Chapter 2
An Abbreviated Survey

Abstract This chapter weaves together a backdrop of related work in music theory, cognitive science, and operations research that has inspired and influenced the design of the Spiral Array and its associated algorithms. The chapter begins with an overview of the Spiral Array model and its core ideas. This is followed by a review of some spatial models for musical pitch that have informed the model's design, and an overview of the Harmonic Network (a.k.a. the *tonnetz*) and some of its applications. The idea of the *center of effect* (CE) is central to the Spiral Array and its associated algorithms. The idea of the CE draws inspiration from interior point methods in linear optimization. The second part of the chapter describes the von Neumann Center of Gravity algorithm and Dantzig's bracketing technique to speed convergence, and then draws analogies between the algorithm and the CEG method.

This book is centered on the Spiral Array model, a spatial construct that represents the interrelations among musical pitches. This chapter gives a brief survey of research that has inspired and influenced the design of the Spiral Array model and its associated tonal analysis algorithms. As motivation, I first give an overview of the intuition behind the Spiral Array model and the Center of Effect Generator (CEG) key-finding algorithm.

This chapter is based, in part, on "Dantzig's Indirect Contribution to Music Research: How the von Neumann Center of Gravity Algorithm Influenced the Center of Effect Generator Key Finding Algorithm" by Elaine Chew, an article in the INFORMS Computing Society Newsletter (Spring 2008), and on the Background (Chapter 2) of "Towards a Mathematical Modeling of Tonality" by Elaine Chew, an MIT PhD dissertation, Cambridge, Massachusetts (2000) https://dspace.mit.edu/handle/1721.1/9139

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2.1 Spiral Array Overview

2.1.1 Tonality in a Nutshell

The Spiral Array is a mathematical model for tonality. More specifically, it is a geometric model that represents elements of the tonal system underlying the music with which we are familiar. These elements include (1) pitches, sounds of a given fundamental frequency; (2) chords, simultaneous sounding of multiple pitches (a chord with three pitches is called a triad); and (3) keys, a collections of pitches, the sequences of which generate particular patterns of perceived stability. The pitch set of a key can be unambiguously defined by three triads. The name of the key is simultaneously the pitch name of the most stable tone. Tonality refers to this set of interrelations among pitches and sets of pitches.

The existence of the tonal system is one of the main reasons why we are able to form expectations, and have them resolved, when listening to music—for example, we can hear when a melody is ended or not (try singing only the first three phrases of “Happy Birthday.") There are many reasons to model tonality—for example, to create a representation on which to base algorithms for computational music analysis, and to understand human perception and cognition. Examples of investigations into mental representations of tonality include [34, 36]. The Spiral Array and associated algorithms are motivated primarily by computational analysis. Algorithms for automated analysis drive systems for automatic accompaniment and computer-assisted composition, and for analysis and synthesis of expressive music performances.

2.1.2 The Spiral Array

The Spiral Array aims to model aspects of tonality. It consists of an array of nested helices that each represents a kind of tonal entity—pitches, major/minor chords, and major/minor keys—in music. Higher level constructs are generated, successively, as the convex sum of their components.

Figure 2.1 shows some of the components of the Spiral Array model. Figure 2.1a depicts the outermost pitch class helix. Pitch representations are spaced evenly at each quarter turn of the helix. Each node represents a pitch class, i.e., a C is not just the middle C on a keyboard, but all Cs at different octaves above and below it. Frequencies of pitches in the same class are related by powers of two. Neighboring pitch classes along the helix contain pitches with frequency ratios of approximately 2:3, and vertical neighbors have ratios of approximately 4:5.

The pitch class spiral is a helical configuration of Longuet-Higgins’ Harmonic Network [29, 30], which is also known in music theoretic circles as the *tonnetz* (tone network) [8, 27]. Section 2.2 describes some spatial representations of musical pitches that preceded the Spiral Array, followed by a more detailed treatment of the Harmonic Network.

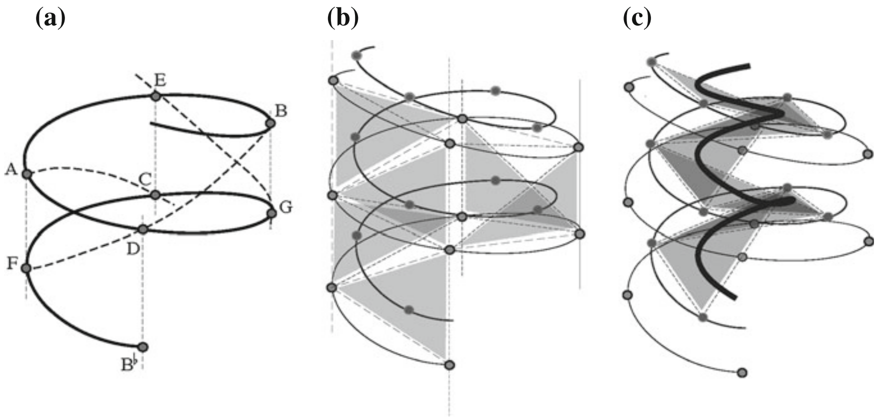

Fig. 2.1 The Spiral Array model: successive generation of major chords from pitches, and major keys from major chords. **a** Pitch class representations. **b** Major chord representations. **c** Major key representations

Distinct from previous models that employ network (or their dual) representations of pitch classes, for example [27], the Spiral Array uses the interior space to define spatial representations of chords and keys as weighted sums of their components. Thus, the Spiral Array captures, spatially and mathematically, the idea that higher level representations are a composite of their lower level elements. Another important trait is that tonal elements from different hierarchical levels reside within the same framework, in the same space.

