}$$

We now define the eccentricity, e, of the ellipse as the ratio of the distance between the foci to the length of the major axis:

$$e = \frac{2f}{2a} = \frac{f}{a} \tag{2.25}$$

Substitution of (2.25) into (2.23) gives

$$b^2 = a^2\left(1 - e^2\right) \tag{2.26}$$

Equations (2.20) and (2.25) then give

$$r_{\max} = a + f = a\left(1 + \frac{f}{a}\right) = a(1 + e) \tag{2.27}$$

$$r_{\min} = a - f = a\left(1 - \frac{f}{a}\right) = a(1 - e) \tag{2.28}$$

$$e = \frac{2f}{2a} = \frac{r_{\max} - r_{mim}}{r_{\max} + r_{mim}} \tag{2.29}$$

If we place the center of the ellipse at the origin of an (x,y) coordinate system as shown in Fig. 2.14, then we can express the equation of the ellipse in terms of x and y as follows. First, from Fig. 2.14 we have

$$\ell_1^2 = (f + x)^2 + y^2 \tag{2.30}$$

$$\ell_2^2 = (f - x)^2 + y^2 \tag{2.31}$$

We can now use (2.19), (2.24), (2.29) and (2.30) to eliminate ℓ_1, ℓ_2 and f and obtain a general equation for an ellipse in terms only of x, y, a and b. First, substitute the square roots of (2.29) and (2.30) into (2.19):

$$\sqrt{(f + x)^2 + y^2} + \sqrt{(f - x)^2 + y^2} = 2a \tag{2.32}$$

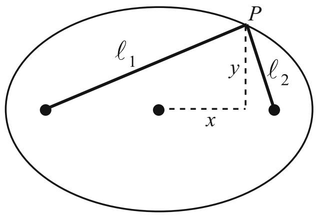


Fig. 2.14 x and y coordinates

then subtract the first term from both sides and square the result:

$$(f-x)^2+y^2=4a^2-4a\sqrt{(f+x)^2+y^2}+(f+x)^2+y^2 \tag{2.33}$$

Take all terms to the LHS except the term with the square root, and expand the squared terms in parentheses to obtain

$$fx+a^2=a\sqrt{f^2+2fx+x^2+y^2} \tag{2.34}$$

Square both sides:

$$f^2x^2+a^4=a^2f^2+a^2x^2+a^2y^2 \tag{2.35}$$

and substitute (2.24) into (2.33):

$$-b^2x^2=-a^2b^2+a^2y^2 \tag{2.36}$$

Finally, divide both sides by a^2b^2 and re-arrange to obtain the desired form of the equation for an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{2.37}$$

We can also write the equation for an ellipse in polar coordinates centered on one focus, as follows. From Fig. 2.15, substitute

$$\begin{aligned} x &= x' - f = r\cos\theta - f \\ y &= r\sin\theta \end{aligned}$$

into (2.37), square the numerators, and rearrange terms to obtain the quadratic equation

$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)-r\left(\frac{2f\cos\theta}{a^2}\right)+\left(\frac{f^2}{a^2}-1\right)=0 \tag{2.38}$$

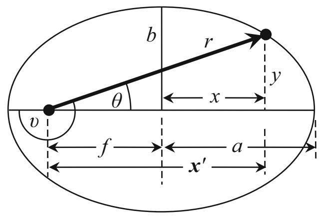


Fig. 2.15 Conversion from (*x*,*y*) to polar coordinates

The solution to (2.38) is

$$r = \frac{\dfrac{2f\cos\theta}{a^2} \pm \sqrt{\left(\dfrac{4f^2\cos^2\theta}{a^4}\right) - 4\left(\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}\right)\left(\dfrac{f^2}{a^2} - 1\right)}}{2\left(\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}\right)}$$

$$= \frac{f\cos\theta \pm a\sqrt{\cos^2\theta + \sin^2\theta\left(\dfrac{a^2 - f^2}{b^2}\right)}}{\cos^2\theta + \dfrac{a^2}{b^2}\sin^2\theta}$$

Now $a^2 - f^2 = b^2$, so the argument of the square root reduces to 1 and r becomes, with the help of (2.29) and (2.26),

$$r = \frac{f\cos\theta \pm a}{\cos^2\theta + \frac{a^2}{b^2}\sin^2\theta} = \frac{f\cos\theta \pm a}{\cos^2\theta + \frac{\sin^2\theta}{1-e^2}} = \frac{a\,(e\cos\theta \pm 1)}{\left(\frac{1-e^2\cos^2\theta}{1-e^2}\right)} \tag{2.39}$$

Regarding the $\pm$ sign, note that $e < 1$ and $\cos\theta \leq 1$. If we choose the negative sign then $(e\cos\theta - 1) < 0$ whereas the denominator is positive. This would make $r < 0$, which is physically impossible. Thus only the + sign is relevant. Equation (2.39) then reduces to,

$$r = \frac{a\,(1 - e^2)}{1 - e\cos\theta} = \frac{r_{\min}\,(1 + e)}{1 - e\cos\theta} \tag{2.40}$$

where $r_{\min} = a(1 - e)$ from (2.28). In celestial mechanics the customary angle used is the *argument of perihelion*, angle υ in Fig. 2.15; then $\cos\upsilon = \cos(\theta + 180°) = -\cos\theta$ and

$$r = \frac{a\,(1 - e^2)}{1 + e\cos\upsilon} = \frac{r_{\min}\,(1 + e)}{1 + e\cos\upsilon} \tag{2.41}$$

