Chapter 2
Fixed Point Results and Convergence of Powers of Operators

In this chapter we establish existence and uniqueness of a fixed point for a generic mapping, convergence of iterates of a generic nonexpansive mapping, stability of the fixed point under small perturbations of a mapping and many other results.

2.1 Convergence of Iterates for a Class of Nonlinear Mappings

Let K be a nonempty, bounded, closed and convex subset of a Banach space $(X, \|\cdot\|)$. We show that the iterates of a typical element (in the sense of Baire's categories) of a class of continuous self-mappings of K converge uniformly on K to the unique fixed point of this typical element.

We consider the topological subspace $K \subset X$ with the relative topology induced by the norm $\|\cdot\|$. Set

$$\operatorname{diam}(K) = \sup\{\|x - y\| : x, y \in K\}. \tag{2.1}$$

Denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to K$ which have the following property:

(P1) For each $\varepsilon > 0$, there exists $x_\varepsilon \in K$ such that

$$\|Ax - x_\varepsilon\| \le \|x - x_\varepsilon\| + \varepsilon \quad \text{for all } x \in K. \tag{2.2}$$

For each $A, B \in \mathcal{A}$, set

$$d(A, B) = \sup\{\|Ax - Bx\| : x \in K\}. \tag{2.3}$$

Clearly, the metric space $(\mathcal{A}, d)$ is complete.

We are now ready to state and prove the following result [149].

Theorem 2.1 *There exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{A}, d)$ and each $A \in \mathcal{F}$ has the following properties*:

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(i) *There exists a unique fixed point* $x_A \in K$ *such that*

$$A^n x \to x_A \quad \text{as } n \to \infty, \text{ uniformly for all } x \in K;$$

(ii)

$$\|Ax - x_A\| \le \|x - x_A\| \quad \text{for all } x \in K;$$

(iii) *For each* $\varepsilon > 0$, *there exist a natural number* n *and a real number* $\delta > 0$ *such that for each integer* $p \ge n$, *each* $x \in K$, *and each* $B \in \mathcal{A}$ *satisfying* $d(B, A) \le \delta$,

$$\left\|B^p x - x_A\right\| \le \varepsilon.$$

The following auxiliary result will be used in the proof of Theorem 2.1.

Proposition 2.2 *Let* $A \in \mathcal{A}$ *and* $\varepsilon \in (0, 1)$. *Then there exist* $\bar{x} \in K$ *and* $B \in \mathcal{A}$ *such that*

$$d(A, B) \le \varepsilon \tag{2.4}$$

and

$$\|\bar{x} - Bx\| \le \|\bar{x} - x\| \quad \text{for all } x \in K. \tag{2.5}$$

Proof Choose a positive number

$$\varepsilon_0 < 8^{-1}\varepsilon^2\big(\mathrm{diam}(K) + 1\big)^{-1}. \tag{2.6}$$

Since $A \in \mathcal{A}$, there exists $\bar{x} \in K$ such that

$$\|Ax - \bar{x}\| \le \|x - \bar{x}\| + \varepsilon_0 \quad \text{for all } x \in K. \tag{2.7}$$

Let $x \in K$. There are three cases:

$$\|Ax - \bar{x}\| < \varepsilon; \tag{2.8}$$

$$\|Ax - \bar{x}\| \ge \varepsilon \quad \text{and} \quad \|Ax - \bar{x}\| < \|x - \bar{x}\|; \tag{2.9}$$

$$\|Ax - \bar{x}\| \ge \varepsilon \quad \text{and} \quad \|Ax - \bar{x}\| \ge \|x - \bar{x}\|. \tag{2.10}$$

First we consider case (2.8). There exists an open neighborhood V_x of x in K such that

$$\|Ay - \bar{x}\| < \varepsilon \quad \text{for all } y \in V_x. \tag{2.11}$$

Define $\psi_x : V_x \to K$ by

$$\psi_x(y) = \bar{x}, \quad y \in V_x. \tag{2.12}$$

Clearly, for all $y \in V_x$,

$$0 = \left\|\psi_x(y) - \bar{x}\right\| \le \|y - \bar{x}\| \quad \text{and} \quad \left\|Ay - \psi_x(y)\right\| = \|Ay - \bar{x}\| < \varepsilon. \tag{2.13}$$

Consider now case (2.9). Since A is continuous, there exists an open neighborhood V_x of x in K such that

$$\|Ay - \bar{x}\| < \|y - \bar{x}\| \quad \text{for all } y \in V_x. \tag{2.14}$$

In this case we define $\psi_x : V_x \to K$ by

$$\psi_x(y) = Ay, \quad y \in V_x. \tag{2.15}$$

Finally, we consider case (2.10). Inequalities (2.10), (2.6) and (2.7) imply that

$$\|x - \bar{x}\| \geq \|Ax - \bar{x}\| - \varepsilon_0 > (7/8)\varepsilon. \tag{2.16}$$

For each $\gamma \in [0, 1]$, set

$$z(\gamma) = \gamma Ax + (1 - \gamma)\bar{x}. \tag{2.17}$$

By (2.17), (2.10) and (2.16), we have

$$\|z(0) - \bar{x}\| = 0 \quad \text{and} \quad \|z(1) - \bar{x}\| = \|Ax - \bar{x}\| \geq \|x - \bar{x}\| > (7/8)\varepsilon. \tag{2.18}$$

By (2.6) and (2.18), there exists $\gamma_0 \in (0, 1)$ such that

$$\|z(\gamma_0) - \bar{x}\| = \|x - \bar{x}\| - \varepsilon_0. \tag{2.19}$$

It now follows from (2.17), (2.19) and (2.7) that

$$\begin{aligned}\gamma_0\big(\|x - \bar{x}\| + \varepsilon_0\big) &\geq \gamma_0\|Ax - \bar{x}\| = \|\gamma_0 Ax + (1 - \gamma_0)\bar{x} - \bar{x}\| \\ &= \|z(\gamma_0) - \bar{x}\| = \|x - \bar{x}\| - \varepsilon_0\end{aligned}$$

and

$$\begin{aligned}\gamma_0 &\geq \big(\|x - \bar{x}\| - \varepsilon_0\big)\big(\|x - \bar{x}\| + \varepsilon_0\big)^{-1} = 1 - 2\varepsilon_0\big(\|x - \bar{x}\| + \varepsilon_0\big)^{-1} \\ &\geq 1 - 2\varepsilon_0\|x - \bar{x}\|^{-1}.\end{aligned} \tag{2.20}$$

Inequalities (2.20) and (2.16) imply that

$$\gamma_0 \geq 1 - 2\varepsilon_0\big((7/8)\varepsilon\big)^{-1}. \tag{2.21}$$

By (2.17), (2.1), (2.21) and (2.6),

$$\begin{aligned}\|z(\gamma_0) - Ax\| &= \|\gamma_0 Ax + (1 - \gamma_0)\bar{x} - Ax\| \\ &= (1 - \gamma_0)\|Ax - \bar{x}\| \leq (1 - \gamma_0)\,\mathrm{diam}(K) \leq 16\varepsilon_0(7\varepsilon)^{-1}\,\mathrm{diam}(K) \\ &\leq 3\varepsilon_0\,\mathrm{diam}(K)\varepsilon^{-1} \leq (3/8)\varepsilon\end{aligned}$$

and

$$\|z(\gamma_0) - Ax\| \leq (3/8)\varepsilon. \tag{2.22}$$

Relations (2.19) and (2.22) imply that there exists an open neighborhood V_x of x in K such that for each $y \in V_x$,

$$\|z(\gamma_0) - Ay\| < \varepsilon \quad \text{and} \quad \|z(\gamma_0) - \bar{x}\| < \|y - \bar{x}\|. \tag{2.23}$$

Define $\psi_x : V_x \to K$ by

$$\psi_x(y) = z(\gamma_0), \quad y \in V_x. \tag{2.24}$$

It is not difficult to see that in all three cases we have defined an open neighborhood V_x of x in K and a continuous mapping $\psi_x : V_x \to K$ such that for each $y \in V_x$,

$$\|Ay - \psi_x(y)\| < \varepsilon \quad \text{and} \quad \|\bar{x} - \psi_x(y)\| \le \|y - \bar{x}\|. \tag{2.25}$$

Since the metric space K with the metric induced by the norm is paracompact, there exists a continuous locally finite partition of unity $\{\phi_i\}_{i \in I}$ on K subordinated to $\{V_x\}_{x \in K}$, where each $\phi_i : K \to [0, 1]$, $i \in I$, is a continuous function such that for each $y \in K$, there is a neighborhood U of y in K such that

$$U \cap \mathrm{supp}(\phi_i) \neq \emptyset$$

only for finite number of $i \in I$;

$$\sum_{i \in I} \phi_i(x) = 1, \quad x \in K;$$

and for each $i \in I$, there is $x_i \in K$ such that

$$\mathrm{supp}(\phi_i) \subset V_{x_i}. \tag{2.26}$$

Here $\mathrm{supp}(\phi)$ is the closure of the set $\{x \in K : \phi(x) \neq 0\}$. Define

$$Bz = \sum_{i \in I} \phi_i(z)\psi_{x_i}(z), \quad z \in K. \tag{2.27}$$

Clearly, $B : K \to K$ is well defined and continuous.

Let $z \in K$. There are a neighborhood U of z in K and $i_1, \dots, i_n \in I$ such that

$$U \cap \mathrm{supp}(\phi_i) = \emptyset \quad \text{for any } i \in I \setminus \{i_1, \dots, i_n\}. \tag{2.28}$$

We may assume without any loss of generality that

$$z \in \mathrm{supp}(\phi_{i_p}), \quad p = 1, \dots, n. \tag{2.29}$$

Then

$$\sum_{p=1}^{n} \phi_{i_p}(z) = 1 \quad \text{and} \quad Bz = \sum_{p=1}^{n} \phi_{i_p}(z)\psi_{x_{i_p}}(z). \tag{2.30}$$

Relations (2.26), (2.29) and (2.25) imply that for $p = 1, \dots, n$ and $z \in V_{x_{i_p}}$,

$$\left\| Az - \psi_{x_{i_p}}(z) \right\| < \varepsilon \quad \text{and} \quad \left\| \bar{x} - \psi_{x_{i_p}}(z) \right\| \le \|\bar{x} - z\|.$$

By the equation above and (2.30),

$$\begin{aligned}
\|Bz - Az\| &= \left\| \sum_{p=1}^{n} \phi_{i_p}(z)\psi_{x_{i_p}}(z) - Az \right\| \\
&\le \sum_{p=1}^{n} \phi_{i_p}(z) \left\| \psi_{x_{i_p}}(z) - Az \right\| < \varepsilon, \\
\|\bar{x} - Bz\| &= \left\| \bar{x} - \sum_{p=1}^{n} \phi_{i_p}(z)\psi_{x_{i_p}}(z) \right\| \\
&\le \sum_{p=1}^{n} \phi_{i_p}(z) \left\| \bar{x} - \psi_{x_{i_p}}(z) \right\| \le \|\bar{x} - z\|,
\end{aligned}$$

and

$$\|Bz - Az\| < \varepsilon, \qquad \|\bar{x} - Bz\| \le \|\bar{x} - z\|.$$

Proposition 2.2 is proved. □

Proof of Theorem 2.1 For each $C \in \mathcal{A}$ and $x \in K$, set $C^0 x = x$. For each natural number n, denote by $\mathcal{F}_n$ the set of all $A \in \mathcal{A}$ which have the following property:

(P2) There exist $\bar{x}$, a natural number q, and a positive number $\delta > 0$ such that

$$\|\bar{x} - Ax\| \le \|\bar{x} - x\| + n^{-1} \quad \text{for all } x \in K,$$

and such that for each $B \in \mathcal{A}$ satisfying $d(B, A) \le \delta$, and each $x \in K$,

$$\left\| B^q x - \bar{x} \right\| \le n^{-1}.$$

Define

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n. \tag{2.31}$$

Lemma 2.3 *Let $A \in \mathcal{F}$. Then there exists a unique fixed point $x_A \in K$ of A such that*

(i) $A^n x \to x_A$ *as* $n \to \infty$, *uniformly on* K;
(ii) $\|Ax - x_A\| \le \|x - x_A\|$ *for all* $x \in K$;

(iii) *For each* $\varepsilon > 0$, *there exist a natural number* q *and* $\delta > 0$ *such that for each* $B \in \mathcal{A}$ *satisfying* $d(B, A) \le \delta$, *each* $x \in K$, *and each integer* $i \ge q$,

$$\left\| B^i x - x_A \right\| \le \varepsilon.$$

Proof Let n be a natural number. Since $A \in \mathcal{F} \subset \mathcal{F}_n$, it follows from property (P2) that there exist $x_n \in K$, an integer $q_n \ge 1$, and a number $\delta_n \ge 0$ such that

$$\|x_n - Ax\| \le \|x_n - x\| + n^{-1} \quad \text{for all } x \in K; \tag{2.32}$$

(P3) For each $B \in \mathcal{A}$ satisfying $d(B, A) \le \delta_n$, and each $x \in K$,

$$\left\| B^{q_n} x - x_n \right\| \le 1/n.$$

Property (P3) implies that for each $x \in K$, $\|A^{q_n} x - x_n\| \le 1/n$. This fact implies, in turn, that for each $x \in K$,

$$\left\| A^i x - x_n \right\| \le 1/n \quad \text{for any integer } i \ge q_n. \tag{2.33}$$

Since n is any natural number, we conclude that for each $x \in K$, $\{A^i x\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists $\lim_{i\to\infty} A^i x$. Inequality (2.33) implies that for each $x \in K$,

$$\left\| \lim_{i\to\infty} A^i x - x_n \right\| \le 1/n. \tag{2.34}$$

Since n is an arbitrary natural number, we conclude that $\lim_{i\to\infty} A^i x$ does not depend on x. Hence there is $x_A \in K$ such that

$$x_A = \lim_{i\to\infty} A^i x \quad \text{for all } x \in K. \tag{2.35}$$

By (2.34) and (2.35),

$$\|x_A - x_n\| \le 1/n. \tag{2.36}$$

Inequalities (2.36) and (2.32) imply that for each $x \in K$,

$$\begin{aligned} \|Ax - x_A\| &\le \|Ax - x_n\| + \|x_n - x_A\| \le 1/n + \|Ax - x_n\| \\ &\le 1/n + \|x - x_n\| + 1/n \le 2/n + \|x - x_A\| + \|x_A - x_n\| \\ &\le \|x - x_A\| + 3/n, \end{aligned}$$

so that

$$\|Ax - x_A\| \le \|x - x_A\| + 3/n.$$

Since n is an arbitrary natural number, we conclude that

$$\|Ax - x_A\| \le \|x - x_A\| \quad \text{for each } x \in K. \tag{2.37}$$

Let $\varepsilon > 0$. Choose a natural number

$$n > 8/\varepsilon. \tag{2.38}$$

Property (P3) implies that

$$\begin{aligned} &\|B^i x - x_n\| \le 1/n \quad \text{for each } x \in K, \text{ each integer } i \ge q_n, \\ &\quad \text{and each } B \in \mathcal{A} \text{ satisfying } d(B, A) \le \delta_n. \end{aligned} \tag{2.39}$$

Inequalities (2.39), (2.36) and (2.38) imply that for each $B \in \mathcal{A}$ satisfying $d(B, A) \le \delta_n$, each $x \in K$, and each integer $i \ge q_n$,

$$\|B^i x - x_A\| \le \|B^i x - x_n\| + \|x_n - x_A\| \le 1/n + 1/n < \varepsilon.$$

This completes the proof of Lemma 2.3. □

Completion of the proof of Theorem 2.1 In order to complete the proof of this theorem, it is sufficient, by Lemma 2.3, to show that for each natural number n, the set $\mathcal{A} \setminus \mathcal{F}_n$ is porous in $(\mathcal{A}, d)$.