Starting from the pitch class spiral, the Spiral Array goes on to define points in the interior of the helix that represent higher level tonal constructs. Chords are represented as points that are the convex combination of their component pitches, a centroid of sorts. For example, each triad is represented as a point on the triangle outlined by its component pitches; Fig. 2.1b shows the major triad representations, which themselves lie on an inner helix. Minor triads and their corresponding helix are defined in a similar fashion.

The major key representations are generated as weighted combinations of their defining triads, which would be three adjacent major triads for a major key, as shown in Fig. 2.1c. The minor key helix is produced in a similar way. Figure 2.1c also depicts the nested helices for pitch classes, major triads, and major keys, in decreasing order of radii. Figure 2.1c is repeated in Fig. 2.2 alongside the corresponding nested helices for pitch classes, major and minor triads, and minor keys.

The weights in the Spiral Array can be calibrated so that model distances concur with particular principles of tonal cognition, as shown in Appendix A. Like the Harmonic Network, the Spiral Array places in the closest proximity pitches related by intervals of a perfect fifth (P5), a major third (M3), and a minor third (m3). As in the work of Shepard [34] and Krumhansl [22], to be reviewed in Sect. 2.2, close tonal relations are mirrored by spatial proximity among their corresponding representations.

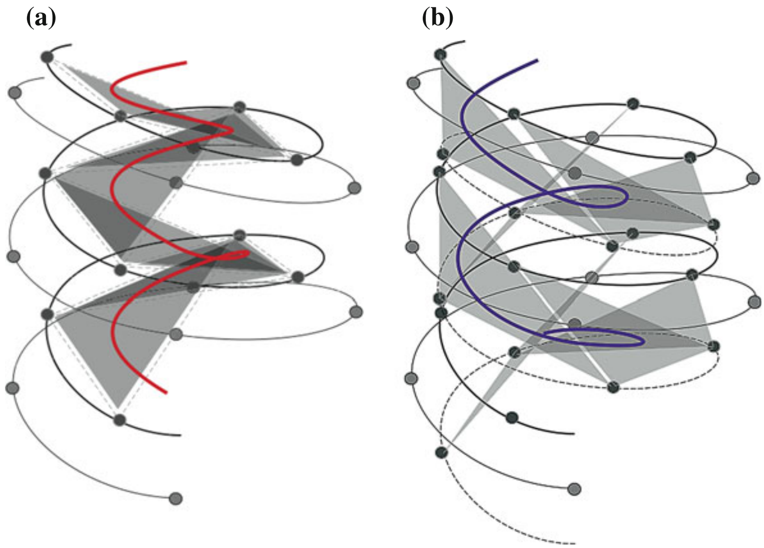

Fig. 2.2 The Spiral Array model: nested helices representing pitch classes, major/minor chords, and major/minor keys (shown separately on two diagrams for clarity). **a** Pitch class, major triad, and major key helices. **b** Pitch class, major and minor triad, and minor key helices

2.1.3 The Center of Effect

This idea of moving off the grid of the Harmonic Network to define points in the interior of the helix is central to the Spiral Array and its algorithms—the key-finding algorithm in Chap. 4, the segmentation algorithms in Chaps. 6 and 7, and the pitch-spelling algorithm in Chap. 8, to name a few.

Any collection of pitch representations can be appropriately weighted to generate a barycenter in the interior of the helix that represents the composite effect of the pitches, called the *center of effect* (CE). For example, in the Center of Effect Generator (CEG) key-finding algorithm (described in Chap. 4), an input music stream is mapped to its pitch representations, with each note weighted by its duration to get a CE; the key is then identified by searching for the key representation nearest to the CE.

The CEG algorithm can be illustrated simply with a melody. A melody consists of a sequence of note events, each note having the properties of pitch and duration. The algorithm generalizes to more complex music with simultaneous tones at any given time. In the Spiral Array, the tonal context of a segment of music is represented by a summary point, the center of effect, CE, of the pitches. A CE of a collection of notes can be generated as the sum of the pitch positions, weighted by their respective durations.

Given a melody, the CEG algorithm successively generates CEs as each note event occurs, thus updating itself as it gravitates toward the key representation. The distance between CE and key need not decrease monotonically; the CE trace can move toward, or away from, a key. The key at any given time is determined by a nearest-neighbor search for the closest key representation.

Figure 2.3 provides a pictorial guide to the CEG algorithm. Figure 2.3a shows the initial CE at the first note of the melody, which is the pitch representation of the first note. At the second note, which is of the same duration as the first, and a perfect fourth up as shown in Fig. 2.3b, the CE moves to the midpoint between the first and second pitches. Suppose the third note is the same as the second, then the CE simply moves closer to the pitch of the second/third note, as shown in Fig. 2.3c. As the iterations continue, a trajectory is traced in the interior of the model. Suppose that, at the state shown in Fig. 2.3i, one wishes to determine the key. The key is found by searching for the nearest key representation. The solution key and the convex hull of its pitch set are shown in Fig. 2.3j.

