

Contents

1	Preliminaries: Sets, Relations, and Functions	1
1.1	Introduction	1
1.2	Membership, Subsets, and Naive Axioms	2
1.3	The Power Set and Set Operations	6
1.4	Ordered Pairs and Relations	8
1.5	Functions	10
1.6	Families and Partitions	13
1.7	Finite and Infinite Sequences and Strings	16
1.8	Partitions and Equivalence Relations	19
1.9	Orders (Linear Orders)	21
 Part I Dedekind: Numbers		
2	The Dedekind–Peano Axioms	29
2.1	Introduction	29
2.2	The Dedekind–Peano Axioms	30
2.3	Addition, Order, and Multiplication	31
2.4	Fractions and Ratios	34
2.5	Order, Addition, and Multiplication of Fractions and Ratios	35
2.6	Properties of Addition and Multiplication of Ratios	37
2.7	Integral Ratios and the Embedding of the Natural Numbers	37
2.8	The Archimedean and Fineness Properties	39
2.9	Irrationality of $\sqrt{2}$ and Density of Square Ratios	40
2.10	Recursive Definitions*	42
3	Dedekind’s Theory of the Continuum	47
3.1	Introduction	47
3.2	Linear Continuum in Geometry	47
3.3	Problems with the Ratios	48
3.4	Irrationals: Dedekind’s Definition of the Continuum	51
3.5	Lengths (Magnitudes)	54

3.6	The Ordered Field \mathbf{R} of Real Numbers	58
3.7	Additional Facts on Ordered Fields*	62
3.8	Alternative Development Routes*	63
3.9	Complex Numbers*	64
4	Postscript I: What Exactly Are the Natural Numbers?	67
4.1	Russell's Absolutism?	67
4.2	Interpretations for the Natural Numbers	69
4.3	Dedekind's Structuralism	70
 Part II Cantor: Cardinals, Order, and Ordinals		
5	Cardinals: Finite, Countable, and Uncountable	77
5.1	Cardinal Numbers	77
5.2	Sum and Product of Cardinal Numbers	81
5.3	Finite Sets and Dedekind Infinite Sets	82
5.4	Natural Numbers and Reflexive Cardinals	87
5.5	The Axiom of Choice vs Effectiveness	90
5.6	\aleph_0 and Countable Sets	94
5.7	The Countable and Dependent Axioms of Choice	99
5.8	$\aleph_0 < \mathfrak{c}$: The Cardinality of the Continuum	101
5.9	CH: The Continuum Hypothesis	105
5.10	More Countable Sets and Enumerations	106
6	Cardinal Arithmetic and the Cantor Set	109
6.1	The Cantor–Bernstein Theorem	109
6.2	Arbitrary Sums and Products of Cardinals	111
6.3	Cardinal Exponentiation: $ \mathbf{P}(A) = 2^{ A }$	114
6.4	Cardinal Arithmetic	115
6.5	The Binary Tree	117
6.6	The Cantor Set \mathbf{K}	119
6.7	The Identity $2^{\aleph_0} = \mathfrak{c}$	123
6.8	Cantor's Theorem: The Diagonal Method	125
6.9	The Cardinal $\mathfrak{f} = 2^{\mathfrak{c}}$ and Beyond	127
6.10	Additional Problems	128
7	Orders and Order Types	131
7.1	Orders, Terminology, and Notation	131
7.2	Some Basic Definitions: Suborders	133
7.3	Isomorphisms, Similarity, and Rearrangements	135
7.4	Order Types and Operations	138
8	Dense and Complete Orders	149
8.1	Limit Points, Derivatives, and Density	149
8.2	Continuums, Completeness, Sup, and Inf	154
8.3	Embeddings and Continuity	156
8.4	Cantor's Theorem on Countable Dense Orders	160

8.5	$\aleph_0 < \mathfrak{c}$: Another Proof of Uncountability of \mathbf{R}	162
8.6	The Order Type of \mathbf{R}	163
8.7	Dedekind Completion	166
8.8	Properties of Complete Orders and Perfect Sets	168
8.9	Connectedness and the Intermediate Value Theorem	173
9	Well-Orders and Ordinals	175
9.1	Well-Orders, Ordinals, Sum, and Product	175
9.2	Limit Points and Transfinite Induction	179
9.3	Well-Orders and Ordinals: Basic Facts	182
9.4	Unique Representation by Initial Sets of Ordinals	184
9.5	Successor, Supremum, and Limit	187
9.6	Operations Defined by Transfinite Recursion	189
9.7	Remainder Ordinals and Ordinal Exponentiation	191
9.8	The Canonical Order on Pairs of Ordinals	195
9.9	The Cantor Normal Form	197
10	Alephs, Cofinality, and the Axiom of Choice	199
10.1	Countable Ordinals, ω_1 , and \aleph_1	199
10.2	The Cardinal \aleph_1	201
10.3	Hartogs' Theorem, Initial Ordinals, and Alephs	203
10.4	Abstract Derivatives and Ranks	206
10.5	AC, Well-Ordering Theorem, Cardinal Comparability	208
10.6	Cofinality: Regular and Inaccessible Cardinals	210
10.7	The Continuum Hypothesis	216
11	Posets, Zorn's Lemma, Ranks, and Trees	221
11.1	Partial Orders	221
11.2	Zorn's Lemma	223
11.3	Some Applications and Examples	225
11.4	Well-Founded Relations and Rank Functions	229
11.5	Trees	234
11.6	König's Lemma and Well-Founded Trees	237
11.7	Ramsey's Theorem	241
12	Postscript II: Infinitary Combinatorics	245
12.1	Weakly Compact Cardinals	245
12.2	Suslin's Problem, Martin's Axiom, and \diamond	247
Part III Real Point Sets		
13	Interval Trees and Generalized Cantor Sets	255
13.1	Intervals, Sup, and Inf	255
13.2	Interval Subdivision Trees	257
13.3	Infinite Branches Through Trees	259
13.4	Cantor Systems and Generalized Cantor Sets	263