Let n be a natural number. Choose a positive number

$$\alpha < (16n)^{-1} 2^{-1} \big((\operatorname{diam}(K) + 1)^2 16 \cdot 8n \big)^{-1}. \tag{2.40}$$

Let

$$A \in \mathcal{A} \quad \text{and} \quad r \in (0, 1]. \tag{2.41}$$

By Proposition 2.2, there exist $A_0 \in \mathcal{A}$ and $\bar{x} \in K$ such that

$$d(A, A_0) \le r/8 \tag{2.42}$$

and

$$\|A_0 x - \bar{x}\| \le \|x - \bar{x}\| \quad \text{for each } x \in K. \tag{2.43}$$

Set

$$\gamma = 8^{-1} r \big(\operatorname{diam}(K) + 1\big)^{-1} \tag{2.44}$$

and choose a natural number q for which

$$1 \le q \big((\operatorname{diam}(K) + 1)^2 16n \cdot 8r^{-1} \big)^{-1} \le 2. \tag{2.45}$$

Define $\bar{A} : K \to K$ by

$$\bar{A}x = (1 - \gamma) A_0 x + \gamma \bar{x}, \quad x \in K. \tag{2.46}$$

Clearly, the mapping $\bar{A}$ is continuous and for each $x \in K$,

$$\begin{aligned} \|\bar{A}x - \bar{x}\| &= \|(1 - \gamma) A_0 x + \gamma \bar{x} - \bar{x}\| \\ &= (1 - \gamma)\|A_0 x - \bar{x}\| \le (1 - \gamma)\|x - \bar{x}\|. \end{aligned} \tag{2.47}$$

Thus $\bar{A} \in \mathcal{A}$. Relations (2.3), (2.46), (2.1), (2.44) and (2.47) imply that

$$\begin{aligned} d(\bar{A}, A_0) &= \sup\{\|\bar{A}x - A_0 x\| : x \in K\} \\ &= \sup\{\gamma\|\bar{x} - A_0 x\| : x \in K\} \le \gamma \operatorname{diam}(K) = r/8. \end{aligned}$$

Together with (2.42) this implies that

$$d(\bar{A}, A) \le d(\bar{A}, A_0) + d(A_0, A) \le r/4. \tag{2.48}$$

Now assume that

$$B \in \mathcal{A} \quad \text{and} \quad d(B, \bar{A}) \le \alpha r. \tag{2.49}$$

Then (2.49), (2.40) and (2.47) imply that for each $x \in K$,

$$\|Bx - \bar{x}\| \le \|Bx - \bar{A}x\| + \|\bar{A}x - \bar{x}\| \le \|x - \bar{x}\| + \alpha r \le \|x - \bar{x}\| + 1/n. \tag{2.50}$$

In addition, (2.49), (2.48) and (2.40) imply that

$$d(B, A) \le d(B, \bar{A}) + d(\bar{A}, A) \le \alpha r + r/4 \le r/2. \tag{2.51}$$

Assume that $x \in K$. We will show that there exists an integer $j \in [0, q]$ such that $\|B^j x - \bar{x}\| \le (8n)^{-1}$. Assume the contrary. Then

$$\|B^i x - \bar{x}\| > (8n)^{-1}, \quad i = 0, \dots, q. \tag{2.52}$$

Let an integer $i \in \{0, \dots, q-1\}$. By (2.49) and (2.47),

$$\begin{aligned} \|B^{i+1}x - \bar{x}\| &= \|B(B^i x) - \bar{x}\| \\ &\le \|B(B^i x) - \bar{A}(B^i x)\| + \|\bar{A}(B^i x) - \bar{x}\| \\ &\le d(B, \bar{A}) + \|\bar{A}(B^i x) - \bar{x}\| \\ &\le \alpha r + (1-\gamma)\|B^i x - \bar{x}\| \end{aligned}$$

and

$$\|B^{i+1}x - \bar{x}\| \le \alpha r + (1-\gamma)\|B^i x - \bar{x}\|.$$

When combined with (2.52), (2.40) and (2.44), this inequality implies that

$$\begin{aligned} \|B^i x - \bar{x}\| - \|B^{i+1}x - \bar{x}\| &\ge \|B^i x - \bar{x}\| - \alpha r - (1-\gamma)\|B^i x - \bar{x}\| \\ &= \gamma\|B^i x - \bar{x}\| - \alpha r > (8n)^{-1}\gamma - \alpha r \ge (16n)^{-1}\gamma, \end{aligned}$$

so that

$$\|B^i x - \bar{x}\| - \|B^{i+1}x - \bar{x}\| \ge (16n)^{-1}\gamma.$$

When combined with (2.1), this inequality implies that

$$\operatorname{diam}(K) \geq \|x - \bar{x}\| - \|B^q x - \bar{x}\| \geq \sum_{i=0}^{q-1} \left(\|B^i x - \bar{x}\| - \|B^{i+1} x - \bar{x}\|\right) \geq q(16n)^{-1}\gamma$$

and

$$q \leq \operatorname{diam}(K) 16n/\gamma,$$

a contradiction (see (2.45)). The contradiction we have reached shows that there exists an integer $j \in [0, \dots, q-1]$ such that

$$\|B^j x - \bar{x}\| \leq (8n)^{-1}. \tag{2.53}$$

It follows from (2.49) and (2.47) that for each integer $i \in \{0, \dots, q-1\}$,

$$\begin{aligned}\|B^{i+1} x - \bar{x}\| &= \|B(B^i x) - \bar{x}\| \leq \|B(B^i x) - \bar{A}(B^i x)\| + \|\bar{A}(B^i x) - \bar{x}\| \\ &\leq d(\bar{A}, B) + \|\bar{A}(B^i x) - \bar{x}\| \leq \alpha r + \|B^i x - \bar{x}\|\end{aligned}$$

and

$$\|B^{i+1} x - \bar{x}\| \leq \|B^i x - \bar{x}\| + \alpha r.$$

This implies that for each integer s satisfying $j < s \leq q$,

$$\|B^s x - \bar{x}\| \leq \|B^j x - \bar{x}\| + \alpha r(s - j) \leq \|B^j x - \bar{x}\| + \alpha r q. \tag{2.54}$$

It follows from (2.53), (2.54), (2.45) and (2.40) that

$$\|B^q x - \bar{x}\| \leq \alpha r q + (8n)^{-1} \leq (2n)^{-1}.$$

Thus we have shown that the following property holds:

For each B satisfying (2.49) and each $x \in K$,

$$\|B^q x - \bar{x}\| \leq (2n)^{-1} \quad \text{and} \quad \|Bx - \bar{x}\| \leq \|x - \bar{x}\| + 1/n$$

(see (2.50)). Thus

$$\{B \in \mathcal{A} : d(B, \bar{A}) \leq \alpha r/2\} \subset \mathcal{F}_n \cap \{B \in \mathcal{A} : d(B, A) \leq r\}.$$

In other words, we have shown that the set $\mathcal{A} \setminus \mathcal{F}_n$ is porous in $(\mathcal{A}, d)$. This completes the proof of Theorem 2.1. □

2.2 Convergence of Iterates of Typical Nonexpansive Mappings

Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, bounded, closed and convex subset of X. In this section we show that the iterates of a typical element (in the sense of Baire category) of a class of nonexpansive mappings which take K to X converge uniformly on K to the unique fixed point of this typical element.

Denote by $\mathcal{M}_{ne}$ the set of all mappings $A : K \to X$ such that

$$\|Ax - Ay\| \leq \|x - y\| \quad \text{for all } x, y \in K.$$

For each $A, B \in \mathcal{M}_{ne}$, set

$$d(A, B) = \sup\big\{\|Ax - Bx\| : x \in K\big\}. \tag{2.55}$$

It is clear that $(\mathcal{M}_{ne}, d)$ is a complete metric space. Denote by $\mathcal{M}_0$ the set of all $A \in \mathcal{M}_{ne}$ such that

$$\inf\big\{\|x - Ax\| : x \in K\big\} = 0. \tag{2.56}$$

In other words, $\mathcal{M}_0$ consists of all those nonexpansive mappings taking K into X which have approximate fixed points. Clearly, $\mathcal{M}_0$ is a closed subset of $\mathcal{M}_{ne}$.

Every nonexpansive self-mapping of K belongs to $\mathcal{M}_0$. In order to exhibit two classes of nonself-mappings of K that are also contained in $\mathcal{M}_0$, we first recall that if $x \in K$, then the inward set $I_K(x)$ of X with respect to K is defined by

$$I_K(x) := \big\{z \in X : z = x + \alpha(y - x) \text{ for some } y \in K \text{ and } \alpha \geq 0\big\}.$$

A mapping $A : K \to X$ is said to be weakly inward if Ax belongs to the closure of $I_K(x)$ for each $x \in K$. Consider now a weakly inward mapping $A \in \mathcal{M}_{ne}$. Fix a point $z \in K$ and $t \in [0, 1)$ and let the mapping $S : K \to X$ be defined by $Sx = tAx + (1 - t)z$, $x \in K$. This strict contraction is also weakly inward and therefore has a unique fixed point $x_t \in K$ by Theorem 2.4 in [118]. Since $\|x_t - Ax_t\| \to 0$ as $t \to 1^-$, we see that $A \in \mathcal{M}_0$.

If K has a nonempty interior $\mathrm{int}(K)$ and a nonexpansive mapping $A : K \to X$ satisfies the Leray-Schauder condition with respect to $w \in \mathrm{int}(K)$, that is, $Ay - w \neq m(y - w)$ for all y in the boundary of K and $m > 1$, then it also belongs to $\mathcal{M}_0$. This is because the strict contraction $S : K \to X$ defined by $Sx = tAx + (1 - t)w$, $x \in K$, also satisfies the Leray-Schauder condition with respect to $w \in \mathrm{int}(K)$ and therefore has a unique fixed point [117].

Set

$$\rho(K) = \sup\big\{\|z\| : z \in K\big\}. \tag{2.57}$$

Our purpose is to show that the iterates of a typical element (in the sense of Baire category) of $\mathcal{M}_0$ converge uniformly on K to the unique fixed point of this typical element. As a matter of fact, we are able to establish a more refined result, involving the notion of porosity.

We are now ready to formulate our result obtained in [152].

Theorem 2.4 *There exists a set* $\mathcal{F} \subset (\mathcal{M}_0, d)$ *such that its complement* $\mathcal{M}_0 \setminus \mathcal{F}$ *is a* σ*-porous subset of* $(\mathcal{M}_0, d)$ *and each* $B \in \mathcal{F}$ *has the following properties*:

1. *There exists a unique point* $x_B \in K$ *such that* $Bx_B = x_B$;

2. *For each* $\varepsilon > 0$, *there exist* $\delta > 0$, *a natural number* q, *and a neighborhood* $\mathcal{U}$ *of* B *in* $(\mathcal{M}_{ne}, d)$ *such that*:
   (a) *if* $C \in \mathcal{U}$, $y \in K$, *and* $\|y - Cy\| \le \delta$, *then* $\|y - x_B\| \le \varepsilon$;
   (b) *if* $C \in \mathcal{U}$, $\{x_i\}_{i=0}^{q} \subset K$, *and* $Cx_i = x_{i+1}$, $i = 0, \ldots, q-1$, *then* $\|x_q - x_B\| \le \varepsilon$.

Although analogous results for the closed subspace of $(\mathcal{M}_0, d)$ comprising all nonexpansive self-mappings of K were established by De Blasi and Myjak in [49, 50], Theorem 2.4 seems to be the first generic result dealing with nonself-mappings. In this connection see also [131, 137].

We begin the proof of Theorem 2.4 with a simple lemma.

Denote by E the set of all $A \in \mathcal{M}_{ne}$ for which there exists $x \in K$ satisfying $Ax = x$. That is, E consists of all those nonexpansive mappings $A : K \to X$ which have a fixed point.

Lemma 2.5 *E is an everywhere dense subset of* $(\mathcal{M}_0, d)$.

Proof Let $A \in \mathcal{M}_0$ and $\varepsilon > 0$. By (2.56), there exists $\bar{x} \in K$ such that

$$\|\bar{x} - A\bar{x}\| < \varepsilon/2.$$

Define

$$By = Ay + \bar{x} - A\bar{x}, \quad y \in K. \tag{2.58}$$

Clearly, $B \in \mathcal{M}_{ne}$ and $B\bar{x} = \bar{x}$. Thus $B \in E$. It is easy to see that $d(A, B) = \|\bar{x} - A\bar{x}\| < \varepsilon$. This completes the proof of Lemma 2.5. □

Proof of Theorem 2.4 For each natural number n, denote by $\mathcal{F}_n$ the set of all those mappings $A \in \mathcal{M}_0$ which have the following property:

(P1) There exist a natural number q, $x_* \in K$, $\delta > 0$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_{ne}$ such that:
   (i) if $B \in \mathcal{U}$ and if $z \in K$ satisfies $\|z - Bz\| \le \delta$, then $\|z - x_*\| \le 1/n$;
   (ii) if $B \in \mathcal{U}$ and if $\{x_i\}_{i=0}^{q} \subset K$ satisfies $x_{i+1} = Bx_i$, $i = 0, \ldots, q-1$, then $\|x_q - x_*\| \le 1/n$.

Set

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n.$$

We intend to prove that $\mathcal{M}_0 \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}_0, d)$. To meet this goal, it is sufficient to show that for each natural number n, the set $\mathcal{M}_0 \setminus \mathcal{F}_n$ is a porous subset of $(\mathcal{M}_0, d)$.

Indeed, let n be a natural number. Choose a positive number

$$\alpha \le 2^{-11}\big(\rho(K) + 1\big)^{-1} n^{-1}. \tag{2.59}$$

Let

$$A \in \mathcal{M}_0 \quad \text{and} \quad r \in (0, 1]. \tag{2.60}$$

By Lemma 2.5, there are $A_0 \in E$ and $x_* \in K$ such that

$$d(A_0, A) < r/8 \quad \text{and} \quad A_0 x_* = x_*. \tag{2.61}$$

Set

$$\gamma = \left[32\left(\rho(K) + 1\right)\right]^{-1} r \tag{2.62}$$

and

$$\delta = (4n)^{-1}\gamma - 2\alpha r. \tag{2.63}$$

By (2.63), (2.62) and (2.56),

$$\delta > 0. \tag{2.64}$$

Now choose an integer $q \geq 4$ such that

$$(1 - \gamma)^q 2\left(\rho(K) + 1\right) < (16n)^{-1}. \tag{2.65}$$

Define

$$A_1 y = (1 - \gamma)A_0 y + \gamma x_*, \quad y \in K. \tag{2.66}$$

Clearly, $A_1 \in \mathcal{M}_{ne}$ and

$$A_1 x_* = x_*. \tag{2.67}$$

By (2.55), (2.66), (2.61) and (2.57),

$$\begin{aligned} d(A_1, A_0) &= \sup\left\{\|A_1 y - A_0 y\| : y \in K\right\} = \sup\left\{\|\gamma A_0 y - \gamma x_*\| : y \in K\right\} \\ &= \gamma \sup\left\{\|A_0 y - A_0 x_*\| : y \in K\right\} \\ &\leq \gamma \sup\left\{\|y - x_*\| : y \in K\right\} \leq 2\gamma\rho(K), \end{aligned}$$

so that

$$d(A_1, A_0) \leq 2\gamma\rho(K). \tag{2.68}$$

By (2.68), (2.61) and (2.62),

$$d(A, A_1) \leq d(A, A_0) + d(A_0, A_1) \leq r/8 + 2\gamma\rho(K) \leq r/4. \tag{2.69}$$

Assume that $B \in \mathcal{M}_{ne}$ satisfies

$$d(B, A_1) \leq 2\alpha r. \tag{2.70}$$

Assume further that

$$z \in K \quad \text{and} \quad \|z - Bz\| \leq \delta. \tag{2.71}$$

By (2.67) and (2.66),

$$\begin{aligned}\|A_1 z - x_*\| &= \|A_1 z - A_1 x_*\| \\ &= (1-\gamma)\|A_0 z - A_0 x_*\| \le (1-\gamma)\|z - x_*\|. \end{aligned} \tag{2.72}$$

By (2.55), (2.70) and (2.72),

$$\begin{aligned}\|Bz - z\| &\ge \|A_1 z - z\| - \|Bz - A_1 z\| \ge \|A_1 z - z\| - d(B, A_1) \\ &\ge \|A_1 z - z\| - 2\alpha r \ge \|z - x_*\| - \|x_* - A_1 z\| - 2\alpha r \\ &\ge \|z - x_*\| - (1-\gamma)\|z - x_*\| - 2\alpha r = \gamma\|z - x_*\| - 2\alpha r.\end{aligned}$$

When combined with (2.71) and (2.63), this inequality implies that

$$\delta \ge \|Bz - z\| \ge \gamma\|z - x_*\| - 2\alpha r$$

and

$$\|z - x_*\| \le \gamma^{-1}(\delta + 2\alpha r) \le (4n)^{-1}.$$

Thus we have shown that

$$\text{if } z \in K \text{ satisfies } \|z - Bz\| \le \delta, \text{ then } \|z - x_*\| \le (4n)^{-1}. \tag{2.73}$$

Now assume that

$$\{x_i\}_{i=0}^{q} \subset K, \qquad Bx_i = x_{i+1}, \quad i = 0, \dots, q-1. \tag{2.74}$$

By (2.74), (2.55), (2.70), (2.66) and (2.61), for $i = 0, \dots, q-1$, there holds

$$\begin{aligned}\|x_{i+1} - x_*\| &= \|Bx_i - x_*\| \le \|Bx_i - A_1 x_i\| + \|A_1 x_i - x_*\| \\ &= \|Bx_i - A_1 x_i\| + \|A_1 x_i - A_1 x_*\| \\ &\le d(B, A_1) + (1-\gamma)\|A_0 x_i - A_0 x_*\| \\ &\le 2\alpha r + (1-\gamma)\|x_i - x_*\|,\end{aligned}$$

that is,

$$\|x_{i+1} - x_*\| \le 2\alpha r + (1-\gamma)\|x_i - x_*\|.$$

In view of this inequality, which is valid for $i = 0, \dots, q-1$, we get

$$\begin{aligned}\|x_q - x_*\| &\le 2\alpha r \sum_{i=0}^{q-1} (1-\gamma)^i + (1-\gamma)^q \|x_0 - x_*\| \\ &\le 2\alpha r\gamma^{-1} + (1-\gamma)^q\|x_0 - x_*\| \le 2\alpha r\gamma^{-1} + 2\rho(K)(1-\gamma)^q.\end{aligned}$$

When combined with (2.62), (2.65) and (2.59), this last inequality implies that

$$\begin{aligned}\|x_q - x_*\| &\le (1-\gamma)^q 2\rho(K) + 2\alpha\bigl[32\bigl(\rho(K)+1\bigr)\bigr] \\ &\le (16n)^{-1} + 64\alpha\bigl[\rho(K)+1\bigr] \le (16n)^{-1} + (32n)^{-1} < (8n)^{-1}.\end{aligned}$$

Thus we have shown that

$$\text{if } \{x_i\}_{i=0}^{q} \subset K \text{ satisfies (2.74), then } \|x_q - x_*\| \le (8n)^{-1}. \tag{2.75}$$

By (2.75), (2.74) and (2.73), each $C \in \mathcal{M}_0$ which satisfies $d(C, A_1) \le \alpha r$ has property (P1). Therefore

$$\bigl\{C \in \mathcal{M}_0 : d(C, A_1) \le \alpha r\bigr\} \subset \mathcal{F}_n.$$

When combined with (2.59) and (2.69), this inclusion implies that

$$\bigl\{C \in \mathcal{M}_0 : d(C, A_1) \le \alpha r\bigr\} \subset \bigl\{B \in \mathcal{M}_0 : d(B, A) \le r\bigr\} \cap \mathcal{F}_n.$$

This means that $\mathcal{M}_0 \setminus \mathcal{F}_n$ is a porous set in $(\mathcal{M}_0, d)$ for all natural numbers n. Therefore $\mathcal{M}_0 \setminus \mathcal{F}$ is a σ-porous set in $(\mathcal{M}_0, d)$.