When tested on a small test set of the fugue subjects of all 24 fugues in Bach's *Well-Tempered Clavier* Book I, ignoring correct answers on the first note, the algorithm

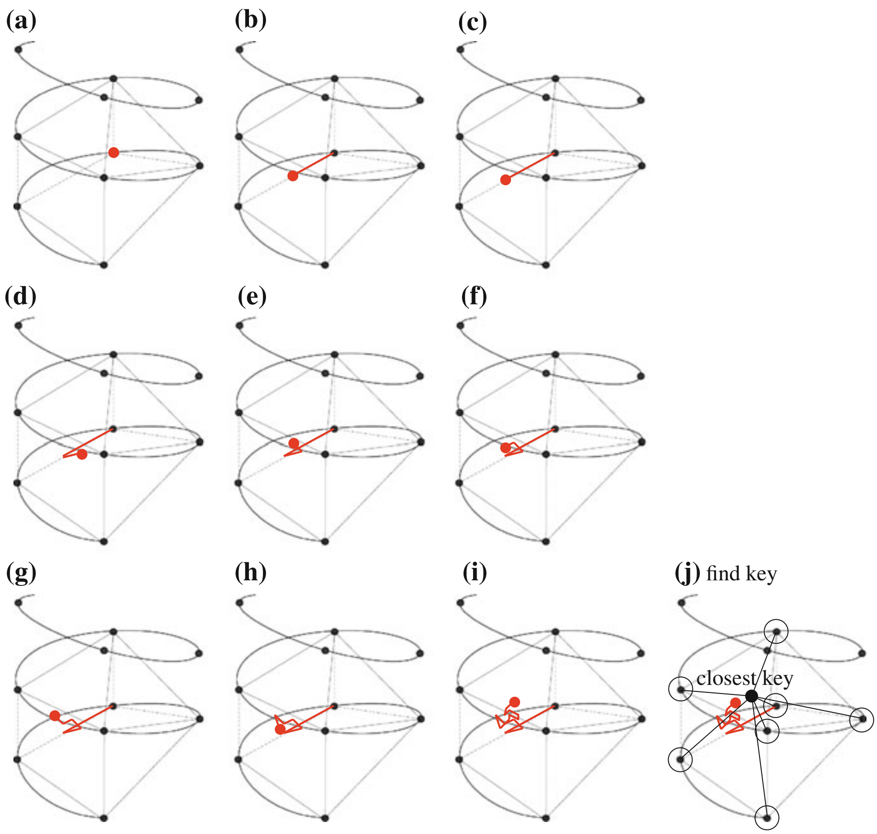

Fig. 2.3 Illustration of the Center of Effect Generator algorithm

found the key of the fugue in 3.75 note events on average, which is on par with key-finding performance by humans, and faster than previous methods.

The inspiration for the idea of the CE stems from interior point algorithms in Operations Research. Interior point algorithms deviate from well-known methods for solving linear optimization problems, namely Dantzig's Simplex Method (see [18] for an introduction), by moving away from the vertices of a polytope to search through the interior of the space for the solution that optimizes an objective function. The CEG algorithm was motivated by my experiences working with George Dantzig on one of the earliest interior point algorithms, the Center of Gravity algorithm [14] proposed by von Neumann. The Center of Gravity algorithm will be described in Sect. 2.3.

2.2 Spatial Models for Pitch Relations

Spatial analogs of physical and psychological phenomena are known to be powerful tools for solving abstract intellectual problems [34]. Biologists have long used geometric models for DNA structure and protein folding. Chemists study structural models for chemical bonding. In mathematics, it was George Dantzig's adroitness at geometry that inspired him to invent the Simplex Method for solving linear optimization models.

Some have argued that problems in music perception and cognition can be reduced to that of finding an optimal data representation [35]. *Computing in Musicology* [17], volume 11, covers an assortment of pitch encodings for computer comparisons of melodies. Tonality describes a highly structured system of pitch relations, but few representations incorporate the functional relations among pitches that generate a tonal center.

2.2.1 Cognitive Representations of Musical Pitch

In 1982, Shepard [34] stated that "the cognitive representation of musical pitch must have properties of great regularity, symmetry, and transformational invariance." Recognizing that perfect fifth (P5) and major third (M3) intervals were perceptually important pitch relations, Shepard sought to design a spatial model in which these relations have direct counterparts in the geometric structure.

In the tradition of spiral models for pitch dating as far back as 1855, Shepard proposed a model that spaced all 12 chromatic pitches equally over one full turn of a spiral (see Fig. 2.4). The equal spacing, distinct from previous models, emphasized the close relationship of pitches related by octave intervals. Further extensions to incorporate perfect fifth interval relations resulted in double-helix structures that still did not provide for the major third.

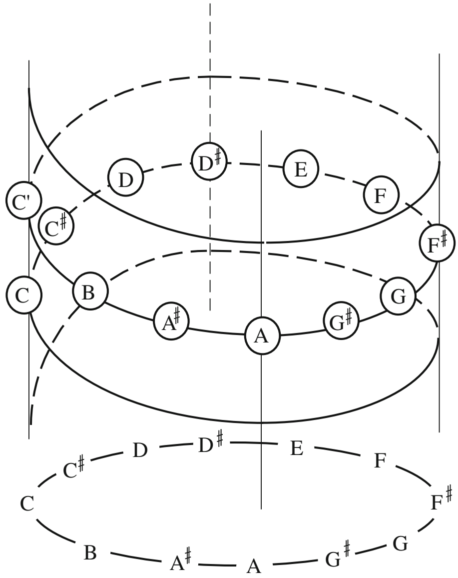

Fig. 2.4 Shepard's spiral model of chromatic pitches

Krumhansl's [20] doctoral thesis attempted to uncover the structure of pitch relations in tonality using experimental data. In this and later publications [21–23], Krumhansl and colleagues used two main ideas in analyzing data garnered from listener-based experiments. The first is the use of Probe Tone Profiles, judgements with respect to how well each of the 12 pitch classes fit into a given key context. The second is the use of Shepard's Multidimensional Scaling technique [32, 33], which maps each profile onto a point in Euclidean space such that high correlation is mirrored by spatial proximity.

