Chapter 2
Infinite-Horizon Theorems

2.1 Introduction

In this chapter, we give infinite-horizon theorems using the tools of the previous chapter. In Sect. 2.2 we present several weak maximum principles which are obtained through the method of reduction to finite horizon. We successively use two additional conditions to obtain results in the infinite-horizon setting from results of the finite-horizon setting. In Sect. 2.3 we present several strong maximum principles which are obtained through the method of reduction to finite horizon. We successively use three additional conditions which permit the extension of finite-horizon results into infinite-horizon results. In Sect. 2.4 we study constrained problems and in Sect. 2.5 multiobjective problems.

2.2 Weak Pontryagin Principles in Infinite Horizon

The first establishment of a Pontryagin principle in infinite horizon in the framework of the continuous time is due to Halkin [34]. The major difficulty to adapt the proof of Halkin to the discrete-time framework is the following one: whereas integrating an ordinary differential equation forward or integrating it backward is the same thing, it is not the same for a difference equation. And the so-called *adjoint equations*, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$, are backward difference equations. To overcome this difficulty, we propose several solutions. Each of the following subsections provides a solution to this difficulty.

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2.2.1 A Condition of Invertibility

This condition concerns the partial differential with respect to the state variable of the vector field of the equation of motion. It is the following: the invertibility of $D_1 f_t(\hat{x}_t, \hat{u}_t)$. This invertibility allows to transform the adjoint equation into a forward difference equation. In this context, this condition appears for the first time in the paper of Blot and Chebbi [16].

We need two lemmas to prove the theorems of this section. First, the following lemma is a way to express the Diagonal Process of Cantor.

Lemma 2.1. *Let Z be a real finite-dimensional normed vector space. For all $(t, T) \in \mathbb{N}_* \times \mathbb{N}_*$ such that $t \leq T$ we consider $z_t^T \in Z$. We assume that, for all $t \in \mathbb{N}_*$, the sequence $T \mapsto z_t^T$ is bounded. Then there exists an increasing function $\sigma : \mathbb{N}_* \to \mathbb{N}_*$ such that, for all $t \in \mathbb{N}_* \to \mathbb{N}_*$ such that, for all $t \in \mathbb{N}_*$, there exists $z_t \in \mathbb{N}_*$.*

A precise proof of this lemma is given in Theorem A.1 in Appendix A. The second lemma expresses a relation between a property related to the norm and an algebraic property.

Lemma 2.2. *Let Z be a real finite-dimensional normed vector space. Let $z_1, \ldots,$ z_k be linearly independent vectors in Z, and for all $j \in \{1, \ldots, k\}$, let $(r_t^j)_{t\in\mathbb{N}}$ be a real sequence. If the vector sequence $\left(\sum_{j=1}^{k} r_t^j v_j\right)_{t\in\mathbb{N}}$ is bounded in Z, then the real sequence $(r_t^j)_{t\in\mathbb{N}}$ is bounded in $\mathbb{R}$ for all $j \in \{1, \ldots, k\}$.*

A precise proof of this lemma is given in [13] p. 48.

First we use the multiplier rule of Halkin to obtain the following result.

Theorem 2.1. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}_a^n)$, or of $(\mathscr{P}_a^s)$, or of $(\mathscr{P}_a^o)$, or of $(\mathscr{P}_a^w)$ for $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, X_t is a nonempty open subset of $\mathbb{R}^n$.
(b) For all $t \in \mathbb{N}$,

$$U_t = \left(\bigcap_{\alpha=1}^{k^i} \{u \in \mathbb{R}^d : g_t^\alpha(u) \geq 0\} \right) \cap \left(\bigcap_{\beta=1}^{k^e} \{u \in \mathbb{R}^d : h_t^\beta(u) = 0\} \right)$$

where $g_t^\alpha : \mathbb{R}^d \to \mathbb{R}$ and $h_t^\beta : \mathbb{R}^d \to \mathbb{R}$ are continuous on a neighborhood of $\hat{u}_t$ and they are differentiable at $\hat{u}_t$, for all $\alpha \in \{1, \ldots, k^i\}$ and for all $\beta \in \{1, \ldots, k^e\}$, and we assume that $U_t \neq \emptyset$.
(c) For all $t \in \mathbb{N}$, the differentials $Dg_t^1(\hat{u}_t), \ldots, Dg_t^{k^i}(\hat{u}_t)$, $Dh_t^1(\hat{u}_t), \ldots, Dh_t^{k^e}(\hat{u}_t)$ are linearly independent.
(d) For all $t \in \mathbb{N}$, ϕ_t and f_t are continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and they are differentiable at $(\hat{x}_t, \hat{u}_t)$.
(e) For all $t \in \mathbb{N}$, the partial differential $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, $(\lambda_{1,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \dots, (\lambda_{k^i,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ $(\mu_{1,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \dots, (\mu_{k^e,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ *which satisfy the following conditions:*

(i) $(\lambda_0, p_1) \neq (0,0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$ *when* $a = i$.
(iv) For all $t \in \mathbb{N}$, *for all* $\alpha \in \{1, \dots, k^i\}$, $\lambda_{\alpha,t} \geq 0$.
(v) For all $t \in \mathbb{N}$, *for all* $\alpha \in \{1, \dots, k^i\}$, $\lambda_{\alpha,t} g_t^\alpha(\hat{u}_t) = 0$.
(vi) For all $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_1\phi_t(\hat{x}_t, \hat{u}_t)$.
(vii) For all $t \in \mathbb{N}$,

$$D_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) + \sum_{\alpha=1}^{k^i} \lambda_{\alpha,t} D g_t^\alpha(\hat{u}_t) + \sum_{\beta=1}^{k^e} \mu_{\beta,t} D h_t^\beta(\hat{u}_t) = 0.$$

Proof. Using Proposition 1.2 we know that for all $T \in \mathbb{N}$, $T \geq 2$, the restriction $(\hat{x}_0, \dots, \hat{x}_{T-1}, \hat{u}_0, \dots, \hat{u}_{T-1})$ is a solution of $(\mathscr{F}^a(T, \eta, \hat{x}_T))$ for $a \in \{e, i\}$. And consequently the conclusions of Proposition 1.3 hold when $a = e$, and the conclusions of Proposition 1.4 hold when $a = i$. Using the assumption (e), the conclusion (iv) of Proposition 1.3 or of Proposition 1.4 which is

$$p_t^T = p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0^T D_1\phi_t(\hat{x}_t, \hat{u}_t),$$

for all $t \in \{0, \dots, T-1\}$, becomes

$$p_{t+1}^T = p_t^T \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1} - \lambda_0^T D_1\phi_t(\hat{x}_t, \hat{u}_t) \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1}. \tag{2.1}$$

Using (2.1), we see that $(\lambda_0^T, p_1^T) = (0,0)$ implies $(\lambda_0^T, p_1^T, \dots, p_T^T) = (0, 0, \dots, 0)$. And so by contraposition we obtain the following relation.

$$(\lambda_0^T, p_1^T, \dots, p_T^T) \neq (0, 0, \dots, 0) \Longrightarrow (\lambda_0^T, p_1^T) \neq (0,0). \tag{2.2}$$

Using assumption (c) and the last assertion of Proposition 1.3 or of Proposition 1.4 we obtain $(\lambda_0^T, p_1^T) \neq (0,0)$. Since the set of all lists of multipliers is a cone, we can normalize these lists of multipliers, and so we can choose:

$$\forall T \in \mathbb{N}_*, \quad \|(\lambda_0^T, p_1^T)\| = 1. \tag{2.3}$$

From (2.1) the sequence $T \mapsto p_1^T$ is bounded in $\mathbb{R}^{n*}$, and the sequence $T \mapsto \lambda_0^T$ is bounded in $\mathbb{R}$. From (2.1) we obtain

$$\|p_2^T\| \leq \|p_1^T\|.\|(D_1 f_1(\hat{x}_1, \hat{u}_1))^{-1}\| + |\lambda_0^T|.\|D_1\phi_1(\hat{x}_1, \hat{u}_1)\|.\|(D_1 f_1(\hat{x}_1, \hat{u}_1))^{-1}\|$$

which implies

$$\sup_{T\geq 2} \|p_2^T\| \leq \|(D_1 f_1(\hat{x}_1, \hat{u}_1))^{-1}\| + \|D_1\phi_1(\hat{x}_1, \hat{u}_1)\|.\|(D_1 f_1(\hat{x}_1, \hat{u}_1))^{-1}\| < +\infty,$$

and proceeding by induction, we obtain

$$\sup_{T\geq t+1} \|p^T_{t+1}\| \leq \sup_{T\geq t} \|p^T_t\|.\|(D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1}\| + \|D_1\phi_t(\hat{x}_t, \hat{u}_t)\|.\|(D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1}\|$$

which implies

$$\forall t \in \mathbb{N}, \quad \sup_{T\geq t} \|p^T_t\| < +\infty. \tag{2.4}$$

From (2.2) we deduce that the sequence $T \mapsto D_2 H_t(\hat{x}_t, \hat{u}_t, p^T_{t+1}, \lambda^T_0)$ is bounded and using the conclusion (v) of Proposition 1.3 or of Proposition 1.4 we obtain that the sequence $T \mapsto \sum_{\alpha=1}^{k^i} \lambda^T_{\alpha,t} Dg^\alpha_t(\hat{u}_t) + \sum_{\beta=1}^{k^e} \mu^T_{\beta,t} Dh^\beta_t(\hat{u}_t)$ is bounded, and by using the assumption (c) and Lemma 2.2 we obtain the following relations:

$$\forall t \in \mathbb{N}, \quad \sup_{T\geq t} |\lambda^T_{\alpha,t}| < +\infty. \tag{2.5}$$

$$\forall t \in \mathbb{N}, \quad \sup_{T\geq t} |\mu^T_{\beta,t}| < +\infty. \tag{2.6}$$

After (2.1)–(2.4), using Lemma 2.1 we know that there exist an increasing function $\sigma : \mathbb{N}_* \to \mathbb{N}_*$, $\lambda_0 \in \mathbb{R}$, $p_t \in \mathbb{R}^{n*}$, $\lambda_{\alpha,t} \in \mathbb{R}$, $\mu_{\beta,t} \in \mathbb{R}$ for all $t \in \mathbb{N}$, for all $\alpha \in \{1, \dots, k^i\}$, for all $\beta \in \{1, \dots, k^e\}$ such that, for all $t \in \mathbb{N}$, for all $\alpha \in \{1, \dots, k^i\}$, for all $\beta \in \{1, \dots, k^e\}$, the following equalities hold:

$$\left.\begin{array}{rl} \lim_{T\to+\infty} \lambda^{\sigma(T)}_0 &= \lambda_0 \\ \lim_{T\to+\infty} p^{\sigma(T)}_{t+1} &= p_{t+1} \\ \lim_{T\to+\infty} \lambda^{\sigma(T)}_{\alpha,t} &= \lambda_{\alpha,t} \\ \lim_{T\to+\infty} \mu^{\sigma(T)}_{\beta,t} &= \mu_{\beta,t}. \end{array}\right\} \tag{2.7}$$

From (2.3), (2.7), and the continuity of the norm we obtain $\|(\lambda_0, p_1)\| = 1$, which implies the conclusion (i). From (2.7) and the conclusions of Proposition 1.3 or of Proposition 1.4, we obtain the conclusions (ii), (iii), (iv), and (v) by taking $T \to +\infty$. □

When we have $\hat{u}_t \in \operatorname{int} U_t$ for all $t \in \mathbb{N}$, using Corollary 1.1 instead of Proposition 1.3 and Corollary 1.2 instead of Proposition 1.4, a similar reasoning allows to establish the following result.

Theorem 2.2. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathcal{P}^n_a)$, or of $(\mathcal{P}^s_a)$, or of $(\mathcal{P}^o_a)$, or of $(\mathcal{P}^w_a)$ for $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, X_t is a nonempty open subset of $\mathbb{R}^n$.
(b) For all $t \in \mathbb{N}$, $\hat{u}_t \in \operatorname{int} U_t$.

(c) For all $t \in \mathbb{N}$, ϕ_t and f_t are continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and they are differentiable at $(\hat{x}_t, \hat{u}_t)$.
(d) For all $t \in \mathbb{N}$, the partial differential $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t \in \mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$ when $a = i$.
(iv) For all $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_1 \phi_t(\hat{x}_t, \hat{u}_t)$.
(v) For all $t \in \mathbb{N}$, $\quad D_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) = 0$.

After the use of the multiplier rule of Halkin, we use the multiplier rule of Clarke.

Theorem 2.3. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}_a^n)$, or of $(\mathscr{P}_a^s)$, or of $(\mathscr{P}_a^o)$, or of $(\mathscr{P}_a^w)$ for $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, ϕ_t is Lipschitzian on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and regular at $(\hat{x}_t, \hat{u}_t)$.
(b) For all $t \in \mathbb{N}$, f_t is strictly differentiable at $(\hat{x}_t, \hat{u}_t)$.
(c) For all $t \in \mathbb{N}$, U_t is closed and Clarke-regular at $\hat{u}_t$.
(d) For all $t \in \mathbb{N}$, the partial differential $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t \in \mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$ when $a = i$.
(iv) For all $t \in \mathbb{N}_*$, $p_t \in \partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) For all $t \in \mathbb{N}$, $\partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) \cap N_{U_t}(\hat{u}_t) \neq \emptyset$, where $N_{U_t}(\hat{u}_t)$ is the normal cone of U_t at $\hat{u}_t$.

