Chapter 2
Review of Signals and Systems

2.1 Signals

Signals correspond to inputs, outputs, and states of systems. Signals are defined in the continuous-time and discrete-time domains.

A continuous-time signal is given as $u \in R^N$. Signal u is a mapping from R to R^N. The domain of u may be a subset of the set of real numbers R; for example, the domain may begin at time $t_0 \in R^+$. A continuous-time signal may be piecewise continuous, as is the case with the output of a DAC, and exemplified in Fig. 2.1 for the case of a converter that holds the value of its output signal until the next update. For brevity, a continuous-time signal u is equivalently written as $u(t)$ to represent both the signal and its value at a particular time instant t.

A discrete-time signal $u(k, T)$ maps an index in Z, the set of integers, for a given sampling period T, to a value in R^N. The first argument in the variable $u(k, T)$ is the time step, while the second argument is the period between two successive values, when viewed from a continuous-time perspective. For example, the discrete-time signal $u(k, T)$ in Fig. 2.1 has a value of $u(0, T)$ at time instant 0, a value of $u(1, T)$ at time instant T, a value of $u(2, T)$ at time instant $2T$, and so on. We are interested in designing digital controllers for continuous-time plants. Dynamic systems, formulated with ordinary differential equations (ODEs), evolve over a time continuum $t \in R$. The signals associated with the digital controllers are defined at time instants given by kT, for $k \in Z$, over the continuum of time, with T time units between two successive values of a discrete-time signal.

Here, we assume uniformity in the time periods, although in reality the signals are not available exactly at integer multiples of T. Fortunately, such assumption is not detrimental in practice. This time-periodic sampling, also called Riemann sampling, is to be contrasted with Lebesgue sampling [3] which, simply put, consists of sampling a signal when it exceeds a given level. Discrete-event systems use Lebesgue sampling.

A discrete-time signal may be interpreted as a sequence. The notation $u(k, T)$ represents both the discrete-time signal and its value for a specific pair (k, T). The infinite sequence can be written as

$$\{u(k, T); k \in Z\} = \{..., u(-2, T), u(-1, T), u(0, T), u(1, T), u(2, T), ...\}. \quad (2.1)$$

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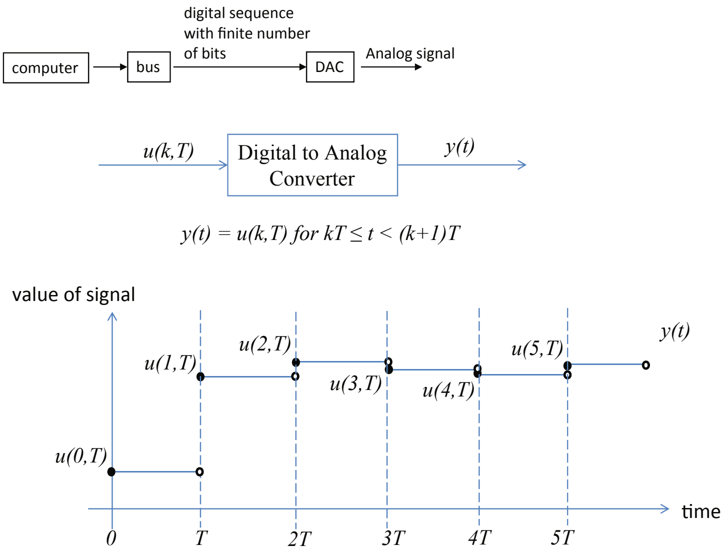

Fig. 2.1 Piecewise-continuous signal

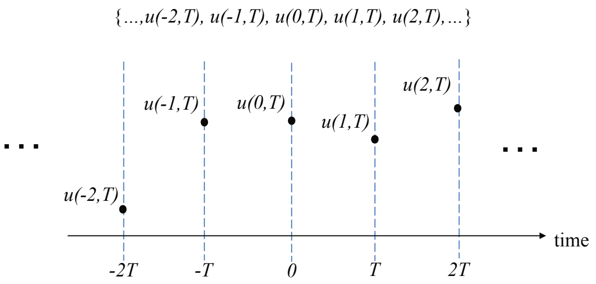

Fig. 2.2 Discrete-time signal as a sequence

The interpretation of a scalar discrete-time signal in the time domain and its representation as a sequence is illustrated in Fig. 2.2. For a vector signal, each entry in the right-hand side of Eq. (2.1) is a vector.