Based on statistical analysis of such experimental data, Krumhansl proposed a conical structure for pitch relations [20]. The pitches of the major triad were located on a plane closest to the vertex, other diatonic pitches on a second plane, and the remaining pitches in the most distant plane. This conical structure does not contradict the Spiral Array arrangement. In fact, the planar version of the key spirals falls neatly out of the Multidimensional Scaling of the listeners' probe tone profile ratings. Lerdahl's Tonal Pitch Space [25] would later have direct parallels with Krumhansl's pitch class cone and key space—see [4] for an exposition and Fig. 2.5 for an illustration comparing Krumhansl's and Lerdahl's pitch cones. In addition, Krumhansl deduced that an established tonal context changes the pitch relationships, thus suggesting that a geometric representation may have to be altered depending on the key context.

In [21, 23], Krumhansl used the Probe Tone Profile method and Multidimensional Scaling to perform studies of listener judgements of triadic proximity, and used empirical data to corroborate the psychological reality of neo-Riemannian transformations, which brings us to the next section on the neo-Reimannian *tonnetz* (tone network).

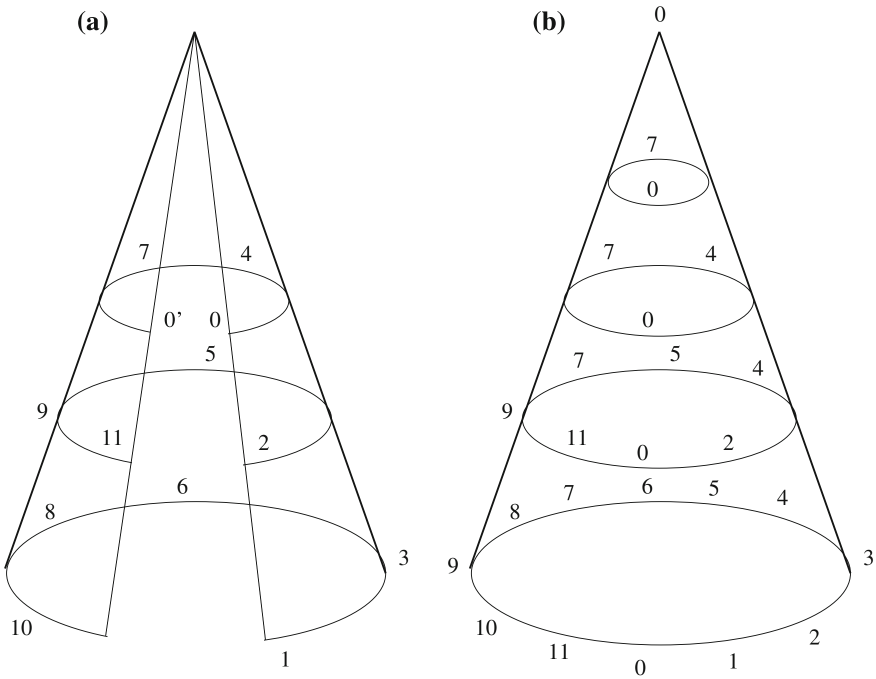

Fig. 2.5 Pitch class cones (figures from [4]): **a** Krumhansl’s pitch cone (inverted); **b** Lerdahl’s pitch class cone

2.2.2 The Tonnetz

Tonality describes a highly structured system of pitch relations, a system that has been studied by theorists over the past centuries. Many theories have been proposed to explain the relations implied in tonal music, and many theorists have chosen to represent these tonal structures geometrically (see review in [24]). One such theory, proposed by the nineteenth-century music theorist Riemann, posits that tonality derives from the establishing of significant tonal relationships through chord functions [10]. This idea has influenced a wide range of research in Music Theory.

Riemann’s theory agrees with Shepard’s intuitions that the most significant interval relations are the perfect fifth (P5) and the major/minor third (M3/m3). Riemann represented these relations in a tone network called the *tonnetz*, shown in Fig. 2.6. Cohn [8] has traced the earliest version of this network of pitches related by perfect fifth and major/minor third intervals to the eighteenth-century mathematician Euler. In the nineteenth century, this representation was appropriated by music theorists such as Oettingen and Riemann.

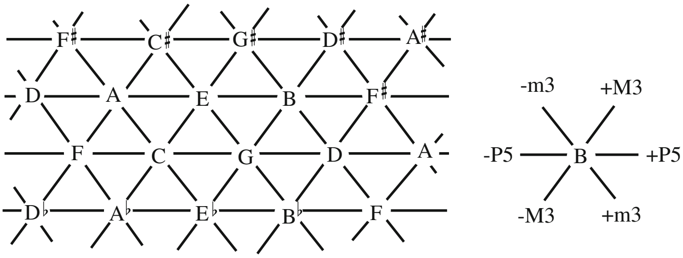

Fig. 2.6 The *Tonnetz* (figure reproduced from [4])

2.2.3 Isomorphic Representations

Longuet-Higgins [29, 30] showed that frequency ratios of musical intervals take the form $2^x.3^y.5^z$, where $x, y, z \in \mathbb{Z}$. He used this relationship to generate a three-dimensional grid of pitches. Since pitches related by powers of two are considered to be equivalent, the projection of the space onto the y–z plane gives the Harmonic Network, shown in Fig. 2.7. The Harmonic Network is effectively the same as the tonnetz, and I shall use the terms Harmonic Network and tonnetz interchangeably.

