Chapter 2
Celestial Mechanics

Patterns of motion in the sky played a significant role in the historical development of mechanics. Briefly reviewing the history lets us see how physical concepts and models emerged from the empirical facts.

2.1 Motions in the Sky

Science often begins when people notice patterns in nature and try to understand what causes them. One well-known pattern is the daily rising and setting of the Sun, Moon, and stars. As the stars move across the sky each night, they look for all the world like points of light on some kind of crystalline sphere rotating around Earth. The Sun seems to move around Earth as well, although the relative positions of the Sun and stars vary throughout the year (the collection of visible stars changes with the season) so there must be two different crystalline spheres. The Moon is a little more complicated because its position and phase both change throughout the month, but both effects can be explained by placing the Moon on a sphere of its own. In other words, most of the obvious motions in the sky can be explained with the intuitive notion that Earth is fixed and objects in the sky move around us. This is the classic **geocentric model** of the universe.

Problems arise, though, when we notice another set of motions in the sky: planets are points of light that seem to "wander" among the stars.[1] Ancient societies knew of five planets (the discovery of others had to await the invention of the telescope). Mercury and Venus always stay fairly close to the Sun, appearing either in the west after sunset or in the east before sunrise. Jupiter and Saturn can be seen across a much wider range of positions, moving from west to east relative to the stars from one night to the next. Mars is a bit like Jupiter and Saturn, but with a twist. Most of

[1]The term "planet" comes from the ancient Greek term *aster planetes*, or "wandering star."

C. Keeton, *Principles of Astrophysics: Using Gravity and Stellar Physics to Explore the Cosmos*, Undergraduate Lecture Notes in Physics, DOI 10.1007/978-1-4614-9236-8_2,

Fig. 2.1 Pictures taken across several months have been combined to illustrate Mars's retrograde motion. Relative to the background stars, Mars usually moves from right to left. However, from November 15, 2007 to January 30, 2008 the planet moved from left to right, producing the loop pattern shown here. In other cases retrograde motion can create a zigzag pattern. (Credit: Tunç Tezel (TWAN), reproduced by permission)

the time it moves from west to east, but every once in a while Mars appears to stop, turn around and go from east to west for a few weeks, then turn around again and resume its "normal" motion (relative to the stars). Today we can see this **retrograde motion** very clearly in composite photographs, as shown in Fig. 2.1.

When scholars in ancient Greece tried to explain the apparent motions of planets, they started with the assumption that the intrinsic motions involve circles. Apollonius (c. 200 BC) constructed a model in which a planet moves on a small circle (called an "epicycle") that itself moves along a larger circle (called the "deferent"). As shown in Fig. 2.2, the composite motion can allow the planet to move backward at certain points in its orbit (depending on the relative sizes and speeds of the epicycle and deferent; see Problem 2.1). As the measurements became more precise, Ptolemy (c. 100 AD) refined the model by shifting the center of the deferent away from Earth and introducing yet a different point (called the "equant") around which the angular speed was defined.[2]

While Ptolemy's model was admittedly complex, its quantitative success kept it successful well into the Renaissance. Nicolaus Copernicus (1473–1543) introduced the first mathematically detailed alternative with the Sun at the center of motion.[3]

[2]There is a common misconception that Ptolemy and his successors added more and more epicycles. They couldn't; even one was hard enough to compute. See Chap. 4 of *The Book Nobody Read* by Owen Gingerich [1].

[3]The geocentric model had been questioned much earlier by Aristarchus (c. 300 BC), but without a fully developed alternative.

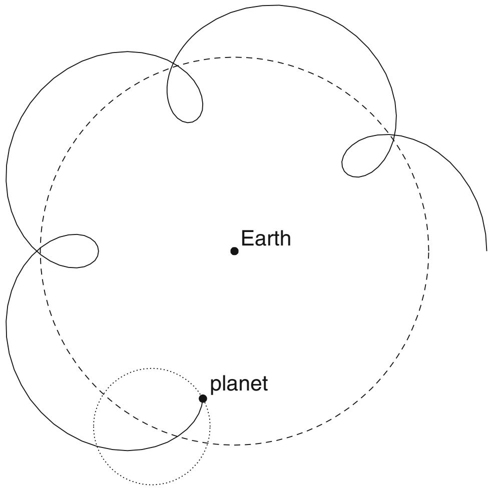

Fig. 2.2 In the geocentric model, a planet moves on an epicycle (*dotted*) whose center moves along a curve called the deferent (*dashed*). The combined motion (*solid*) can cause the planet to move backward as viewed from Earth. In the full Ptolemaic model, the deferent was not perfectly centered on Earth