Now let $A \in \mathcal{F}$ and $\varepsilon > 0$. Choose a natural number

$$n > 8\bigl(\min\{1, \varepsilon\}\bigr)^{-1}. \tag{2.76}$$

Since $A \in \mathcal{F}_n$, property (P1) implies that there exist a natural number q_n, a number $\delta_n > 0$, a neighborhood $\mathcal{U}_n$ of A in $\mathcal{M}_{ne}$, and a point $x_n \in K$ such that the following property holds:

(P2) (i) if $B \in \mathcal{U}_n$, $z \in K$, and $\|z - Bz\| \le \delta_n$, then $\|z - x_n\| \le 1/n$;
(ii) if $B \in \mathcal{U}_n$, $\{z_i\}_{i=0}^{q_n} \subset K$, and $z_{i+1} = Bz_i$, $i = 0, \dots, q_n - 1$, then $\|z_{q_n} - x_n\| \le 1/n$.

Since $A \in \mathcal{M}_0$, there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset K$ such that

$$\lim_{i\to\infty} \|y_i - Ay_i\| = 0. \tag{2.77}$$

Hence there exists a natural number i_0 such that

$$\|y_i - Ay_i\| \le \delta_n \quad \text{for all integers } i \ge i_0.$$

When combined with (P2)(i), this implies that

$$\|x_n - y_i\| \le 1/n \quad \text{for all integers } i \ge i_0. \tag{2.78}$$

In view of (2.78), for each pair of integers $i, j \ge i_0$,

$$\|y_i - y_j\| \le \|y_i - x_n\| + \|x_n - y_j\| \le 2/n < \varepsilon.$$

Since ε is an arbitrary positive number, we conclude that $\{y_i\}_{i=1}^{\infty}$ is a Cauchy sequence and therefore there exists

$$x_A = \lim_{i\to\infty} y_i. \tag{2.79}$$

Clearly, $Ax_A = x_A$. It is easy to see that x_A is the unique fixed point of A. Indeed, if it were not unique, then we would be able to construct a nonconvergent sequence $\{y_i\}_{i=0}^{\infty}$ satisfying (2.77).

By (2.78) and (2.79),

$$\|x_A - x_n\| \le 1/n. \tag{2.80}$$

Now assume that

$$B \in \mathcal{U}_n, \qquad z \in K, \quad \text{and} \quad \|z - Bz\| \le \delta_n. \tag{2.81}$$

By (P2)(i) and (2.81),

$$\|z - x_n\| \le 1/n.$$

When combined with (2.80) and (2.76), this inequality implies that

$$\|z - x_A\| \le \|z - x_n\| + \|x_n - x_A\| \le 2/n < \varepsilon.$$

Finally, suppose that

$$B \in \mathcal{U}_n, \qquad \{z_i\}_{i=0}^{q_n} \subset K, \quad \text{and} \quad Bz_i = z_{i+1}, \quad i = 0, \dots, q_n - 1. \tag{2.82}$$

Then by (P2)(ii) and (2.82),

$$\|z_{q_n} - x_n\| \le 1/n.$$

When combined with (2.80) and (2.76), this last inequality implies that

$$\|z_{q_n} - x_A\| \le \|z_{q_n} - x_n\| + \|x_n - x_A\| \le 2/n < \varepsilon.$$

This completes the proof of Theorem 2.4. □

2.3 A Stability Result in Fixed Point Theory

Let $K \subset X$ be a nonempty, compact and convex subset of a Banach space $(X, \|\cdot\|)$. In this section, which is based on [153], we consider a complete metric space of all the continuous self-mappings of K and show that a typical element of this space (in the sense of Baire's categories) has a fixed point which is stable under small perturbations of the mapping.

Denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to K$. For each $A, B \in \mathcal{A}$, set

$$d(A, B) = \sup\bigl\{\|Ax - Bx\| : x \in K\bigr\}.$$

Clearly, $(\mathcal{A}, d)$ is a complete metric space. By Schauder's fixed point theorem, for each $A \in \mathcal{A}$ there exists $x_* \in K$ such that $Ax_* = x_*$. We begin with the following simple result.

Proposition 2.6 *Let $A \in \mathcal{A}$, $\Omega = \{x \in K : Ax = x\}$, and let $\varepsilon > 0$. Then there exists a positive number δ such that for each $B \in \mathcal{A}$ satisfying $d(A, B) \leq \delta$ and each $x \in K$ satisfying $Bx = x$, there exists $y \in \Omega$ such that $\|x - y\| \leq \varepsilon$.*

Proof Assume the contrary. Then there exist a sequence $\{B_n\}_{n=1}^{\infty} \subset \mathcal{A}$ satisfying

$$d(A, B_n) \leq 1/n \quad \text{for all integers } n \geq 1, \tag{2.83}$$

and a sequence $\{x_n\}_{n=1}^{\infty} \subset K$ such that for each integer $n \geq 1$,

$$B_n x_n = x_n \quad \text{and} \quad \inf\bigl\{\|x_n - y\| : y \in \Omega\bigr\} \geq \varepsilon. \tag{2.84}$$

Since K is compact, we may assume without loss of generality that there exists

$$x_* = \lim_{n \to \infty} x_n. \tag{2.85}$$

It follows from (2.85), (2.84), (2.83) and the continuity of A that

$$\begin{aligned}\|Ax_* - x_*\| &\leq \|Ax_* - Ax_n\| + \|B_n x_n - Ax_n\| + \|B_n x_n - x_n\| + \|x_n - x_*\| \\ &\leq \|Ax_* - Ax_n\| + 1/n + \|x_n - x_*\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus $Ax_* = x_*$, $x_* \in \Omega$, and (2.85) contradicts (2.84). The contradiction we have reached proves Proposition 2.6. □

In view of this result, it is natural to ask if, given $A \in \mathcal{A}$, there is a fixed point $x_* \in K$ of A with the following property:

For each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $B \in \mathcal{A}$ satisfying $d(A, B) \leq \delta$, there exists $y \in K$ such that $By = y$ and $\|y - x_*\| \leq \varepsilon$.

Example 2.7 Let $X = R^1$, $K = [0, 1]$ and $Ax = x$, $x \in K$. Clearly, the set of fixed points of A is the interval $[0, 1]$. For each integer $n \geq 1$, define

$$A_n x = (1 - 1/n)x, \qquad B_n x = \min\{x + 1/n, 1\} \quad \text{for all } x \in [0, 1].$$

Clearly, $B_n, A_n \to A$ as $n \to \infty$. It is easy to see that for each $n \geq 1$, the set of fixed points of A_n is the singleton $\{0\}$ while the set of fixed points of B_n is the interval $[1 - 1/n, 1]$.

This example shows that in general the answer to our question is negative. Nevertheless, we show in this section that for a typical $A \in \mathcal{A}$ (in the sense of Baire's categories) the answer is positive.

Let $K \subset X$ be a nonempty, closed and convex subset of a Banach space $(X, \|\cdot\|)$. Denote by $\tilde{\mathcal{A}}$ the family of all continuous mappings $A : K \to K$ such that the closure of $A(K)$ is a compact set in the norm topology. It is well known [171] that for each $A \in \tilde{\mathcal{A}}$ there is $x_A \in K$ such that $Ax_A = x_A$.

For each $A, B \in \tilde{\mathcal{A}}$ set

$$d(A, B) = \sup\bigl\{\|Ax - Bx\| : x \in K\bigr\}. \tag{2.86}$$

It is not difficult to see that $(\tilde{\mathcal{A}}, d)$ is a complete metric space.

Theorem 2.8 *There exists a subset $\mathcal{F} \subset \tilde{\mathcal{A}}$ which is a countable intersection of open everywhere dense subsets of $(\tilde{\mathcal{A}}, d)$ such that for each $A \in \mathcal{F}$, there exists $x_* \in K$ such that*

(i) $Ax_* = x_*$;
(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $B \in \tilde{\mathcal{A}}$ satisfies $d(A, B) \le \delta$, then there is $z \in K$ which satisfies $Bz = z$ and $\|z - x_*\| \le \varepsilon$.*

Two auxiliary propositions will precede the proof of Theorem 2.8.

Proposition 2.9 *Let $A \in \tilde{\mathcal{A}}$, $\varepsilon > 0$ and let $x_* \in K$ satisfy $Ax_* = x_*$. Then there exist $B \in \tilde{\mathcal{A}}$ and $\delta > 0$ such that $d(B, A) \le \varepsilon$ and $Bz = x_*$ for each $z \in K$ satisfying $\|z - x_*\| \le \delta$.*

Proof There exists $\delta > 0$ such that for each $z \in K$ satisfying $\|z - x_*\| \le 4\delta$, the following inequality holds:

$$\|Az - x_*\| \le \varepsilon/4. \tag{2.87}$$

By Urysohn's theorem, there exists a continuous function $\lambda : X \to [0, 1]$ such that

$$\lambda(z) = 1 \quad \text{for each } z \in X \text{ satisfying } \|z - x_*\| \le \delta \tag{2.88}$$

and

$$\lambda(z) = 0 \quad \text{for each } z \in X \text{ satisfying } \|z - x_*\| \ge 2\delta. \tag{2.89}$$

Define

$$Bz = \lambda(z)x_* + \bigl(1 - \lambda(z)\bigr)Az \tag{2.90}$$

for all $z \in K$.

Clearly, $B : K \to K$ is continuous, $B(K)$ is contained in a compact subset of X, and

$$Bx_* = x_*. \tag{2.91}$$

By (2.90), (2.88) and (2.89), for each $z \in K$ satisfying $\|z - x_*\| \le \delta$, we have

$$Bz = x_*, \tag{2.92}$$

and for each $z \in K$ satisfying $\|z - x_*\| \geq 2\delta$,

$$Bz = Az. \tag{2.93}$$

It follows from (2.90) and the choice of δ (see (2.87)) that for each $z \in K$ satisfying $\|z - x_*\| \leq 2\delta$,

$$\begin{aligned}\|Bz - Az\| &= \big\|\lambda(z)x_* + \big(1 - \lambda(z)\big)Az - Az\big\| \\ &\leq \|x_* - Az\| \leq \varepsilon/4.\end{aligned}$$

This completes the proof of Proposition 2.9. □

Proposition 2.10 *Let $A \in \tilde{\mathcal{A}}$, $\varepsilon > 0$, let $x_* \in K$ be a fixed point of A, and let $B \in \tilde{\mathcal{A}}$, $\delta > 0$ be as guaranteed by Proposition 2.9. Then for each $C \in \tilde{\mathcal{A}}$ satisfying $d(C, B) \leq \delta$, there is $y \in K$ such that*

$$Cy = y \quad \textit{and} \quad \|y - x_*\| \leq d(C, B).$$

Proof By Proposition 2.9,

$$d(A, B) \leq \varepsilon \tag{2.94}$$

and

$$Bz = x_* \quad \text{for each } z \in K \text{ satisfying } \|z - x_*\| \leq \delta. \tag{2.95}$$

Assume that $C \in \tilde{\mathcal{A}}$ satisfies

$$d(C, B) \leq \delta. \tag{2.96}$$

Set

$$\Omega = \big\{z \in K : \|z - x_*\| \leq d(C, B)\big\}. \tag{2.97}$$

Clearly, Ω is a closed and convex set. It follows from (2.97), (2.96) and (2.95) that for each $z \in \Omega$,

$$\|x_* - Cz\| \leq \|x_* - Bz\| + \|Bz - Cz\| = \|Bz - Cz\| \leq d(C, B)$$

and $Cz \in \Omega$. Thus $C(\Omega) \subset \Omega$. Clearly $C(\Omega) \subset C(X)$ is contained in a compact subset of X. By Schauder's theorem there is $y \in \Omega$ such that $Cy = y$. Proposition 2.10 is proved. □

Proof of Theorem 2.8 Let $A \in \tilde{\mathcal{A}}$ and $\varepsilon \in (0, 1)$. By Propositions 2.9 and 2.10, there exist

$$A_\varepsilon \in \tilde{\mathcal{A}}, \qquad x_{A,\varepsilon} \in K \quad \text{and} \quad \delta_{A,\varepsilon} \in (0, 1)$$

such that

$$d(A, A_\varepsilon) \le \varepsilon, \tag{2.98}$$

$$A_\varepsilon z = x_{A,\varepsilon} \quad \text{for each } z \in K \text{ satisfying } \|z - x_{A,\varepsilon}\| \le \delta_{A,\varepsilon}, \tag{2.99}$$

and the following property holds:

(P) For each $C \in \tilde{\mathcal{A}}$ satisfying $d(C, A_\varepsilon) \le \delta_{A,\varepsilon}$, there is $y \in K$ such that

$$Cy = y, \qquad \|y - x_{A,\varepsilon}\| \le d(C, A_\varepsilon).$$

For each integer $i \ge 1$, set

$$\mathcal{U}(A, \varepsilon, i) = \{C \in \tilde{\mathcal{A}} : d(C, A_\varepsilon) < \delta_{A,\varepsilon}/i\}. \tag{2.100}$$

Define

$$\mathcal{F} = \bigcap_{i=1}^{\infty} \bigcup \{\mathcal{U}(A, \varepsilon, i) : A \in \tilde{\mathcal{A}}, \varepsilon \in (0, 1)\}. \tag{2.101}$$

Clearly, $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of $(\tilde{\mathcal{A}}, d)$.

Let $B \in \mathcal{F}$. For each integer $i \ge 1$, there are $A_i \in \tilde{\mathcal{A}}$ and $\varepsilon_i \in (0, 1)$ such that

$$B \in \mathcal{U}(A_i, \varepsilon_i, i). \tag{2.102}$$

It follows from (2.102), (2.100) and property (P) that for each integer $i \ge 1$, there $y_i \in K$ such that

$$By_i = y_i \tag{2.103}$$

and

$$\|y_i - x_{A_i,\varepsilon_i}\| \le d\big(A, (A_i)_{\varepsilon_i}\big) \le \delta_{A_i,\varepsilon_i}/i. \tag{2.104}$$

Since $\{y_i\}_{i=1}^{\infty} \subset B(K)$, there is a subsequence $\{y_{i_k}\}_{k=1}^{\infty}$ which converges to $x_* \in K$. Clearly, $Bx_* = x_*$.