Proof. Using Proposition 1.2, the restriction $(\hat{x}_0, \ldots, \hat{x}_{T-1}, \hat{u}_0, \ldots, \hat{u}_{T-1})$ is a solution of $(\mathscr{F}^a(T, \eta, \hat{x}_T))$, for $a \in \{e, i\}$. Consequently using Proposition 1.5 when $a = e$ and Proposition 1.6 when $a = i$, we know that, for all $T \in N$, $T \geq 2$, there exist $\lambda_0^T \in \mathbb{R}$, $p_{t+1}^T \in \mathbb{R}^{n*}$, when $t \in \{0, \ldots, T-1\}$, which satisfy the following conditions:

$$\left(\lambda_0^T, p_1^T, \ldots, p_T^T\right) \neq (0, 0, \ldots, 0) \tag{2.8}$$

$$\lambda_0^T \geq 0 \tag{2.9}$$

$$\forall t \in \{0, \ldots, T-1\}, \exists \varphi_t^T \in \partial_1 \phi_t(\hat{x}_t, \hat{u}_t) \text{ s.t. } p_t^T = \lambda_0^T \varphi_t^T + p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t) \tag{2.10}$$

$$\left.\begin{array}{r} \forall t \in \{0, \ldots, T-1\}, \exists \psi_t^T \in \partial_2 \phi_t(\hat{x}_t, \hat{u}_t) \text{ s.t.} \\ \forall v_t \in T_{U_t}(\hat{u}_t), \langle \lambda_0^T \psi_t^T + p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t), v_t \rangle \leq 0 \end{array}\right\} \tag{2.11}$$

Using assumption (d) we can transform (2.10) into the following relation:

$$p^T_{t+1} = p^T_t \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1} - \lambda^T_0 \varphi^T_t \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1}. \tag{2.12}$$

Reasoning as in the proof of Theorem 2.2, from (2.8) and (2.12) we deduce that $(\lambda^T_0, p^T_1) \neq (0, 0)$ and we can choose (λ^T_0, p^T_1) such that $\|(\lambda^T_0, p^T_1)\| = 1$. And so the sequences $T \mapsto \lambda^T_0$ and $T \mapsto p^T_1$ are bounded, and from (2.12) we deduce by induction that, for all $t \in \mathbb{N}$, all the sequences $T \mapsto p^T_{t+1}$ are bounded. Since the Clarke differentials at a point are compact [38], we obtain that, for all $t \in \mathbb{N}$, the sequences $T \mapsto \varphi^T_t$ and $T \mapsto \psi^T_t$ are bounded. Using Lemma 2.1, there exist an increasing function $\sigma : \mathbb{N}_* \to \mathbb{N}_*$, $\lambda_0 \in \mathbb{R}$, $p_{t+1} \in \mathbb{R}^{n*}$, $\varphi_t \in \partial_1 \phi_t(\hat{x}_t, \hat{u}_t)$ and $\psi_t \in \partial_2 \phi_t(\hat{x}_t, \hat{u}_t)$ such that the following equalities hold for all $t \in \mathbb{N}$:

$$\left.\begin{array}{l} \lim_{T\to+\infty} \lambda_0^{\sigma(T)} = \lambda_0 \\ \lim_{T\to+\infty} p_{t+1}^{\sigma(T)} = p_{t+1} \\ \lim_{T\to+\infty} \varphi_t^{\sigma(T)} = \varphi_t \\ \lim_{T\to+\infty} \psi_t^{\sigma(T)} = \psi_t. \end{array}\right\} \tag{2.13}$$

Taking $T \to +\infty$, from $\|(\lambda^{\sigma(T)}_0, p^{\sigma(T)}_1)\| = 1$ and from (2.13), we obtain $\|(\lambda_0, p_1)\| = 1$ that ensures the conclusion (i).

From (2.9) and (2.13) we obtain the conclusion (ii). From (2.10) and (2.13) we obtain the conclusion (iii). From (1.13) and (2.13) we obtain

$$p_{t+1} = p_t \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1} - \lambda_0 \varphi_t \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1}$$

which implies the conclusion (iv).

From (2.11) and 2.13) we obtain, for all $v_t \in T_{U_t}(\hat{u}_t)$,

$$\langle \lambda_0 \psi_t + p_{t+1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t), v_t \rangle \leq 0$$

which implies $\lambda_0 \psi_t + p_{t+1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t) \in N_{U_t}(\hat{u}_t)$ with $\psi_t \in \partial_2 \phi_t(\hat{x}_t, \hat{u}_t)$, which implies the conclusion (v). □

When $\hat{u}_t \in \operatorname{int} U_t$, using Corollary 1.3 instead of Proposition 1.5 and Corollary 1.4 instead of Proposition 1.6 and proceeding as in the proof of Theorem 2.3 we obtain the following result.

Theorem 2.4. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathcal{P}^n_a)$, or of $(\mathcal{P}^s_a)$, or of $(\mathcal{P}^o_a)$, or of $(\mathcal{P}^w_a)$ for $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, ϕ_t is Lipschitzian on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and regular at $(\hat{x}_t, \hat{u}_t)$.
(b) For all $t \in \mathbb{N}$, f_t is strictly differentiable at $(\hat{x}_t, \hat{u}_t)$.
(c) For all $t \in \mathbb{N}$, $\hat{u}_t \in \operatorname{int} U_t$.

(d) For all $t \in \mathbb{N}$, the partial differential $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t \in \mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$ when $a = i$.
(iv) For all $t \in \mathbb{N}_*$, $p_t \in \partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) For all $t \in \mathbb{N}$, $0 \in \partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.

2.2.2 A Condition of Positivity

All the weak Pontryagin principles of the previous subsection use the condition of invertibility of $D_1 f_t(\hat{x}_t, \hat{u}_t)$. In this subsection, to avoid this condition of invertibility, we introduce the following positivity condition:

$$\left.\begin{array}{ll} \forall i, j \in \{1, \ldots, n\}, & \dfrac{\partial f_t^j(\hat{x}_t, \hat{u}_t)}{\partial x^i} \geq 0 \\ \forall j \in \{1, \ldots, n\}, & \dfrac{\partial f_t^j(\hat{x}_t, \hat{u}_t)}{\partial x^j} > 0. \end{array}\right\} \tag{2.14}$$

Note that this condition does not imply the invertibility of $D_1 f_t(\hat{x}_t, \hat{u}_t)$ when $n > 1$. To see that it suffices to consider the case where $\frac{\partial f_t^j(\hat{x}_t, \hat{u}_t)}{\partial x^i} = 1$ for all i, j, the condition (2.14) is fulfilled, and $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is not invertible since its rank is equal to 1. In this context, this condition was introduced for the first time in the paper of Blot [11].

The following elementary lemma will be very useful.

Lemma 2.3. *Under (2.14), setting $\varrho_t := \min_{1 \leq j \leq n} \frac{\partial f_t^j(\hat{x}_t, \hat{u}_t)}{\partial x^j} > 0$, the following assertions hold:*

(i) For all $y \in \mathbb{R}^n_+$, $D_1 f_t(\hat{x}_t, \hat{u}_t).y \geq \varrho_t y$.
(ii) For all $\pi \in \mathbb{R}^{n*}_+$, $\pi \circ D_1 f_t(\hat{x}_t, \hat{u}_t) \geq \varrho_t \pi$.

Proof. $(e_i)_{1 \leq i \leq n}$ denotes the canonical basis of R^n, and $(e_i^*)_{1 \leq i \leq n}$ its dual basis.

(i) When $y \in \mathbb{R}^n_+$, and $j \in \{1, \ldots, n\}$ we have

$$\langle e_j^*, D_1 f_t(\hat{x}_t, \hat{u}_t).y \rangle = \sum_{i=1}^n \frac{\partial f_t^j(\hat{x}_t, \hat{u}_t)}{\partial x^i} y^i \geq \frac{\partial f_t^j(\hat{x}_t, \hat{u}_t)}{\partial x^j} y^j + 0 \geq \varrho_t y^j,$$

that means $D_1 f_t(\hat{x}_t, \hat{u}_t).y \geq \varrho_t y$ for the natural order of $\mathbb{R}^n$.

(ii) Let $\pi \in \mathbb{R}^{n*}_{+}$. For all $y \in \mathbb{R}^n_+$, after (i), we have $D_1 f_t(\hat{x}_t, \hat{u}_t).y \geq \varrho_t y$, and since $\pi \geq 0$, we have

$$\pi \circ D_1 f_t(\hat{x}_t, \hat{u}_t).y = \pi(D_1 f_t(\hat{x}_t, \hat{u}_t).y) \geq \pi(\varrho_t y) = \varrho_t \pi(y),$$

that means $\pi \circ D_1 f_t(\hat{x}_t, \hat{u}_t) \geq \varrho_t \pi$ for the order of $\mathbb{R}^{n*}$. □

The following remark contains elementary facts on the orders of $\mathbb{R}^n$ and of $\mathbb{R}^{n*}$ which will be very useful.

Remark 2.1. When $\|.\|$ is one of the usual norms of $\mathbb{R}^n$, $\|x\|_\infty := \max_{1 \leq i \leq n} |x^i|$, $\|x\|_1 := \sum_{i=1}^{n} |x^i|$, $\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x^i|^2}$, it satisfies the following property:

$$\forall x, y \in \mathbb{R}^n, 0 \leq x \leq y \Longrightarrow \|x\| \leq \|y\|.$$

It is easy to verify this property by using elementary calculations.

When $\mathbb{R}^n$ is endowed with the norm $\|.\|_\infty$, and $\mathbb{R}^{n*}$ is endowed with the norm $\|\varphi\|_* := \sup\{|\langle \varphi, x \rangle| : x \in \mathbb{R}^n, \|x\|_\infty \leq 1\}$, it is easy to see that $\|\varphi\|_* = \sum_{i=1}^{n} |\langle \varphi, e_i \rangle|$. The following property holds:

$$\forall \varphi, \psi \in \mathbb{R}^{n*}, 0 \leq \varphi \leq \psi \Longrightarrow \|\varphi\|_* \leq \|\psi\|_*.$$

To verify that, noting that $e_i \geq 0$ for all $i \in \{1, \ldots, n\}$, and then we have $0 \leq \varphi \leq \psi \Longrightarrow 0 \leq \varphi(e_i) \leq \psi(e_i)$, for all $i \in \{1, \ldots, n\}$, which implies

$$\|\varphi\|_* = \sum_{i=1}^{n} |\langle \varphi, e_i \rangle| = \sum_{i=1}^{n} \langle \varphi, e_i \rangle \leq \sum_{i=1}^{n} \langle \psi, e_i \rangle = \sum_{i=1}^{n} |\langle \psi, e_i \rangle| = \|\psi\|_*.$$

Now we can establish a weak Pontryagin principle.

Theorem 2.5. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}_i^n)$, or of $(\mathscr{P}_i^s)$, or of $(\mathscr{P}_i^o)$, or of $(\mathscr{P}_i^w)$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, X_t is a nonempty open subset of $\mathbb{R}^n$.
(b) For all $t \in \mathbb{N}$,

$$U_t = \left(\bigcap_{\alpha=1}^{k^i} \{u \in \mathbb{R}^d : g_t^\alpha(u) \geq 0\} \right) \cap \left(\bigcap_{\beta=1}^{k^e} \{u \in \mathbb{R}^d : h_t^\beta(u) = 0\} \right)$$

where $g_t^\alpha : \mathbb{R}^d \to \mathbb{R}$ and $h_t^\beta : \mathbb{R}^d \to \mathbb{R}$ are continuous on a neighborhood of $\hat{u}_t$ and they are differentiable at $\hat{u}_t$, for all $\alpha \in \{1, \ldots, k^i\}$ and for all $\beta \in \{1, \ldots, k^e\}$, and we assume that $U_t \neq \emptyset$.

(c) *For all* $t \in \mathbb{N}$, *the differentials* $Dg_t^1(\hat{u}_t), \ldots, Dg_t^{k^i}(\hat{u}_t)$, $Dh_t^1(\hat{u}_t), \ldots, Dh_t^{k^e}(\hat{u}_t)$ *are linearly independent.*

(d) *For all* $t \in \mathbb{N}$, ϕ_t *and* f_t *are continuous on a neighborhood of* $(\hat{x}_t, \hat{u}_t)$, *and they are differentiable at* $(\hat{x}_t, \hat{u}_t)$.

(e) *For all* $t \in \mathbb{N}$, *the positivity condition (2.14) is fulfilled.*

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, $(\lambda_{1,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \ldots, (\lambda_{k^i,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ $(\mu_{1,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \ldots, (\mu_{k^e,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ *which satisfy the following conditions:*

(i) $(\lambda_0, p_1) \neq (0, 0)$.

(ii) $\lambda_0 \geq 0$.

(iii) *For all* $t \in \mathbb{N}$, $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.

(iv) *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1, \ldots, k^i\}$, $\lambda_{\alpha,t} \geq 0$.

(v) *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1, \ldots, k^i\}$, $\lambda_{\alpha,t} g_t^\alpha(\hat{u}_t) = 0$.

(vi) *For all* $t \in \mathbb{N}$, $p_t = p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_1\phi_t(\hat{x}_t, \hat{u}_t)$.