Figure 2.3 shows four types of signals. The signals found in a digital control system are shown in Fig. 2.3a, b and d. An analog signal is shown in Fig. 2.3a. An analog signal is continuous in amplitude and over time. The signal shown in Fig. 2.3b may correspond to the output of a DAC, for instance. It is a piecewise constant signal whose values belong to a finite set of real numbers. A discrete-time signal $u \in R^N$, with $N = 1$, is shown in Fig. 2.3c. There is no truncation or rounding of the amplitude of a signal in Fig. 2.3c. A digital signal is represented in Fig. 2.3d. A digital signal is discrete in magnitude (quantized values) and over time, such as the output of an ADC, for example. The values for such a signal belong to a finite set of real numbers.

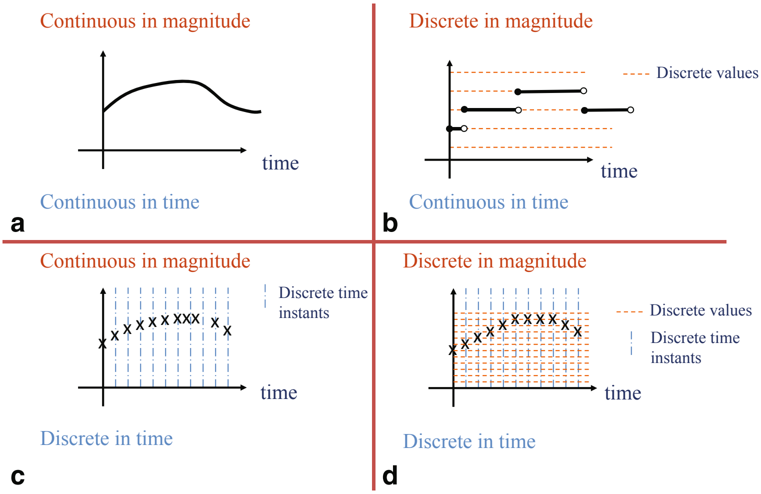

Fig. 2.3 Four types of signals

2.2 Systems

Systems present in a digital control loop evolve either in the continuous-time or in the discrete-time domain.

Definition 2.1 Linear, Time-invariant Continuous-time System A system with inputs, outputs and states defined in the continuous-time domain; namely, for any time $t \in R$. A linear, time-invariant continuous-time system may be represented (1) as a transfer function in the complex variable s, and (2) as a finite-dimensional state-space given by

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.2}$$

where x is the state vector of dimension $n \times 1$, u is the input of dimension $m \times 1$, y is the output of dimension $r \times 1$, and the matrices and vectors are $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{r\times n}$, and $D \in R^{r\times m}$.

A nonlinear, time-invariant continuous-time system is given in state-space form as

$$\begin{aligned} \frac{dx}{dt} &= f(x,u) \\ y &= g(x,u). \end{aligned} \tag{2.3}$$

Definition 2.2 Discrete-time System A system with inputs, outputs, and states defined in the discrete-time domain; that is, at discrete instants of time, also known as the sampling instants. A linear system is represented as a transfer function in z or γ. Finite dimensional linear and nonlinear discrete-time systems may be represented in a state-space, in either the shift or the delta form.

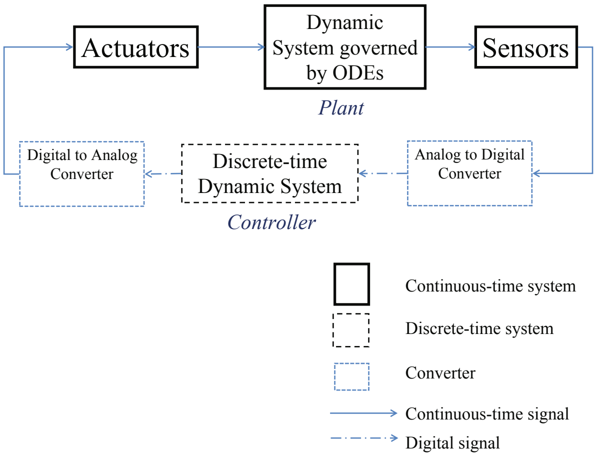

Fig. 2.4 Basic control system with continuous-time and discrete-time signals and systems

Figure 2.4 shows a basic digital control loop, where the two types of systems and signals are present. The part of the system comprising the dynamic system under control with its actuators and sensors evolves in the continuous-time domain. The digital control scheme implemented on digital hardware evolves in the digital domain; namely, with finite wordlength computations and fixed- or floating-point arithmetic. DACs and ADCs ensure the transition between the two domains.