Let $\varepsilon > 0$. There exists a natural number k such that

$$i_k^{-1} < 8^{-1}\varepsilon \quad \text{and} \quad \|y_{i_k} - x_*\| \le \varepsilon/8. \tag{2.105}$$

It follows from (2.104) and (2.105) that

$$\|y_{i_k} - x_{A_{i_k},\varepsilon_{i_k}}\| \le 1/i_k < \varepsilon/8. \tag{2.106}$$

Inequalities (2.105) and (2.106) imply that

$$\|x_* - x_{A_{i_k},\varepsilon_{i_k}}\| \le \|x_* - y_{i_k}\| + \|y_{i_k} - x_{A_{i_k},\varepsilon_{i_k}}\| \le \varepsilon/4. \tag{2.107}$$

Let

$$C \in \mathcal{U}(A_{i_k}, \varepsilon_{i_k}, i_k). \tag{2.108}$$

It follows from (2.108), (2.100), (2.105) and property (P) that there exists a point $z \in K$ such that

$$Cz = z \quad \text{and} \quad \|z - x_{A_{i_k}, \varepsilon_{i_k}}\| \le d\big(C, (A_{i_k})_{\varepsilon_{i_k}}\big) \le 1/i_k \le \varepsilon/8.$$

When combined with (2.107), this implies that

$$\|z - x_*\| \le \|z - x_{A_{i_k}, \varepsilon_{i_k}}\| + \|x_{A_{i_k}, \varepsilon_{i_k}} - x_*\| \le \varepsilon/2.$$

Theorem 2.8 is proved. □

2.4 Well-Posed Null and Fixed Point Problems

The notion of well-posedness is of great importance in many areas of mathematics and its applications. In this section we consider two complete metric spaces of continuous mappings and establish generic well-posedness of certain null and fixed point problems. Our results, which were obtained in [154], are a consequence of the variational principle established in [74]. For other related results concerning the well-posedness of fixed point problems see [50, 139].

Let $(X, \|\cdot\|, \ge)$ be a Banach space ordered by a closed convex cone $X_+ = \{x \in X : x \ge 0\}$ such that $\|x\| \le \|y\|$ for each pair of points $x, y \in X_+$ satisfying $x \le y$. Let (K, ρ) be a complete metric space. Denote by $\mathcal{M}$ the set of all continuous mappings $A : K \to X$. We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$E(\varepsilon) = \big\{(A, B) \in \mathcal{M} \times \mathcal{M} : \|Ax - Bx\| \le \varepsilon \text{ for all } x \in K\big\}, \tag{2.109}$$

where $\varepsilon > 0$. It is not difficult to see that this uniform space is metrizable (by a metric d) and complete.

Denote by $\mathcal{M}_p$ the set of all $A \in \mathcal{M}$ such that

$$Ax \in X_+ \quad \text{for all } x \in K \tag{2.110}$$

and

$$\inf\big\{\|Ax\| : x \in K\big\} = 0. \tag{2.111}$$

It is not difficult to see that $\mathcal{M}_p$ is a closed subset of $(\mathcal{M}, d)$.

We can now state and prove our first result.

Theorem 2.11 *There exists an everywhere dense G_δ subset $\mathcal{F} \subset \mathcal{M}_p$ such that for each $A \in \mathcal{F}$, the following properties hold*:

1. *There is a unique $\bar{x} \in K$ such that $A\bar{x} = 0$.*
2. *For any $\varepsilon > 0$, there exist $\delta > 0$ and a neighborhood U of A in $\mathcal{M}_p$ such that if $B \in U$ and if $x \in K$ satisfies $\|Bx\| \le \delta$, then $\rho(x, \bar{x}) \le \varepsilon$.*

Proof We obtain this theorem as a realization of the variational principle established in Theorem 2.1 of [74] with $f_A(x) = \|Ax\|$, $x \in K$. In order to prove our theorem by using this variational principle we need to prove the following assertion:

(A) For each $A \in \mathcal{M}_p$ and each $\varepsilon > 0$, there are $\bar{A} \in \mathcal{M}_p$, $\delta > 0$, $\bar{x} \in K$ and a neighborhood W of $\bar{A}$ in $\mathcal{M}_p$ such that

$$(A, \bar{A}) \in E(\varepsilon),$$

and if $B \in W$ and $z \in K$ satisfy $\|Bz\| \leq \delta$, then

$$\rho(z, \bar{x}) \leq \varepsilon.$$

Let $A \in \mathcal{M}_p$ and $\varepsilon > 0$. Choose $\bar{u} \in X_+$ such that

$$\|\bar{u}\| = \varepsilon/4, \tag{2.112}$$

and $\bar{x} \in K$ such that

$$\|A\bar{x}\| \leq \varepsilon/8. \tag{2.113}$$

Since A is continuous, there is a positive number r such that

$$r < \min\{1, \varepsilon/16\} \tag{2.114}$$

and

$$\|Ax - A\bar{x}\| \leq \varepsilon/8 \quad \text{for each } x \in K \text{ satisfying } \rho(x, \bar{x}) \leq 4r. \tag{2.115}$$

By Urysohn's theorem, there is a continuous function $\phi : K \to [0, 1]$ such that

$$\phi(x) = 1 \quad \text{for each } x \in K \text{ satisfying } \rho(x, \bar{x}) \leq r \tag{2.116}$$

and

$$\phi(x) = 0 \quad \text{for each } x \in K \text{ satisfying } \rho(x, \bar{x}) \geq 2r. \tag{2.117}$$

Define

$$\bar{A}x = \big(1 - \phi(x)\big)(Ax + \bar{u}), \quad x \in K. \tag{2.118}$$

It is clear that $\bar{A} : K \to X$ is continuous. Now (2.116)–(2.118) imply that

$$\bar{A}x = 0 \quad \text{for each } x \in K \text{ satisfying } \rho(x, \bar{x}) \leq r \tag{2.119}$$

and

$$\bar{A}x \geq \bar{u} \quad \text{for each } x \in K \text{ satisfying } \rho(x, \bar{x}) \geq 2r. \tag{2.120}$$

It is not difficult to see that $\bar{A} \in \mathcal{M}_p$. We claim that $(A, \bar{A}) \in E(\varepsilon)$.

Let $x \in K$. There are two cases: either

$$\rho(x, \bar{x}) \geq 2r \tag{2.121}$$

or

$$\rho(x, \bar{x}) < 2r. \tag{2.122}$$

Assume first that (2.121) holds. Then it follows from (2.121), (2.117), (2.118) and (2.112) that

$$\|Ax - \bar{A}x\| = \|\bar{u}\| = \varepsilon/4.$$

Now assume that (2.122) holds. Then by (2.122), (2.118) and (2.112),

$$\begin{aligned}\|\bar{A}x - Ax\| &= \big\|\big(1 - \phi(x)\big)(Ax + \bar{u}) - Ax\big\| \le \|\bar{u}\| + \|Ax\| \\ &\le \varepsilon/4 + \|Ax\|.\end{aligned}$$

It follows from this inequality, (2.122), (2.115) and (2.113) that

$$\|\bar{A}x - Ax\| \le \varepsilon/4 + \|Ax\| < \varepsilon/2.$$

Therefore in both cases $\|\bar{A}x - Ax\| \le \varepsilon/2$. Since this inequality holds for any $x \in K$, we conclude that

$$(A, \bar{A}) \in E(\varepsilon). \tag{2.123}$$

Consider now an open neighborhood U of $\bar{A}$ in $\mathcal{M}_p$ such that

$$U \subset \big\{B \in \mathcal{M}_p : (\bar{A}, B) \in E(\varepsilon/16)\big\}. \tag{2.124}$$

Let

$$B \in U, \qquad z \in K \tag{2.125}$$

and

$$\|Bz\| \le \varepsilon/16. \tag{2.126}$$

Relations (2.126), (2.125), (2.124) and (2.109) imply that

$$\|\bar{A}z\| \le \|Bz\| + \|\bar{A}z - Bz\| \le \varepsilon/16 + \varepsilon/16. \tag{2.127}$$

We claim that

$$\rho(z, \bar{x}) \le \varepsilon. \tag{2.128}$$

Assume the contrary. Then by (2.114),

$$\rho(z, \bar{x}) > \varepsilon \ge 2r.$$

When combined with (2.120), this implies that

$$\bar{A}z \ge \bar{u}.$$

It follows from this inequality, the monotonicity of the norm, (2.125), (2.124), (2.109) and (2.112) that

$$\|Bz\| \geq \|\bar{A}z\| - \varepsilon/16 \geq \|\bar{u}\| - \varepsilon/16 = \varepsilon/4 - \varepsilon/16 = 3\varepsilon/16.$$

This, however, contradicts (2.126). The contradiction we have reached proves (2.128) and Theorem 2.11 itself. □

Now assume that the set K is a subset of X and

$$\rho(x, y) = \|x - y\|, \quad x, y \in K.$$

Denote by $\mathcal{M}_n$ the set of all mappings $A \in \mathcal{M}$ such that

$$Ax \geq x \quad \text{for all } x \in K$$

and

$$\inf\{\|Ax - x\| : x \in K\} = 0.$$

Clearly, $\mathcal{M}_n$ is a closed subset of $(\mathcal{M}, d)$. Define a map $J : \mathcal{M}_n \to \mathcal{M}_p$ by

$$J(A)x = Ax - x \quad \text{for all } x \in K$$

and all $A \in \mathcal{M}_n$. Clearly, there exists $J^{-1} : \mathcal{M}_p \to \mathcal{M}_n$, and both J and its inverse J^{-1} are continuous. Therefore Theorem 2.11 implies the following result regarding the generic well-posedness of the fixed point problem for $A \in \mathcal{M}_n$.

Theorem 2.12 *There exists an everywhere dense G_δ subset $\mathcal{F} \subset \mathcal{M}_n$ such that for each $A \in \mathcal{F}$, the following properties hold*:

1. *There is a unique $\bar{x} \in K$ such that $A\bar{x} = \bar{x}$.*
2. *For any $\varepsilon > 0$, there exist $\delta > 0$ and a neighborhood U of A in $\mathcal{M}_n$ such that if $B \in U$ and if $x \in K$ satisfies $\|Bx - x\| \leq \delta$, then $\|x - \bar{x}\| \leq \varepsilon$.*

2.5 Mappings in a Finite-Dimensional Euclidean Space

In this section we study the existence and stability of fixed points of continuous mappings in finite-dimensional Euclidean spaces. Our results [156] establish generic existence and stability of fixed points for a class of nonself-mappings defined on certain closed (but not necessarily either convex or bounded) subsets of a finite-dimensional Euclidean space. In these results, we endow the relevant space of mappings with two topologies, one weaker than the other. In the first result we find an open (in the weak topology) and everywhere dense (in the strong topology) set such that each mapping in it possesses a fixed point. In the second result we construct a countable intersection of open (in the weak topology) and everywhere dense (in the strong topology) sets such that each mapping in this intersection has a stable fixed point.

Let $K \subset R^n$ be a nonempty, closed subset of the n-dimensional Euclidean space $(R^n, \|\cdot\|)$. We assume that K is the closure of its nonempty interior $\mathrm{int}(K)$.

For each $x \in R^n$ and each $r > 0$, set $B(x,r) = \{y \in R^n : \|x - y\| \le r\}$ and fix $\theta \in K$.

Denote by $\mathcal{M}$ the set of all continuous mappings $A : K \to R^n$. We equip the space $\mathcal{M}$ with the uniformity determined by the base

$$\mathcal{E}_w(N,\varepsilon) = \big\{(A,B) \in \mathcal{M} \times \mathcal{M} : \|Ax - Bx\| \le \varepsilon \\ \text{for all } x \in B(\theta, N) \cap K\big\}, \tag{2.129}$$

where $N, \varepsilon > 0$.

Clearly, the space $\mathcal{M}$ with this uniformity is metrizable and complete. We equip the space $\mathcal{M}$ with the topology induced by this uniformity. This topology will be called the weak topology.

We also equip the space $\mathcal{M}$ with the uniformity determined by the base

$$\mathcal{E}_s(\varepsilon) = \big\{(A,B) \in \mathcal{M} \times \mathcal{M} : \|Ax - Bx\| \le \varepsilon \text{ for all } x \in K\big\}, \tag{2.130}$$

where $\varepsilon > 0$. Clearly, the space $\mathcal{M}$ with this uniformity is also metrizable and complete. The topology induced by this uniformity on $\mathcal{M}$ will be called the strong topology.

Denote by $\mathcal{M}_f$ the set of all $A \in \mathcal{M}$ which have approximate fixed points. In other words, the set $\mathcal{M}_f$ consists of all $A \in \mathcal{M}$ such that

$$\inf\big\{\|x - Ax\| : x \in K\big\} = 0. \tag{2.131}$$

It is clear that $\mathcal{M}_f$ is a closed subset of $\mathcal{M}$ with the strong topology.

Note that if the set K is bounded, then $\mathcal{M}_f$ consists of all those elements of $\mathcal{M}$ which have fixed points. Every self-mapping of K which is a strict contraction, that is, has a Lipschitz constant strictly less than one, clearly belongs to $\mathcal{M}_f$.

If K is bounded and convex and a continuous mapping $A : K \to R^n$ satisfies the Leray-Schauder condition with respect to $w \in \mathrm{int}(K)$, that is, $Ay - w \ne m(y - w)$ for all y on the boundary of K and $m > 1$, then it also belongs to $\mathcal{M}_f$. If such an A is a strict contraction, then this continues to be true even if K is neither bounded nor convex.

We endow the topological subspace $\mathcal{M}_f \subset M$ with both the relative weak and strong topologies.

The following two results were obtained in [156].

Theorem 2.13 *Let $\gamma \in (0,1)$. There exists an open (in the weak topology) and everywhere dense (in the strong topology) set $\mathcal{F}_\gamma \subset \mathcal{M}_f$ such that for each $A \in \mathcal{F}_\gamma$, there are $x_A \in \mathrm{int}(K)$, $r_A \in (0,1)$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that*

$$B(x_A, r_A) \subset K \quad \textit{and} \quad Ax_A = x_A,$$

and for each $C \in \mathcal{U}$, there is $x_C \in K$ such that $Cx_C = x_C$ and $\|x_C - x_A\| \le \gamma r_A$.

Theorem 2.14 *There exists a set $\mathcal{F} \subset \mathcal{M}_f$ which is a countable intersection of open* (*in the weak topology*) *and everywhere dense* (*in the strong topology*) *subsets of $\mathcal{M}_f$ such that for each $A \in \mathcal{F}$ and each $\gamma \in (0,1)$, there exist $x_A \in \mathrm{int}(K)$, $r_A \in (0,1)$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that*

$$B(x_A, r_A) \subset K \quad \text{and} \quad Ax_A = x_A,$$

and for each $C \in \mathcal{U}$ there is $x_C \in K$ such that $Cx_C = x_C$ and $\|x_C - x_A\| \le \gamma r_A$.

Example 2.15 Let $n = 1$, $K = \bigcup_{j=0}^{\infty}[2j, 2j+1]$, and define, for each integer $j \ge 1$ and each $x \in [2j, 2j+1]$, $Ax = x + 2^{-j}$. Clearly, $\inf\{|x - Ax| : x \in K\} = 0$ but A is fixed point free.