In this **heliocentric model**, retrograde motion is an illusion that occurs when fast-moving Earth overtakes a slower-moving outer planet (see Problem 2.2); planets never actually move backward in space. Offering a simple explanation of retrograde motion is not all that Copernicus's model had going for it. The heliocentric model also explained why the observed planets fall into two categories: Mercury and Venus are never seen far from the Sun because their orbits are smaller than Earth's; while Mars, Jupiter, and Saturn can be seen near the Sun, on the opposite side of the sky, or anywhere in between because their orbits are larger than Earth's. Last but not least, Copernicus's model revealed a simple pattern in the quantitative relation between a planet's distance from the Sun and its orbital period. To Copernicus, this was a striking success: "In no other way," he wrote, "do we find a wonderful commensurability and a sure harmonious connection between the size of the orbit and the planet's period" (quoted by Gingerich [1, p. 54]).

That said, the original heliocentric model was not without fault. Like the Greeks, Copernicus assumed that planetary motion involved circles. While he was able to eliminate equants and large epicycles, he still needed small epicycles to make the model fit the data. That made Copernicus's model about as mathematically complex as Ptolemy's, even if it was conceptually simpler.

Copernicus's model made an important prediction: Earth moves in space. If that is true, then our perspective on the stars should change as Earth travels from one side of its orbit to the other (e.g., from January to July). Tycho Brahe (1546–1601), who was perhaps the world's greatest naked-eye astronomer, set out to test this prediction. He amassed years' worth of careful measurements of planet and star

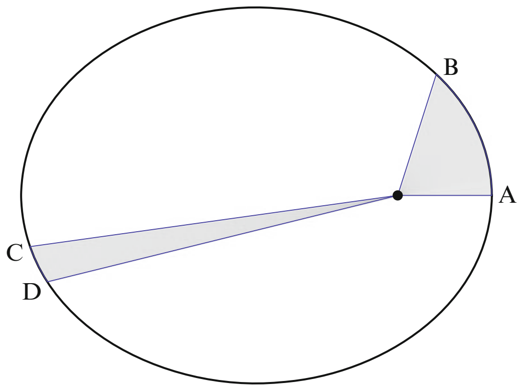

Fig. 2.3 Illustration of Kepler's first and second laws of planetary motion. I. The ellipse indicates the orbit, and the *dot* indicates the Sun at one focus. II. The time it takes the planet to travel from A to B is the same as the time to travel from C to D, so the areas of the *two shaded regions* are the same. This is a rather extreme example; the orbits of planets in our Solar System are much less elongated

positions in an attempt to measure **parallax**, or small shifts in the apparent positions of stars that should arise when we look from different sides of Earth's orbit. Tycho failed to find clear evidence for parallax, although now we know that stars are so far away that parallax can only be detected with a good telescope. Tycho's efforts did ultimately provide support for the heliocentric model, although not in the way he expected.

Shortly before he died, Tycho hired Johannes Kepler (1571–1630) as an assistant. Kepler combed through Tycho's measurements of planet positions and tried to find a geometric model to explain the motion. He initially adopted Copernicus's heliocentric model with circular orbits modified by epicycles. Kepler found, though, that the model could not quite reproduce Tycho's high-quality data, notably for Mars. Once he considered more general forms of motion, Kepler discovered that he could fit the data using elliptical orbits. Working through the details, he eventually extracted three **laws of planetary motion**:

I. Planets move in elliptical orbits, with the Sun at one focus.
II. A line that connects a planet to the Sun sweeps out equal areas in equal times (see Fig. 2.3).
III. A planet's orbital period P (in years) and average distance from the Sun a (in AU) are related by $P^2 = a^3$.

Suddenly the heliocentric model had an attractive and powerful quantitative framework. Still, people continued to struggle with the notion of a moving Earth.