Table 2.1 Eccentricities of ellipses and related curves

Curve	Eccentricity
Circle	0
Ellipse	$0 < e < 1$
Parabola	1
Hyperbola	>1

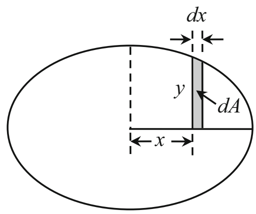


Fig. 2.16 Finding the area of an ellipse

The left equation of (2.40) describes an elliptical orbit. Kepler's first "law" is that planets move in elliptic orbits with the Sun at one focus. We now look at different limiting cases of the eccentricity. First, if we set $f = 0$ then the two foci coincide at the center of the ellipse and $e = f/a = 0$. Then (2.23) and (2.26) both give $b = a$ (i.e., the semi-major and semi-minor axes are equal), and from (2.37) we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \quad \text{or} \quad x^2 + y^2 = a^2,$$

which is the equation for a circle. Thus a circle is an ellipse with an eccentricity of zero. If, on the other hand, we let $f \to \infty$ at constant $r_{\min}$, then from (2.25) and (2.28),

$$e = \frac{f}{a} = \frac{f}{f + r_{\min}} \to 1$$

The resulting curve is a parabola. If $e > 1$ then we have a hyperbola. These results are summarized in Table 2.1.

The area of an ellipse can be found by integration, as intimated in Fig. 2.16:

$$A = 4\int_{x=0}^{a} dA = 4\int_{x=0}^{a} y\,\mathrm{d}x \tag{2.42}$$

where, by (2.37), we have

$$y = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \tag{2.43}$$

This can be integrated with a trigonometric substitution,

$$x = a \ \sin\theta \tag{2.44}$$

to obtain

$$A = \pi ab = \pi a^2\sqrt{1 - e^2} \tag{2.45}$$

Challenges

[2.1] Compute the great circle distance and the initial and final bearings (=azimuths) for a voyage to Singapore ($\lambda = 103^{\circ}\!.85$ E, $\phi = 1^{\circ}\!.28$ N) from Vancouver ($\lambda = 123^{\circ}\!.2$ W, $\phi = 49^{\circ}\!.3^{\circ}$N).

[2.2] Use spherical trigonometry to derive the equations of transformation between the equatorial and ecliptic systems. Figure 2.8 may be helpful.

[2.3] Compute the celestial longitude and latitude of an object at $\alpha = 15^{\mathrm{h}}\ 39^{\mathrm{m}}\ 40^{\mathrm{s}}$, $\delta = -5^{\circ}\ 17.8'$.

[2.4] Derive the equations of transformation between the horizon (A, h) and (H, δ) equatorial system to obtain the Hour Angle, H, and the declination, δ, in terms of the altitude, h, the azimuth, A, and the observer's latitude, ϕ.

[2.5] For a site with latitude $\lambda = 38^{\circ}$ N, what are the equatorial coordinates of:

(a) a star located at the NCP?
(b) a star on the horizon at the South point at 12^{h} local sidereal time?
(c) a star overhead at local midnight on March 21?

[2.6] For a site with latitude $\lambda = 38^{\circ}$S, what are the equatorial coordinates of:

(a) a star located at the SCP?
(b) a star on the horizon at the North point at 12^{h} local sidereal time?
(c) a star overhead at local midnight on March 21?

[2.7] Prove that the altitude of the North Celestial Pole in Fig. 2.7 is the latitude of the observer.

[2.8] Figure 2.17 shows certain aspects of Fig. 2.8. Prove that

(a) the great circle arc EC between the north ecliptic pole and the north celestial pole, labeled θ in Fig. 2.17, is equal to the obliquity of the ecliptic, ε; and
(b) angles x and y are both equal to 90°. [Hint: Both the sine and cosine laws may be helpful.]

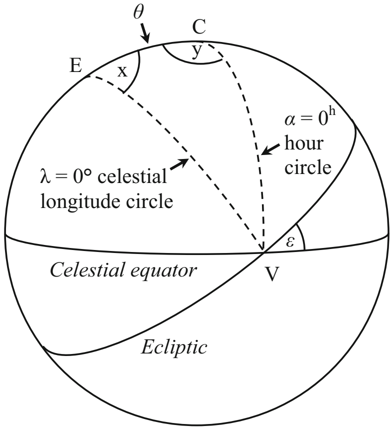


Fig. 2.17 Diagram for Challenge [2.8]. E = North Ecliptic Pole; C = North Celestial Pole; V = Vernal Equinox; ε = obliquity of the ecliptic; θ = arc EC

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