(vii) *For all* $t \in \mathbb{N}$,

$$D_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) + \sum_{\alpha=1}^{k^i} \lambda_{\alpha,t} Dg_t^\alpha(\hat{u}_t) + \sum_{\beta=1}^{k^e} \mu_{\beta,t} Dh_t^\beta(\hat{u}_t) = 0.$$

Proof. Using Proposition 1.2 and Proposition 1.4, we obtain, for all $T \in \mathbb{N}$, $T \geq 2$, the existence of $\lambda_0^T \in \mathbb{R}$ and, for all $t \in \{0, \ldots, T-1\}$, the existence of a list of elements of $p_{t+1}^T \in \mathbb{R}^{n*}$ and the existence of two lists of real numbers $(\lambda_{\alpha,t}^T)_{1\leq\alpha\leq k^i}$, $(\mu_{\beta,t}^T)_{1\leq\beta\leq k^e}$ which satisfy the following conditions:

$$(\lambda_0^T, p_1^T, \ldots, p_T^T, \lambda_{1,t}^T, \ldots, \lambda_{k^i,T-1}^T, \mu_{1,t}^T, \ldots, \mu_{k^e,T-1}^T) \neq (0, \ldots, 0). \tag{2.15}$$

$$\lambda_0^T \geq 0. \tag{2.16}$$

$$p_{t+1}^T \geq 0. \tag{2.17}$$

$$\lambda_{\alpha,t}^T \geq 0, \lambda_{\alpha,t}^T g_t^\alpha(\hat{u}_t) = 0. \tag{2.18}$$

$$p_t^T = p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0^T D_1\phi_t(\hat{x}_t, \hat{u}_t). \tag{2.19}$$

$$\left.\begin{aligned} &p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0^T D_2\phi_t(\hat{x}_t, \hat{u}_t) \\ &+ \sum_{\alpha=1}^{k^i} \lambda_{\alpha,t}^T Dg_t^\alpha(\hat{u}_t) + \sum_{\beta=1}^{k^e} \mu_{\beta,t}^T Dh_t^\beta(\hat{u}_t) = 0. \end{aligned}\right\} \tag{2.20}$$

From (2.19) we obtain $p_{t+1}^T \circ (\hat{x}_t, \hat{u}_t) = p_t^T - \lambda_0^T D_1\phi_t(\hat{x}_t, \hat{u}_t)$, and after Lemma 2.3 we have $p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t) \geq \varrho_t p_{t+1}^T$. And so we have

$$0 \leq \varrho_t p_{t+1}^T \leq p_t^T - \lambda_0^T D_1\phi_t(\hat{x}_t, \hat{u}_t),$$

which implies (cf. Remark 2.1)

$$\varrho_t \|p_{t+1}^T\|_* = \|\varrho_t p_{t+1}^T\|_* \leq \|p_t^T - \lambda_0^T D_1\phi_t(\hat{x}_t, \hat{u}_t)\|_* \leq \|p_t^T\|_* + \lambda_0^T \|D_1\phi_t(\hat{x}_t, \hat{u}_t)\|_*.$$

And so we obtain, for all $t \in \{0, \dots, T-1\}$,

$$\|p^T_{t+1}\|_* \leq \frac{1}{\varrho_t}\|p^T_t\|_* + \lambda^T_0 \frac{1}{\varrho_t}\|D_1\phi_t(\hat{x}_t, \hat{u}_t)\|_*. \tag{2.21}$$

We set

$$a_t := \frac{1}{\prod_{s=1}^{t} \varrho_s} \in (0, +\infty)$$

and

$$b_t := \sum_{s=1}^{t} \frac{1}{\prod_{k=s}^{t} \varrho_k} \|D_1\phi_s(\hat{x}_t, \hat{u}_s)\|_* \in (0, +\infty)$$

and we proceed by induction to obtain from (2.21) the following assertion:

$$\left.\begin{array}{l} \forall t \in \mathbb{N}, \exists a_t \in (0, +\infty), \exists b_t \in (0, +\infty), \forall T > t, \\ \|p^T_{t+1}\|_* \leq a_t \|p^T_1\|_* + b_t \lambda^T_0. \end{array}\right\} \tag{2.22}$$

From this last assertion we obtain that $(\lambda^T_0, p^T_1) = (0, 0)$ implies $(\lambda^T_0, p^T_1, \dots, p^T_T) = (0, 0, \dots, 0)$, and using (2.20) we obtain

$$\sum_{\alpha=1}^{k^i} \lambda^T_{\alpha,t} Dg^\alpha_t(\hat{u}_t) + \sum_{\beta=1}^{k^e} \mu^T_{\beta,t} Dh^\beta_t(\hat{u}_t) = 0,$$

and using assumption (c), this last equality implies that $\lambda^T_{\alpha,t} = 0$ and $\mu^T_{\beta,t} = 0$ for all $\alpha \in \{1, \dots, k^i\}$, for all $\beta \in \{1, \dots, k^e\}$ and for all $t \in \{0, \dots, T-1\}$. And so we have proven that $(\lambda^T_0, p^T_1) = (0, 0)$ implies the negation of (2.15). Then using (2.15) and the contraposition we have proven that $(\lambda^T_0, p^T_1) \neq (0, 0)$. Using the property of cone of the set of all lists of multipliers, we can normalize and obtain, for all $T \in \mathbb{N}$, $T \geq 2$:

$$\|(\lambda^T_0, p^T_1)\| = 1. \tag{2.23}$$

Then, from (2.22) and (2.23) we obtain

$$\forall t \in \mathbb{N}, \exists a_t \in (0, +\infty), \exists b_t \in (0, +\infty), \forall T > t, \|p^T_{t+1}\|_* \leq a_t + b_t$$

which implies

$$\forall t \in \mathbb{N}, \quad \sup_{T>t} \| p^T_{t+1} \|_* \leq +\infty. \tag{2.24}$$

From (2.20) we obtain

$$\begin{cases} \sum_{\alpha=1}^{k^i} \lambda^T_{\alpha,t} Dg^\alpha_t(\hat{u}_t) + \sum_{\beta=1}^{k^e} \mu^T_{\beta,t} Dh^\beta_t(\hat{u}_t) \\ = -p^T_{t+1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t) - \lambda^T_0 D_2\phi_t(\hat{x}_t, \hat{u}_t), \end{cases}$$

and using (2.23) and (2.24) we deduce that the sequence $T \mapsto \sum_{\alpha=1}^{k^i} \lambda^T_{\alpha,t} Dg^\alpha_t(\hat{u}_t) + \sum_{\beta=1}^{k^e} \mu^T_{\beta,t} Dh^\beta_t(\hat{u}_t)$ is bounded for all $t \in \mathbb{N}$, and then using assumption (c) and Lemma 2.2, we obtain

$$\forall t \in \mathbb{N}, \forall \alpha \in \{1, \ldots, k^i\}, \forall \beta \in \{1, \ldots, k^e\}, \sup_{T>t} |\lambda^T_{\alpha,t}| < +\infty, \sup_{T>t} |\mu^T_{\beta,t}| < +\infty. \tag{2.25}$$

And then we can conclude as in the proof of Theorem 2.1. □

When $\hat{u}_t \in \operatorname{int} U_t$, using Corollary 1.2 instead of Proposition 1.4, we obtain the following result.

Theorem 2.6. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}^n_i)$, or of $(\mathscr{P}^s_i)$, or of $(\mathscr{P}^o_i)$, or of $(\mathscr{P}^w_i)$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, X_t is a nonempty open subset of $\mathbb{R}^n$.
(b) For all $t \in \mathbb{N}$, $\hat{u}_t \in \operatorname{int} U_t$.
(c) For all $t \in \mathbb{N}$, ϕ_t and f_t are continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and they are differentiable at $(\hat{x}_t, \hat{u}_t)$.
(d) For all $t \in \mathbb{N}$, the condition (2.14) holds.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$.
(iv) For all $t \in \mathbb{N}$, $p_t = p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_1\phi_t(\hat{x}_t, \hat{u}_t)$.
(v) For all $t \in \mathbb{N}$, $D_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) = 0$.

After the use of the multiplier rule of Halkin, we use the multiplier rule of Clarke.

Theorem 2.7. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}^n_i)$, or of $(\mathscr{P}^s_i)$, or of $(\mathscr{P}^o_i)$, or of $(\mathscr{P}^w_i)$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, ϕ_t is Lipschitzian on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and regular at $(\hat{x}_t, \hat{u}_t)$.
(b) For all $t \in \mathbb{N}$, f_t is strictly differentiable at $(\hat{x}_t, \hat{u}_t)$.
(c) For all $t \in \mathbb{N}$, U_t is closed and Clarke-regular at $\hat{u}_t$.
(d) For all $t \in \mathbb{N}$, the positivity condition (2.14) holds.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t - \hat{x}_{t+1}\rangle = 0$.
(iv) For all $t \in \mathbb{N}_*$, $p_t \in \partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) For all $t \in \mathbb{N}$, $\partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) \cap N_{U_t}(\hat{u}_t) \neq \emptyset$, where $N_{U_t}(\hat{u}_t)$ is the normal cone of U_t at $\hat{u}_t$.

Proof. Using Proposition 1.2 and Proposition 1.6 we obtain the assertions (2.15), (2.16), and (2.17) inside the proof of Theorem 2.5 and the assertions (2.10) and (2.11) inside the proof of Theorem 2.3.

Since the Clarke differentials $\partial_1\phi_t(\hat{x}_t, \hat{u}_t)$ and $\partial_2\phi_t(\hat{x}_t, \hat{u}_t)$ are compact sets, they are bounded sets, and so we have

$$c_t := \sup\{\|\varphi\|_* : \varphi \in \partial_1\phi_t(\hat{x}_t, \hat{u}_t)\} < +\infty$$
$$d_t := \sup\{\|\psi\|_* : \psi \in \partial_2\phi_t(\hat{x}_t, \hat{u}_t)\} < +\infty.$$

Then, from (2.10), we obtain $p^T_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t) = p^T_t - \lambda^T_0 \varphi^T_t$ which implies $\varrho_t p^T_{t+1} \leq p^T_t - \lambda^T_0 \varphi^T_t$, where ϱ_t is defined in the proof of Theorem 2.5, which implies (cf. Remark 2.1) $\varrho_t \|p^T_{t+1}\|_* \leq \|p^T_t\|_* + \lambda^T_0 c_t$, i.e.,

$$\|p^T_{t+1}\|_* \leq \frac{1}{\varrho_t}\|p^T_t\|_* + \frac{c_t}{\varrho_t}\lambda^T_0.$$

And after that, we can proceed as in the proof of Theorem 2.5. □

When $\hat{u}_t \in \operatorname{int} U_t$, using Corollary 1.4 instead of Proposition 1.6 and proceeding as in the proof of Theorem 2.7 we obtain the following result.

Theorem 2.8. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathcal{P}^n_i)$, or of $(\mathcal{P}^s_i)$, or of $(\mathcal{P}^o_i)$, or of $(\mathcal{P}^w_i)$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, ϕ_t is Lipschitzian on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and regular at $(\hat{x}_t, \hat{u}_t)$.
(b) For all $t \in \mathbb{N}$, f_t is strictly differentiable at $(\hat{x}_t, \hat{u}_t)$.
(c) For all $t \in \mathbb{N}$, $\hat{u}_t \in \operatorname{int} U_t$.
(d) For all $t \in \mathbb{N}$, the positivity condition (2.14) holds.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0,0)$.
(ii) $\lambda_0 \geq 0$.
(iii) *For all* $t \in \mathbb{N}$, $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$.
(iv) *For all* $t \in \mathbb{N}_*$, $p_t \in \partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) *For all* $t \in \mathbb{N}$, $0 \in \partial_2 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.

2.3 Strong Pontryagin Principles in Infinite Horizon

In this section, to establish strong Pontryagin principles in infinite horizon, in a first subsection, we use the invertibility condition, in a second subsection we use the positivity condition, and in a third subsection, we use a new condition that we call a condition of partial submersion.

2.3.1 *The Invertibility Condition*

In a first time we use a consequence of a result of Michel, Proposition 1.7 and Proposition 1.8.

Theorem 2.9. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* $(\mathscr{P}^n_a)$*, or of* $(\mathscr{P}^s_a)$*, or of* $(\mathscr{P}^o_a)$*, or of* $(\mathscr{P}^w_a)$ *when* $a \in \{e, i\}$*. We assume that the following conditions are fulfilled:*

(a) *For all* $t \in \mathbb{N}$, X_t *is a nonempty open convex subset of* $\mathbb{R}^n$ *and* U_t *is a nonempty subset of* $\mathbb{R}^d$.
(b) *For all* $t \in \mathbb{N}$, *the functions* ϕ_t *and* f_t *are differentiable with respect to the first vector variable.*
(c) *For all* $t \in \mathbb{N}$, *for all* $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, $co A'_t(x_t, x_{t+1}) \subset B'_t(x_t, x_{t+1})$.
(d) *For all* $t \in \mathbb{N}$, *the partial differential* $D_1 f_t(\hat{x}_t, \hat{u}_t)$ *is invertible.*

Then there exist $\lambda_0^T \in \mathbb{R}$ *and* $(p_{t+1})_{t \in \mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$ *which satisfy the following conditions:*

(i) $(\lambda_0^T, p_1) \neq (0,0)$.
(ii) $\lambda_0^T \geq 0$.
(iii) *For all* $t \in \mathbb{N}_*$, $p_t^T = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(iv) *For all* $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u \in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0^T)$.

Proof. **The case** $a = e$. From Proposition 1.2 we can use Proposition 1.7 which provides, for all $T \in \mathbb{N}$, $T \geq 2$, a real number λ_0^T and elements of the dual space of $\mathbb{R}^n$, $p_1^T, \ldots, p_T^T$, which satisfy the following conditions:

$$\left(\lambda_0^T, p_1^T, \ldots, p_T^T\right) \neq (0, 0, \ldots, 0). \tag{2.26}$$

$$\lambda_0^T \geq 0. \tag{2.27}$$

$$\forall t \in \{1, \dots, T-1\}, p_t^T = p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \lambda_0^T D_1 \phi_t(\hat{x}_t, \hat{u}_t). \tag{2.28}$$

$$\forall t \in \{1, \dots, T-1\}, \forall u \in U_t, H_t\left(\hat{x}_t, \hat{u}_t, p_{t+1}^T, \lambda_0^T\right) \geq H_t\left(\hat{x}_t, u, p_{t+1}^T, \lambda_0^T\right). \tag{2.29}$$

Proceeding as in the proof of Theorem 2.1, we obtain the relations (2.3) and (2.4) which say that the sequences $T \mapsto \lambda_0^T$ and $T \mapsto p_{t+1}^T$ (for all $t \in \mathbb{N}$) are bounded with the additional condition $\|(\lambda_0^T, p_1^T)\| = 1$. And so we can use Lemma 2.1 and we can assert that there exist an increasing function $\sigma : \mathbb{N}_* \to \mathbb{N}_*$, $\lambda_0 \in \mathbb{R}$, $p_{t+1} \in \mathbb{R}^{n*}$ for all $t \in \mathbb{N}$, such that the following relations hold:

$$\lim_{T \to +\infty} \lambda_0^{\sigma(T)} = \lambda_0, \quad \lim_{T \to +\infty} p_{t+1}^{\sigma(T)} = p_{t+1}$$

for all $t \in \mathbb{N}$. Using the continuity of the norm we obtain $\|(\lambda_0, p_1)\| = 1$. Using the continuity of the functions inside the relations (2.27)–(2.29), we obtain the conclusions of the theorem.