That situation finally began to change thanks to the work of Galileo Galilei (1564–1642), who was arguably the first great experimental physicist. Using the newly-invented telescope, Galileo made two key discoveries related to planetary motion. First, he observed that Venus has phases just like the Moon. In the geocentric model, Venus would always stay between Earth and the Sun so it could

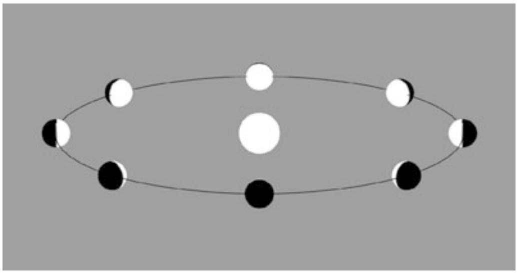

Fig. 2.4 Phases of Venus in the heliocentric model (not to scale). Full and gibbous phases can occur only if Venus travels to the far side of the Sun

only have new and crescent phases. Galileo saw that Venus has quarter and gibbous phases as well, which implies that Venus can go "behind" the Sun (as seen from Earth; see Fig. 2.4). In other words, if we know that Venus is closer to the Sun than Earth is, and the planet has a full cycle of phases, then it must orbit the Sun. Second, Galileo discovered four moons orbiting Jupiter. While this did not directly prove that planets orbit the Sun, it did demonstrate that objects can orbit something other than Earth. On the basis of his evidence, Galileo argued strongly in favor of the heliocentric model, most famously in his book *Dialogue on the Two Chief World Systems*. The work violated dictates from the Catholic Church, causing the book to be banned by the Roman Inquisition and Galileo to be placed under house arrest. More than three and a half centuries later, Pope John Paul II renounced the Church's condemnation of Galileo.

2.2 Laws of Motion

All of those ideas set the stage for Isaac Newton (1642–1727) to devise the fields we now know as theoretical physics and calculus (among other accomplishments). In 1665, Newton graduated from Cambridge but the university then closed because of the plague. He went home and, working alone, entered a period of remarkable intellectual creativity.[4] Newton started with mathematics, inventing the idea of

[4]Historical aside: In 1665–1666 Newton solved the problems of motion and gravity to his satisfaction, keeping a detailed notebook but not publishing his work. In 1684, Edmund Halley visited Newton to pose the question: If gravity has an inverse square force law, what curve will a planet follow? Newton knew the answer was an ellipse (see Sect. 3.1), but only after battling Robert Hooke for some time did he finally decide to write his famous work *Philosophiae Naturalis Principia Mathematica*, or "Mathematical Principles of Natural Philosophy." Newton's introduction of *mathematical* principles was profoundly important for the further development of physics and astrophysics. See *Isaac Newton* by James Gleick [2] for more about the life and work of this fascinating figure.

plotting solutions of equations as curves (a topic now known as algebraic geometry). He developed calculus so he could analyze curves, using derivatives to represent tangent lines and integrals to compute areas. Then Newton began to think about curves representing trajectories of objects in motion. Before he could apply his mathematical tools to motion, though, Newton had to introduce some new physical concepts that became his famous **laws of motion**:

I. **Inertia.** An object will remain at rest or in uniform motion in a straight line unless acted on by an unbalanced force.

II. **Force and acceleration.** A net force acting on an object produces an acceleration in the same direction as the applied force. The acceleration and force are related by

$$\mathbf{F} = m\mathbf{a} = m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{2.1}$$

III. **Equal/opposite reaction.** If object #1 exerts a force on object #2, then object #2 exerts an equal and opposite force back on object #1: $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

These laws are general; they are not specific to planets. In fact, to explain planetary motion Newton had to add one more law specifying the force. We will come to the law of gravity in Sect. 2.3.

While they are often introduced as above, Newton's laws of motion can be restated in terms of quantities that do not change with time. Think of a rod: the (x, y, z) coordinates of the endpoints depend on whether the rod is moving or rotating, but the *distance* between the two endpoints is always the same. A quantity that is "conserved" is usually thought to represent some fundamental property of a system (such as the length of the rod). Stating physical theories in terms of **conservation laws** can often help us find the simplest expressions of those theories. Let's see a few examples that are probably familiar but nonetheless valuable.

