

Chapter 2

Examples of Lyapunov Exponents in Two-Dimensional Systems

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Abstract In this paper, we construct two examples of two-dimensional dynamical systems. In both examples, we have computed the Lyapunov exponents for the forward trajectory of $(0,0)$. In the first, such a forward trajectory has positive Lyapunov exponent but does not have sensitive dependence on initial conditions, while the second example has negative Lyapunov exponent but sensitive dependence on initial conditions.

2.1 Introduction

In recent years, Lyapunov exponents have been a powerful tool for understanding chaotic behavior and predictability in nonlinear discrete dynamical systems and time series obtained from models. But this tool introduces a host of difficulties of interpretation in the sense of deciding whether chaos and random effects are present or whether it is possible to determine the existence of external influences.

At the same time, it is a widespread practice, especially in experimental and applied dynamics, to associate the presence of a positive Lyapunov exponent with instability and a negative Lyapunov exponent with stability of orbits of a dynamical system. Stability and instability of orbits are defined in topological terms, while a Lyapunov exponent is a numerical characteristic calculated throughout the orbit. However, these interpretations have no firm mathematical foundation unless some restrictions on the maps describing the system are introduced. To illustrate this, in [3, 4], two interval maps have been constructed: one has an orbit with positive Lyapunov exponent but is Lyapunov stable, while the other has an orbit with negative

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Lyapunov exponent and sensitive dependence on initial conditions (s.d.i.c.). Such examples have been improved in some sense in [1].

Using the examples from [1], in this paper we construct examples of two discrete two-dimensional dynamical systems with similar behaviors to the former. Before constructing them, it is necessary to begin with some introductory definitions.

2.2 Definitions and Preliminary Results

Given a dynamical systems (X, f) , where X is a compact space and f a continuous map from X into itself, the trajectory of a point $x \in X$ is the sequence $(f^n(x))_{n=0}^\infty = (x_n)_{n=0}^\infty$, where as usual, f^n denotes the n -iteration of f acting on x . Two trajectories starting from nearby initial conditions can diverge or converge in certain directions under f as n grows. Lyapunov exponents provide a means of quantifying the expansion or contraction of such trajectories in different directions.

Let $X \subset \mathbb{R}^m$ and let d be any metric on it. If $(x_n)_{n=0}^\infty$ and $(x'_n)_{n=0}^\infty$ are two trajectories starting from nearby initial states x_0 and x'_0 , we write $\delta x_n = x'_n - x_n$. If f has continuous partial derivatives in every x_i , then iterating the map, we have a linear approximation in which $Df(x)$ denotes the differential of the map f at the point x with the (i, j) element of this matrix given by $\frac{\partial f_i}{\partial x_j}$, where f_i and x_j are the components of f and x in local coordinates on X , and

$$Df^n(x_0) = Df(x_{n-1})Df(x_{n-2}) \cdots Df(x_1)Df(x_0),$$

where δx_n represents the separation of these orbits after n iterations of the map f :

$$\delta x_n \simeq Df^n(x_0)\delta x_0 = \prod_{i=0}^{n-1} Df(x_i)\delta x_0.$$

The matrix $(Df^n(x_0)^t)(Df^n(x_0))$ has m eigenvalues given by $\mu_i(n, x_0)$, where $i = 1, 2, \dots, m$, such that $\mu_1(n, x_0) \geq \mu_2(n, x_0) \geq \cdots \geq \mu_m(n, x_0)$.

Definition 1. The i th local Lyapunov exponent at x_0 is defined by

$$\lambda_i(x_0) = \lim_{n \rightarrow \infty} \frac{1}{2n} \log(|\mu_i(n, x_0)|)$$

if this limit exists.

In [5] are stated conditions for the existence of such a limit. Now we recall the notions of instability and stability in the Lyapunov sense.

Definition 2 (Sensitive dependence or Lyapunov instability). The forward orbit $(x_n)_{n=0}^\infty$ has sensitive dependence on initial conditions (s.d.i.c.) if there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists y with $d(x_0, y) < \delta$ and $N \geq 0$ such that $d(f^N(y), f^N(x_0)) \geq \varepsilon$.