14	Real Sets and Functions	265
14.1	Open Sets	265
14.2	Limit Points, Isolated Points, and Derived Sets	266
14.3	Closed, Dense-in-Itself, and Perfect Sets	268
14.4	Dense, Discrete, and Nowhere Dense Sets	270
14.5	Continuous Functions and Homeomorphisms	275
15	The Heine–Borel and Baire Category Theorems	281
15.1	The Heine–Borel Theorem	281
15.2	Sets of Lebesgue Measure Zero	285
15.3	Lebesgue Measurable Sets	287
15.4	F_σ and G_δ Sets	290
15.5	The Baire Category Theorem	291
15.6	The Continuum Hypothesis for G_δ Sets	293
15.7	The Banach–Mazur Game and Baire Property	295
15.8	Vitali and Bernstein Sets	297
16	Cantor–Bendixson Analysis of Countable Closed Sets	301
16.1	Homeomorphisms of Orders and Sets	301
16.2	The Cantor–Bendixson Theorem and Perfect Sets	303
16.3	Ordinal Analysis of Countable Closed Bounded Sets	305
16.4	Cantor and Uniqueness of Trigonometric Series	310
17	Brouwer’s Theorem and Sierpinski’s Theorem	313
17.1	Brouwer’s Theorem	313
17.2	Homeomorphic Permutations of the Cantor Set	315
17.3	Sierpinski’s Theorem	318
17.4	Brouwer’s and Sierpinski’s Theorems in General Spaces	319
18	Borel and Analytic Sets	321
18.1	Sigma-Algebras and Borel Sets	321
18.2	Analytic Sets	324
18.3	The Lusin Separation Theorem	331
18.4	Measurability and Baire Property of Analytic Sets	333
18.5	The Perfect Set Property for Analytic Sets	335
18.6	A Non-Borel Analytic Set	338
19	Postscript III: Measurability and Projective Sets	345
19.1	The Measure Problem and Measurable Cardinals	345
19.2	Projective Sets and Lusin’s Problem	352
19.3	Measurable Cardinals and PCA (Σ^1_2) Sets	354

Part IV Paradoxes and Axioms

20	Paradoxes and Resolutions	361
20.1	Some Set Theoretic Paradoxes	361
20.2	Russell’s Theory of Types	364
20.3	Zermelo’s Axiomatization	366

21	Zermelo–Fraenkel System and von Neumann Ordinals	369
21.1	The Formal Language of ZF	369
21.2	The First Six ZF Axioms	370
21.3	The Replacement Axiom	376
21.4	The von Neumann Ordinals	377
21.5	Finite Ordinals and the Axiom of Infinity	382
21.6	Cardinal Numbers and the Transfinite	385
21.7	Regular Sets and Ranks	390
21.8	Foundation and the Set Theoretic Universe V	393
21.9	Other Formalizations of Set Theory	395
21.10	Further Reading	398
22	Postscript IV: Landmarks of Modern Set Theory	399
22.1	Gödel’s Axiom of Constructibility	399
22.2	Cohen’s Method of Forcing	402
22.3	Gödel’s Program and New Axioms	404
22.4	Large Cardinal Axioms	405
22.5	Infinite Games and Determinacy	407
22.6	Projective Determinacy	409
22.7	Does the Continuum Hypothesis Have a Truth Value?	411
22.8	Further References	412

Appendices

A	Proofs of Uncountability of the Reals	413
A.1	Order-Theoretic Proofs	413
A.2	Proof Using Cantor’s Diagonal Method	415
A.3	Proof Using Borel’s Theorem on Interval Lengths	416
B	Existence of Lebesgue Measure	419
C	List of ZF Axioms	421
	References	423
	List of Symbols and Notations	427
	Index	431

Set Theory

With an Introduction to Real Point Sets

Dasgupta, A.

2014, XV, 444 p. 17 illus., Hardcover

ISBN: 978-1-4614-8853-8

A product of Birkhäuser Basel