The case $a = i$. Our strategy is to use the first case. For all $t \in \mathbb{N}$ we introduce the function $\hat{f}_t : X_t \times U_t \to X_{t+1}$ by setting

$$\hat{f}_t(x_t, u_t) := f_t(x_t, u_t) + (\hat{x}_{t+1} - f_t(\hat{x}_t, \hat{u}_t)). \tag{2.30}$$

We denote by $\mathrm{Adm}_\eta^e(\hat{f})$ the set of all processes $(\underline{x}, \underline{u}) \in \prod_{t \in \mathbb{N}} X_t \times \prod_{t \in \mathbb{N}}$ such that $x_{t+1} = \hat{f}_t(x_t, u_t)$ for all $t \in \mathbb{N}$, and we denote by Adm_η^i the set of all processes $(\underline{x}, \underline{u}) \in \prod_{t \in \mathbb{N}} X_t \times \prod_{t \in \mathbb{N}}$ such that $x_{t+1} \leq f_t(x_t, u_t)$ for all $t \in \mathbb{N}$. Since $\hat{x}_{t+1} \leq f_t(\hat{x}_t, \hat{u}_t)$, we have, for all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta^e(\hat{f})$, $x_{t+1} = \hat{f}_t(x_t, u_t) \leq f_t(x_t, u_t)$, which implies

$$\mathrm{Adm}_\eta^e(\hat{f}) \subset \mathrm{Adm}_\eta^i. \tag{2.31}$$

We denote by $\mathrm{Dom}_\eta^e(J, \hat{f})$ the set of all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta^e(\hat{f})$ which belong to $\mathrm{Dom}_\eta^e(J)$ (cf. Sect. 1.2). Using (2.34), it is clear that we have

$$\mathrm{Dom}_\eta^e(J, \hat{f}) \subset \mathrm{Dom}_\eta^i(J). \tag{2.32}$$

Note that $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{Adm}_\eta^e(\hat{f})$, and consequently that $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{Dom}_\eta^e(J, \hat{f})$ when $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{Dom}_\eta^i(J)$.

We fix $k \in \{n, s, o, w\}$ and we denote by $(\mathscr{P}_e^k(\hat{f}))$ the problem $(\mathscr{P}_e^k)$ where we have replaced (DE) by $x_{t+1} = \hat{f}_t(x_t, u_t)$. Note that the criterion of this problem is the same as the criterion of $(\mathscr{P}_i^k)$. And so, using the previous inclusions, we see that if $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of $(\mathscr{P}_i^k)$ then it is also a solution of $(\mathscr{P}_e^k(\hat{f}))$. We see that the assumptions on f_t imply the same assumptions on $\hat{f}_t$, and so we can apply the first case to $(\mathscr{P}_e^k(\hat{f}))$. After that, it suffices to translate the conclusions on $(\mathscr{P}_e^k(\hat{f}))$ into conclusions on $(\mathscr{P}_i^k)$. If we denote by $\hat{H}_t$ the Hamiltonian of $(\mathscr{P}_e^k(\hat{f}))$, H_t being the Hamiltonian of $(\mathscr{P}_i^k)$, we see that the difference $\hat{H}_t - H_t$ is a constant which is independent of x_t and u_t, which implies that the adjoint equation of $(\mathscr{P}_e^k(\hat{f}))$ is exactly the adjoint equation of $(\mathscr{P}_i^k)$, and the strong maximum principle of $(\mathscr{P}_e^k(\hat{f}))$ implies this one of $(\mathscr{P}_i^k)$. □

This theorem was established in [16]. There exist other versions in [13]. In this last paper an analogous version for systems governed by (DI) is stated. But the proof given for the case of (DI) is not very explicative. And so, we provide an original proof of the theorem in the case of (DI).

Theorem 2.10. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}_a^n)$, or of $(\mathscr{P}_a^s)$, or of $(\mathscr{P}_a^o)$, or of $(\mathscr{P}_a^w)$, when $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, X_t is a nonempty open subset of $\mathbb{R}^n$, and U_t is a nonempty subset of $\mathbb{R}^d$.
(b) For all $t \in \mathbb{N}$, $\phi_t \in C^0(X_t \times U_t, \mathbb{R})$ and, for all $(x, u) \in \mathbb{R}^n \times U_t$, the partial differential $D_1\phi_t(x, u)$ exists and $D_1\phi_t \in C^0(X_t \times U_t, \mathbb{R}^{n*})$.
(c) For all $t \in \mathbb{N}$, $f_t \in C^0(X_t \times U_t, \mathbb{R}^n)$ and, for all $(x, u) \in \mathbb{R}^n \times U_t$, the partial differential $D_1 f_t(x, u)$ exists and $D_1\phi_t \in C^0(X_t \times U_t, \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n))$.
(d) For all $t \in \mathbb{N}$, for all $x \in X_t$, for all $u, v \in U_t$, for all $r \in [0, 1]$, there exists $w \in U_t$ such that

$$\begin{cases} \phi_t(x, w) \geq (1 - r)\phi_t(x, u) + r\phi_t(x, v) \\ f_t(x, w) = (1 - r) f_t(x, u) + r f_t(x, v). \end{cases}$$

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_{t+1})_{t \in \mathbb{N}} \in (\mathbb{R}^{n*})^N$ which satisfy the following conditions:

(i) λ_0 and $(p_{t+1})_{t \in \mathbb{N}}$ are not simultaneously equal to zero.
(ii) $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$ for all $t \in \mathbb{N}$.
(iii) $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) = \max_{u \in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$ for all $t \in \mathbb{N}$.

Proof. The proof is similar to this one of Theorem 2.9 replacing the use of Proposition 1.7 by the use of Proposition 1.9 when $a = e$ and the use of Proposition 1.8 by the use of Proposition 1.10 when $a = i$. □

2.3.2 A Condition of Positivity

In this subsection, we use the positivity condition already used in Sect. 2.2 to obtain weak Pontryagin principles, in order to obtain strong Pontryagin principles.

Theorem 2.11. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* $(\mathscr{P}_i^n)$*, or of* $(\mathscr{P}_i^s)$*, or of* $(\mathscr{P}_i^o)$*, or of* $(\mathscr{P}_i^w)$*. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$*,* X_t *is nonempty and convex,* $\hat{x}_t \in \operatorname{int} X_t$*, and* U_t *is nonempty.*
(b) For all $t \in \mathbb{N}$*, the partial functions* $\phi_t(., \hat{u}_t)$ *and* $f_t(., \hat{u}_t)$ *are continuous on a neighborhood of* $\hat{x}_t$ *and differentiable at* $\hat{x}_t$*.*
(c) For all $t \in \mathbb{N}$*, for all* $(x_t, x_{t+1}) \in X_t \times X_{t+1}$*,* $coA_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$*.*
(d) For all $t \in \mathbb{N}$*, for all* $i, j \in \{1, \dots, n\}$*,* $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} \geq 0$*, and for all* $j \in \{1, \dots, n\}$*,* $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} > 0$*.*

Then there exist $\lambda_0 \in \mathbb{R}$*,* $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^N$ *which satisfy the following conditions:*

(i) $(\lambda_0, p_1) \neq (0, 0)$*.*
(ii) $\lambda_0 \geq 0$*.*
(iii) For all $t \in \mathbb{N}$*,* $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$*.*
(iv) For all $t \in \mathbb{N}_*$*,* $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$*.*
(v) For all $t \in \mathbb{N}$*,* $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u\in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$*.*

Proof. Using Propositions 1.2 and 1.8, we obtain the existence, for all $T \in \mathbb{N}$, $T \geq 2$, of $\lambda_0^T \in \mathbb{R}$ and of $p_1^T, \dots, p_T^T \in \mathbb{R}^{n*}$ which satisfy the conclusions of Proposition 1.8. Then using Lemma 2.3 and reasoning as in the proof of Theorem 2.5, we obtain the relations (2.22), (2.23), and (2.24). Then using Lemma 2.1, we obtain the existence of a strictly increasing function $\sigma : \mathbb{N}_* \to \mathbb{N}_*$, of $\lambda_0 \in \mathbb{R}$ and of a sequence $(p_{t+1})_{t\in\mathbb{N}}$ in $\mathbb{R}^{n*}$ such that $\lim_{T\to+\infty} \lambda_0^{\sigma(T)} = \lambda_0$, and $\lim_{T\to+\infty} p_{t+1}^{\sigma(T)} = p_{t+1}$ for all $t \in \mathbb{N}$. And then, from the conclusions of Proposition 1.8, we obtain the conclusion of this theorem by taking $T \to +\infty$. □

In the previous theorem, we have only considered problems which are governed by (DI). In the following theorems, we consider problems governed by (DE).

Theorem 2.12. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* $(\mathscr{P}_e^o)$*. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$*,* X_t *is nonempty and convex,* $\hat{x}_t \in \operatorname{int} X_t$*, and* U_t *is nonempty.*
(b) For all $t \in \mathbb{N}$*, the partial functions* $\phi_t(., \hat{u}_t)$ *and* $f_t(., \hat{u}_t)$ *are continuous on a neighborhood of* $\hat{x}_t$ *and differentiable at* $\hat{x}_t$*.*
(c) For all $t \in \mathbb{N}$*, for all* $(x_t, x_{t+1}) \in X_t \times X_{t+1}$*,* $coA_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$*.*
(d) For all $t \in \mathbb{N}$*, for all* $i, j \in \{1, \dots, n\}$*,* $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} \geq 0$*, and for all* $j \in \{1, \dots, n\}$*,* $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} > 0$*.*
(e) For all $t \in \mathbb{N}$*, for all* $u_t \in U_t$*, the partial function* $\phi_t(., u_t)$ *is increasing.*
(f) For all $t \in \mathbb{N}$*, for all* $u_t \in U_t$*, the partial function* $f_t(., u_t)$ *is increasing.*

Then there exist $\lambda_0 \in \mathbb{R}$*,* $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^N$ *which satisfy the following conditions:*

(i) $(\lambda_0, p_1) \neq (0,0)$.
(ii) $\lambda_0 \geq 0$.
(iii) *For all* $t \in \mathbb{N}$, $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.
(iv) *For all* $t \in \mathbb{N}_*$, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) *For all* $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u \in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$.

Proof. Noting that (CA1) = (e) and that (CA2) = (f), we can use Theorem 1.1 and we can assert that $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of $(\mathscr{P}_i^o)$. And then we conclude by using Theorem 2.11. □

Theorem 2.13. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* $(\mathscr{P}_e^o)$. *We assume that the following conditions are fulfilled:*

(a) *For all* $t \in \mathbb{N}$, X_t *is nonempty and convex,* $\hat{x}_t \in \operatorname{int} X_t$, *and* U_t *is nonempty.*
(b) *For all* $t \in \mathbb{N}$, *the partial functions* $\phi_t(., \hat{u}_t)$ *and* $f_t(., \hat{u}_t)$ *are continuous on a neighborhood of* $\hat{x}_t$ *and differentiable at* $\hat{x}_t$.
(c) *For all* $t \in \mathbb{N}$, *for all* $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, $co A_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$.
(d) *For all* $t \in \mathbb{N}$, *for all* $i, j \in \{1, \dots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} \geq 0$, *and for all* $j \in \{1, \dots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} > 0$.
(e) *For all* $t \in \mathbb{N}$, *for all* $x_t \in X_t$, *the partial function* $\phi_t(x_t, .)$ *is increasing.*
(f) *For all* $t \in \mathbb{N}$, *for all* $(y_{t+1}, y_t, u_t \in X_{t+1} \times X_t \times U_t$ *such that* $y_{t+1} \leq f_t(y_t, u_t)$, *there exists* $v_t \in U_t$ *such that* $v_t \geq u_t$ *and* $y_{t+1} = f_t(y_t, v_t)$.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_{t+1})_{t \in \mathbb{N}} \in (\mathbb{R}^{n*})^N$ *which satisfy the following conditions:*

(i) $(\lambda_0, p_1) \neq (0,0)$.
(ii) $\lambda_0 \geq 0$.
(iii) *For all* $t \in \mathbb{N}$, $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.
(iv) *For all* $t \in \mathbb{N}_*$, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) *For all* $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u \in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$.

Proof. Noting that (CA4) = (f) and that (CA5) = (e), we can use Theorem 1.2 and we can assert that $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of $(\mathscr{P}_i^o)$. And then we conclude by using Theorem 2.11. □

Theorem 2.14. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* $(\mathscr{P}_e^n)$. *We assume that the following conditions are fulfilled:*

(a) *For all* $t \in \mathbb{N}$, X_t *is nonempty and convex,* $\hat{x}_t \in \operatorname{int} X_t$, *and* U_t *is nonempty.*
(b) *For all* $t \in \mathbb{N}$, *the partial functions* $\phi_t(., \hat{u}_t)$ *and* $f_t(., \hat{u}_t)$ *are continuous on a neighborhood of* $\hat{x}_t$ *and differentiable at* $\hat{x}_t$.
(c) *For all* $t \in \mathbb{N}$, *for all* $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, $co A_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$.
(d) *For all* $t \in \mathbb{N}$, *for all* $i, j \in \{1, \dots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} \geq 0$, *and for all* $j \in \{1, \dots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} > 0$.
(e) *For all* $t \in \mathbb{N}$, *for all* $u_t \in U_t$, *the partial function* $\phi_t(., u_t)$ *is increasing.*

(f) *For all* $t \in \mathbb{N}$, *for all* $u_t \in U_t$, *the partial function* $f_t(., u_t)$ *is increasing.*
(g) *For all* $t \in \mathbb{N}$, $\phi_t \geq 0$.
(h) *For all* $t \in \mathbb{N}$, *for all* $z_t \in X_t$, *there exists* $s \in \mathbb{N}_*$ *and there exists* $(v_t, \dots, v_{t+s-1}) \in \prod_{j=0}^{s-1} U_{t+j}$ *such that by setting* $z_{t+j+1} := f_{t+j}(z_{t+j}, v_{t+j})$ *for* $j \in \{0, \dots, s-1\}$ *we have* $z_{t+s} = \hat{x}_{t+s}$.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^N$ *which satisfy the following conditions:*

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) *For all* $t \in \mathbb{N}$, $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.
(iv) *For all* $t \in \mathbb{N}_*$, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) *For all* $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u\in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$.