In order to prove Theorem 2.13 we need two auxiliary results.
Denote by $\mathcal{E}$ the set of all $A \in \mathcal{M}_f$ for which there exist

$$x_A \in \mathrm{int}(K) \quad \text{and} \quad r_A \in (0,1) \tag{2.132}$$

such that

$$B(x_A, r_A) \subset K \quad \text{and} \quad Ay = x_A \quad \text{for all } y \in B(x_A, r_A/4). \tag{2.133}$$

Lemma 2.16 *The set $\mathcal{E}$ is an everywhere dense subset of $\mathcal{M}_f$ with the strong topology.*

Proof Let $A \in \mathcal{M}_f$ and $\varepsilon > 0$. By the definition of $\mathcal{M}_f$ (see (2.131)), there exists $x_0 \in K$ such that

$$\|Ax_0 - x_0\| < \varepsilon/16. \tag{2.134}$$

Since K is the closure of $\mathrm{int}(K)$ and A is continuous, there is $x_1 \in \mathrm{int}(K)$ such that

$$\|x_1 - x_0\| < \varepsilon/16 \quad \text{and} \quad \|Ax_1 - Ax_0\| < \varepsilon/16. \tag{2.135}$$

Set

$$A_1 y = Ay - Ax_1 + x_1, \quad y \in K. \tag{2.136}$$

Clearly, $A_1 \in \mathcal{M}$. In view of (2.136),

$$A_1 x_1 = x_1. \tag{2.137}$$

By (2.136), (2.135) and (2.134), for each $y \in K$,

$$\begin{aligned} \|Ay - A_1 y\| &= \|Ax_1 - x_1\| \le \|Ax_1 - Ax_0\| + \|Ax_0 - x_0\| + \|x_0 - x_1\| \\ &< 3\varepsilon/16. \end{aligned} \tag{2.138}$$

Since A_1 has a fixed point (see (2.137)), it is clear that $A_1 \in \mathcal{M}_f$. Since A_1 is continuous and $x_1 \in \mathrm{int}(K)$, there exists $r_1 \in (0, 1)$ such that

$$B(x_1, r_1) \subset K \quad \text{and} \quad \|A_1 x - A_1 x_1\| \le \varepsilon/16 \quad \text{for all } x \in B(x_1, r_1). \tag{2.139}$$

Define

$$\begin{gathered} \psi(t) = 1, \quad t \in [0, r_1/2], \qquad \psi(t) = 0, \quad t \in [r_1, \infty), \\ \psi(t) = 2(r_1 - t)r_1^{-1}, \quad t \in (r_1/2, r_1), \end{gathered} \tag{2.140}$$

and

$$By = \psi\big(\|y - x_1\|\big)x_1 + \big(1 - \psi\big(\|y - x_1\|\big)\big)A_1 y, \quad y \in K. \tag{2.141}$$

Clearly, $B \in \mathcal{M}$. It follows from (2.141) and (2.140) that for each $y \in B(x_1, r_1/2)$,

$$By = x_1. \tag{2.142}$$

Therefore $B \in \mathcal{E}$. We will now show that

$$\|By - Ay\| \le \varepsilon \quad \text{for all } x \in K.$$

Indeed, let $y \in K$. There are two cases to be considered:

$$\|x_1 - y\| \le r_1; \tag{2.143}$$

$$\|x_1 - y\| > r_1. \tag{2.144}$$

If (2.144) holds, then (2.144), (2.141), (2.140) and (2.138) imply that

$$By = A_1 y \quad \text{and} \quad \|By - Ay\| = \|A_1 y - Ay\| < \varepsilon/4. \tag{2.145}$$

Let (2.143) hold. Then by (2.143), (2.141), (2.140), (2.137) and (2.139),

$$\|By - A_1 y\| = \big\|\psi\big(\|y - x_1\|\big)(x_1 - A_1 y)\big\| \le \|x_1 - A_1 y\| = \|A_1 x_1 - A_1 y\| < \varepsilon/16.$$

When combined with (2.138), this inequality implies that

$$\|By - Ay\| \le \|By - A_1 y\| + \|A_1 y - Ay\| \le \varepsilon/16 + 3\varepsilon/16 = \varepsilon/4.$$

Thus

$$\|By - Ay\| \le \varepsilon/4 \quad \text{for all } y \in K.$$

This completes the proof of Lemma 2.16. □

Lemma 2.17 *Let $A \in \mathcal{E}$, $x_A \in \mathrm{int}(K)$, $r_A \in (0, 1)$ satisfy (2.133) and let $\gamma \in (0, 1)$. Then there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that for each $B \in \mathcal{U}$, there is $x_B \in K$ such that $\|x_B - x_A\| \le \gamma r_A/4$ and $Bx_B = x_B$.*

Proof Set

$$\Delta = \gamma r_A/4 \tag{2.146}$$

and put

$$\mathcal{U} = \big\{B \in \mathcal{M}_f : \|Bz - Az\| \le \Delta \text{ for each } z \in B(x_A, r_A)\big\}. \tag{2.147}$$

Clearly, $\mathcal{U}$ is a neighborhood of A in $\mathcal{M}_f$ with the weak topology.

Let $B \in \mathcal{U}$. It follows from (2.147), (2.133) and (2.146) that for each $z \in B(x_A, \gamma r_A/4)$,

$$\|Bz - x_A\| \le \|Bz - Az\| + \|Az - x_A\| \le \Delta + \|Az - x_A\| = \Delta = \gamma r_A/4.$$

Thus

$$B\big(B(x_A, \gamma r_A/4)\big) \subset B(x_A, \gamma r_A/4).$$

Since the mapping B is continuous, there is $x_B \in B(x_A, \gamma r_A/4)$ such that

$$Bx_B = x_B.$$

Lemma 2.17 is proved. □

Proof of Theorem 2.13 Let $A \in \mathcal{E}$. There exist $x_A \in \mathrm{int}(K)$ and $r_A \in (0, 1)$ such that (2.133) holds. By Lemma 2.17, there exists an open neighborhood $\mathcal{U}(A)$ of A in $\mathcal{M}_f$ with the weak topology such that the following property holds:

(P1) For each $B \in \mathcal{U}(f)$, there is $x_B \in K$ such that

$$Bx_B = x_B \quad \text{and} \quad \|x_B - x_A\| \le \gamma r_A/8. \tag{2.148}$$

Set

$$\mathcal{F}_\gamma = \bigcup \big\{\mathcal{U}(A) : A \in \mathcal{E}\big\}. \tag{2.149}$$

By Lemma 2.16, $\mathcal{F}_\gamma$ is an open (in the weak topology) and everywhere dense (in the strong topology) subset of $\mathcal{M}_f$.

Let $B \in \mathcal{F}_\gamma$. By (2.149), there is $A \in \mathcal{E}$ such that

$$B \in \mathcal{U}(A). \tag{2.150}$$

By property (P1), for each $C \in \mathcal{U}(A)$, there is $x_C \in K$ such that

$$Cx_C = x_C \quad \text{and} \quad \|x_C - x_A\| \le \gamma r_A/8. \tag{2.151}$$

Clearly,

$$\|x_B - x_A\| \le \gamma r_A/8. \tag{2.152}$$

It follows from (2.152) and (2.135) that

$$B(x_B, r_A/2) \subset B(x_A, r_A) \subset K. \tag{2.153}$$

By (2.151) and (2.152), for each $C \in \mathcal{U}(A)$,

$$\|x_C - x_B\| \le \|x_C - x_A\| + \|x_A - x_B\| \le \gamma r_A/8 + \gamma r_A/8 = \gamma r_A/4.$$

This completes the proof of Theorem 2.13. □

Proof of Theorem 2.14 For each integer $n \ge 1$, let $\mathcal{F}_n$ be as guaranteed in Theorem 2.13 with $\gamma = (2n)^{-1}$. Set

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n. \tag{2.154}$$

Clearly, $\mathcal{F}$ is a countable intersection of open (in the weak topology), everywhere dense (in the strong topology) subsets of $\mathcal{M}_f$.

Let $A \in \mathcal{F}$ and $\gamma \in (0, 1)$. Choose a natural number n such that

$$n^{-1} < \gamma/8. \tag{2.155}$$

Since $A \in \mathcal{F}_n$ and the assertion of Theorem 2.13 holds with $\gamma = (2n)^{-1}$ and $\mathcal{F}_\gamma = \mathcal{F}_n$, there are $x_A \in \mathrm{int}(K)$, $r_A \in (0, 1)$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that $B(x_A, r_A) \subset K$, $Ax_A = x_A$, and for each $C \in \mathcal{U}$, there is $x_C \in K$ such that $Cx_C = x_C$ and

$$\|x_C - x_A\| \le r_A(2n)^{-1} < r_A\gamma.$$

Thus Theorem 2.14 is also proved. □

2.6 Approximate Fixed Points

Let (K, ρ) be a complete metric space such that

$$\sup\{\rho(x, y) : x, y \in K\} = \infty,$$

and let $(X, \|\cdot\|, \ge)$ be a Banach space ordered by a closed convex cone

$$X_+ = \{x \in X : x \ge 0\}.$$

We assume that $\|x\| \le \|y\|$ for each $x, y \in X_+$ which satisfy $x \le y$.

Denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to X_+$. We equip the set $\mathcal{A}$ with the uniformity determined by the following base:

$$E_s(\varepsilon) = \{(A, B) \in \mathcal{A} \times \mathcal{A} : \|Ax - Bx\| \le \varepsilon \text{ for all } x \in K\}, \tag{2.156}$$

where $\varepsilon > 0$ [80]. Clearly, the uniform space obtained in this way is metrizable and complete. The uniformity determined by (2.156) induces a topology on $\mathcal{A}$ which is called the strong topology.

Denote by $\mathcal{F}_0$ the set of all $A \in \mathcal{A}$ for which

$$\inf\{\|Ax\| : x \in K\} > 0.$$

Theorem 2.18 *The set $\mathcal{F}_0$ is an open everywhere dense subset of $\mathcal{A}$ with the strong topology.*

Proof Let $A \in \mathcal{F}_0$. There is $r > 0$ such that

$$\|Ax\| \geq r \quad \text{for all } x \in K. \tag{2.157}$$

Set

$$U = \{B \in \mathcal{A} : (B, A) \in E_s(r/4)\}. \tag{2.158}$$

Clearly, U is a neighborhood of A in $\mathcal{A}$ with the strong topology. Assume that $B \in U$. Then it follows from (2.157) and (2.158) that for each $x \in K$,

$$\begin{aligned}\|Bx\| &\geq \|Ax\| - \|Ax - Bx\| \\ &\geq r - \|Ax - Bx\| \geq r - r/4 = 3r/4.\end{aligned}$$

Thus $B \in \mathcal{F}_0$. This implies that $U \subset \mathcal{F}_0$. In other words, we have shown that $\mathcal{F}_0$ is an open subset of $\mathcal{F}_0$ with the strong topology.

Now we show that $\mathcal{F}_0$ is an everywhere dense subset of $\mathcal{A}$ with the strong topology. Let $A \in \mathcal{F}_0$ and $\varepsilon > 0$. Choose $u \in X$ such that

$$u \in X_+ \quad \text{and} \quad \|u\| = \varepsilon/2, \tag{2.159}$$

and set

$$Bx = Ax + u, \quad x \in K. \tag{2.160}$$

By (2.159) and (2.160), for each $x \in K$,

$$\|Bx\| = \|Ax + u\| \geq \|u\| = \varepsilon/2.$$

Thus $B \in \mathcal{F}_0$. In view of (2.160), (2.159) and (2.156), $(A, B) \in E_s(\varepsilon)$. Therefore $\mathcal{F}_0$ is an everywhere dense subset of $\mathcal{A}$ with the strong topology. Theorem 2.18 is proved. □

Now we equip the set $\mathcal{A}$ with a topology which will be called the weak topology. Fix $\theta \in K$. For each $\varepsilon, n > 0$, set

$$\begin{aligned}E_w(\varepsilon, n) = \{&(A, B) \in \mathcal{A} \times \mathcal{A} : \|Ax - Bx\| \leq \varepsilon \\ &\text{for each } x \in K \text{ satisfying } \rho(\theta, x) \leq n\}.\end{aligned} \tag{2.161}$$

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$E_w(\varepsilon, n), \quad \varepsilon, n > 0.$$

Clearly, the uniform space obtained in this way is metrizable and complete. The uniformity determined by (2.161) induces in the set $\mathcal{A}$ a topology which is called the weak topology.

Theorem 2.19 *There exists a set $\mathcal{F}_1 \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ with the weak topology such that for each $A \in \mathcal{F}_1$,*

$$\inf\{\|Ax\| : x \in K\} = 0. \tag{2.162}$$

Proof Denote by $\mathcal{E}$ the set of all $A \in \mathcal{A}$ for which there is $x \in K$ such that $Ax = 0$. First we show that $\mathcal{E}$ is an everywhere dense subset of $\mathcal{A}$ with the weak topology. Let $A \in \mathcal{A}$ and $\varepsilon, n > 0$. Choose $\bar{x} \in K$ such that

$$\rho(\theta, \bar{x}) \geq 4n + 4. \tag{2.163}$$

By Urysohn's theorem there is a continuous function $\phi : K \to [0, 1]$ such that

$$\phi(x) = 1 \quad \text{if } \rho(x, \bar{x}) \leq 1$$

and

$$\phi(x) = 0 \quad \text{if } \rho(x, \bar{x}) \geq 2. \tag{2.164}$$

Set

$$Bx = \big(1 - \phi(x)\big)Ax, \quad x \in K. \tag{2.165}$$

Clearly, $B \in \mathcal{A}$. In view of (2.164) and (2.165),

$$\phi(\bar{x}) = 1 \quad \text{and} \quad B\bar{x} = 0.$$

Thus $B \in \mathcal{E}$. Let $x \in K$ satisfy

$$\rho(x, \theta) \leq n. \tag{2.166}$$

It follows from (2.166) and (2.163) that

$$\begin{aligned} \rho(\bar{x}, x) &\geq \rho(\bar{x}, \theta) - \rho(\theta, x) \\ &\geq 4n + 4 - n = 3n + 4. \end{aligned}$$

When combined with (2.164) and (2.165), this implies that

$$\phi(x) = 0 \quad \text{and} \quad Bx = Ax.$$

Thus $Bx = Ax$ for each $x \in K$ satisfying (2.166). The definition of the base E_w (see (2.161)) implies that $(A, B) \in E_w(\varepsilon, n)$. In other words we have shown that $\mathcal{E}$ is an everywhere dense subset of $\mathcal{A}$ with the weak topology.

Let $A \in \mathcal{E}$ and let $n \geq 1$ be an integer. There is $x_A \in K$ such that

$$Ax_A = 0. \tag{2.167}$$

Since A is continuous, there is $r \in (0, 1)$ such that

$$\|Ax\| \le (4n)^{-1} \quad \text{for each } x \in K \text{ satisfying } \rho(x, x_A) \le r. \tag{2.168}$$

Choose an open neighborhood $\mathcal{U}(A, n)$ of A in $\mathcal{A}$ with the weak topology such that

$$\mathcal{U}(A, n) \subset \bigl\{B \in \mathcal{A} : (A, B) \in E_w\bigl((4n)^{-1}, n + 4 + \rho(\theta, x_A)\bigr)\bigr\}. \tag{2.169}$$

Let

$$B \in \mathcal{U}(A, n), \qquad x \in K, \qquad \rho(x, x_A) \le r. \tag{2.170}$$

By (2.170) and (2.168),

$$\|Ax\| \le (4n)^{-1}. \tag{2.171}$$

In view of (2.170) and since $r < 1$,

$$\begin{aligned} \rho(\theta, x) &\le \rho(\theta, x_A) + \rho(x_A, x) \\ &\le \rho(\theta, x_A) + r < \rho(\theta, x_A) + 1. \end{aligned}$$

Together with (2.169), (2.170) and (2.161), this inequality implies that

$$\|Ax - Bx\| \le (4n)^{-1}.$$

When combined with (2.171), this inequality implies that $\|Bx\| \le 1/n$. Thus we have shown that the following property holds:

(P0) For each $B \in \mathcal{U}(A, n)$, $\inf\{\|Bz\| : z \in K\} \le 1/n$.

Set

$$\mathcal{F}_1 = \bigcap_{n=1}^{\infty} \bigcup \bigl\{\mathcal{U}(A, n) : A \in \mathcal{E}\bigr\}. \tag{2.172}$$

Clearly, $\mathcal{F}_1$ is a countable intersection of open everywhere dense (in the weak topology) subsets of $\mathcal{A}$.

Let $B \in \mathcal{F}_1$ and $\varepsilon > 0$. Choose a natural number n such that

$$8/n < \varepsilon. \tag{2.173}$$

By (2.172), there is $A \in \mathcal{E}$ such that

$$B \in \mathcal{U}(A, n).$$

It follows from this inclusion, property (P0) and (2.173) that

$$\inf\bigl\{\|Bz\| : z \in K\bigr\} \le 1/n < \varepsilon.$$

Since ε is an arbitrary positive number, we conclude that

$$\inf\bigl\{\|Bz\| : z \in K\bigr\} = 0.$$

Theorem 2.19 is proved. □

Assume now that K is a subset of X and that

$$\rho(x, y) = \|x - y\|, \quad x, y \in K.$$

Denote by $\mathcal{B}$ the set of all continuous mappings $A : K \to X$ such that

$$Ax \geq x \quad \text{for all } x \in K.$$

For each $A \in \mathcal{B}$, denote by $J(A)$ the mapping defined by

$$J(A)x = Ax - x, \quad x \in K.$$

Clearly, $J(\mathcal{B}) = \mathcal{A}$, and if $A_1, A_2 \in \mathcal{B}$ are such that

$$J(A_1) = J(A_2),$$

then $A_1 = A_2$. We equip the set $\mathcal{B}$ with the uniformity determined by the following base:

$$\mathcal{E}_s(\varepsilon) = \big\{(A, B) \in \mathcal{B} \times \mathcal{B} : \|Ax - Bx\| \leq \varepsilon \text{ for all } x \in K\big\},$$

where $\varepsilon > 0$. It is not difficult to see that the space $\mathcal{B}$ with this uniformity is metrizable and complete. This uniformity induces in $\mathcal{B}$ a topology which is called the strong topology. It is easy to see that the mapping J is a homeomorphism of the spaces $\mathcal{B}$ and $\mathcal{A}$ with the strong topologies. Thus Theorem 2.18 implies the following result.

Corollary 2.20 *The set of all $A \in \mathcal{B}$ for which*

$$\inf\big\{\|Ax - x\| : x \in K\big\} > 0$$

is an open everywhere dense subset of $\mathcal{B}$ with the strong topology.

We also equip the set $\mathcal{B}$ with the uniformity determined by the following base:

$$\begin{aligned}\mathcal{E}_w(\varepsilon, n) = \big\{&(A, B) \in \mathcal{B} \times \mathcal{B} : \|Ax - Bx\| \leq \varepsilon \\ &\text{for each } x \in K \text{ satisfying } \|\theta - x\| \leq n\big\}\end{aligned}$$

where $n, \varepsilon > 0$. It is not difficult to see that the space $\mathcal{B}$ with this uniformity is metrizable and complete. This uniformity induces in $\mathcal{B}$ a topology which is called the weak topology. It is easy to see that the mapping J is a homeomorphism of the spaces $\mathcal{B}$ and $\mathcal{A}$ with the weak topologies.