Definition 3. The forward orbit $(x_n)_{n=0}^{\infty}$ does not have s.d.i.c. or it is Lyapunov stable if for every $\varepsilon > 0$, there is $\delta > 0$ such that whenever $d(x_0, y) < \delta$, then $d(f^n(y), f^n(x_0)) < \varepsilon$ for all $n \geq 0$.

2.3 Two Two-Dimensional Examples

We propose two dynamical systems, one defined in $[0, 1]^2$ that has a forward trajectory with a positive Lyapunov exponent but not having s.d.i.c., and other defined in $[0, 1]^2$ that has a forward trajectory with a negative Lyapunov exponent but having s.d.i.c. The examples are two-dimensional versions of those mentioned in the introduction. The maps we are using are examples of permutation maps introduced in [2].

Example 1. We are going to obtain a continuous function $F = (f, g)$ in $[0, 1]^2$ such that the forward trajectory of $(0, 0)$ has a positive Lyapunov exponent but does not have s.d.i.c.

(a) The first component of F is the map $f : [0, 1] \rightarrow [0, 1]$ that was introduced in [3]:

$$f(x) = \begin{cases} 2x - 1 + \frac{1}{2^{n+1}}, & a_n < x \leq b_n, x \in [0, 1], \\ \frac{5^{n+2} - 22}{2 \cdot 5^{n+2} - 11}(x - b_n) + 1 + \frac{2}{10^{n+1}} - \frac{1}{2^{n+1}}, & b_n < x \leq a_{n+1}, \\ 1, & x = 1, \end{cases}$$

with $a_n = 1 - 2^{-n} - 10^{-n-1}$, $b_n = 1 - 2^{-n} + 10^{-n-1}$, $n = 0, 1, 2, \dots$

(b) The second component of F is $g : [0, 1] \rightarrow [0, 1]$ such that

$$g(x) = \begin{cases} 3x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{15}, \\ \frac{6}{127}x + \frac{7}{10} - \frac{2}{635}, & \frac{1}{15} < x \leq \frac{1}{2} - \frac{1}{100}, \\ 3x + \frac{1}{2} - \frac{5}{2^{n+1}}(2^n - 1), & a_n < x \leq b_n, \\ \frac{5^{n+2} - 33}{2 \cdot 5^{n+2} - 11}(x - b_n) + 1 + \frac{3}{10^{n+1}} - \frac{1}{2^{n+1}}, & b_n < x \leq a_{n+1}, \\ 1, & x = 1, \end{cases}$$

with $a_n = 1 - 2^{-n} - 10^{-n-1}$, $b_n = 1 - 2^{-n} + 10^{-n-1}$, $n = 1, 2, \dots$

The map $F(x, y) = (f(y), g(x))$ is continuous in $[0, 1]^2$ because f and g are continuous in $[0, 1]$. We consider the trajectory of $(0, 0)$:

$$\{(0, 0), (x_1, y_1), (x_2, y_2), \dots\} = \left\{ \left(1 - \frac{1}{2^k}, 1 - \frac{1}{2^k} \right) \right\}_{k=0}^{\infty}$$

The map is differentiable at every point of this trajectory (except $(0, 0)$). Since f and g are differentiable maps to the right of 0, we define

$$DF(0^+, 0^+) = \begin{pmatrix} 0 & \lim_{y \rightarrow 0^+} f(y) \\ \lim_{x \rightarrow 0^+} g(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$$

and

$$DF(x_1, y_1) = DF^2(0, 0) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix},$$

$$DF(x_2, y_2) = DF^3(0, 0) = \begin{pmatrix} 0 & 12 \\ 18 & 0 \end{pmatrix},$$

that is, in general,

$$DF^{2n}(0, 0) = \begin{pmatrix} 6^n & 0 \\ 0 & 6^n \end{pmatrix} \quad DF^{2n-1}(0, 0) = \begin{pmatrix} 0 & 2 \cdot 6^{n-1} \\ 3 \cdot 6^{n-1} & 0 \end{pmatrix}$$