Momentum is defined by

$$\mathbf{p} \equiv m\mathbf{v}$$

We can use this to rewrite Eq. (2.1) as

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

While this might seem trivial, it is actually a nice generalization of Newton's second law. It helps us see that when there is no net force, momentum does not change. Thus, Newton's first law is fundamentally a statement of conservation of momentum.

Angular momentum is defined by

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) \tag{2.2}$$

We will sometimes use the **specific angular momentum**, defined to be the angular momentum per unit mass:

$$\boldsymbol{\ell} \equiv \frac{\mathbf{L}}{m} = \mathbf{r} \times \mathbf{v} \tag{2.3}$$

Let's take the derivative of angular momentum with respect to time:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \\ &= \mathbf{v} \times (m\,\mathbf{v}) + \mathbf{r} \times \mathbf{F} \\ &= \mathbf{r} \times \mathbf{F}\end{aligned}$$

(The cross product of a vector with itself is zero, so the first term vanishes.) Clearly if there is no net force then angular momentum is conserved. More interesting is a situation in which the force is purely radial, $\mathbf{F} = F(r)\,\hat{\mathbf{r}}$. In this case,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times [F(r)\,\hat{\mathbf{r}}] = 0$$

We see that if a force is applied but there is no angular component to the force, then angular momentum is conserved.

Energy. If a force acts on an object, it takes "work" to move the object against the force. The amount of work required to go from some initial position $\mathbf{r}_i$ to final position $\mathbf{r}_f$ can be calculated as

$$\Delta U = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{2.4}$$

We call this **potential energy** because it is energy that would be released if the object were to move back to the initial position. We include a minus sign because the work acts against the force $\mathbf{F}$, and we write ΔU to emphasize that this is an energy difference. If desired, we can pick a reference point at which the potential is defined to be zero and thus obtain a potential energy function $U(\mathbf{r})$. Then Eq. (2.4) can be inverted to say the force is obtained by differentiating the potential energy:

$$\mathbf{F} = -\nabla U \tag{2.5}$$

(This is independent of the choice of zeropoint because any additive constant vanishes in the derivative.) Now let's return to Eq. (2.4) and use Newton's second law along with $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ to see what we can learn:

$$\Delta U = -\int_{t_i}^{t_f} \left(m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \right) \cdot (\mathbf{v}\,\mathrm{d}t)$$

$$
\begin{aligned}
&= -\int_{t_i}^{t_f} m\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,|\mathbf{v}|^2\right)\mathrm{d}t \\
&= -\left(\frac{1}{2}\,m\,v_f^2 - \frac{1}{2}\,m\,v_i^2\right) \\
&= -(K_f - K_i) \\
&= -\Delta K
\end{aligned}
$$

Toward the end we identify $K = (1/2)\,m\,v^2$ as the **kinetic energy**, or energy of motion. Trivially rewriting the final equation gives

$$\Delta U + \Delta K = 0$$

or

$$\Delta E_{\text{tot}} = 0 \qquad \text{where} \qquad E_{\text{tot}} = U + K$$

This is the statement of conservation of energy. Note that potential and kinetic energy are not separately conserved; in fact, one can be traded for the other. But the combination—the total energy—is conserved. This is true for any force, at least in the context of Newtonian physics.

2.3 Law of Gravity

In order to apply his general laws of motion to planets, Newton had to specify the force that acts on planets to generate their motion. We saw in Chap. 1 how he used Kepler's third law to motivate the inverse square law form. To give a precise formulation, let's suppose that an object of mass M exerts a gravitational force on a second object of mass m whose position relative to the first object is given by the vector $\mathbf{r}$. If the objects are both point masses, Newton's law of gravity in vector form reads

$$\mathbf{F}_{\text{grav}}(\mathbf{r}) = -\frac{GMm}{r^2}\,\hat{\mathbf{r}} \tag{2.6}$$

where $\hat{\mathbf{r}}$ reminds us that the force is radial, and the minus signs indicates that gravity is an attractive force.

What if the two objects are not point masses? One of Newton's triumphs was to show that the gravitational force outside a spherically symmetric object of mass M is the *same* as that from a point mass M at the center of the object. Also, the gravitational force inside a spherical shell is zero. To understand these results, consider the setup in Fig. 2.5. Let's use spherical coordinates[5] but modify them

[5]See Sect. A.2 for a review.