Proof. Note that (CA 1) = (e), (CA 2) = (f), (CA 3) = (g), and (CA, $(\underline{\hat{x}}, \underline{\hat{u}})$) = (h). And then we can use Theorem 1.3 to assert that $(\underline{\hat{x}}, \underline{\hat{u}})$ is also a solution of $(\mathscr{P}_i^n)$. We conclude by using Theorem 2.11. □

Theorem 2.15. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* $(\mathscr{P}_e^n)$. *We assume that the following conditions are fulfilled:*

(a) *For all* $t \in \mathbb{N}$, X_t *is nonempty and convex,* $\hat{x}_t \in \operatorname{int} X_t$, *and* U_t *is nonempty.*
(b) *For all* $t \in \mathbb{N}$, *the partial functions* $\phi_t(., \hat{u}_t)$ *and* $f_t(., \hat{u}_t)$ *are continuous on a neighborhood of* $\hat{x}_t$ *and differentiable at* $\hat{x}_t$.
(c) *For all* $t \in \mathbb{N}$, *for all* $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, $coA_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$.
(d) *For all* $t \in \mathbb{N}$, *for all* $i, j \in \{1, \dots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} \geq 0$, *and for all* $j \in \{1, \dots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} > 0$.
(e) *For all* $t \in \mathbb{N}$, $\phi_t \geq 0$.
(f) *For all* $t \in \mathbb{N}$, *for all* $(y_{t+1}, y_t, u_t) \in X_{t+1} \times X_t \times U_t$ *such that* $y_{t+1} \leq f_t(y_t, u_t)$, *there exists* $v_t \in U_t$ *such that* $v_t \geq u_t$ *and* $y_{t+1} = f_t(y_t, v_t)$.
(g) *For all* $t \in \mathbb{N}$, *for all* $x_t \in X_t$, *the partial function* $\phi_t(x_t, .)$ *is increasing.*
(h) *For all* $t \in \mathbb{N}$, *for all* $z_t \in X_t$, *there exists* $s \in \mathbb{N}_*$ *and there exists* $(v_t, \dots, v_{t+s-1}) \in \prod_{j=0}^{s-1} U_{t+j}$ *such that by setting* $z_{t+j+1} := f_{t+j}(z_{t+j}, v_{t+j})$ *for* $j \in \{0, \dots, s-1\}$ *we have* $z_{t+s} = \hat{x}_{t+s}$.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^N$ *which satisfy the following conditions:*

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) *For all* $t \in \mathbb{N}$, $p_{t+1} \geq 0$ *and* $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.
(iv) *For all* $t \in \mathbb{N}_*$, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) *For all* $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u\in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$.

Proof. Note that (CA 3) = (e), (CA 4) = (f), (CA 5) = (g), and (CA, $(\underline{\hat{x}}, \underline{\hat{u}})$) = (h). Then we can use Theorem 1.4 to assert that $(\underline{\hat{x}}, \underline{\hat{u}})$ is also a solution of $(\mathscr{P}_i^n)$. We conclude by using Theorem 2.11. □

Remark 2.2. These theorems, from Theorem 2.11 until Theorem 2.15, appear in the paper of Blot [11]. It is useful to note that in Theorems 2.12–2.15 the adjoint variables p_{t+1} are positive although the problem is governed by (DE).

In all the results of this subsection, we have used the condition of Michel. If we use the condition of Ioffe and Tihomirov, we obtain the following result which is new.

Theorem 2.16. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}_i^n)$, or of $(\mathscr{P}_i^s)$, or of $(\mathscr{P}_i^o)$, or of $(\mathscr{P}_i^w)$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, $\phi_t(., \hat{u}_t)$ and $f_t(., \hat{u}_t)$ are of class C^1 at $\hat{x}$.
(b) For all $t \in \mathbb{N}$, there exists a neighborhood V_t of $\hat{x}_t$ in X_t such that, for all $x \in V_t$, for all $u_1, u_2 \in U_t$, for all $\theta \in [0, 1]$, there exists $u_3 \in U_t$ such that

$$\begin{cases} \phi_t(x, u_3) \geq (1-\theta)\phi_t(x, u_1) + \theta\phi_t(x, u_2) \\ f_t(x, u_3) \geq (1-\theta) f_t(x, u_1) + \theta f_t(x, u_2). \end{cases}$$

(c) For all $t \in \mathbb{N}$, for all $i, j \in \{1, \ldots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} \geq 0$, and

for all $j \in \{1, \ldots, n\}$, $\frac{\partial f^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^j} > 0$.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_{t+1})_{t \in \mathbb{N}} \in (\mathbb{R}^{n*})^N$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.
(iv) For all $t \in \mathbb{N}_*$, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) For all $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u \in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$.

Proof. Using Propositions 1.2 and 1.10, we obtain, for all $T \in \mathbb{N}$, $T \geq 2$, $\lambda_0^T \in \mathbb{R}$ and $p_1^T, \ldots, p_T^T \in \mathbb{R}^{n*}$ which satisfy the conclusion of Proposition 1.10. Then we conclude as in the proof of Theorem 2.11. □

Remark 2.3. Proceeding as we do to establish the results from Theorem 2.12 until Theorem 2.15, we can obtain strong Pontryagin principles for the problems $(\mathscr{P}_e^o)$ and $(\mathscr{P}_e^n)$ where the part of the assumption which comes from the result of Michel is replaced by assumptions which come from Ioffe and Tihomirov.

2.3.3 A Condition of Partial Submersion

To avoid the invertibility condition, beside the positivity condition, we introduce another condition on the vector field of the dynamical system.

In this subsection, $\mathbb{R}^n$ is endowed with its usual inner product which is denoted by $(. \mid .)$. Following [81] (p. 410), when E and F are two Hilbert spaces, and when $T \in \mathscr{L}(E, F)$, the adjoint of T is $T^* \in \mathscr{L}(F, E)$ characterized by $(T.x \mid y) = (x \mid T^*.y)$. And so, in our problems, we will use $D_1 f_t(\hat{x}_t, \hat{u}_t)^* \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^*)$. When $\pi \in \mathbb{R}^{n*}$, we associate to π the vector $\pi^* \in \mathbb{R}^n$ characterized by $(\pi^* \mid y) = \langle \pi, y \rangle$ for all $y \in \mathbb{R}^n$. When $\pi \in \mathbb{R}^{n*}$ and $L \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, for all $y \in \mathbb{R}^n$, we have $\langle \pi \circ L, y \rangle = \pi(L.y) = \langle \pi, L.y \rangle = (\pi^* \mid L.y) = (L^*.\pi^* \mid y)$. Also recall that the gradient of a real-valued differentiable function is the vector in the primal space which represents the differential which belongs to the dual space. And so, in our problems, we will use of the partial gradient of ϕ_t, $(\nabla_1 \phi_t(x_t, u_t) \mid y) = \langle D_1 \phi_t(x_t, u_t), y \rangle$ for all $y \in \mathbb{R}^n$. And so, using these notions, the adjoint equation can be rewritten as

$$p_t^* = D_1 f_t(\hat{x}_t, \hat{u}_t)^*.p_{t+1}^* + \lambda_0 \nabla_1 \phi_t(\hat{x}_t, \hat{u}_t) \tag{2.33}$$

and the Hamiltonian can be written as

$$H_t(x_t, u_t, p_{t+1}, \lambda_0) = \lambda_0 \phi_t(x_t, u_t) + (p_{t+1}^* \mid f_t(x_t, u_t)). \tag{2.34}$$

Assuming the existence of the partial differential with respect to the state variable, we introduce the two following subspaces:

$$\left. \begin{array}{l} M_t := \mathrm{Im} D_1 f_t(\hat{x}_t, \hat{u}_t) \\ N_t := M_t^{\perp} = \mathrm{Ker} D_1 f_t(\hat{x}_t, \hat{u}_t)^* \end{array} \right\} \tag{2.35}$$

where $\perp$ denotes the orthogonal subspace. π_{M_t} and π_{N_t} denote the orthogonal projectors on M_t and on N_t. We also use the notation $S_{N_t}(x, \rho) := \{z \in N_t : \|z - x\| = \rho\}$ and $B_{N_t}(x, \rho) := \{z \in N_t : \|z - x\| \leq \rho\}$.

Now we can introduce our new condition.

$$\left. \begin{array}{l} \quad \forall t \in \mathbb{N}, \exists P_t \subset U_t, P_t \neq \emptyset \text{ s.t.} \\ (\alpha)\ \exists \varrho_t > 0, \pi_{N_t}(f_t(\{\hat{x}_t\} \times P_t)) \supset (S_{N_t}(0, \varrho_t) + \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t)) \\ (\beta)\ \pi_{M_t}(f_t(\{\hat{x}_t\} \times P_t)) \text{ is bounded} \\ (\gamma)\ \phi_t(\{\hat{x}_t\} \times P_t) \text{ is bounded.} \end{array} \right\} \tag{2.36}$$

We also consider another condition which is simpler than (2.36).

$$\left.\begin{array}{l} \quad \forall t \in \mathbb{N}, \\ (1)\ \phi_t(\hat{x}_t, .)\ \text{ and }\ f_t(\hat{x}_t, .)\ \text{ are continuous on }\ U_t \\ (2)\ D_1 f_t(\hat{x}_t, \hat{u}_t)\ \text{ exists} \\ (3)\ \hat{u}_t \in \operatorname{int} U_t \\ (4)\ f_t(\hat{x}_t, .)\ \text{ is of class }\ C^1\ \text{ at }\ \hat{u}_t \\ (5)\ \operatorname{Im}\pi_{N_t} \circ D_2 f_t(\hat{x}_t, \hat{u}_t) = N_t \end{array}\right\} \tag{2.37}$$

Remark 2.4. The condition (2.37) implies the condition (2.36). To justify that, note that, using condition (5), since $D_2(\pi_{N_t} \circ f_t)(\hat{x}_t, \hat{u}_t) = \pi_{N_t} \circ D_2 f_t(\hat{x}_t, \hat{u}_t)$ is surjective from $\mathbb{R}^d$ onto N_t, using a theorem of Graves ([64] p. 397), there exists a closed ball $P_t = \{u \in U_t : \|u - \hat{u}_t\| \leq r_t\}$ such that

$$\pi_{N_t} \circ f_t(\{\hat{x}_t\} \times P_t) \supset B_{N_t}(f_t(\hat{x}_t, \hat{u}_t), \varrho_t) \supset S_{N_t}(0, \varrho_t) + f_t(\hat{x}_t, \hat{u}_t),$$

and so the condition (α) of (2.36) is fulfilled. Since $\dim\mathbb{R}^d < +\infty$, P_t is compact. The continuities in condition (1) imply the conditions (β) and (γ).

Theorem 2.17. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{P}^n_a)$, or of $(\mathscr{P}^s_a)$, or of $(\mathscr{P}^o_a)$, or of $(\mathscr{P}^w_a)$ when $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, X_t is a nonempty open convex subset of $\mathbb{R}^n$ and U_t is a nonempty subset of $\mathbb{R}^d$.
(b) For all $t \in \mathbb{N}$, the functions ϕ_t and f_t are differentiable with respect to the first vector variable.
(c) For all $t \in \mathbb{N}$, for all $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, $coA'_t(x_t, x_{t+1}) \subset B'_t(x_t, x_{t+1})$.
(d) Condition (2.36) holds.

Then there exist $\lambda_0 \in \mathbb{R}$ and $(p_{t+1})_{t \in \mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$ when $a = i$.
(iv) For all $t \in \mathbb{N}_*$, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) For all $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0) = \max_{u \in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$.

Proof. **The case a = e.** Using Propositions 1.2 and 1.7, we can assert that, for all $T \in \mathbb{N}$, $T \geq 2$, there exist $\lambda_0^T \in \mathbb{R}$ and $p_1^T, \ldots, p_T^T \in \mathbb{R}^{n*}$ which satisfy the conclusions of Proposition 1.8.

Since M_t is the orthogonal to $\operatorname{Ker} D_1 f_t(\hat{x}_t, \hat{u}_t)^*$, for all $z \in M_t$, we have $D_1 f_t(\hat{x}_t, \hat{u}_t)^*.z \neq 0$. Using the compactness of the unit sphere of M_t and the continuity of $D_1 f_t(\hat{x}_t, \hat{u}_t)^*$, we have $a_t := \inf\{\|D_1 f_t(\hat{x}_t, \hat{u}_t)^*.z\| : z \in M_t, \|z\| = 1\} > 0$. And so we have

$$\exists a_t \in (0, +\infty), \forall z \in M_t, \|D_1 f_t(\hat{x}_t, \hat{u}_t)^*.z\| \geq a_t.\|z\|. \tag{2.38}$$

Using the vector translation (2.33) of the third conclusion of Proposition 1.7, we obtain

$$\begin{aligned} p_t^{T*} &= D_1 f_t(\hat{x}_t, \hat{u}_t)^*.p_{t+1}^{T*} + \lambda_0^T \nabla_1 \phi_t(\hat{x}_t, \hat{u}_t) \\ &= D_1 f_t(\hat{x}_t, \hat{u}_t)^*.\pi_{M_t}(p_{t+1}^{T*}) + D_1 f_t(\hat{x}_t, \hat{u}_t)^*.\pi_{N_t}(p_{t+1}^{T*}) + \lambda_0^T \nabla_1 \phi_t(\hat{x}_t, \hat{u}_t) \\ &= D_1 f_t(\hat{x}_t, \hat{u}_t)^*.\pi_{M_t}(p_{t+1}^{T*}) + \lambda_0^T \nabla_1 \phi_t(\hat{x}_t, \hat{u}_t) \end{aligned}$$

which implies $p_t^{T*} - \lambda_0^T \nabla_1 \phi_t(\hat{x}_t, \hat{u}_t) = D_1 f_t(\hat{x}_t, \hat{u}_t)^*.\pi_{M_t}(p_{t+1}^{T*})$, and therefore, using (2.38), we obtain

$$\begin{aligned} \|p_t^{T*}\| + \lambda_0^T \|\nabla_1 \phi_t(\hat{x}_t, \hat{u}_t)\| &\geq \|p_t^{T*} - \lambda_0^T \nabla_1 \phi_t(\hat{x}_t, \hat{u}_t)\| \\ = \|D_1 f_t(\hat{x}_t, \hat{u}_t)^*.\pi_{M_t}(p_{t+1}^{T*})\| &\geq a_t.\|\pi_{M_t}(p_{t+1}^{T*})\| \end{aligned}$$

from which we have

$$\forall T > t, \qquad \|\pi_{M_t}(p_{t+1}^{T*})\| \leq \frac{1}{a_t}\|p_t^{T*}\| + \lambda_0^T \frac{1}{a_t}\|\nabla_1 \phi_t(\hat{x}_t, \hat{u}_t)\|. \tag{2.39}$$