for $n = 1, 2, \dots$

Now we compute the eigenvalues $(DF^k)^t(DF^k)$, where $k = 2n$ or $k = 2n - 1$ with $n = 1, 2, \dots$. Then we have, when $k = 2n - 1$,

$$(DF^{2n-1}(0, 0))^t DF^{2n-1}(0, 0) = \begin{pmatrix} 3^2 \cdot 6^{2(n-1)} & 0 \\ 0 & 2^2 \cdot 6^{2(n-1)} \end{pmatrix},$$

and the largest eigenvalue of such a matrix is $\mu_1(2n - 1, (0, 0)) = 3^2 6^n$, and when $k = 2n$,

$$(DF^{2n}(0, 0))^2 = \begin{pmatrix} 6^{2n} & 0 \\ 0 & 6^{2n} \end{pmatrix},$$

whose eigenvalue is $\mu_1(2n, (0, 0)) = 6^{2n}$.

Therefore, the Lyapunov exponent of the forward orbit with initial condition $(0, 0)$ is

$$\lambda_1(0, 0) = \lim_{k \rightarrow \infty} \frac{1}{2k} \log(|\mu_1(k, (0, 0))|) = \log 6 > 0.$$

It is left to prove that the forward trajectory of $(0, 0)$ does not have s.d.i.c.

To see this, let $\varepsilon > 0$. We can choose $k > 0$ such that $1/2^k < \varepsilon$. Since f' and g' are positive on the components of the forward orbit of $(0, 0)$, with the maximum distance denoted by $|\cdot|$, we have that

$$|F^k(x, y) - F^k(0, 0)| \leq |(\frac{1}{2^k}, \frac{1}{2^k})| = \frac{1}{2^k} < \varepsilon$$

for $n \geq k$ and $0 < |(x, y)| < \bar{\delta}$. It remains to prove that the last inequality holds for $n < k$. This is made by the continuity of F^j , since given $\varepsilon > 0$ there exists δ_j such that if $0 < |(x, y)| < \delta_j$, then $|F^j(x, y) - F^j(0, 0)| < \varepsilon$ for $j = 1, \dots, n-1$. Then if we take

$$\delta = \min \{ \delta_1, \dots, \delta_{n-1}, \bar{\delta} \} \text{ and } 0 < |(x, y)| < \delta,$$

it follows that

$$|F^k(x, y) - F^k(0, 0)| < \varepsilon$$

for all $k > 0$.

Example 2. We are going to obtain a continuous function $G = (f^2, g)$ in $[0, 1]^2$ such that the forward trajectory of $(0, 0)$ has a negative Lyapunov exponent but it does not have s.d.i.c.

- (a) The first component of G is f^2 , where the map $f : [0, 1] \rightarrow [0, 1]$ is defined in [3] by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & 0 \leq x < 7/16 \text{ or } a_n \leq x < b_n \\ (2^{n+1} - 4^{n+1} - 2^{-1})(x + 2^{-n} - 2 \cdot 4^{-n-1} - 1) & b_n \leq x < c_n \\ \frac{1 - 2^{-n-2} - 2 \cdot 4^{-n-3}}{2^{-n-1} - 9 \cdot 4^{-n-2}}(x + 2^{-n} - 2 \cdot 4^{-n-1} - 1) & c_n \leq x < a_{n+1} \end{cases}$$

where $a_n = 1 - 2^{-n} - 4^{-n-1}$, $b_n = 1 - 2^{-n} + 4^{-n-1}$, $c_n = 1 - 2^{-n} + 2 \cdot 4^{-n-1}$ for $n = 1, 2, \dots$.