Therefore Theorem 2.19 implies the following corollary.

Corollary 2.21 *There exists a set $\mathcal{F} \subset \mathcal{B}$ which is a countable intersection of open and everywhere dense subsets of $\mathcal{B}$ with the weak topology such that for each $A \in \mathcal{F}$,*

$$\inf\big\{\|Ax - x\| : x \in K\big\} = 0.$$

The results of this section were obtained in [157].

2.7 Generic Existence of Small Invariant Sets

In this section we consider generic properties of mappings with approximate fixed points. More precisely, let K be a closed and convex subset of a Banach space $(X, \|\cdot\|)$. We consider a complete metric space of all the continuous self-mappings of K with approximate fixed points. We show that a typical element of this space (in the sense of Baire's categories) has invariant balls of arbitrarily small radii. This result was obtained in [146].

Denote by $\mathcal{A}$ the set of all mappings $A : K \to K$ such that

$$\inf\big\{\|x - Ax\| : x \in K\big\} = 0. \tag{2.174}$$

We equip the set $\mathcal{A}$ with the uniformity determined by the following base:

$$E(\varepsilon) = \big\{(A, B) \in \mathcal{A} \times \mathcal{A} : \|Ax - Bx\| \le \varepsilon \text{ for all } x \in K\big\}, \tag{2.175}$$

where $\varepsilon > 0$. It is easy to see that the uniform space $\mathcal{A}$ is metrizable (by a metric d).

We first observe that $(\mathcal{A}, d)$ is a complete metric space.

Proposition 2.22 *The metric space* $(\mathcal{A}, d)$ *is complete.*

Proof Let $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ be a Cauchy sequence. Then for any $\varepsilon > 0$, there is a natural number i_ε such that

$$\|A_i x - A_j x\| \le \varepsilon \quad \text{for all integers } i, j \ge i_\varepsilon \text{ and all } x \in K. \tag{2.176}$$

This implies that for each $x \in K$, $\{A_i x\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists

$$Ax := \lim_{i\to\infty} A_i x. \tag{2.177}$$

Let $\varepsilon > 0$ and let a natural number i_ε satisfy (2.176). Relations (2.176) and (2.177) imply that for each integer $j \ge i_\varepsilon$ and each $x \in K$,

$$\|Ax - A_j x\| = \lim_{i\to\infty} \|A_i x - A_j x\| \le \varepsilon.$$

Thus

$$\|Ax - A_j x\| \le \varepsilon \quad \text{for each integer } j \ge i_\varepsilon \text{ and each } x \in K. \tag{2.178}$$

In order to complete the proof of Proposition 2.22, it is sufficient to show that the mapping A satisfies (2.174).

Let $\delta > 0$. Then in view of (2.178) there is a natural number i_0 such that

$$\|Ax - A_{i_0} x\| \le \delta/4 \quad \text{for all } x \in K. \tag{2.179}$$

Since $A_{i_0} \in \mathcal{A}$, there is $y \in K$ such that

$$\|A_{i_0} y - y\| \le \delta/4.$$

When combined with (2.179), this inequality implies that

$$\|Ay - y\| \le \|Ay - A_{i_0}y\| + \|A_{i_0}y - y\| \le \delta/4 + \delta/4 = \delta/2.$$

Since δ is any positive number, we conclude that $A \in \mathcal{A}$. This completes the proof of Proposition 2.22. □

Denote by $\mathcal{A}_c$ the set of all continuous $A \in \mathcal{A}$. Clearly, $\mathcal{A}_c$ is a closed subset of $(\mathcal{A}, d)$.

Theorem 2.23 *There exists a set $\mathcal{F} \subset \mathcal{A}_c$ which is a countable intersection of open and everywhere dense subsets of $\mathcal{A}_c$ such that each $A \in \mathcal{F}$ has the following property*:

For each $\gamma \in (0, 1)$, there are $x_\gamma \in K$, $r \in (0, 1]$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}_c$ such that for each $C \in \mathcal{U}$,

$$C\big(\{z \in K : \|z - x_\gamma\| \le r\}\big) \subset \{z \in K : \|z - x_\gamma\| \le \gamma r\}. \tag{2.180}$$

Corollary 2.24 *Assume that for each $x \in K$, the set $\{z \in K : \|z - x\| \le 1\}$ is compact. Let $\mathcal{F}$ be as guaranteed by Theorem 2.23, and let $A \in \mathcal{F}$, $\gamma \in (0, 1)$.*

Then there are $x_A \in K$ and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}_c$ such that for each $C \in \mathcal{U}$, there is a point $z \in K$ so that $\|z - x_A\| \le \gamma$ and $Cz = z$.

Corollary 2.25 *Assume that X is finite-dimensional. Then the assertion of Corollary 2.24 holds.*

Corollary 2.26 *Assume that the assumptions of Corollary 2.24 hold, and that $A \in \mathcal{F}$ and $\varepsilon > 0$. Then there are $\bar{x} \in K$ and $r \in (0, 1]$ such that*

$$A\bar{x} = \bar{x} \quad \text{and} \quad A\big(\{z \in K : \|z - \bar{x}\| \le r\}\big) \subset \{z \in K : \|z - \bar{x}\| \le \varepsilon r\}.$$

Proof Choose a positive number γ such that

$$\gamma < 1/2 \quad \text{and} \quad \gamma < \varepsilon/8. \tag{2.181}$$

By Theorem 2.23, there are $x_\gamma \in K$ and $r \in (0, 1]$ such that (2.180) holds with $C = A$. By Schauder's theorem, there is $\bar{x} \in K$ such that

$$\|\bar{x} - x_\gamma\| \le \gamma r \quad \text{and} \quad A\bar{x} = \bar{x}. \tag{2.182}$$

We have, by (2.182),

$$\{z \in K : \|z - x_\gamma\| \le \gamma r\} \subset \{z \in K : \|z - \bar{x}\| \le 2\gamma r\}.$$

When combined with (2.180) (with $C = A$), this inclusion implies that

$$A\big(\{z \in K : \|z - x_\gamma\| \le r\}\big) \subset \{z \in K : \|z - \bar{x}\| \le 2\gamma r\}. \tag{2.183}$$

On the other hand, by (2.181) and (2.182),

$$\{z \in K : \|z - \bar{x}\| \le r/2\} \subset \{z \in K : \|z - x_\gamma\| \le r\}. \tag{2.184}$$

It now follows from (2.184), (2.183) and (2.181) that

$$\begin{aligned} A(\{x \in K : \|z - \bar{x}\| \le r/2\}) &\subset A(\{x \in K : \|z - x_\gamma\| \le r\}) \\ &\subset \{z \in K : \|z - \bar{x}\| \le \varepsilon r/4\}. \end{aligned}$$

Corollary 2.26 is proved. □

Corollary 2.27 *Assume that X is finite-dimensional. Then the assertion of Corollary 2.26 holds.*

Corollary 2.28 *Let K be compact. Then $\mathcal{A}_c$ is the set of all continuous mappings $A : K \to K$ and the assertion of Corollary 2.26 holds.*

We begin the proof of Theorem 2.23 with the following lemma.

Lemma 2.29 *Let $A \in \mathcal{A}_c$ and $\varepsilon > 0$. Then there are $x_* \in K$, $r > 0$, and $B \in \mathcal{A}_c$ such that*

$$\|Ax - Bx\| \le \varepsilon \quad \text{for all } x \in K,$$

$$Bx = x_* \quad \text{for all } x \in K \text{ satisfying } \|x - x_*\| \le r.$$

Proof Since $A \in \mathcal{A}_c$ (see (2.174)), there is $x_* \in K$ such that

$$\|Ax_* - x_*\| \le \varepsilon/8. \tag{2.185}$$

There also is a number $r \in (0, 1)$ such that

$$\|Ax - Ax_*\| \le \varepsilon/8 \quad \text{for each } x \in K \text{ such that } \|x - x_*\| \le 2r. \tag{2.186}$$

By Urysohn's theorem, there exists a continuous function $\phi : K \to [0, 1]$ such that

$$\phi(x) = 1, \quad x \in \{z \in K : \|z - x_*\| \le r\} \tag{2.187}$$

and

$$\phi(x) = 0, \quad x \in K \text{ and } \|x - x_*\| \ge 2r.$$

Set

$$Bx = \phi(x)x_* + (1 - \phi(x))Ax, \quad x \in K. \tag{2.188}$$

Clearly, $B : K \to K$ is continuous, and

$$Bx = x_* \quad \text{for all } x \in K \text{ such that } \|x - x_*\| \le r. \tag{2.189}$$

Now we show that

$$\|Bx - Ax\| \le \varepsilon \quad \text{for all } x \in K.$$

Let $x \in K$. There are two cases: (1) $\|x - x_*\| \le 2r$; (2) $\|x - x_*\| > 2r$.

Consider the first case. Then (2.188), (2.185) and (2.186) imply that

$$\begin{aligned}\|Ax - Bx\| &= \big\|Ax - \phi(x)x_* - \big(1 - \phi(x)\big)Ax\big\| \\ &= \phi(x)\|x_* - Ax\| \le \|x_* - Ax\| \le \|x_* - Ax_*\| + \|Ax_* - Ax\| \\ &\le \varepsilon/8 + \varepsilon/8 = \varepsilon/4.\end{aligned}$$

Consider now the second case. Then by (2.188) and (2.187),

$$\|Ax - Bx\| = \|Ax - Ax\| = 0.$$

Thus $\|Ax - Bx\| \le \varepsilon$ for all $x \in K$. Lemma 2.29 is proved. □

Proof of Theorem 2.23 Denote by $\mathcal{E}$ the set of all $A \in \mathcal{A}_c$ with the following property:

There are $x_* \in K$ and $r > 0$ such that $Ax = x_*$ for all $x \in K$ satisfying $\|x - x_*\| \le r$.

By Lemma 2.29, $\mathcal{E}$ is an everywhere dense subset of $\mathcal{A}_c$.

Let $A \in \mathcal{E}$ and let n be a natural number. There are $x_A \in K$ and $r_A \in (0, 1)$ such that

$$Ax = x_A \quad \text{for all } x \in K \text{ satisfying } \|x - x_A\| \le r_A. \tag{2.190}$$

Denote by $\mathcal{U}(A, n)$ the open neighborhood of A in $\mathcal{A}_c$ such that

$$\mathcal{U}(A, n) \subset \big\{B \in \mathcal{A}_c : (A, B) \in E(r_A/n)\big\}. \tag{2.191}$$

Let $B \in \mathcal{U}(A, n)$. Clearly,

$$\|By - Ay\| \le r_A/n \le 1/n \quad \text{for all } y \in K. \tag{2.192}$$

By (2.190) and (2.192), for all $y \in K$ such that $\|y - x_A\| \le r_A$,

$$\|By - x_A\| \le \|By - Ay\| + \|Ay - x_A\| \le \|By - Ay\| \le r_A/n.$$

Thus

$$\|By - x_A\| \le r_A/n \quad \text{for all } y \in K \text{ such that } \|y - x_A\| \le r_A. \tag{2.193}$$

We have shown that the following property holds:

(P1) For each $B \in \mathcal{U}(A, n)$, (2.193) is true.

Define

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \bigcup \{\mathcal{U}(A, n) : A \in \mathcal{E}\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of $\mathcal{A}_c$.

Let $B \in \mathcal{F}$ and $\gamma \in (0, 1)$. Choose a natural number n such that $8/n < \gamma$. By the definition of $\mathcal{F}$, there are $A \in \mathcal{E}$ such that

$$B \in \mathcal{U}(A, n). \tag{2.194}$$

It follows from property (P1) and (2.193) that for each $C \in \mathcal{U}(A, n)$,

$$\begin{aligned} C\big(\{z \in K : \|z - x_A\| \le r_A\}\big) &\subset \{z \in K : \|z - x_A\| \le r_A/n\} \\ &\subset \{z \in K : \|z - x_A\| \le \gamma r_A\}. \end{aligned}$$

This completes the proof of Theorem 2.23. □

2.8 Many Nonexpansive Mappings Are Strict Contractions

Let K be a nonempty, bounded, closed and convex subset of a Banach space $(X, \|\cdot\|)$. In this section we consider the space of all nonexpansive self-mappings of K equipped with an appropriate complete metric d and prove that the complement of the subset of strict contractions is porous. This result was established in [150].

Set

$$\operatorname{rad}(K) = \sup\{\|x\| : x \in K\} \tag{2.195}$$

and

$$d(K) = \sup\{\|x - y\| : x, y \in K\}.$$

For each $A : K \to X$, let

$$\operatorname{Lip}(A) = \sup\{\|Ax - Ay\|/\|x - y\| : x, y \in K, x \ne y\} \tag{2.196}$$

be the Lipschitz constant of A. Denote by $\mathcal{A}$ the set of all nonexpansive mappings $A : K \to K$, that is, all self-mappings of K with $\operatorname{Lip}(A) \le 1$, or equivalently, all self-mappings of K which satisfy

$$\|Ax - Ay\| \le \|x - y\| \quad \text{for all } x, y \in K. \tag{2.197}$$

We say that a self-mapping $A : K \to K$ is a strict contraction if $\operatorname{Lip}(A) < 1$. Our new metric is defined by

$$d(A, B) = \sup\{\|Ax - Bx\| : x \in K\} + \operatorname{Lip}(A - B), \tag{2.198}$$

where $A, B \in \mathcal{A}$. It is not difficult to see that the metric space $(\mathcal{A}, d)$ is complete.

Theorem 2.30 *Denote by $\mathcal{F}$ the set of all strict contractions $A \in \mathcal{A}$. Then $\mathcal{A} \setminus \mathcal{F}$ is porous.*

Proof Fix a number $\alpha > 0$ such that

$$\alpha < \big(1 + 2\,\mathrm{rad}(K)\big)^{-1} 32^{-1} \tag{2.199}$$

and fix $\theta \in K$. Let $A \in \mathcal{A}$ and let $r \in (0, 1]$. Set

$$\gamma = \big(1 + 2\,\mathrm{rad}(K)\big)^{-1} r/8 \tag{2.200}$$

and put

$$A_\gamma x = (1 - \gamma)Ax + \gamma\theta, \quad x \in K. \tag{2.201}$$

Clearly, $A_\gamma \in \mathcal{A}$ and for each $x, y \in K$,

$$\|A_\gamma x - A_\gamma y\| = (1 - \gamma)\|Ax - Ay\| \le (1 - \gamma)\|x - y\|. \tag{2.202}$$

By (2.201), (2.195), (2.196) and (2.198), for each $x \in K$,

$$\begin{aligned}
\|A_\gamma x - Ax\| &= \big\|(1 - \gamma)Ax + \gamma\theta - Ax\big\| = \gamma\|\theta - Ax\| \\
&\le 2\gamma\,\mathrm{rad}(K), \\
\mathrm{Lip}(A_\gamma - A) &= \sup\big\{\big\|(A_\gamma - A)x - (A_\gamma - A)y\big\|/\|x - y\| : x, y \in K, x \ne y\big\} \\
&= \sup\big\{\big\|(\gamma\theta - \gamma Ax) - (\gamma\theta - \gamma Ay)\big\|/\|x - y\| : x, y \in K, x \ne y\big\} \\
&= \gamma \sup\big\{\|Ax - Ay\|/\|x - y\| : x, y \in K, x \ne y\big\} \le \gamma,
\end{aligned}$$

and

$$d(A, A_\gamma) \le 2\gamma\,\mathrm{rad}(K) + \gamma = \gamma\big(1 + 2\,\mathrm{rad}(K)\big). \tag{2.203}$$

Relations (2.200) and (2.203) imply that

$$d(A, A_\gamma) \le r/8. \tag{2.204}$$

Assume that $B \in \mathcal{A}$,

$$d(B, A_\gamma) \le \alpha r. \tag{2.205}$$

In view of (2.205), (2.198), (2.202) and (2.200), we see that

$$\begin{aligned}
\mathrm{Lip}(B) &\le \mathrm{Lip}(A_\gamma) + \mathrm{Lip}(B - A_\gamma) \le \mathrm{Lip}(A_\gamma) + d(B, A_\gamma) \\
&\le \mathrm{Lip}(A_\gamma) + \alpha r \le (1 - \gamma) + \alpha r \\
&= 1 - (r/8)\big(1 + 2\,\mathrm{rad}(K)\big)^{-1} + r(32\big(1 + 2\,\mathrm{rad}(K)\big)^{-1} \\
&\le 1 - (r/16)\big(1 + 2\,\mathrm{rad}(K)\big)^{-1} < 1
\end{aligned}$$

and so $B \in \mathcal{F}$. Clearly, by (2.205), (2.204) and (2.199),

$$d(B, A) \le d(B, A_\gamma) + d(A_\gamma, A) \le \alpha r + r/8 \le r.$$

Thus for each $B \in \mathcal{A}$ satisfying (2.205), $B \in \mathcal{F}$ and $d(B, A) \le r$. This completes the proof of Theorem 2.30. □

Now let F be a nonempty closed convex subset of K. For each $x \in K$, set

$$\rho(x, F) = \inf\big\{\|x - y\| : y \in F\big\}. \tag{2.206}$$

Assume that there exists $P \in \mathcal{A}$ such that

$$P(K) = F, \qquad Px = x, \quad x \in F. \tag{2.207}$$

Denote by $\mathcal{A}^{(F)}$ the set of all $A \in \mathcal{A}$ such that

$$Ax = x, \quad x \in F. \tag{2.208}$$

Clearly, $\mathcal{A}^{(F)}$ is a closed subset of $(\mathcal{A}, d)$.