Now we introduce the following notation:

$$\begin{cases} \Delta\phi_t(u_t) := \phi_t(\hat{x}_t, \hat{u}_t) - \phi_t(\hat{x}_t, u_t) \\ \Delta f_t(u_t) := f_t(\hat{x}_t, \hat{u}_t) - f_t(\hat{x}_t, u_t). \end{cases}$$

Using (2.34), the fourth conclusion of Proposition 1.7 implies, for all $u_t \in U_t$, $\lambda_0^T \Delta\phi_t(u_t) + (p_{t+1}^{T*} \mid \Delta f_t(u_t)) \geq 0$, which implies by using the orthogonality between M_t and N_t,

$$\lambda_0^T \Delta\phi_t(u_t) + (\pi_{M_t}(p_{t+1}^{T*}) \mid \pi_{M_t}(\Delta f_t(u_t))) + (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(\Delta f_t(u_t))) \geq 0$$

which implies

$$\begin{cases} \lambda_0^T \Delta\phi_t(u_t) + (\pi_{M_t}(p_{t+1}^{T*}) \mid \pi_{M_t}(\Delta f_t(u_t))) \\ \geq (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, u_t))) - (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))). \end{cases}$$

Using the Cauchy–Schwarz–Buniakovski inequality, we obtain

$$\begin{cases} \lambda_0^T |\Delta\phi_t(u_t)| + \|\pi_{M_t}(p_{t+1}^{T*})\|.\|\pi_{M_t}(\Delta f_t(u_t))\| \\ \geq (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, u_t))) - (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))). \end{cases}$$

Using conditions (β) and (γ) of the assumption (2.36) and the fact that the norm of an orthogonal projector is less than 1, we know that

$$\xi_t := \sup_{u_t \in U_t} |\Delta\phi_t(u_t)| < +\infty, \qquad \zeta_t := \sup_{u_t \in U_t} \|\pi_{M_t}(\Delta f_t(u_t))\| < +\infty.$$

And then using the previous inequalities, we obtain by taking the sup on the $u_t \in U_t$,

$$
\begin{aligned}
&\lambda_0^T.\xi_t + \zeta_t.\|\pi_{M_t}(p_{t+1}^{T*})\| \\
&\quad \geq \sup_{u_t \in U_t} (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, u_t))) - (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))) \\
&\quad \geq \sup_{z_t \in S_{N_t}(0,\varrho_t)} (\pi_{N_t}(p_{t+1}^{T*}) \mid z_t + \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))) - (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))) \\
&\quad = \sup_{z_t \in S_{N_t}(0,\varrho_t)} (\pi_{N_t}(p_{t+1}^{T*}) \mid z_t) + (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))) \\
&\qquad - (\pi_{N_t}(p_{t+1}^{T*}) \mid \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))) \\
&\quad = \sup_{z_t \in S_{N_t}(0,\varrho_t)} (\pi_{N_t}(p_{t+1}^{T*}) \mid z_t) \\
&\quad = \varrho_t. \sup_{w_t \in S_{N_t}(0,1)} (\pi_{N_t}(p_{t+1}^{T*}) \mid w_t) \\
&\quad = \varrho_t.\|\pi_{N_t}(p_{t+1}^{T*})\|,
\end{aligned}
$$

and so we have proven the following property:

$$
\forall T > t, \|\pi_{N_t}(p_{t+1}^{T*})\| \leq \frac{\xi_t}{\varrho_t}\lambda_0^T + \frac{\zeta_t}{\varrho_t}\|\pi_{M_t}(p_{t+1}^{T*})\|. \tag{2.40}
$$

Using (2.39) in (2.40), we obtain the following inequalities, for all $T > t$:

$$
\begin{cases}
\|\pi_{M_t}(p_{t+1}^{T*})\| \leq \dfrac{\|\nabla_1\phi_t(\hat{x}_t, \hat{u}_t)\|}{a_t}\lambda_0^T + \frac{1}{a_t}\|p_t^{T*}\| \\
\|\pi_{N_t}(p_{t+1}^{T*})\| \leq \left(\dfrac{\xi_t}{\varrho_t} + \dfrac{\zeta_t.\|\nabla_1\phi_t(\hat{x}_t, \hat{u}_t)\|}{\varrho_t.a_t}\right)\lambda_0^T + \dfrac{\zeta_t}{\varrho_t.a_t}\|p_t^{T*}\|,
\end{cases}
$$

from which we deduce

$$
\begin{aligned}
&\|p_{t+1}^T\|_* = \|p_{t+1}^{T*}\| = \|\pi_{M_t}(p_{t+1}^{T*}) + \pi_{N_t}(p_{t+1}^{T*})\| = \|\pi_{M_t}(p_{t+1}^{T*})\| + \|\pi_{N_t}(p_{t+1}^{T*})\| \\
&\quad \leq \left(\frac{\xi_t}{\varrho_t} + \frac{(\zeta_t + \varrho_t)\|\nabla_1\phi_t(\hat{x}_t, \hat{u}_t)\|}{\varrho_t.a_t}\right)\lambda_0^T + \frac{\zeta_t + \varrho_t}{\varrho_t.a_t}\|p_t^{T*}\|
\end{aligned}
$$

and so using the normalization $\|(\lambda_0^T, p_1^T)\| = 1$, from the previous inequality, by induction we obtain that, for all $t \in \mathbb{N}$, the sequence $T \mapsto p_t^T$ is bounded, and we can conclude as in the proof of Theorem 2.1.

The case a = i. The reasoning is similar using Proposition 1.8 instead of Proposition 1.7. □

To finish this subsection, we use the condition of Ioffe and Tihomirov.

Theorem 2.18. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathcal{P}_i^n)$, or of $(\mathcal{P}_i^s)$, or of $(\mathcal{P}_i^o)$, or of $(\mathcal{P}_i^w)$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, $\phi_t(., \hat{u}_t)$ and $f_t(., \hat{u}_t)$ are of class C^1 at $\hat{x}$.

(b) For all $t \in \mathbb{N}$, there exists a neighborhood V_t of $\hat{x}_t$ in X_t such that, for all $x \in V_t$, for all $u_1, u_2 \in U_t$, for all $\theta \in [0,1]$, there exists $u_3 \in U_t$ such that

$$\begin{cases} \phi_t(x,u_3) \geq (1-\theta)\phi_t(x,u_1) + \theta\phi_t(x,u_2) \\ f_t(x,u_3) \geq (1-\theta) f_t(x,u_1) + \theta f_t(x,u_2). \end{cases}$$

(c) The condition (2.36) holds.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^N$ which satisfy the following conditions:

(i) $(\lambda_0, p_1) \neq (0,0)$.
(ii) $\lambda_0 \geq 0$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.
(iv) For all $t \in \mathbb{N}_*$, $p_t = D_1 H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0)$.
(v) For all $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0^T) = \max_{u \in U_t} H_t(\hat{x}_t, u, p_{t+1}, \lambda_0)$.

Proof. Using Propositions 1.2 and 1.9 when $a = e$ or Proposition 1.10 when $a = i$, we obtain $\lambda_0^T \in \mathbb{R}$, $p_1^T, \ldots, p_T^T \in \mathbb{R}^{n*}$ which satisfy the conclusions of Proposition 1.9 when $a = e$ or of Proposition 1.10 when $a = i$. And then we conclude as in the proof of Theorem 2.17. □

Remark 2.5. Proceeding as we do to establish the results from Theorem 2.12 until Theorem 2.15, we can obtain strong Pontryagin principles for the problems $(\mathscr{P}_e^o)$ and for $(\mathscr{P}_e^n)$ where the part of the assumption which comes from the result of Michel is replaced by assumptions which come from Ioffe and Tihomirov.

2.4 Constrained Problems

In this section we still consider systems governed by (DE) or (DI). We consider constraints which possess the following form, for all $t \in \mathbb{N}$, when $x_t \in X_t$:

$$\begin{aligned} \mathscr{U}_t(x_t) := \{u_t \in U_t : \forall j \in \{1, \ldots, d^i\}, g_t^j(x_t, u_t) \\ \geq 0, \forall k \in \{1, \ldots, d^e\}, h_t^k(x_t, u_t) = 0\}. \end{aligned} \tag{2.41}$$

The terminology varies when we speak of such constraints. Following [6] (p. 221) these constraints represent a "feedback perfect state information": the value of the state variable x_t modifies the set of all admissible values of the control variable u_t. We define the admissible processes which satisfy these constraints, when $a \in \{e, i\}$.

$$\mathrm{Adm}_{\eta,c}^a := \{(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta^a : \forall t \in \mathbb{N}, u_t \in \mathscr{U}_t(x_t)\}. \tag{2.42}$$

We define the problems where these constraints are present, when $a \in \{e, i\}$.

$(\mathscr{C}_a^n)$ Maximize $J(\underline{x}, \underline{u})$ when $(\underline{x}, \underline{u}) \in \mathrm{Dom}_\eta^a(J) \cap \mathrm{Adm}_{\eta,c}^a$.

$(\mathscr{C}_a^s)$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Dom}_\eta^a(J) \cap \mathrm{Adm}_{\eta,c}^a$ such that, for all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_{\eta,c}^a$,

$$J(\hat{\underline{x}}, \hat{\underline{u}}) \geq \limsup_{T \to +\infty} \sum_{t=0}^{T} \phi_t(x_t, u_t).$$

$(\mathscr{C}_a^o)$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Adm}_{\eta,c}^a$ such that, for all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_{\eta,c}^a$,

$$\liminf_{T \to +\infty} \sum_{t=0}^{T} (\phi_t(\hat{x}_t, \hat{u}_t) - \phi_t(x_t, u_t)) \geq 0.$$

$(\mathscr{C}_a^w)$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Adm}_{\eta,c}^a$ such that, for all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_{\eta,c}^a$,

$$\limsup_{T \to +\infty} \sum_{t=0}^{T} (\phi_t(\hat{x}_t, \hat{u}_t) - \phi_t(x_t, u_t)) \geq 0.$$

Besides the Hamiltonian H_t defined in Chap. 1, we consider the Lagrangian $L_t : X_t \times U_t \times \mathbb{R}^{n*} \times \mathbb{R} \times \mathbb{R}^{d^i *} \times \mathbb{R}^{d^e *} \to \mathbb{R}$ by setting

$$L_t(x, u, p, \lambda, \mu, \nu) := H_t(x, u, p, \lambda) + \langle \mu, g_t(x, u) \rangle + \langle \nu, h_t(x, u) \rangle. \tag{2.43}$$

where $g_t := (g_t^1, \dots, g_t^{d^i})$ and $h_t := (h_t^1, \dots h_t^{d^e})$.

Theorem 2.19. *Let $(\hat{\underline{x}}, \hat{\underline{u}})$ be a solution of $(\mathscr{C}_a^n)$, or of $(\mathscr{C}_a^s)$, or of $(\mathscr{C}_a^o)$, or of $(\mathscr{C}_a^w)$ where $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(1) For all $t \in \mathbb{N}$, X_t is nonempty open and convex, and the functions ϕ_t, f_t, g_t, h_t are continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and differentiable at $(\hat{x}_t, \hat{u}_t)$.
(2) For all $t \in \mathbb{N}$, $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.
(3) Setting $S_t^j := D_1 g_t^j(\hat{x}_t, \hat{u}_t) \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t) - D_2 g_t^j(\hat{x}_t, \hat{u}_t)$ and $M_t^k := D_1 h_t^k(\hat{x}_t, \hat{u}_t) \circ (D_1 f_t(\hat{x}_t, \hat{u}_t))^{-1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t) - D_2 h_t^k(\hat{x}_t, \hat{u}_t)$ for all $t \in \mathbb{N}$, the family $((S_t^j)_{1 \leq j \leq d^i}, (M_t^k)_{1 \leq k \leq d^e})$ is linearly independent.

Then there exist $\lambda_0 \in \mathbb{R}$, $(p_{t+1})_{t \in \mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$, $(\mu_t)_{t \in \mathbb{N}} \in (\mathbb{R}^{d^i *})^{\mathbb{N}}$ and $(\nu_t)_{t \in \mathbb{N}} \in (\mathbb{R}^{d^e *})^{\mathbb{N}}$ which satisfy the following conditions:

(i) $(\lambda_0, p_1, \mu_0, \nu_0) \neq (0, 0, 0, 0)$.
(ii) $\lambda_0 \geq 0$, $\mu_t \geq 0$ and $\langle \mu_t, g_t(\hat{x}_t, \hat{u}_t) \rangle = 0$ for all $t \in \mathbb{N}$.
(iii) For all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$ when $a = i$.
(iv) For all $t \in \mathbb{N}$, $p_t = D_1 L_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0, \mu_t, \nu_t)$.
(v) For all $t \in \mathbb{N}$, $D_2 L_t(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0, \mu_t, \nu_t) = 0$.