- (b) The second component is $g : [0, 1] \rightarrow [0, 1]$, defined by

$$g(x) = \begin{cases} 3x + \frac{1}{2} & 0 \leq x \leq \frac{1}{15} \\ \frac{6}{127}x + \frac{7}{10} - \frac{2}{635} & \frac{1}{15} < x \leq \frac{1}{2} - \frac{1}{100} \\ 3x + \frac{1}{2} - \frac{5}{2^{n+1}}(2^n - 1) & a_n < x \leq b_n \\ \frac{5^{n+2} - 33}{2 \cdot 5^{n+2} - 11}(x - b_n) + 1 + \frac{3}{10^{n+1}} - \frac{1}{2^{n+1}} & b_n < x \leq a_{n+1} \end{cases}$$

where $a_n = 1 - 2^{-n} - 10^{-n-1}$, $b_n = 1 - 2^{-n} + 10^{-n-1}$, for $n = 1, 2, \dots$.

The map $G(x, y) = (f^2(y), g(x))$ is continuous in $[0, 1]^2$, since f and g are continuous in $[0, 1]$. Let us consider the trajectory of $(0, 0)$, denoted by

$$\{(0, 0), (x_1, y_1), (x_2, y_2), \dots\},$$

where

$$(x_{2n}, y_{2n}) = \left(1 - \frac{1}{2^{3n}}, 1 - \frac{1}{2^{3n}}\right) \quad \text{and} \quad (x_{2n-1}, y_{2n-1}) = \left(1 - \frac{1}{2^{3n-1}}, 1 - \frac{1}{2^{3n-2}}\right)$$

for $n = 1, 2, \dots$.

Similarly to the previous example, we have

$$DG(0,0) = \begin{pmatrix} 0 & 1/4 \\ 3 & 0 \end{pmatrix}, \quad DG(x_1, y_1) = DG^2(0,0) = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix},$$

$$DG(x_2, y_2) = DG^3(0,0) = \begin{pmatrix} 0 & 3/42 \\ 32/4 & 0 \end{pmatrix},$$

and in general we have

$$DG^{2n}(0,0) = \begin{pmatrix} (3/4)^n & 0 \\ 0 & (3/4)^n \end{pmatrix}, \quad DG^{2n-1}(0,0) = \begin{pmatrix} 0 & \frac{3^{n-1}}{4^n} \\ \frac{3^n}{4^{n-1}} & 0 \end{pmatrix},$$

for $n = 1, 2, \dots$.

Now we compute the eigenvalues of $(DG^k)^t \cdot DG^k$ when $k = 2n$ or $k = 2n - 1$ with $n = 1, 2, \dots$. Then when $k = 2n - 1$, we have

$$(DG^{2n-1}(0,0))^t DG^{2n-1}(0,0) = \begin{pmatrix} \frac{3^{2n}}{4^{2(n-1)}} & 0 \\ 0 & \frac{3^{2(n-1)}}{4^{2n}} \end{pmatrix}$$

$$\text{and } \mu_1(2n-1, (0,0)) = \frac{3^{2n}}{4^{2(n-1)}}.$$

$$\text{And when } k = 2n, \text{ we have that } \mu_1(2n, (0,0)) = \frac{3^{2n}}{4^{2n}}.$$

Then the Lyapunov exponent of the orbit through $(0,0)$ is

$$\lambda_1(0,0) = \lim_{k \rightarrow \infty} \frac{1}{2k} \log(|\mu_1(k, (0,0))|) = \log\left(\frac{3}{4}\right) < 0.$$

It is left to prove that the forward trajectory of $(0,0)$ has s.d.i.c. We use the maximum distance, which is represented by $|\cdot|$.

To do it, fix $\varepsilon = 3/8$ and take $k \geq 2$ such that $f^k < 1/2$ and $f^k(0) < 7/8$. Since for every $\delta > 0$ there exists (x,y) with $|(x,y)| < \delta$, we have

$$|G^k(0,0) - G^k(x,y)| > \varepsilon$$

Remark 1. Using similar procedures in a current work in progress, we are extending the constructions made on I^2 or $[0, 1) \times [0, 1)$ to I^n or $[0, 1)^n$ for $n > 2$, using permutation maps introduced in [2].

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