2.3 Operators in the Discrete-time Domain

Two operators are of interest in this book: the shift and delta operators. They act upon discrete-time signals. The numerical properties of these operators depend on the value of T. The context dictates whether one is preferred over the other.

To present the main characteristics of shift and delta operators, consider a discrete-time signal $f : N \to R^n$. The signal is also represented as $f(k, T)$, where $T \in R^+$ is the sampling or update period and $k \in N$ is the time step. Signal $f(k, T)$ can be written as the following sequence of vectors in R^n:

$$\{..., f(0, T), f(1, T), f(2, T), ...\}. \tag{2.4}$$

2.3.1 Shift Operator

The shift operator q acting on signal f given as

$$f(k, T) = \{..., \underset{k=0}{f(0, T)}, \underset{k=1}{f(1, T)}, \underset{k=2}{f(2, T)}, ...\}. \tag{2.5}$$

results in

$$qf(k,T) \quad = \quad \{..., \underset{k=0}{f(1,T)}, \underset{k=1}{f(2,T)}, \underset{k=2}{f(3,T)}, ...\}. \tag{2.6}$$

Operator q acts on a discrete-time signal by shifting the whole signal to the left along the time axis by one period T.

For a linear, time-invariant discrete-time system G with state $x(k,T) \in R^n$, input $u(k,T) \in R^m$ and output $y(k,T) \in R^p$, the state-space equation in the shift form is given as

$$\begin{aligned} x(k+1,T) &= Ax(k,T) + Bu(k,T) \\ y(k,T) &= Cx(k,T) + Du(k,T) \end{aligned} \tag{2.7}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$ and $D \in R^{p\times m}$. Equation (2.7) is the classical expression for discrete-time systems.

The right-shift operator q^{-1} acts on a discrete-time signal $f(k,T)$ as $q^{-1}f(k,T) = f(k-1,T)$. The operation q^{-1} therefore shifts the discrete-time signal to the right by one sampling interval.

The zero-state response of a linear, time-invariant (LTI) system, expressed in the shift form with Eq. (2.7), subject to a unit discrete-time pulse given as

$$u(k,T) = \begin{cases} 1, k = 0 \\ 0, k \neq 0 \end{cases} \tag{2.8}$$

is as follows

$$y(k,T) = \begin{cases} CA^{k-1}B, k \geq 1 \\ D, k = 0 \end{cases} \tag{2.9}$$

In general, using the convolution sum [65], one writes the zero-state response as a function of the state-space components of the discrete-time system and input $u(k,T)$ as

$$y(k,T) = C\sum_{i=0}^{k-1} A^{k-i-1}Bu(i,T) + Du(k,T). \tag{2.10}$$

The shift operator is well-known and thoroughly employed in the discrete-time control literature. There exist other ways, however, to operate on discrete-time signals and to represent discrete-time systems. These representations may be used for analysis, design, and implementation of a control law. The delta operator is an alternative to the shift operator.

2.3.2 Delta Operator

It is well-known that a discrete-time system can be expressed in the so-called delta form [31]. Using again the discrete-time signal $f(k,T)$, the delta operator is defined as

$$\delta f(k, T) = \frac{f(k+1, T) - f(k, T)}{T} \tag{2.11}$$

which is reminiscent of the numerical approximation to the derivative using a forward slope.

The delta operator can be expressed in terms of the shift operator as $\delta = (q-1)/T$. The delta operator produces a weighted difference between a shifted version of a signal and the signal itself.