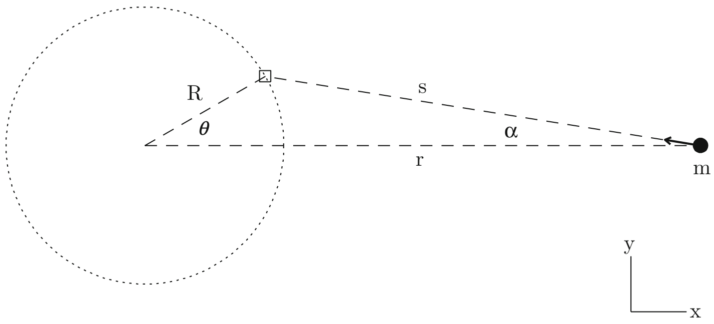

Fig. 2.5 Setup for computing the gravitational force from an extended spherical object

so θ is measured from the x-axis while ϕ is in the direction perpendicular to the page. Then complete the triangle by defining the side s and angle α as shown. By symmetry, the net force on m is in the x-direction. The contribution to F_x from a small volume element $\mathrm{d}V$ at r and θ is

$$\mathrm{d}F_x = -\frac{G\,m\,\rho\,\mathrm{d}V}{s^2}\ \cos\alpha \tag{2.7}$$

We would like to rewrite this in terms of R and θ. From the law of cosines,

$$s^2 = r^2 + R^2 - 2rR\cos\theta \tag{2.8}$$

and from the law of sines,

$$\frac{\sin\alpha}{R} = \frac{\sin\theta}{s} \quad \Rightarrow \quad \sin\alpha = \frac{R\sin\theta}{(r^2 + R^2 - 2rR\cos\theta)^{1/2}}$$

Then the familiar trigonometric identity $\cos^2\alpha + \sin^2\alpha = 1$ yields

$$\cos\alpha = \frac{r - R\cos\theta}{(r^2 + R^2 - 2rR\cos\theta)^{1/2}}$$

Putting the pieces together, we can write Eq. (2.7) as

$$\mathrm{d}F_x = -Gm\,\frac{r - R\cos\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}}\,\rho\,\mathrm{d}V$$

We obtain the net force by integrating, using the spherical volume element $\mathrm{d}V = R^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$:

$$F_x = -Gm\int \mathrm{d}R\ R^2\rho(R)\int_0^\pi \mathrm{d}\theta\ \sin\theta\int_0^{2\pi}\mathrm{d}\phi\ \frac{r - R\cos\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}}$$

(We discuss the limits for the R integral below.) The ϕ integral gives 2π. To evaluate the θ integral, change integration variables to s using Eq. (2.8). This yields

$$\begin{aligned} F_x &= -2\pi G m \int \mathrm{d}R\; R^2 \rho(R) \int_{|r-R|}^{r+R} \mathrm{d}s\; \frac{r^2 - R^2 + s^2}{2Rr^2 s^2} \\ &= -\frac{\pi G m}{r^2} \int \mathrm{d}R\; R\,\rho(R) \left[-\frac{r^2 - R^2}{s} + s \right]_{s=|r-R|}^{s=r+R} \end{aligned} \tag{2.9}$$

Because of the absolute value, the value of the quantity in square brackets depends on whether $r - R$ is positive or negative:

$$\begin{aligned} R < r \quad &\Rightarrow \quad \left[-\frac{r^2 - R^2}{s} + s \right]_{s=r-R}^{s=r+R} = 4R \\ R > r \quad &\Rightarrow \quad \left[-\frac{r^2 - R^2}{s} + s \right]_{s=R-r}^{s=r+R} = 0 \end{aligned}$$

The second result says there is no contribution to the integral in Eq. (2.9) from the region with $R > r$. In other words, mass outside of r does not contribute to the gravitational force at r (given spherical symmetry). Using the first result in Eq. (2.9) lets us write

$$F_x = -\frac{Gm}{r^2} \int_0^r 4\pi R^2\, \rho(R)\, \mathrm{d}R \tag{2.10}$$

This integral gives $M(r)$, or the total mass enclosed within radius r, Thus, we can write the gravitational force from an extended, spherically-symmetric object (now in vector form) as

$$\mathbf{F}_{\text{grav}}(\mathbf{r}) = -\frac{GM(r)m}{r^2}\, \hat{\mathbf{r}} \tag{2.11}$$