Theorem 2.31 *Denote by $\mathcal{F}$ the set of all $A \in \mathcal{A}^{(F)}$ which have the following property*:

There is a number $q \in (0, 1)$ such that

$$\rho(Ax, F) \le q\rho(x, F) \quad \textit{for all } x \in K.$$

Then $\mathcal{A}^{(F)} \setminus \mathcal{F}$ is a porous subset of $(\mathcal{A}^{(F)}, d)$.

Proof Fix a number $\alpha > 0$ such that

$$\alpha < \big(1 + 2\,\mathrm{rad}(K)\big)^{-1} 32^{-1}. \tag{2.209}$$

Let $A \in \mathcal{A}^{(F)}$ and $r \in (0, 1]$. Set

$$\gamma = \big(1 + 2\,\mathrm{rad}(K)\big)^{-1} r/8 \tag{2.210}$$

and put

$$A_\gamma x = (1 - \gamma)Ax + \gamma Px, \quad x \in K. \tag{2.211}$$

Clearly, $A_\gamma \in \mathcal{A}$,

$$A_\gamma x = x, \quad x \in F, \quad \text{and} \quad A_\gamma \in \mathcal{A}^{(F)}. \tag{2.212}$$

For each $x \in K$ and $y \in F$, we have by (2.211),

$$\begin{aligned}\rho(A_\gamma x, F) &= \rho\big((1-\gamma)Ax + \gamma Px, F\big)\\ &\le \big\|(1-\gamma)Ax + \gamma Px - \big((1-\gamma)y + \gamma Px\big)\big\|\\ &= (1-\gamma)\|Ax - y\| \le (1-\gamma)\|x - y\|.\end{aligned}$$

Hence

$$\rho(A_\gamma x, F) \le (1-\gamma)\inf\big\{\|x - y\| : y \in F\big\} = (1-\gamma)\rho(x, F).$$

Thus

$$\rho(A_\gamma x, F) \le (1-\gamma)\rho(x, F), \quad x \in K. \tag{2.213}$$

By (2.211), (2.195), and (2.199), we have for $x \in K$,

$$\begin{aligned}\|A_\gamma x - Ax\| = \big\|(1-\gamma)Ax + \gamma Px - Ax\big\| = \gamma\|Px - Ax\| \le 2\gamma\,\mathrm{rad}(K),\\ \mathrm{Lip}(A_\gamma - A) = \mathrm{Lip}\big((1-\gamma)A + \gamma P - A\big)\\ = \mathrm{Lip}(\gamma P - \gamma A) \le 2\gamma\end{aligned}$$

and

$$d(A, A_\gamma) \le 2\gamma\,\mathrm{rad}(K) + 2\gamma = 2\gamma\big(\mathrm{rad}(K) + 1\big). \tag{2.214}$$

It follows from (2.214) and (2.210) that

$$d(A, A_\gamma) \le r/4. \tag{2.215}$$

Assume now that

$$B \in \mathcal{A}^{(F)}$$

and

$$d(B, A_\gamma) \le \alpha r. \tag{2.216}$$

Then by (2.216), (2.215) and (2.209),

$$d(B, A) \le d(B, A_\gamma) + d(A_\gamma, A) \le \alpha r + r/4 \le r. \tag{2.217}$$

Let $x \in K$ and $y \in F$. It follows from (2.208), (2.212), (2.211), (2.196), (2.198), (2.216), (2.209) and (2.210) that

$$\begin{aligned}\rho(Bx, F) &\le \big\|Bx - \big((1-\gamma)y + \gamma Px\big)\big\|\\ &\le \|Bx - A_\gamma x\| + \big\|A_\gamma x - \big[(1-\gamma)y + \gamma Px\big]\big\|\\ &\le \big\|(Bx - By) - (A_\gamma x - A_\gamma y)\big\| + (1-\gamma)\|Ax - y\|\\ &\le \big\|(B - A_\gamma)x - (B - A_\gamma)y\big\| + (1-\gamma)\|x - y\|\\ &\le \mathrm{Lip}(B - A_\gamma)\|x - y\| + (1-\gamma)\|x - y\|\end{aligned}$$

$$\leq \alpha r\|x-y\| + (1-\gamma)\|x-y\| = \|x-y\|(\alpha r + 1 - \gamma)$$
$$\leq \|x-y\|\big(1 - \big(1 + 2\,\mathrm{rad}(K)\big)^{-1}/16\big).$$

Therefore

$$\rho(Bx, F) \leq \big(1 - \big(1 + 2\,\mathrm{rad}(K)\big)^{-1}/16\big) \inf\big\{\|x-y\| : y \in F\big\}$$
$$= \big(1 - \big(1 + 2\,\mathrm{rad}(K)\big)^{-1}/16\big)\rho(x, F).$$

Thus $B \in \mathcal{F}$. This completes the proof of Theorem 2.31. □

2.9 Krasnosel'skii-Mann Iterations of Nonexpansive Operators

In this section we study the convergence of Krasnosel'skii-Mann iterations of nonexpansive operators on a closed and convex, but not necessarily bounded, subset of a hyperbolic space. More precisely, we show that in an appropriate complete metric space of nonexpansive operators, there exists a subset which is a countable intersection of open and everywhere dense sets such that each operator belonging to this subset has a (necessarily) unique fixed point and the Krasnosel'skii-Mann iterations of the operator converge to it.

Let (X, ρ, M) be a complete hyperbolic space and let K be a closed and ρ-convex subset of X. Denote by $\mathcal{A}$ the set of all operators $A : K \to K$ such that

$$\rho(Ax, Ay) \leq \rho(x, y) \quad \text{for all } x, y \in K. \tag{2.218}$$

Fix some $\theta \in K$ and for each $s > 0$, set

$$B(s) = \big\{x \in K : \rho(x, \theta) \leq s\big\}. \tag{2.219}$$

For the set $\mathcal{A}$ we consider the uniformity determined by the following base:

$$E(n) = \big\{(A, B) \in \mathcal{A} \times \mathcal{A} : \rho(Ax, Bx) \leq n^{-1} \text{ for all } x, y \in B(n)\big\}, \tag{2.220}$$

where n is a natural number. Clearly the uniform space $\mathcal{A}$ is metrizable and complete.

A mapping $A : K \to K$ is called regular if there exists a necessarily unique $x_A \in K$ such that

$$\lim_{n\to\infty} A^n x = x_A \quad \text{for all } x \in K.$$

A mapping $A : K \to K$ is called super-regular if there exists a necessarily unique $x_A \in K$ such that for each $s > 0$,

$$A^n x \to x_A \quad \text{as } n \to \infty \text{ uniformly on } B(s).$$

Denote by I the identity operator. For each pair of operators $A, B : K \to K$ and each $r \in [0, 1]$, define an operator $rA \oplus (1-r)B$ by

$$\big(rA \oplus (1-r)B\big)(x) = rAx \oplus (1-r)Bx, \quad x \in K.$$

In this section we prove the following three results [132].

Theorem 2.32 *Let $A : K \to K$ be super-regular and let ε, s be positive numbers. Then there exist a neighborhood U of A in $\mathcal{A}$ and an integer $n_0 \geq 2$ such that for each $B \in U$, each $x \in B(s)$ and each integer $n \geq n_0$, the following inequality holds: $\rho(x_A, B^n x) \leq \varepsilon$.*

Theorem 2.33 *There exists a set $\mathcal{F}_0 \subset \mathcal{A}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}$ such that each $A \in \mathcal{F}_0$ is super-regular.*

Let $\{\bar{r}_n\}_{n=1}^{\infty}$ be a sequence of positive numbers from the interval $(0, 1)$ such that

$$\lim_{n\to\infty} \bar{r}_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \bar{r}_n = \infty.$$

Theorem 2.34 *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}$ such that each $A \in \mathcal{F}$ is super-regular and the following assertion holds:*

Let $x_A \in K$ be the unique fixed point of $A \in \mathcal{F}$ and let $\delta, s > 0$. Then there exist a neighborhood U of A in $\mathcal{A}$ and an integer $n_0 \geq 1$ such that for each sequence of positive numbers $\{r_n\}_{n=1}^{\infty}$ satisfying $r_n \in [\bar{r}_n, 1]$, $n = 1, 2, \ldots$, and each $B \in U$ the following relations hold:

(i)

$$\rho\Big(\big(r_n B \oplus (1-r_n)I\big)\cdots\big(r_1 B \oplus (1-r_1)I\big)x,$$
$$\big(r_n B \oplus (1-r_n)I\big)\cdots\big(r_1 B \oplus (1-r_1)I\big)y\Big) \leq \delta$$

for each integer $n \geq n_0$ and each $x, y \in B(s)$;

(ii) *if $B \in U$ is regular, then*

$$\rho\Big(\big(r_n B \oplus (1-r_n)I\big)\cdots\big(r_1 B \oplus (1-r_1)I\big)x, x_A\Big) \leq \delta$$

for each integer $n \geq n_0$ and each $x \in B(s)$.

Proof of Theorem 2.32 We may assume that $\varepsilon \in (0, 1)$. Recall that x_A is the unique fixed point of A. There exists an integer $n_0 \geq 4$ such that for each $x \in B(2s + 2 + 2\rho(x_A, \theta))$ and each integer $n \geq n_0$,

$$\rho\big(x_A, A^n x\big) \leq 8^{-1}\varepsilon. \tag{2.221}$$

Set

$$U = \{B \in \mathcal{A} : \rho(Ax, Bx) \le (8n_0)^{-1}\varepsilon, x \in B(8s + 8 + 8\rho(x_A, \theta))\}. \tag{2.222}$$

Let $B \in U$. It is easy to see that for each $x \in K$ and all integers $n \ge 1$,

$$\begin{aligned} \rho(A^n x, B^n x) &\le \rho(A^n x, AB^{n-1}x) + \rho(AB^{n-1}x, B^n x) \\ &\le \rho(A^{n-1}x, B^{n-1}x) + \rho(AB^{n-1}x, B^n x) \end{aligned} \tag{2.223}$$

and

$$\begin{aligned} \rho(B^n x, x_A) &\le \rho(B^n x, A^n x) + \rho(A^n x, x_A) \le \rho(B^n x, A^n x) + \rho(x, x_A) \\ &\le \rho(B^n x, A^n x) + \rho(x, \theta) + \rho(\theta, x_A). \end{aligned} \tag{2.224}$$

Using (2.222), (2.223) and (2.224) we can show by induction that for all $x \in B(4s + 4 + 4\rho(x_A, \theta))$, and for all $n = 1, 2, \dots, n_0$,

$$\rho(A^n x, B^n x) \le (8n_0)^{-1}\varepsilon n \tag{2.225}$$

and

$$\rho(B^n x, \theta) \le 2\rho(x_A, \theta) + \rho(x, \theta) + \frac{1}{2}.$$

Let $y \in B(s)$. We intend to show that $\rho(x_A, B^n y) \le \varepsilon$ for all integers $n \ge n_0$. Indeed, by (2.225),

$$\rho(\theta, B^m y) \le \frac{1}{2} + 2\rho(x_A, \theta) + s, \quad m = 1, \dots, n_0. \tag{2.226}$$

By (2.225) and (2.221),

$$\rho(x_A, B^{n_0} y) \le \varepsilon/2. \tag{2.227}$$

Now we are ready to show by induction that for all integers $m \ge n_0$,

$$\rho(x_A, B^m y) \le \varepsilon. \tag{2.228}$$

By (2.227), inequality (2.228) is valid for $m = n_0$.

Assume that an integer $k \ge n_0$ and that (2.228) is valid for all integers $m \in [n_0, k]$. Together with (2.226) this implies that

$$\rho(\theta, B^i y) \le \frac{1}{2} + 2\rho(x_A, \theta) + s, \quad i = 1, \dots, k. \tag{2.229}$$

Set

$$j = 1 + k - n_0 \quad \text{and} \quad x = B^j y. \tag{2.230}$$

By (2.229), (2.230), (2.221) and (2.225),

$$\rho(A^{n_0}x, B^{n_0}x) \le \varepsilon/8, \qquad \rho(x_A, A^{n_0}x) \le \varepsilon/8 \quad \text{and} \quad \rho(x_A, B^{k+1}y) \le \varepsilon/4.$$

This completes the proof of Theorem 2.32. □

Proof of Theorem 2.33 For each $A \in \mathcal{A}$ and $\gamma \in (0,1)$, define $A_\gamma : K \to K$ by

$$A_\gamma x = (1-\gamma)Ax \oplus \gamma\theta, \quad x \in K.$$

Let $A \in \mathcal{A}$ and $\gamma \in (0,1)$. Clearly,

$$\rho(A_\gamma x, A_\gamma y) \le (1-\gamma)\rho(Ax, Ay) \le (1-\gamma)\rho(x,y), \quad x, y \in K.$$

Therefore there exists $x(A,\gamma) \in K$ such that

$$A_\gamma\big(x(A,\gamma)\big) = x(A,\gamma).$$

Evidently, A_γ is super-regular and the set $\{A_\gamma : A \in \mathcal{A}, \gamma \in (0,1)\}$ is everywhere dense in $\mathcal{A}$. By Theorem 2.32, for each $A \in \mathcal{A}$, each $\gamma \in (0,1)$ and each integer $i \ge 1$, there exist an open neighborhood $U(A,\gamma,i)$ of A_γ in $\mathcal{A}$ and an integer $n(A,\gamma,i) \ge 2$ such that the following property holds:

(i) for each $B \in U(A,\gamma,i)$, each $x \in B(4^{i+1})$ and each $n \ge n(A,\gamma,i)$,

$$\rho\big(x(A,\gamma), B^n x\big) \le 4^{-i-1}.$$

Define

$$\mathcal{F}_0 = \bigcap_{q=1}^{\infty} \bigcup \big\{U(A,\gamma,i) : A \in U, \gamma \in (0,1), i = q, q+1, \ldots\big\}.$$

Clearly, $\mathcal{F}_0$ is a countable intersection of open and everywhere dense sets in $\mathcal{A}$.

Let $A \in \mathcal{F}_0$. There exist sequences $\{A_q\}_{q=1}^\infty \subset \mathcal{A}$, $\{\gamma_q\}_{q=1}^\infty \subset (0,1)$ and a strictly increasing sequence of natural numbers $\{i_q\}_{q=1}^\infty$ such that

$$A \in U(A_q, \gamma_q, i_q), \quad q = 1, 2, \ldots. \tag{2.231}$$

By property (i) and (2.231), for each $x \in B(4^{i_q+1})$ and each integer $n \ge n(A_q, \gamma_q, i_q)$,

$$\rho\big(x(A_q,\gamma_q), A^n x\big) \le 4^{-i_q-1}.$$

This implies that A is super-regular. Theorem 2.33 is proved. □

In order to prove Theorem 2.34 we need the following auxiliary results.