Noting that pitches in a key are positioned in a compact neighborhood on the Harmonic Network—see for example, the C major pitches highlighted in bold in Fig. 2.7—Longuet-Higgins and Steedman [31] proposed a Shape-Matching Algorithm (SMA) for key-finding using this table of harmonic relations. The SMA is closely related to the Spiral Array Center of Effect Generator (CEG) key-finding algorithm, and will be described in greater detail in Chap. 5, where it will be compared to the CEG.

Longuet-Higgins' work has inspired further mathematical analyses of the Harmonic Network, such as that of Balzano [1], which favors the major third-minor third (M3/m3) axes as opposed to the perfect fifth-major third (P5/M3) arrangement. However, one can demonstrate that the perfect fifth-major third (P5/M3), major

E	B	F♯	C♯	G♯	D♯	A♯	E♯	B♯
C	G	D	**A**	**E**	**B**	F♯	C♯	G♯
A♭	E♭	B♭	**F**	**C**	**G**	**D**	A	E
F♭	C♭	G♭	D♭	A♭	E♭	B♭	F	C
D♭♭	A♭♭	E♭♭	B♭♭	F♭	C♭	G♭	D♭	A♭

Fig. 2.7 The Harmonic Network

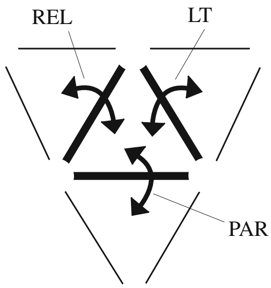

Fig. 2.8 Transformations on the Harmonic Network

third-minor third (M3/m3), and the perfect fifth-minor third (P5/m3) representations are equivalent.

Observe that the pitches displayed on the Harmonic Network repeat periodically. Rolling up the planar network so that the repeated pitch names are aligned one with another, one gets the pitch class helix of the Spiral Array. Like the Harmonic Network, neighboring pitches (on the Spiral Array's pitch class helix) are a perfect fifth (P5) apart, and pitches vertically above each other are a major third (M3) apart. If one considers equal temperament, e.g. A$\sharp$ = B$\flat$, the spiral would close to form a torus.

The added dimension (above the planar configuration) provided by the spiral arrangement allows computationally efficient and cognitively accurate algorithms to be designed that use the interior of the helix to succinctly represent different pitch collections. These will be described in subsequent chapters. Prior to the Spiral Array, researchers have alluded to the inherent toroid structure of the Harmonic Network but only used the planar versions in applications to music analysis.

2.2.4 Transformational Theory

Transformational Theory, also known as Neo-Riemannian Theory, is a branch of music theory that views transitions between chords as group theoretic transformations. In the 1980s, Lewin [26, 27] was instrumental in reviving the use of the tonnetz in music analysis, thus planting the seed for the emerging field of Transformational Theory for the analysis of triadic post-tonal music [9].

In the Harmonic Network, similar triads (where similarity is measured in terms of common pitches and parsimonious voice leading) form triangles that have common sides and are consequently near each other. Lewin [26] proposed a set of transformations on the Harmonic Network that mapped similar triads one to another.

The three main types of transformations are parallel (PAR), relative (REL), and leading tone exchange (LT), as shown in Fig. 2.8. PAR maps a major (minor) triad to its parallel minor (major) triad. REL maps a major (minor) triad to its relative minor (major) triad. And, LT lowers the root of the triad to its leading tone to form a new triad; in this case the minor triad is rooted on the mediant, the third scale degree ($\hat{3}$), of the major triad, and the major triad is based on the submediant, the sixth scale degree ($\hat{6}$), of the minor triad.

Each transformation links two triads that differ from each other only by one pitch. The PAR transformation links two triads that share the root and the fifth. In the REL transformation, the root and third of the major triad is the third and fifth of the minor triad. In the LT transformation, the third and fifth of the major triad is the root and third of the minor triad. This minimal change property of the transformations results in parsimonious voice leading; thus, smooth harmonic movements in a piece of music trace continuous paths in this space. This same principle of distance-minimizing motion is exploited in the Spiral Array space.

Paths and cycles in the Harmonic Network that are a result of a string of transformations defined by Lewin correspond to triadic movements in tonal music. They can also be viewed as trajectories in the dual tonnetz space, see Fig. 2.9. An entire body of literature has emerged based on such transformations in the dual space.

In 1996, Cohn focused on the parsimonious voice-leading property [7, 8] of the transformations to abstract hexatonic and octatonic transformation cycles on the tonnetz. In his essays, he showed how these patterns corresponded to triadic movements in chromatic music of the nineteenth century. Such models have been generalized [19, 28], and extended to meta-cycles (cycles of cycles) [15] and to tetrachords [5, 15, 16]. Computationally, the finding of transformation cycles was recently automated using finite state transducers [2]. Neo-Riemannian theory has also been applied to the analysis [3] of, and to the generating of tonally idiomatic chord progressions [6] in, pop-rock music.

These mathematical approaches are primarily graph-theoretic in nature. Several allude to the inherent toroid structure of the tonnetz; none has explored the use of the helical configuration of the Harmonic Network.

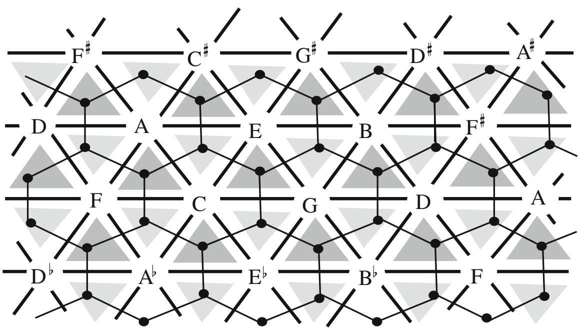

Fig. 2.9 Network showing the dual graph of the tonnetz (figure from [4])

Back to the Spiral Array

The Spiral Array preserves the Harmonic Network's preference for perfect fifth (P5) and major/minor third (M3/m3) intervals. Similar triads are still physically close in the Spiral Array. At the same time, the new configuration offers the benefits of modeling pitch relations in a higher dimension.