Proof. We do the proof in the case $a = e$. The case $a = i$ is similar. We use the method of reduction to the finite horizon. For all $T \in \mathbb{N}$, $T \geq 2$, the restriction $(\hat{x}_0, \dots, \hat{x}_T, \hat{u}_0, \dots, \hat{u}_{T-1})$ is a solution of the problem

$$(\mathscr{F}C_e(T,\eta,\hat{x}_T)) \begin{cases} \text{maximize } J_T(x_0,\dots,x_T,u_0,\dots,u_{T-1}) \\ \quad \text{when} \quad \forall t \in \{0,\dots,T-1\}, x_{t+1} = f_t(x_t,u_t) \\ \qquad\qquad \forall t \in \{0,\dots,T-1\}, u_t \in \mathscr{U}_t(x_t) \\ \qquad\qquad x_0 = \eta, x_T = \hat{x}_T. \end{cases}$$

Note that x_0 and x_T are not variables of this problem. As in Sect. 1.4, we translate this problem into a problem of static optimization on which we can use the multiplier rule of Halkin that permits to obtain $\lambda_0^T \in \mathbb{R}$, $p_1^T, \dots, p_T^T \in \mathbb{R}^{n*}$, $\mu_0^T, \dots, \mu_{T-1}^T \in \mathbb{R}^{d^i}*$, $\nu_0^T, \dots, \nu_{T-1}^T \in \mathbb{R}^{d^e}*$ such that the following properties hold:

$$(\lambda_0^T, p_1^T, \dots, p_T^T, \mu_0^T, \dots, \mu_{T-1}^T, \nu_0^T, \dots, \nu_{T-1}^T) \neq 0. \tag{2.44}$$

$$\lambda_0^T, \forall t \in \{0,\dots,T-1\}, \mu_t^T \geq 0, \langle \mu_t^T, g_t(\hat{x}_t,\hat{u}_t)\rangle = 0. \tag{2.45}$$

$$\forall t \in \{1,\dots,T-1\}, p_t^T = D_1 L_t(\hat{x}_t,\hat{u}_t,p_{t+1}^T,\lambda_0^T,\mu_t^T,\nu_t^T). \tag{2.46}$$

$$\forall t \in \mathbb{N}, D_2 L_t(\hat{x}_t,\hat{u}_t,p_{t+1}^T,\lambda_0^T,\mu_t^T,\nu_t^T) = 0. \tag{2.47}$$

We want to prove the following assertion:

$$(\lambda_0^T, p_1^T, \mu_0^T, \nu_0^T) \neq 0. \tag{2.48}$$

To abridge the writing, we set $\hat{\phi}_t := \phi_t(\hat{x}_t,\hat{u}_t)$, $\hat{f}_t := f_t(\hat{x}_t,\hat{u}_t)$, $\hat{g}_t := g_t(\hat{x}_t,\hat{u}_t)$, $\hat{h}_t := h_t(\hat{x}_t,\hat{u}_t)$. We proceed by contradiction, we assume that $(\lambda_0^T, p_1^T, \mu_0^T, \nu_0^T) = 0$. Then using (2.46) and (2.47) for $t = 1$ we obtain $0 = p_2^T \circ D_1\hat{f}_t + \mu_1^T \circ D_1\hat{g}_t + \nu_1^T \circ D_1\hat{h}_t$ and $0 = p_2^T \circ D_2\hat{f}_t + \mu_1^T \circ D_2\hat{g}_t + \nu_1^T \circ D_2\hat{h}_t$ from which we deduce $-p_2^T = \mu_1^T \circ D_1\hat{g}_t \circ (D_1\hat{f}_t)^{-1} + \nu_1^T \circ D_1\hat{h}_t \circ (D_1\hat{f}_t)^{-1}$ and $-p_2^T \circ D_2\hat{f}_t = \mu_1^T \circ D_2\hat{g}_t + \nu_1^T \circ D_2\hat{h}_t$, which implies

$$\begin{aligned} -p_2^T \circ D_2\hat{f}_t &= \mu_1^T \circ D_1\hat{g}_t \circ (D_1\hat{f}_t)^{-1} \circ D_2\hat{f}_t + \nu_1^T \circ D_1\hat{h}_t \circ (D_1\hat{f}_t)^{-1} \circ D_2\hat{f}_t \\ &= \mu_1^T \circ D_2\hat{g}_t + \nu_1^T \circ D_2\hat{h}_t \end{aligned}$$

Denoting $S_t := (S_t^1, \dots, S_t^{d^i})$ and $M_t := (M_t^1, \dots, M_t^{d^e})$, we deduce from the last relation

$$\mu_1^T \circ S_t + \nu_1^T \circ M_t = 0,$$

and using the coordinates and the assumption (3) we obtain $\mu_1^T = 0$ and $\nu_1^T = 0$. Then (2.46) for $t = 1$ implies $p_2^T \circ D_1\hat{f}_t = 0$, and assumption (2) implies $p_2^T = 0$. And so we have proven that $(\lambda_0^T, p_1^T, \mu_0^T, \nu_0^T) = 0$ implies $(\lambda_0^T, p_2^T, \mu_1^T, \nu_1^T) = 0$. Iterating this reasoning we obtain, for all $t \in \mathbb{N}$, $p_{t+1}^T = 0$, $\mu_t^T = 0$, and $\nu_t^T = 0$, which is a contradiction with (2.44). And so (2.48) is proven. Using a normalization, i.e., multiplying all the multipliers by $\|(\lambda_0^T, p_1^T, \mu_0^T, \nu_0^T)\|$, we can assume that $\|(\lambda_0^T, p_1^T, \mu_0^T, \nu_0^T)\| = 1$. Consequently the sequences $T \mapsto \lambda_0^T$,

$T \mapsto p_1^T$, $T \mapsto \mu_0^T$, and $T \mapsto \nu_0^T$ are bounded. Then from (2.46) we obtain $p_2^T = (p_1^T - \lambda_0^T D_1\hat{\phi}_1 - \mu_1^T \circ D_1\hat{g}_1 - \nu_1^T \circ D_1\hat{h}_1) \circ (D_1\hat{f}_1)^{-1}$ which implies $p_2^T \circ D_2\hat{f}_1 = (p_1^T - \lambda_0^T D_1\hat{\phi}_1 - \mu_1^T \circ D_1\hat{g}_1 - \nu_1^T \circ D_1\hat{h}_1) \circ (D_1\hat{f}_1)^{-1} \circ D_2\hat{f}_1$, and from (2.47) we obtain $p_2^T \circ D_2\hat{f}_1 = -\lambda_0^T D_2\hat{\phi}_1 - \mu_1^T \circ D_2\hat{g}_1 - \nu_1^T \circ D_2\hat{h}_1$. From these two equalities we deduce

$$\mu_1^T \circ S_1 + \nu_1^T \circ M_1 = p_1^T \circ (D_1\hat{f}_1)^{-1} \circ D_2\hat{f}_1 + \lambda_0^T (D_2\hat{\phi}_1 - D_1\hat{\phi}_1 \circ (D_1\hat{f}_1)^{-1} \circ D_2\hat{f}_1.$$

The right-hand term is bounded as function of T, and consequently $T \mapsto \mu_1^T \circ S_1 + \nu_1^T \circ M_1$ is bounded. Then, translating this expression in terms of coordinates, using assumption (3) and Lemma 2.2, we obtain that the sequences $T \mapsto \mu_1^T$ and $T \mapsto \nu_1^T$ are bounded. And then, from $p_2^T = (p_1^T - \lambda_0^T D_1\hat{\phi}_1 - \mu_1^T \circ D_1\hat{g}_1 - \nu_1^T \circ D_1\hat{h}_1) \circ (D_1\hat{f}_1)^{-1}$, we obtain that $T \mapsto p_2^T$ is bounded. Iterating this reasoning, we obtain that, for all $t \in \mathbb{N}$, the sequences $T \mapsto p_t^T$, $T \mapsto \mu_t^T$ and $T \mapsto \nu_t^T$ are bounded. And then we can use Lemma 2.1 and conclude as in the proof of Theorem 2.3. □

This result appears in the paper of Blot [12]. To finish this section we give a strong Pontryagin principle. We consider the following simplified constraints:

$$\mathscr{U}_t^1(x_t) := \{u_t \in U_t : \forall j \in \{1, \dots, d^i\}, g_t^j(x_t, u_t) \geq 0\}. \tag{2.49}$$

For $\ell \in \{n, s, o, w\}$, we denote by $(\mathscr{C}_e^{\ell,1})$ the problem obtained by replacing $\mathscr{U}_t(x_t)$ by $\mathscr{U}_t^1(x_t)$ into $(\mathscr{C}_a^n)$. For these simplified constraints, the Lagrangian becomes $L_t^1(x, u, p, \lambda, \mu) := \lambda\phi_t(x, u) + \langle p, f_t(x, u)\rangle + \langle \mu, g_t(x, u)\rangle$. To use the condition of Michel, we ought to consider $A_t(x_t, x_{t+1})$ as the set of all $(r_t, \zeta_t, \xi_t) \in U_t \times \mathbb{R}^n \times \mathbb{R}^{d^i}$ for which there exists $u_t \in U_t$ satisfying $r_t \leq \phi_t(x_t, u_t)$, $\zeta_t = f_t(x_t, u_t) - x_{t+1}$ and $\xi_t \leq g_t(x_t, u_t)$. $B_t(x_t, x_{t+1})$ is the set of all $(r_t, \zeta_t, \xi_t) \in U_t \times \mathbb{R}^n \times \mathbb{R}^{d^i}$ for which there exists $(u_t, \alpha_t, \beta_t) \in U_t \times \mathbb{R}^n \times \mathbb{R}^{d^i}$ satisfying $r_t \leq \phi_t(x_t, u_t)$, $\alpha_t^k \zeta_t^k = f_t^k(x_t, u_t) - x_{t+1}^k$ for all $k \in \{1, \dots, n\}$and $\beta_t^j \xi_t^j \leq g_t^j(x_t, u_t)$ for all $j \in \{1, \dots, d^i\}$.

Theorem 2.20. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{C}_e^{n,1})$, or of $(\mathscr{C}_e^{s,1})$, or of $(\mathscr{C}_e^{o,1})$, or of $(\mathscr{C}_e^w)$. We assume that the following conditions are fulfilled:*

(1) For all $t \in \mathbb{N}$, X_t is nonempty open and convex, the functions ϕ_t, f_t, g_t are continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and differentiable at $(\hat{x}_t, \hat{u}_t)$.
(2) For all $t \in \mathbb{N}$, $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.
(3) For all $t \in \mathbb{N}$, for all $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, $coA_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$.
(4) For all $t \in \mathbb{N}$, there exists $\tilde{u}_t \in U_t(\hat{x}_t)$ such that $f_t(\hat{x}_t, \tilde{u}_t) = f_t(\hat{x}_t, \hat{u}_t)$ and $g_t^j(\hat{x}_t, \tilde{u}_t) > 0$, for all $j \in \{1, \dots, d^i\}$.

Then there exist $\lambda_0 \in R$, $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$, $(\mu_t)_{t\in\mathbb{N}} \in (\mathbb{R}^{d^i*})^{\mathbb{N}}$ which satisfy the following conditions:

(i) $(\lambda, p_1) \neq (0, 0)$.
(ii) $\lambda_0 \geq 0$, $\mu_t \geq 0$ and $\langle \mu_t, g_t(\hat{x}_t, \hat{u}_t)\rangle = 0$ for all $t \in \mathbb{N}$.

(iii) *For all* $t \in \mathbb{N}$, $p_t = D_1 L_t^1(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0, \mu_t)$.
(iv) *For all* $t \in \mathbb{N}$, $L_t^1(\hat{x}_t, \hat{u}_t, p_{t+1}, \lambda_0, \mu_t) = \max_{u_t \in U_t} L_t^1(\hat{x}_t, u_t, p_{t+1}, \lambda_0, \mu_t)$.

This result appears in the paper of Blot and Hayek [23] where a proof is given. This paper contains other results on the constrained problems.

2.5 Multiobjective Problems

Results of the previous sections are extended to multiobjective problems by using similar methods. All the results of this section are due to Hayek [51] and [50]. The controlled dynamical systems are still (DE) and (DI). The difference with the previous sections is that we replace ϕ_t by several functions $\phi_{1,t}, \ldots, \phi_{m,t}$ from $X_t \times U_t$ into $\mathbb{R}$. We define $J_j(\underline{x}, \underline{u}) := \sum_{t=0}^{+\infty} \phi_{j,t}(x_t, u_t)$ when the series converges in $\mathbb{R}$. And we define $\mathrm{Dom}_\eta^a(J_j)$ as the set of all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta$ such that the series $\sum_{t=0}^{+\infty} \phi_{j,t}(x_t, u_t)$ converges in $\mathbb{R}$. We introduce the notation $\mathrm{Dom}_\eta^a((J_j)_{1 \le j \le m}) := \bigcap_{j=1}^{m} \mathrm{Dom}_\eta^a(J_j)$. The notions of optimality are notions of Pareto optimality and of weak Pareto optimality. Precisely the considered problems are the following ones.

$(\mathcal{V}_a^n)$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Dom}_\eta^a((J_j)_{1 \le j \le m})$ such that there does not exist any $(\underline{x}, \underline{u}) \in \mathrm{Dom}_\eta^a((J_j)_{1 \le j \le m})$ such that $J_j(\underline{x}, \underline{u}) \ge J_j(\hat{\underline{x}}, \hat{\underline{u}})$ for all $j \in \{1, \ldots, m\}$ and $J_h(\underline{x}, \underline{u}) > J_h(\hat{\underline{x}}, \hat{\underline{u}})$ for some $h \in \{1, \ldots, m\}$.

$(\mathcal{V}_a^{n,w})$ Find $(\hat{\underline{x}}, \hat{\underline{u}} \in \mathrm{Dom}_\eta^a((J_j)_{1 \le j \le m})$ such that there does not exist any $(\underline{x}, \underline{u}) \in \mathrm{Dom}_\eta^a((J_j)_{1 \le j \le m})$ such that $J_j(\underline{x}, \underline{u}) > J_j(\hat{\underline{x}}, \hat{\underline{u}})$ for all $j \in \{1, \ldots, m\}$.

$(\mathcal{V}_a^o)$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Adm}_\eta^a$ such that there does not exist any $(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta^a$ such that $\limsup_{T \to +\infty} \sum_{t=0}^{T} (\phi_{j,t}(x_t, u_t) - \phi_{j,t}(\hat{x}_t, \hat{u}_t)) \ge 0$ for all $j \in \{1, \ldots, m\}$ and $\limsup_{T \to +\infty} \sum_{t=0}^{T} (\phi_{h,t}(x_t, u_t) - \phi_{h,t}(\hat{x}_t, \hat{u}_t)) > 0$ for some $h \in \{1, \ldots, m\}$.

$(\mathcal{V}_a^{o,w})$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Adm}_\eta^a$ such that there does not exist any $(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta^a$ such that $\limsup_{T \to +\infty} \sum_{t=0}^{T} (\phi_{j,t}(x_t, u_t) - \phi_{j,t}(\hat{x}_t, \hat{u}_t)) > 0$ for all $j \in \{1, \ldots, m\}$.

$(\mathcal{V}_a^w)$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Adm}_\eta^a$ such that there does not exist any $(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta^a$ such that $\liminf_{T \to +\infty} \sum_{t=0}^{T} (\phi_{j,t}(x_t, u_t) - \phi_{j,t}(\hat{x}_t, \hat{u}_t)) \ge 0$ for all $j \in \{1, \ldots, m\}$ and $\liminf_{T \to +\infty} \sum_{t=0}^{T} (\phi_{h,t}(x_t, u_t) - \phi_{h,t}(\hat{x}_t, \hat{u}_t)) > 0$ for some $h \in \{1, \ldots, m\}$.