Using Eq. (2.4), we can now determine the **gravitational potential energy** for point masses:

$$\begin{aligned} \Delta U_{\text{grav}} &= -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}_{\text{grav}} \cdot \mathrm{d}\mathbf{r} \\ &= GMm \int_{\mathbf{r}_i}^{\mathbf{r}_f} \frac{1}{r^2}\, \hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{r} \\ &= GMm \int_{r_i}^{r_f} \frac{1}{r^2}\, \mathrm{d}r \\ &= -GMm \left(\frac{1}{r_f} - \frac{1}{r_i} \right) \end{aligned}$$

This is also the potential energy outside any spherical object with total mass M. As noted above, we must pick a reference point in order to define the full potential energy function. The most common choice in astrophysics is to put the reference point at infinity and define the potential energy to be zero there. This yields

$$U(r) = -\frac{GMm}{r} \tag{2.12}$$

It can be valuable to factor out m:

$$\Phi(r) = \frac{U(r)}{m} = -\frac{GM}{r} \tag{2.13}$$

This function is independent of m, so it describes the gravitational field around M in a general way. We call it the **gravitational potential** of M. To see its utility, consider:

$$m\,\mathbf{a} = \mathbf{F} = -\nabla U = -m\nabla\Phi \quad \Rightarrow \quad \mathbf{a} = -\nabla\Phi$$

All objects at a given position in the gravitational field of M experience the same acceleration, regardless of their mass.

If we focus attention near the surface of Earth (as in introductory physics courses), it may be convenient to adopt a different convention and let the reference point for the potential be Earth's surface. Then the potential energy at a height h above the surface is written as

$$U(h) = -GM_{\oplus}m\left(\frac{1}{R_{\oplus}+h} - \frac{1}{R_{\oplus}}\right)$$

If $h \ll R_{\oplus}$, we can make a Taylor series expansion and find

$$U(h) \approx mgh \qquad \text{where} \qquad g = \frac{GM_{\oplus}}{R_{\oplus}^2} = 9.80\,\mathrm{m\,s^{-2}}$$

Remember that this is valid only near the surface of Earth.

Application: Escape

In the next chapter we will see how Newton's laws of motion and gravity come together to explain Kepler's laws. First, though, it is useful to do a short example that illustrates how conservation laws can help us analyze certain problems quickly and easily.

"What goes up must come down," according to the common saying, but Newton begged to differ. He discerned that the force causing an apple to fall from a tree is the same force keeping the Moon in orbit around Earth; the key difference is that the Moon's forward motion keeps it from crashing into the ground. In principle, if we could throw an apple hard enough we could give it enough motion to go up

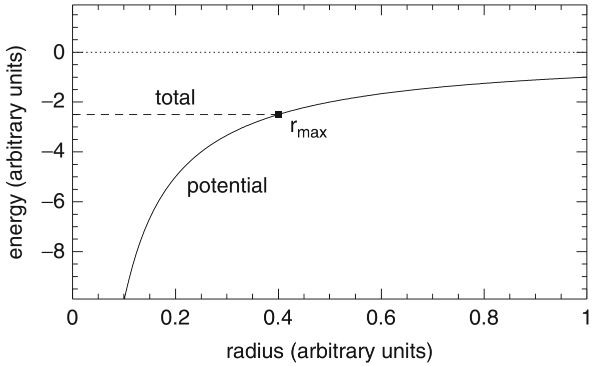

Fig. 2.6 The *solid curve* shows the gravitational potential energy; the *dashed horizontal line* shows the total energy (which is conserved); and the difference between the two gives the kinetic energy. Since kinetic energy cannot be negative, the object can never go beyond $r_{\max}$

and never come back down. (This works better with rockets than apples.) How hard would we have to throw it?

To find out, suppose an object with mass m is at radius r and moving with speed v in the gravitational field around mass M. Is there any limit on how far the small object can go? If so, what is the maximum radius ($r_{\max}$) it can reach? How fast do we need to make the object move if we want it to escape?