For a discrete-time system G, to obtain the state-space equations in the delta form, one uses Eqs. (2.7) and (2.11), yielding

$$\begin{aligned} \delta x(k, T) &= \underbrace{\frac{(A-I)}{T}}_{=A_\delta} x(k, T) + \underbrace{\frac{B}{T}}_{=B_\delta} u(k, T) \\ y(k, T) &= C_\delta x(k, T) + D_\delta u(k, T) \end{aligned} \tag{2.12}$$

where $C_\delta = C$ and $D_\delta = D$. Inputs and outputs are unaffected by the choice of either the shift or the delta form. Recall that (2.7) and (2.11) are two expressions for the same system G. The difference lies in the expression for the update of the state.

The inverse delta operator δ^{-1} corresponds to area summation in the discrete-time domain and is written as

$$\delta^{-1} f(k, T) = \sum_{i=0}^{k-1} f(i, T) T. \tag{2.13}$$

In Eq. (2.13), it is assumed that $f(k, T)$ is defined for $k \geq 0$.

For the details on the numerical and the analytical properties of the delta form representation, the reader is referred to [31], [59] and [78]. Briefly, in general, the delta operator is favored over the shift operator in the representation of discrete-time systems as the sampling or control update period T is relatively close to zero.

2.4 Transforms

For continuous-time signals and systems, the Laplace transform enables analysis and design. In the discrete-time domain, various transforms exist. In this book, we are interested in the Z and Γ transforms. The former relates to the shift operator, whereas the latter pertains to the delta operator.

2.4.1 Z Transform and Transfer Function in z

The classical definition of the Z transform of a discrete-time signal $f(k, T)$, defined for $k = 0, 1, 2, ...$, is given as

$$Z\{f(k, T)\} = \sum_{k=0}^{\infty} f(k, T) z^{-k} \tag{2.14}$$

where z is a complex variable [47], [72]. Clearly, Eq. (2.14) is a series in a complex variable, and therefore has a region of convergence in the z-plane.

The transfer function of a single-input, single-output discrete-time system G with input $u(k,T)$ and output $y(k,T)$ is given as

$$\begin{aligned} G(z) &= \frac{Z\{y(k,T)\}}{Z\{u(k,T)\}} \\ &= \frac{Y(z)}{U(z)} \\ &= \frac{b_n z^n + b_{n-1}z^{n-1} + \ldots + b_1 z + b_0}{a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} \end{aligned} \tag{2.15}$$

where $a_i, b_i \in R$. The extension to multi-input, multi-output systems is straightforward. For a discrete-time system expressed in the shift state-space form as

$$\begin{aligned} x(k+1,T) &= A_q x(k,T) + B_q u(k,T) \\ y(k,T) &= C_q x(k,T) + D_q u(k,T) \end{aligned}$$

with $x(k,T) \in R^{n\times 1}$, $u(k,T) \in R^{r\times 1}$, $y(k,T) \in R^{m\times 1}$, $A_q \in R^{n\times n}$, $B_q \in R^{n\times r}$, $C_q \in R^{m\times n}$, $D_q \in R^{m\times r}$ and initial condition $x(0,T)$, the relationship between input, output and state in the complex variable z is written as

$$\begin{aligned} X(z) &= (zI - A_q)^{-1}B_q u(z) + z(zI - A_q)^{-1}x(0,T) \\ Y(z) &= C_q((zI - A_q)^{-1}B_q + D_q)u(z) + C_q z(zI - A_q)^{-1}x(0,T). \end{aligned}$$

When the sampling period associated with a transfer function is ambiguous, for example in case of dual-rate systems, the identifier of a transfer function may include the sampling period as its second argument. For instance, we may write $X(z,T)$ instead of $X(z)$, without loss of generality, to make sure that it is understood that the discrete-time transfer function is obtained with period T.

2.4.2 *Z* Transform and *q* Operator

Let the discrete-time signal $x(k,T)$ start at $k = 0$, the initial time. The relationship between z and q is given as

$$\begin{aligned} Z\{qx(k,T)\} &= \sum_{k=0}^{\infty} x(k+1,T)z^{-k} \\ &= z\sum_{m=1}^{\infty} x(m,T)z^{-m} \\ &= z\left(\sum_{m=0}^{\infty} x(m,T)z^{-m} - x(0,T)\right) \\ &= zX(z) - zx(0,T). \end{aligned} \tag{2.16}$$

2.4.3 Γ Transform and Transfer Function in γ

The Γ transform of $f(k,T)$, defined for $k = 0, 1, 2, ...$, is given as

$$\Gamma\{f(k,T)\} = \sum_{k=0}^{\infty} f(k,T)(T\gamma + 1)^{-k} T \tag{2.17}$$

where γ is a complex variable [31]. As is the case with the Z transform, the series in the complex variable γ is well-defined for some region in the complex γ plane.