The Spiral Array allows one to calculate barycenters for any pitch collection that are distinct from the pitches themselves, in the interior of the helical space. As a result, one can define representations for higher level elements such as intervals, triads, tetrachords, and keys. Similar elements—where similarity is measured in terms of common pitches and perfect fifth or major/minor third interval relations—are still physically close in space.

Note that elements that are spatially close in the Spiral Array are not necessarily close in space on a keyboard or fingerboard. Pitches that are near one to another on a keyboard, such as two pitches a half step apart, are represented by points in space that are relatively far apart. The Spiral Array (and the Harmonic Network) stresses harmonic relations (perfect fifth and major/minor third intervals), and not linear relations such as pitches separated by a half step.

2.3 An Early Interior Point Algorithm

Leaving behind the music theory and psychology music domains, we now transition to the field of operations research. In the summer of 1991, I was fortunate to work with George Dantzig on an undergraduate research project focusing on von Neumann's Center of Gravity algorithm, an early interior point method for solving linear optimization problems.

The Simplex Method

Linear programming problems are typically solved using the Simplex Method invented by Dantzig (see [18] for an introduction). A typical linear programming problem is expressed as a system of equations or inequalities consisting of a linear objective function that must be maximized (or minimized) subject to a number of linear constraints.

Suppose there are n variables and m constraints. If the problem is well-behaved, and a feasible solution exists, the problem translates to one of finding the variables that maximize (or minimize) the objective function value within a feasible region in n-dimensional space, typically a polytope with m sides that is bounded by hyperplanes defined by the constraints. The optimal solution resides in a corner of the polytope, i.e., is one of the corner point feasible solutions (CPFSs).

The Simplex Method can be illustrated using a simple three-dimensional example. Starting with a known CPFS—perhaps the origin, as depicted in Fig. 2.10a—as the

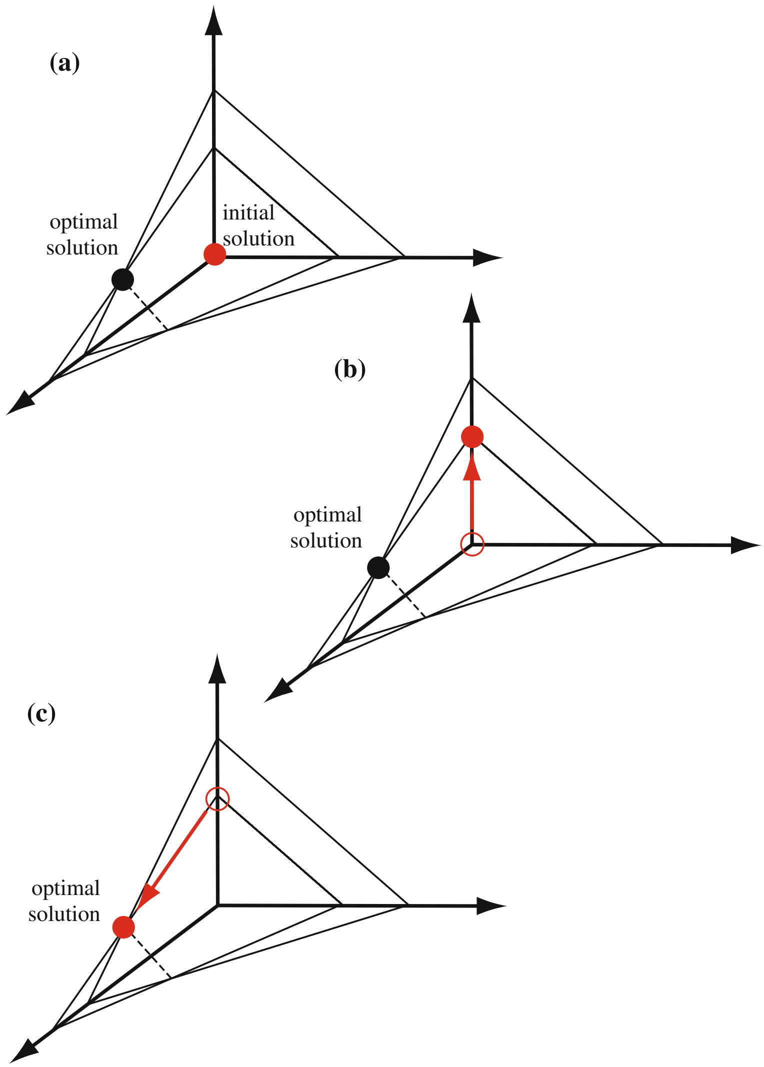

Fig. 2.10 Illustration of the Simplex Method: **a** iteration 0: initial solution; **b** iteration 1; **c** iteration 2: optimal solution found

current (suboptimal) solution, each Simplex pivot identifies an edge incident on the current point that improves the objective function value at the fastest rate, then travels along that edge to reach the next solution, as shown in Fig. 2.10b. The algorithm stops when traveling down any of the incident edges cannot improve the objective value function. The optimal solution is thus found, as in Fig. 2.10c.