If we wanted to work with the original version of Newton's laws of motion, we would have to solve the differential equation $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2 = -(GM/r^2)\hat{\mathbf{r}}$ for all trajectories that originate at radius r with speed v, and then we would have to search among those trajectories to find $r_{\max}$. That does not sound like a simple task. But the analysis gets *much* easier if we turn to conservation of energy. At any given r, the total energy is the sum of the potential and kinetic terms,

$$E = -\frac{GMm}{r} + \frac{1}{2}mv^2 \tag{2.14}$$

We can think about this in terms of an energy diagram as in Fig. 2.6. The total energy must be independent of radius. Since the kinetic term is non-negative, the potential energy can never exceed the total energy. The maximum allowed radius is the place where the kinetic energy vanishes and the potential energy equals the total energy,

$$E = -\frac{GMm}{r_{\max}} \tag{2.15}$$

Equating (2.14) and (2.15) lets us solve for $r_{\max}$:

$$r_{\max} = \left(\frac{1}{r} - \frac{v^2}{2GM}\right)^{-1} \tag{2.16}$$

Notice that we reached this answer in just three lines of algebra; we did not have to specify the direction of motion or examine specific trajectories, or deal with vectors and differential equations at all. Applying conservation of energy is a powerful approach to this problem.

We can now ask how fast the object would have to be moving when it is at radius r in order to escape the gravitational field altogether. This is the speed that allows $r_{\max}$ to become infinity, and it is given by the value of v that causes the quantity in parentheses in Eq. (2.16) to vanish:

$$v_{\rm esc} = \left(\frac{2GM}{r}\right)^{1/2} \tag{2.17}$$

We call this the **escape velocity** at a distance r from an object of mass M.

Problems

2.1. Consider a geocentric model for retrograde motion. Suppose the deferent has radius R and angular speed Ω, while the epicycle has radius $a < R$ and angular speed ω (about its center). Find the velocity vector in polar coordinates centered on Earth. By analyzing the tangential velocity at the innermost points, show that the condition to have retrograde motion is $a\omega > R\Omega$.

2.2. Here is a way to understand retrograde motion in the heliocentric model using geometric reasoning (no equations required). Consider a system with two planets orbiting the Sun along circles in the same plane. Suppose the outer planet takes twice as long as the inner planet to orbit the star. Let $t = 0$ be the time when the two planets are lined up on one side of the star.

(a) Sketch the orbits, and add some distant stars. Suppose both planets orbit and spin in the counterclockwise direction. Indicate the directions in the star field that an observer on the inner planet would identify as "east" and "west."
(b) Sketch the positions of the planets a little before and after $t = 0$. In which direction across the sky does the outer planet appear to move, as viewed from the inner planet?
(c) Repeat part (b) at times when the planets are not lined up (for example, when the inner planet has completed 1/4 or 1/2 of its orbit).

2.3. To practice/review working with vectors, compute the specific angular momentum for straight line motion $\mathbf{r}(t) = vt\,\hat{\mathbf{x}} + b\,\hat{\mathbf{y}}$. Is angular momentum conserved? Should it be?

2.4. Consider conservation of energy and angular momentum as applied to an elliptical orbit.

(a) At what point in an elliptical orbit does a planet move fastest? Slowest?

(b) Sketch the kinetic and potential energies as a function of time for a planet in an elliptical orbit.

2.5. Consider a uniform sphere with mass M and radius R. Compute the gravitational force on a particle of mass m at any radius $0 < r < \infty$. Then compute the corresponding gravitational potential. You make take the potential to be zero at the center of the sphere.

2.6. Consider a particle of mass m released from rest at a distance r_0 from a point mass M (and assume $M \gg m$ so M does not move). Use conservation of energy to find the speed v, which is also $\mathrm{d}r/\mathrm{d}t$. Then compute the acceleration and show that the motion satisfies Newton's laws.

2.7. For a sufficiently small object, compute the radius at which the escape velocity equals the speed of light. Since nothing can go faster than the speed of light, this is the "Schwarzschild radius" for the event horizon of a black hole. What is the Schwarzschild radius of a black hole the mass of Earth? Of the Sun?

2.8. Could you jump off an asteroid? Let's find out.

(a) Estimate the velocity you achieve when you jump straight up on Earth. Hint: use the height you reach to estimate the change in your potential energy, and then use conservation of energy to estimate your initial kinetic energy.
(b) Now estimate the size of the largest asteroid you could escape from by jumping. You will need to make an assumption about the asteroid's density; just explain your reasoning.

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