The transfer function in the complex variable γ for a single-input, single-output discrete-time system G with input $u(k,T)$ and output $y(k,T)$ is

$$\begin{aligned} G(\gamma) &= \frac{\Gamma\{y(k,T)\}}{\Gamma\{u(k,T)\}} \\ &= \frac{Y(\gamma)}{U(\gamma)} \\ &= \frac{d_n\gamma^n + d_{n-1}\gamma^{n-1} + ... + d_1\gamma + d_0}{c_n\gamma^n + c_{n-1}\gamma^{n-1} + ... + c_1\gamma + c_0} \end{aligned}$$

where $c_i, d_i \in R$.

For a discrete-time system expressed in the delta state-space form as

$$\begin{aligned} \delta x(k,T) &= A_\delta x(k,T) + B_\delta u(k,T) \\ y(k,T) &= C_\delta x(k,T) + D_\delta u(k,T) \end{aligned}$$

with $x(k,T) \in R^{n\times 1}$, $u(k,T) \in R^{r\times 1}$, $y(k,T) \in R^{m\times 1}$, $A_\delta \in R^{n\times n}$, $B_\delta \in R^{n\times r}$, $C_\delta \in R^{m\times n}$, $D_\delta \in R^{m\times r}$ and initial condition $x(0,T)$, the relationship between input, output, and state in the complex variable γ is written as

$$\begin{aligned} X(\gamma) &= (\gamma I - A_\delta)^{-1} B_\delta u(\gamma) + (T\gamma + 1)(\gamma I - A_\delta)^{-1} x(0,T) \\ Y(\gamma) &= C_\delta((\gamma I - A_\delta)^{-1} B_\delta + D_\delta) u(\gamma) + C_\delta(T\gamma + 1)(\gamma I - A_\delta)^{-1} x(0,T). \end{aligned}$$

2.4.4 Γ Transform and Delta Operator

Let the discrete-time signal $x(k,T)$ start at $k = 0$. The Γ transform of $\delta x(k,T)$ is given as

$$\begin{aligned} \Gamma\{\delta x(k,T)\} &= \sum_{k=0}^{\infty} \delta x(k,T)\,(T\gamma + 1)^{-k} T \\ &= \sum_{k=0}^{\infty} x(k+1,T)\,(T\gamma + 1)^{-k} - \sum_{k=0}^{\infty} x(k,T)\,(T\gamma + 1)^{-k} \\ &= (T\gamma + 1)\left(\sum_{m=0}^{\infty} x(m,T)\,(T\gamma + 1)^{-m} - x(0,T)\right) - \frac{1}{T}X(\gamma) \\ &= (T\gamma + 1)\frac{1}{T}X(\gamma) - (T\gamma + 1)\,x(0,T) - \frac{1}{T}X(\gamma) \\ &= \gamma X(\gamma) - (T\gamma + 1)\,x(0,T). \end{aligned} \tag{2.18}$$

2.4.5 Transfer Functions in Complex Variables z and γ

The transfer function is unique for a given system, when expressed with a particular complex variable. Therefore, a discrete-time system G has a unique transfer function in z which can be mapped to a unique transfer function in γ, and vice-versa. To go from one transfer function to the other, one has to substitute for the complex variable by means of the relationships $\gamma = (z - 1)/T$ and $z = 1 + T\gamma$. When the sampling period T is changed, so are the values of the coefficients of the transfer function.

The principles of multiplicity of state representations, similarity transformations, realization, controllability, and observability for linear state-space discrete-time systems readily apply to the shift and delta state-space forms. A detailed study can be found in [4], [31] and [72].

2.5 Instantaneous Sampler and Holds

Since the plant is a continuous-time system and the controllers are implemented in a discrete-time form, the feedback system relies on converters to go from one domain to the other.