Although the Simplex Method is theoretically exponential in computational time, the algorithm is hardly ever this inefficient.

Interior Point Methods

In the late 1980s, interior point methods were popularized by Karmarkar's much-publicized discovery of a polynomial time algorithm for linear programming. In the wake of this discovery, George Dantzig himself was revisiting the first interior point algorithm, the von Neumann Center of Gravity Algorithm, communicated verbally to him in 1948 by John von Neumann. Dantzig had just documented proof of convergence for the von Neumann Center of Gravity Algorithm, and his proposed extension of the algorithm that had a guaranteed polynomial bound [11, 12].

The von Neumann Center of Gravity Algorithm solves the following problem: given n points on a unit sphere centered on the origin in m dimensions, find nonnegative weights so that the weighted sum of the n points is the origin. Each von Neumann iteration could be calculated in relatively few computational steps. Dantzig showed that the algorithm had lower polynomial complexity (degree 2) than Karmarkar's, but a much higher constant based on the required precision.

While the von Neumann Center of Gravity Algorithm was guaranteed to converge, it did so very slowly after the initial steps. Dantzig's modification—to bracket the target—aimed to speed the convergence. A description of, and theorems related to, the von Neumann Center of Gravity Algorithm would later be incorporated into the chapter on Early Interior Point Methods in Dantzig and Mukund Thapa's *Linear Programming 2* [14].

In tests I conducted [13], the bracketing method was shown to work well for most problems, yielding a solution with very few von Neumann iterations. The bracketing method performed best when solving linear programs having many more variables than constraints, and a large radius of feasibility. When the radius of feasibility is very small and the number of constraints large, the iterations converged slowly, like the original Center of Gravity algorithm.

The following sections describe von Neumann's Center of Gravity algorithm and Dantzig's bracketing technique.

2.3.1 von Neumann's Center of Gravity Algorithm

The von Neumann Center of Gravity Algorithm solves linear optimization problems that have been recast in the form:

$$\begin{aligned}\sum_{j=1}^{n} P_j x_j &= 0,\\ \sum_{j=1}^{n} x_j &= 1,\\ \|P_j\|_2 &= 1 \quad \forall\, j = 1 \ldots n,\\ x_j &\geq 0 \quad \forall\, j = 1 \ldots n.\end{aligned}$$

The P_j's are points on a unit ball centered on the origin, and the goal is to find the combination of weights, x_j's, on these points so that they sum to the origin.

A two-dimensional example demonstrates the von Neumann Center of Gravity Algorithm. Assume that $n = 5$ and that the five P_j's are situated on the unit circle as shown in Fig. 2.11a. Any of these five points on the circumference can serve as the initial solution, A^0, such as the P_j that is colored red in Fig. 2.11a.

At iteration t, draw a line from the current solution, A^t, through the center of the circle, as shown in Fig. 2.11b. Find the P_j that makes the smallest acute angle, θ, with this line, P_{acute}, as shown in Fig. 2.11c. If there is no P_j on the opposite side of the circle, i.e., $\theta > \pi/2$, then the problem is infeasible. Drop a perpendicular from the origin to the line through A^t and P_{acute} to get the new solution point, A^{t+1}. Update the weights on the P_j's accordingly. This step is shown in Fig. 2.11d.

Dantzig [12] proved that, independent of the number of rows, m, and columns, n, in the problem, a precision of ε can be guaranteed with less than $1/\varepsilon^2$ iterations, where ε is the distance between the solution and the origin. Thus, convergence can be slow if the solution must be very close to the origin.

2.3.2 Dantzig's Bracketing Technique

The bracketing technique proposed in [11] aims to speed the convergence of the von Neumann Center of Gravity Algorithm by providing larger targets that are centered on the vertices of a simplex inside the convex hull of the P_j's.

This bracketing technique is shown in two dimensions in Fig. 2.12. Instead of targeting the origin, the bracketing technique applies the von Neumann Center of Gravity Algorithm $m + 1$ times, with each vertex of a simplex inside the unit ball as target. Such a simplex is shown in Fig. 2.12. For each vertex, the algorithm iterates until the approximate solution converges to a point within a given radius of the vertex.

The vertices, and the corresponding radius defining the required precision for these targets, are chosen so that the balls circumscribing each vertex fall within the circle (or sphere) that lies inside the convex hull of the P_j's. One such circle that resides inside the convex hull is indicated by the dotted line in Fig. 2.12. In practice, one does not know the radius of the dotted circle, and different radii were tested empirically in the experiments that I ran.

Once approximate solutions inside the distributed targets are found, the origin is bracketed by these solutions, and one can then solve for the origin with straightforward linear operations.

In an experiment involving 47 data files, with matrices varying in size from 3×4 to 28×40, it was found that the von Neumann Center of Gravity Algorithm with bracketing works well for most problems, yielding a solution with relatively few von Neumann steps. For problems that have a small radius of feasibility, the iterations converged slowly, and the bracketing extension does little to improve on the original von Neumann Center of Gravity Algorithm.