$(\mathscr{V}_a^{w,w})$ Find $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{Adm}_\eta^a$ such that there does not exist any $(\underline{x}, \underline{u}) \in \mathrm{Adm}_\eta^a$ such that $\liminf_{T \to +\infty} \sum_{t=0}^{T} (\phi_{j,t}(x_t, u_t) - \phi_{j,t}(\hat{x}_t, \hat{u}_t)) > 0$ for all $j \in \{1, \dots, m\}$.

A solution of $(\mathscr{V}_a^n)$ (respectively $(\mathscr{V}_a^o)$, respectively $(\mathscr{V}_a^w)$) is called a Pareto optimal solution (respectively an overtaking Pareto optimal solution, respectively a weak overtaking Pareto optimal solution). For the solutions of $(\mathscr{V}_a^{n,w})$, $(\mathscr{V}_a^{o,w})$, $(\mathscr{V}_a^{w,w})$ we replace Pareto by weak Pareto optima.

We start with a first result of necessary conditions for weak Pareto optima in the form of a weak Pontryagin principle.

Theorem 2.21. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{V}_a^{n,w})$, or of $(\mathscr{V}_a^{o,w})$, or of $(\mathscr{V}_a^{w,w})$ when $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, $\hat{u}_t \in \operatorname{int} U_t$, $\phi_{j,t}$, and f_t are of class C^1 at $(\hat{x}_t, \hat{u}_t)$ for all $j \in \{1, \dots, m\}$.
(b) For all $t \in \mathbb{N}$, $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.

Then there exist $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and $(p_{t+1})_{t \in \mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$ which satisfy the following conditions:

(i) $(\lambda_1, \dots, \lambda_m, p_1) \neq (0, \dots, 0, 0)$.
(ii) For all $j \in \{1, \dots, m\}$, $\lambda_j \geq 0$, and when $a = i$, for all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$.
(iii) For all $t \in \mathbb{N}_*$, $p_t = \sum_{j=1}^{m} \lambda_j D_1 \phi_{j,t}(\hat{x}_t, \hat{u}_t) + p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t)$.
(iv) For all $t \in \mathbb{N}$, $\sum_{j=1}^{m} \lambda_j D_2 \phi_{j,t}(\hat{x}_t, \hat{u}_t) + p_{t+1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t) = 0$.

The proof of this result uses the method of reduction to finite horizon. Since the associated finite-horizon problems are now multiobjective problems while they were single-objective problems in the previous sections, the multiplier rules of static optimization (of Halkin or Clarke) are replaced by a multiplier rule which is special to static multiobjective problems and based on a theorem of Motzkin [51]. After that, the question is to extract the multipliers of the infinite-horizon problem from the sequences of multipliers of the finite-horizon problems, and the reasoning is similar to the reasoning of the previous sections.

Remark 2.6. When $a = i$, there exists in [51] a theorem where the condition of invertibility is replaced by the positivity condition as defined in Sect. 2.1.2 for single-objective problems. Moreover in the previous theorem, if in addition we assume that $D_2 f_0(\eta, \hat{u}_0)$ is onto, we have $(\lambda_1, \dots, \lambda_m) \neq (0, \dots, 0)$.

After a weak Pontryagin principle, we state a result in the form of a strong Pontryagin principle.

Theorem 2.22. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(\mathscr{V}_a^{n,w})$, or of $(\mathscr{V}_a^{o,w})$, or of $(\mathscr{V}_a^{w,w})$ when $a \in \{e, i\}$. We assume that the following conditions are fulfilled:*

(a) For all $t \in \mathbb{N}$, X_t is convex, and for all $j \in \{1, \dots, m\}$, for all $u_t \in U_t$, $\phi_{j,t}(., u_t)$ and $f_t(., u_t)$ are of class C^1 at $\hat{x}_t$.
(b) For all $t \in \mathbb{N}$, $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.
(c) For all $t \in \mathbb{N}$, for all $x_t \in X_t$, for all $u'_t, u''_t \in U_t$, for all $\theta \in [0, 1]$, there exists $u_t \in U_t$ such that, for all $j \in \{1, \dots, m\}$, $\phi_{j,t}(x_t, u_t) \geq (1-\theta)\phi_{j,t}(x_t, u'_t) + \theta\phi_{j,t}(x_t, u''_t)$ and $f_t(x_t, u_t) = (1-\theta) f_t(x_t, u'_t) + \theta f_t(x_t, u''_t)$.

Then there exist $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$ which satisfy the following conditions:

(i) $(\lambda_1, \dots, \lambda_m, p_1) \neq (0, \dots, 0, 0)$.
(ii) For all $j \in \{1, \dots, m\}$, $\lambda_j \geq 0$, and when $a = i$, for all $t \in \mathbb{N}$, $p_{t+1} \geq 0$ and $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.
(iii) For all $t \in \mathbb{N}_*$, $p_t = \sum_{j=1}^{m} \lambda_j D_1\phi_{j,t}(\hat{x}_t, \hat{u}_t) + p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t)$.
(iv) For all $t \in \mathbb{N}$, for all $u_t \in U_t$, $\sum_{j=1}^{m} \lambda_j \phi_{j,t}(\hat{x}_t, \hat{u}_t) + \langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t)\rangle \geq \sum_{j=1}^{m} \lambda_j \phi_{j,t}(\hat{x}_t, u_t) + \langle p_{t+1}, f_t(\hat{x}_t, u_t)\rangle$.

The proof of this result also uses the method of the reduction to finite horizon. The tool of static multiobjective optimization which is used is a theorem of Khanh and Nuong (Theorem 2.2 in [58]). We recognize in assumption (c) a generalization of the condition of Ioffe and Tihomirov. The end of the proof is similar to this one of strong Pontryagin principles of the previous sections.

Remark 2.7. If moreover we assume that $d \geq n$ and that $f_0(\{\eta\} \times U_0) - \hat{x}_1$ is a neighborhood of 0 in $\mathbb{R}^n$ or that there exists $u'_0 \in U_0$ such that $\hat{x}_1^k < f_0^k(\eta, u'_0)$ for all $k \in \{1, \dots, m\}$, then we have $(\lambda_1, \dots, \lambda_m) \neq (0, \dots, 0)$. In the previous theorem, when $a = i$, we can replace the invertibility condition by the positivity condition as in Sect. 2.2.2.

The following result is a result of sufficient conditions.

Theorem 2.23. *Let $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{Dom}^e_\eta((J_j)_{1\leq j\leq m})$. We assume that there exist $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$ which satisfy the following conditions:*

(i) $(\lambda_1, \dots, \lambda_m, p_1) \neq (0, \dots, 0, 0)$.
(ii) For all $j \in \{1, \dots, m\}$, $\lambda_j \geq 0$.
(iii) For all $t \in \mathbb{N}_*$, $p_t = \sum_{j=1}^{m} \lambda_j D_1\phi_{j,t}(\hat{x}_t, \hat{u}_t) + p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t)$.
(iv) For all $t \in \mathbb{N}$, for all $u_t \in U_t$, $\sum_{j=1}^{m} \lambda_j \phi_{j,t}(\hat{x}_t, \hat{u}_t) + \langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t)\rangle \geq \sum_{j=1}^{m} \lambda_j \phi_{j,t}(\hat{x}_t, u_t) + \langle p_{t+1}, f_t(\hat{x}_t, u_t)\rangle$.
(v) For all $t \in \mathbb{N}$, $X_t \times U_t$ is convex and the function

$$(x_t, u_t) \mapsto \sum_{j=1}^{m} \lambda_j \phi_{j,t}(x_t, u_t) + \langle p_{t+1}, f_t(x_t, u_t)\rangle \text{ is concave.}$$

(vi) *For all* $x_t \in X_t$, $\lim_{t\to+\infty} \langle p_{t+1}, x_t - \hat{x}_t\rangle = 0$.

Then $(\underline{\hat{x}}, \underline{\hat{u}})$ *is a solution of* $(\mathcal{V}_e^{n,w})$*, and moreover if* $\lambda_j > 0$ *for all* $j \in \{1, \dots, m\}$ *then* $(\underline{\hat{x}}, \underline{\hat{u}})$ *is a solution of* $(\mathcal{V}_e^{n})$*.*

The proof of this result uses the well-known fact that is: if $(\underline{\hat{x}}, \underline{\hat{u}})$ maximizes the weighted functional $\sum_{j=1}^{m} \theta_j J_j(\underline{x}, \underline{u})$ where $\theta_j \geq 0$ for all $j \in \{1, \dots, m\}$, then $(\underline{\hat{x}}, \underline{\hat{u}})$ is a weak Pareto optimum, i.e., a solution of $(\mathcal{V}_e^{n,w})$. The concavity condition permits to transform necessary conditions of optimality on the weighted functional into sufficient conditions of optimality. The assumption (vi) is called a sufficient condition of transversality at infinity.

Remark 2.8. Using the function

$$(x_t, p_{t+1}, \lambda_1, \dots, \lambda_m) \mapsto \max_{u_t \in U_t} \left(\sum_{j=1}^{m} \lambda_j \phi_{j,t}(x_t, u_t) + \langle p_{t+1}, f_t(x_t, u_t)\rangle \right)$$

it is possible to state an additional theorem of sufficient conditions, [51].

To finish this section, we provide a strong Pontryagin principle in presence of constraints in the form $\mathcal{U}_t^1(x_t)$ as defined in (2.49). $\mathrm{Adm}_{\eta,c}^a$ is defined by replacing U_t by $\mathcal{U}_t^1(x_t)$ in Adm_{η}^a, $\mathrm{Dom}_{\eta,c}^a(J_j)$ is defined by replacing U_t by $\mathcal{U}_t^1(x_t)$ in $\mathrm{Dom}_{\eta}^a(J_j)$, and $\mathrm{Dom}_{\eta,c}^a((J_j)_{1\leq j\leq m}) := \bigcap_{1\leq j\leq m} \mathrm{Dom}_{\eta,c}^a(J_j)$. When $\ell \in \{n, o, w\}$ and $a \in \{e, i\}$, $(\mathcal{V}_a^{\ell,c})$ and $(\mathcal{V}_a^{\ell,w,c})$ are obtained by replacing Adm_{η}^a by $\mathrm{Adm}_{\eta,c}^a$ and $\mathrm{Dom}_{\eta}^a((J_j)_{1\leq j\leq m})$ by $\mathrm{Dom}_{\eta,c}^a((J_j)_{1\leq j\leq m})$ in $(\mathcal{V}_a^{\ell})$ and $(\mathcal{V}_a^{\ell,w})$. In the conditions of Michel, the sets $A_t(x_t, x_{t+1})$ and $B_t(x_t, x_{t+1})$ are defined as before in Theorem 2.20 in the previous section.

Theorem 2.24. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a solution of* $(\mathcal{V}_a^{n,w,c})$ *or of* $(\mathcal{V}_a^{o,w,c})$ *or of* $(\mathcal{V}_a^{w,w,c})$*. We assume that the following conditions are fulfilled:*

(a) *For all* $t \in \mathbb{N}$, X_t *is nonempty open and convex, and for all* $u_t \in U_t$*, for all* $j \in \{1, \dots, m\}$, $\phi_{j,t}(., u_t)$, $f_t(., u_t)$ *and* $g_t(., u_t)$ *are of class* C^1 *on* X_t*.*
(b) *For all* $t \in \mathbb{N}$*, for all* $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, $coA_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$*.*
(c) *For all* $t \in \mathbb{N}$, $D_1 f_t(\hat{x}_t, \hat{u}_t)$ *is invertible.*
(d) *For all* $t \in \mathbb{N}$*, there exists* $u'_t \in U_t$ *such that* $f_t(\hat{x}_t, u'_t) = f_t(\hat{x}_t, \hat{u}_t)$ *and* $g_t^h(\hat{x}_t, u'_t) > 0$ *for all* $h \in \{1, \dots, d^i\}$*.*

Then there exist $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$, $(p_{t+1})_{t\in\mathbb{N}} \in (\mathbb{R}^{n*})^{\mathbb{N}}$ *and* $(q_t)_{t\in\mathbb{N}} \in \mathbb{R}^{d^i *}$ *which satisfy the following conditions:*

(i) $(\lambda_1, \dots, \lambda_m, p_1) \neq (0, \dots, 0, 0)$.
(ii) $\lambda_j \geq 0$ *for all* $j \in \{1, \dots, m\}$*, and* $q_t \geq 0$ *for all* $t \in \mathbb{N}$*.*

(iii) $p_t = \sum_{j=1}^{m} \lambda_j D_1 \phi_{j,t}(\hat{x}_t, \hat{u}_t) + p_{t+1}, \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + q_t \circ D_1 g_t(\hat{x}_t, \hat{u}_t)$ *for all* $t \in \mathbb{N}$.

(iv) $\sum_{j=1}^{m} \lambda_j \phi_{j,t}(\hat{x}_t, \hat{u}_t) + \langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t)\rangle + \langle q_t, g_t(\hat{x}_t, \hat{u}_t)\rangle \geq$
$\sum_{j=1}^{m} \lambda_j \phi_{j,t}(\hat{x}_t, u_t) + \langle p_{t+1}, f_t(\hat{x}_t, u_t)\rangle + \langle q_t, g_t(\hat{x}_t, u_t)\rangle$ *for all* $u_t \in U_t$, *for all* $t \in \mathbb{N}$.

Moreover, if in addition we assume that $f_0(\{\eta\} \times U_0) - \hat{x}_1$ *is a neighborhood of 0 in* $\mathbb{R}^n$, *and if there exists* $u_0'' \in U_0$ *such that* $f_0(\eta, u_0'') - \hat{x}_1 = 0$ *and* $g_t^h(\eta, u_0'') > 0$ *for all* $h \in \{1, \ldots, d^i\}$, *then we have* $(\lambda_1, \ldots, \lambda_m) \neq (0, \ldots, 0)$.

The proof of this theorem also uses the reduction to finite horizon. In addition it uses a generalization of the parametrized static optimization Theorem 1.6 for single-objective problems to the multiobjective case. This generalization to weak Pareto optima can be found in Hayek [50].

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