2.5.1 Instantaneous Sampler

An example of instantaneous sampler, or IS, is shown in Fig. 2.5. IS maps a signal in the continuous-time domain to one evolving in the discrete-time domain. In Fig. 2.5, IS outputs a discrete-time signal whose values correspond to those of the continuous-time input at the sampling instants. Thus, IS in an idealization of the ADC, where conversion is assumed to occur infinitely fast and with infinite resolution. There is no quantization effect with IS. IS is represented by the symbol S in a block diagram. As shown in Fig. 2.5, the input to S is a continuous-time signal, while the output is a discrete-time signal with label X. For a continuous-time signal $u(t)$, the output of IS is $y(k, T)$ and is given by $y(k, T) = u(kT)$, $k > 0$. This means signal y has the value of signal u at each sampling instant.

A well-known phenomenon arising with IS is aliasing. Briefly, aliasing pertains to the loss of information. Aliasing is therefore undesirable, and a designer typically tries to reduce its impact on the performance of the system. Aliasing may be viewed in the time and the frequency domains. In the time domain, multiple continuous-time sinusoidal signals end up as the same discrete-time sinusoidal signal once sampled. In the frequency domain, the spectrum folds back within the primary strip [4], [31].

To reduce the effect of aliasing, a designer may (1) precede the IS by a low-pass continuous-time filter, thereby attenuating the high-frequency content of the signal being sampled, and (2) design and implement a feedback control system at a shorter sampling period T. The former is a straightforward design process, whereas the latter may not always be possible. In practice, limitations in computing power constrain the choice for T.

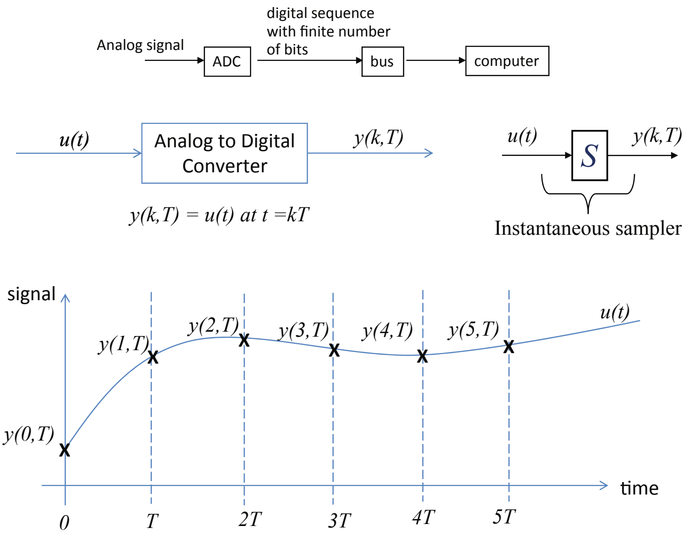

Fig. 2.5 Instantaneous sampling

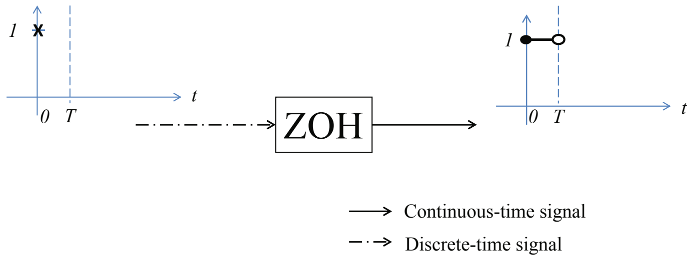

Fig. 2.6 Response of ZOH to unit discrete-time pulse

2.5.2 *Zero-order Hold*

The zero-order hold, or ZOH, is the typical model of the DAC process used by control system designers. ZOH is an idealization of DAC with a conversion assumed to occur infinitely fast. ZOH maps a signal in the discrete-time domain to one evolving in continuous-time. Figure 2.1 shows the piecewise-constant output of a ZOH.

The output of the ZOH is written as $y(t) = u(k, T)$ for $kT \leq t < (k+1)T$, from Fig. 2.1.

A hold is denoted with symbol H in the block diagrams. We use the symbol ZOH for the zero-order hold.

A hold is typically defined in terms of its response to a unit discrete-time pulse. This input-output relationship is shown in Fig. 2.6 for the ZOH.

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