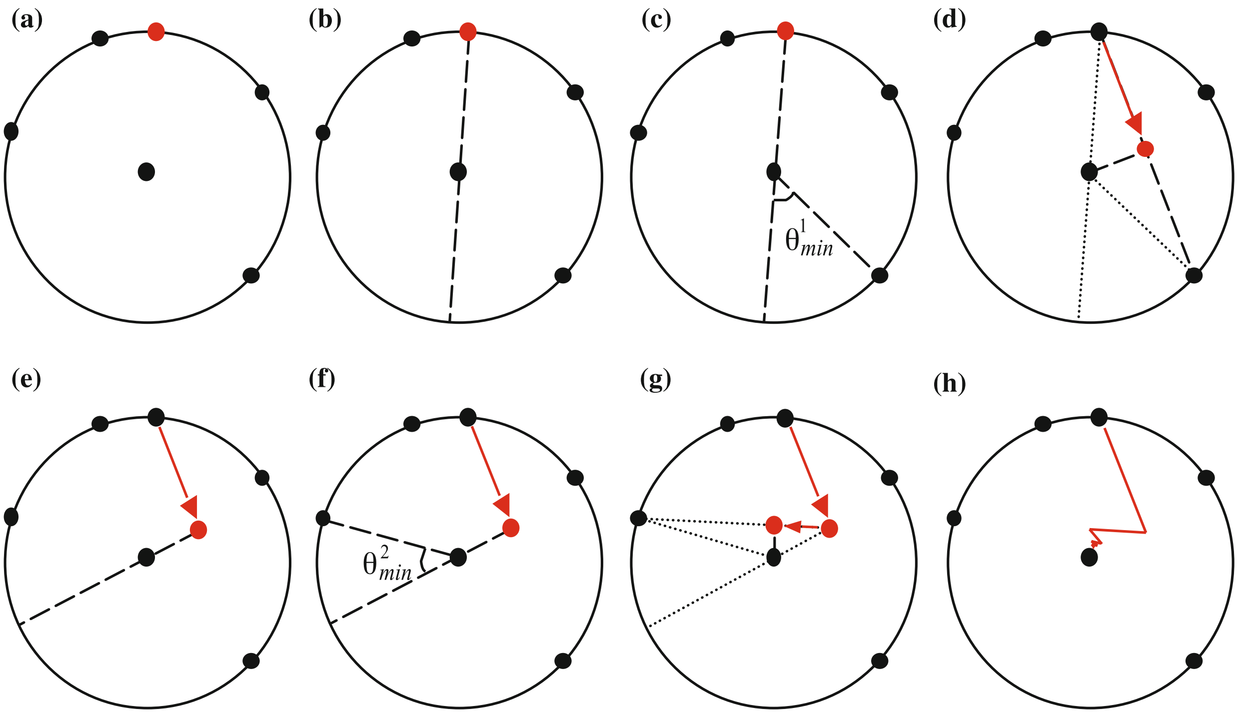

Fig. 2.11 An illustration of the von Neumann Center of Gravity Algorithm: **a** initial solution; **b** shoot line through center; **c** find point making smallest angle; **d** drop perpendicular; **e** shoot line through center; **f** find point making smallest angle; **g** drop perpendicular; **h** iterate

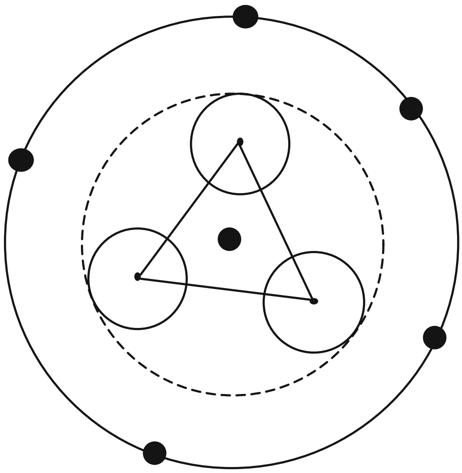

Fig. 2.12 Dantzig's bracketing technique

Back to the Center of Effect

The Spiral Array model represents tonal entities in music—pitches, chords, and keys—as points on nested helices in the same three-dimensional space. Higher level constructs are generated by successive aggregation, as weighted sums of their components. Central to the model and its algorithms is the idea of the center of effect, CE. The tonal context of a segment of music is represented by a summary point, the CE, a weighted sum of the pitch set. As more music is heard, the position of the CE generally gravitates toward the key, and the key is computed through a nearest-neighbor search. The CEG method was shown to converge faster, than other existing methods to the key (in 3.75 steps, which is on par with human hearing) when applied to a classical test set.

While I did not set out to deliberately employ ideas from the von Neumann algorithm and its extension, in retrospect, the Spiral Array model and CEG algorithm was very much influenced by the kind of geometric and interior point approach embodied in the von Neumann Center of Gravity Algorithm.

The most obvious similarity between the von Neumann Center of Gravity Algorithm and the Spiral Array CEG algorithm is the geometric and interior point approach that underlies both methods. The von Neumann Center of Gravity Algorithm is an early interior point algorithm, while the Spiral Array CEG algorithm can be loosely considered an interior point approach to key-finding. A difference is that, in the CEG algorithm, convergence is not guaranteed; the goal of music is not to converge monotonically to a key, but to create interest through the varying of distances to different keys.

The von Neumann algorithm works best in problems where the number of variables is extremely large, compared to the number of constraints. The Spiral Array is presently defined only in three dimensions, thus there exists a fixed limit to the problem size. The CEG algorithm is thus highly amenable to real-time applications, as in the MuSA.RT analysis and visualization system, described in Chap. 9.

Direct implementation of the idea of bracketing in key-finding still requires some thought. The idea exists in music cognition: notes that sound imply certain chords, and these chords in turn point to the key context. The challenge in employing parallel optimization to multiple targets in music analysis is the management of time. Music unfolds in a single stream that is experienced over time. It is unclear whether the iterations toward multiple targets should be implemented simultaneously or in series in a real-time system that mimics human key-finding abilities. A possible CEG algorithm with bracketing could be to determine the recent chords (the distributed targets) and use them to determine the key (the bracketed ultimate target).

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