Let

$$\bar{r}_n \in (0,1), \quad n = 1, 2, \ldots, \qquad \lim_{n\to\infty} \bar{r}_n = 0, \qquad \sum_{n=1}^{\infty} \bar{r}_n = 1. \tag{2.232}$$

Lemma 2.35 *Let $A \in \mathcal{A}$, $S_1 > 0$ and let $n_0 \ge 2$ be an integer. Then there exist a neighborhood U of A in $\mathcal{A}$ and a number $S_* > S_1$ such that for each $B \in U$, each sequence $\{r_i\}_{i=1}^{n_0-1} \subset (0,1]$ and each sequence $\{x_i\}_{i=1}^{n_0} \subset K$ satisfying*

$$x_1 \in B(S_1), \qquad x_{i+1} = r_i B x_i \oplus (1-r_i)x_i, \quad i = 1, \ldots, n_0 - 1, \tag{2.233}$$

the following relations hold:

$$x_i \in B(S_*), \quad i = 1, \ldots, n_0.$$

Proof Set

$$S_{i+1} = 2S_i + 2 + 2\rho(\theta, A\theta), \quad i = 1, \ldots, n_0 - 1, \quad \text{and} \quad S_* = S_{n_0}. \tag{2.234}$$

Set

$$U = \{B \in \mathcal{A} : \rho(Ax, Bx) \leq 1, x \in B(S_*)\}. \tag{2.235}$$

Assume that $B \in U$, $\{r_i\}_{i=1}^{n_0-1} \subset (0, 1]$, $\{x_i\}_{i=1}^{n_0} \subset K$ and that (2.233) holds. We will show that

$$\rho(\theta, x_i) \leq S_i, \quad i = 1, \ldots, n_0. \tag{2.236}$$

Clearly, (2.236) is valid for $i = 1$. Assume that the integer $m \in [1, n_0 - 1]$ and that (2.236) holds for all integers $i = 1, \ldots, m$. Then by (2.236) with $i = m$, (2.233), (2.235) and (2.234),

$$\begin{aligned}
\rho(\theta, x_{m+1}) &= \rho\big(\theta, r_m B(x_m) \oplus (1 - r_m)x_m\big) \\
&\leq \rho\big(r_m B(\theta) \oplus (1 - r_m)x_m, r_m B(x_m) \oplus (1 - r_m)x_m\big) \\
&\quad + \rho\big(\theta, r_m B(\theta) \oplus (1 - r_m)x_m\big) \\
&\leq r_m \rho(\theta, x_m) + \rho\big(\theta, B(\theta)\big) + \rho\big(B(\theta), r_m B(\theta) \oplus (1 - r_m)x_m\big) \\
&\leq S_m + \rho\big(\theta, A(\theta)\big) + \rho\big(A(\theta), B(\theta)\big) + \rho\big(B(\theta), x_m\big) \\
&\leq S_m + \rho\big(\theta, A(\theta)\big) + 1 + \rho(x_m, \theta) + \rho(\theta, A\theta) + \rho\big(A(\theta), B(\theta)\big) \\
&\leq 2S_m + 2\rho\big(\theta, A(\theta)\big) + 2 = S_{m+1}.
\end{aligned}$$

Lemma 2.35 is proved. □

For each $A \in \mathcal{A}$ and each $\gamma \in (0, 1)$, define $A_\gamma : K \to K$ by

$$A_\gamma x = (1 - \gamma)Ax \oplus \gamma\theta, \quad x \in K. \tag{2.237}$$

Let $A \in \mathcal{A}$ and $\gamma \in (0, 1)$. Clearly,

$$\rho(A_\gamma x, A_\gamma y) \leq (1 - \gamma)\rho(x, y), \quad x, y \in K. \tag{2.238}$$

There exists $x(A, \gamma) \in K$ such that

$$A_\gamma\big(x(A, \gamma)\big) = x(A, \gamma).$$

Clearly, $\mathcal{A}_\gamma$ is super-regular and the set $\{A_\gamma : A \in \mathcal{A}, \gamma \in (0, 1)\}$ is everywhere dense in $\mathcal{A}$.

Lemma 2.36 *Let $A \in \mathcal{A}$, $\gamma \in (0, 1)$, $r \in (0, 1]$ and $x, y \in X$. Then*

$$\rho\big(rA_\gamma x \oplus (1-r)x, rA_\gamma y \oplus (1-r)y\big) \le (1-\gamma r)\rho(x, y).$$

Proof By (2.238),

$$\begin{aligned}\rho\big(rA_\gamma x \oplus (1-r)x, rA_\gamma y \oplus (1-r)y\big) &\le r\rho(A_\gamma x, A_\gamma y) + (1-r)\rho(x, y)\\ &\le (1-r)\rho(x, y) + r(1-\gamma)\rho(x, y)\\ &= \rho(x, y)(1-\gamma r).\end{aligned}$$

Lemma 2.36 is proved. □

Lemma 2.37 *Let $A \in \mathcal{A}$, $\gamma \in (0, 1)$ and $\delta, S > 0$. Then there exist a neighborhood U of A_γ in $\mathcal{A}$ and an integer $n_0 \ge 4$ such that for each $B \in U$, each sequence of numbers $r_i \in [\bar{r}_i, 1]$, $i = 1, \dots, n_0 - 1$, and each $x, y \in B(S)$, the following inequality holds*:

$$\begin{aligned}&\rho\big(\big(r_{n_0-1}B \oplus (1-r_{n_0-1})I\big)\cdots\big(r_1 B \oplus (1-r_1)I\big)x,\\ &\quad \big(r_{n_0-1}B \oplus (1-r_{n_0-1})I\big)\cdots\big(r_1 B \oplus (1-r_1)I\big)y\big) \le \delta.\end{aligned}$$

Proof Choose a number

$$\gamma_0 \in (0, \gamma). \tag{2.239}$$

Clearly, $\prod_{i=1}^{\infty}(1-\gamma_0\bar{r}_i) \to 0$ as $n \to \infty$. Therefore there exists an integer $n_0 \ge 4$ such that

$$(2S+2)\prod_{i=1}^{n_0-1}(1-\gamma_0\bar{r}_i) < \delta/2. \tag{2.240}$$

By Lemma 2.35, there exist a neighborhood U_1 of A_γ in $\mathcal{A}$ and a number $S_* > 0$ such that for each $B \in U_1$, each sequence $\{r_i\}_{i=1}^{n_0-1} \subset (0, 1]$, and each sequence $\{x_i\}_{i=1}^{n_0} \subset X$ satisfying

$$x_1 \in B(S), \qquad x_{i+1} = r_i B x_i \oplus (1-r_i)x_i, \quad i = 1, \dots, n_0 - 1, \tag{2.241}$$

the following relations hold:

$$x_i \in B(S_*), \quad i = 1, \dots, n_0. \tag{2.242}$$

Choose a natural number m_1 such that

$$m_1 > 2S_* + 2 \quad \text{and} \quad 8m_1^{-1} < \delta(\gamma - \gamma_0)r_i, \quad i = 1, \dots, n_0 - 1, \tag{2.243}$$

and define

$$U = \big\{B \in U_1 : \rho(A_\gamma x, Bx) < m_1^{-1}, x \in B(m_1)\big\}. \tag{2.244}$$

Assume that $B \in U$, $r_i \in [\bar{r}_i, 1]$, $i = 1, \dots, n_0 - 1$, and

$$x, y \in B(S). \tag{2.245}$$

Set

$$\begin{aligned} x_1 = x, \qquad y_1 = y, \qquad x_{i+1} = r_i B x_i \oplus (1 - r_i) x_i, \\ y_{i+1} = r_i B y_i \oplus (1 - r_i) y_i, \quad i = 1, \dots, n_0 - 1. \end{aligned} \tag{2.246}$$

It follows from the definition of U_1 (see (2.241) and (2.242)) that

$$y_i, x_i \in B(S_*), \quad i = 1, \dots, n_0. \tag{2.247}$$

To prove the lemma it is sufficient to show that

$$\rho(x_{n_0}, y_{n_0}) \le \delta. \tag{2.248}$$

Assume the contrary. Then

$$\rho(x_i, y_i) > \delta, \quad i = 1, \dots, n_0. \tag{2.249}$$

Fix $i \in \{1, \dots, n_0 - 1\}$. It follows from (2.246), (2.247), (2.243), (2.244) and (2.237) that

$$\begin{aligned} \rho(x_{i+1}, y_{i+1}) &= \rho\big(r_i B x_i \oplus (1 - r_i) x_i, r_i B y_i \oplus (1 - r_i) y_i\big) \\ &\le \rho\big(r_i A_\gamma x_i \oplus (1 - r_i) x_i, r_i A_\gamma y_i \oplus (1 - r_i) y_i\big) \\ &\quad + \rho(A_\gamma x_i, B x_i) + \rho(A_\gamma y_i, B y_i) \\ &\le r_i \rho(A_\gamma x_i, A_\gamma y_i) + (1 - r_i) \rho(x_i, y_i) + 2m_1^{-1} \\ &\le 2m_1^{-1} + (1 - r_i) \rho(x_i, y_i) + r_i \rho(x_i, y_i)(1 - \gamma) \\ &\le 2m_1^{-1} + \rho(x_i, y_i)\big(1 - r_i + r_i(1 - \gamma)\big) \\ &= 2m_1^{-1} + \rho(x_i, y_i)(1 - \gamma r_i). \end{aligned} \tag{2.250}$$

By (2.250), (2.243) and (2.249),

$$\rho(x_{i+1}, y_{i+1}) \le \rho(x_i, y_i)(1 - \gamma_0 r_i),$$

and since this inequality holds for all $i \in \{1, \dots, n_0 - 1\}$, it follows from (2.245) and (2.240) that

$$\rho(x_{n_0}, y_{n_0}) \le 2S \prod_{i=1}^{n_0 - 1} (1 - \gamma_0 r_i) < \delta/2.$$

This contradicts (2.249) and proves Lemma 2.37. □

Proof of Theorem 2.34 Let

$$\{\bar{r}_n\}_{n=1}^{\infty} \subset (0,1), \qquad \lim_{n\to\infty} \bar{r}_n = 0, \qquad \sum_{n=1}^{\infty} \bar{r}_n = \infty. \tag{2.251}$$

By Theorem 2.33, there exists a set $\mathcal{F}_0 \subset \mathcal{A}$ which is a countable intersection of open and everywhere dense sets such that each $A \in \mathcal{F}_0$ is super-regular.

For each $A \in \mathcal{A}$ and each $\gamma > 0$, define $A_\gamma \in \mathcal{A}$ by

$$A_\gamma x = (1-\gamma)Ax \oplus \gamma\theta, \quad x \in K.$$

Clearly, A_γ is super-regular, and for each $A \in \mathcal{A}$ and $\gamma \in (0,1)$, there exists $x(A,\gamma) \in K$ for which

$$A_\gamma\big(x(A,\gamma)\big) = x(A,\gamma). \tag{2.252}$$

Let $A \in \mathcal{A}$, $\gamma \in (0,1)$ and let $i \geq 1$ be an integer. By Lemma 2.37, there exist an open neighborhood $U_1(A,\gamma,i)$ of A_γ in $\mathcal{A}$ and an integer $n_0(A,\gamma,i) \geq 4$ such that the following property holds:

(a) for each $B \in U_1(A,\gamma,i)$, each sequence of numbers

$$r_j \in [\bar{r}_j, 1], \quad j = 1,\dots,n_0(A,\gamma,i)-1,$$

and each pair of sequences $\{x_i\}_{i=1}^{n_0(A,\gamma,i)}, \{y_i\}_{i=1}^{n_0(A,\gamma,i)} \subset X$ satisfying

$$x_1, y_1 \in B\Big(8^{i+1}\big(4 + 4\rho\big(x(A,\gamma),\theta\big)\big)\Big), \tag{2.253}$$

$$\begin{aligned} &x_{i+1} = r_i B x_i \oplus (1-r_i)x_i, \qquad y_{i+1} = r_i B y_i \oplus (1-r_i)y_i, \\ &\quad i = 1,\dots,n_0(A,\gamma,i)-1, \end{aligned} \tag{2.254}$$

the following inequality holds:

$$\rho(x_{n_0(A,\gamma,i)}, y_{n_0(A,\gamma,i)}) \leq 8^{-i-1}. \tag{2.255}$$

Since A_γ is super-regular, by Theorem 2.32 there is an open neighborhood $U(A,\gamma,i)$ of A_γ in $\mathcal{A}$ and an integer $n(A,\gamma,i)$, such that

$$U(A,\gamma,i) \subset U_1(A,\gamma,i), \qquad n(A,\gamma,i) \geq n_0(A,\gamma,i), \tag{2.256}$$

and the following property holds:

(b) for each $B \in U(A,\gamma,i)$, each $x \in B(8^{i+1}(2 + 2\rho(x(A,\gamma),\theta)))$ and each integer $m \geq n(A,\gamma,i)$,

$$\rho\big(x(A,\gamma), B^m x\big) \leq 8^{-1-i}. \tag{2.257}$$

Define

$$\mathcal{F} = \mathcal{F}_0 \cap \left[\bigcap_{q=1}^{\infty} \bigcup \big\{U(A,\gamma,i) : A \in \mathcal{A}, \gamma \in (0,1), i = q, q+1, \dots\big\}\right].$$

Clearly, $\mathcal{F}$ is a countable intersection of open and everywhere dense sets in $\mathcal{A}$.

Let $A \in \mathcal{F}$. Then $A \in \mathcal{F}_0$ and it is super-regular. There exists $x(A) \in K$ such that

$$A\bigl(x(A)\bigr) = x(A). \tag{2.258}$$

There also exist sequences $\{A_q\}_{q=1}^{\infty} \subset \mathcal{A}$, $\{\gamma_q\}_{q=1}^{\infty} \subset (0,1)$ and a strictly increasing sequence of natural numbers $\{i_q\}_{q=1}^{\infty}$ such that

$$A \in U(A_q, \gamma_q, i_q), \quad q = 1, 2, \ldots. \tag{2.259}$$

Let $\delta, s > 0$. Choose a natural number q such that

$$2^q > 16(s+1) \quad \text{and} \quad 2^{-q} < 8^{-1}\delta, \tag{2.260}$$

and consider the open set $U(A_q, \gamma_q, i_q)$.

Let $r_j \in [\bar{r}_j, 1]$, $j = 1, 2, \ldots,$ and $B \in U(A_q, \gamma_q, i_q)$. By property (a), the first part the theorem (assertion (i)) is valid.

To prove assertion (ii), assume, in addition, that B is regular. Then there is $x(B) \in K$ such that

$$B\bigl(x(B)\bigr) = x(B). \tag{2.261}$$

By property (b),

$$\rho\bigl(x(A_q, \gamma_q), x(A)\bigr), \rho\bigl(x(A_q, \gamma_q), x(B)\bigr) \le 8^{-i_q - 1}. \tag{2.262}$$

Let $x_1 \in B(s)$ and

$$x_{j+1} = r_j B x_j \oplus (1 - r_j) x_j, \quad j = 1, 2, \ldots.$$

It follows from property (a) and (2.261) that

$$\rho\bigl(x_j, x(B)\bigr) \le 8^{-i_q - 1} \quad \text{for all integers } j \ge n(A_q, \gamma_q, i_q).$$

Together with (2.262) and (2.260), this implies that for all integers $j \ge n(A_q, \gamma_q, i_q)$,

$$\rho\bigl(x_j, x(A)\bigr) \le 3 \cdot 8^{-i_q - 1} < \delta.$$

This completes the proof of Theorem 2.34. □

2.10 Power Convergence of Order-Preserving Mappings

In this section we study the asymptotic behavior of the iterations of those order-preserving mappings on an interval $\langle 0, u_* \rangle$ in an ordered Banach space X for which the origin is a fixed point. Here u_* is an interior point of the cone of positive elements X_+ of the space X. Such classes of order-preserving mappings arise, for example, in mathematical economics. We show that for a generic mapping there exists a fixed point which belongs to the interior of X_+ such that the iterations of the mapping with an initial point in the interior of X_+ converge to it.

Let $(X, \|\cdot\|)$ be a Banach space ordered by a closed cone X_+ with a nonempty interior such that $\|x\| \le \|y\|$ for each $x, y \in X_+$ satisfying $x \le y$. For each $u, v \in X$ such that $u \le v$ denote

$$\langle u, v\rangle = \{x \in X : u \le x \le v\}.$$

Let u_* be an interior point of X_+. Define

$$\|x\|_* = \inf\bigl\{r \in [0, \infty) : -ru_* \le x \le ru_*\bigr\}, \quad x \in X. \tag{2.263}$$

Clearly, $\|\cdot\|_*$ is a norm on X which is equivalent to the norm $\|\cdot\|$.

An operator $A : \langle 0, u_*\rangle \to \langle 0, u_*\rangle$ is called monotone if

$$Ax \le Ay \quad \text{for each } x, y \in \langle 0, u_*\rangle \text{ such that } x \le y. \tag{2.264}$$

Denote by $\mathcal{M}$ the set of all monotone continuous operators $A : \langle 0, u_*\rangle \to \langle 0, u_*\rangle$ such that

$$A(0) = 0 \tag{2.265}$$

and

$$A(\alpha z) \ge \alpha Az \quad \text{for all } z \in \langle 0, u_*\rangle \text{ and } \alpha \in [0, 1]. \tag{2.266}$$

Geometrically, (2.266) means that the hypograph of A is star-shaped with respect to the origin.

For the space $\mathcal{M}$ we define a metric $\rho : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by

$$\rho(A, B) = \sup\bigl\{\|Ax - Bx\|_* : x \in \langle 0, u_*\rangle\bigr\}, \quad A, B \in \mathcal{M}. \tag{2.267}$$

It is easy to see that the metric space $\mathcal{M}$ is complete.

An operator $A : \langle 0, u_*\rangle \to \langle 0, u_*\rangle$ is called concave if for all $x, y \in \langle 0, u_*\rangle$ and $\alpha \in [0, 1]$,

$$A\bigl(\alpha x + (1-\alpha)y\bigr) \ge \alpha Ax + (1-\alpha)Ay. \tag{2.268}$$

We denote by $\mathcal{M}_{co}$ the set of all concave operators $A \in \mathcal{M}$. Clearly, $\mathcal{M}_{co}$ is a closed subset of $\mathcal{M}$. We consider the topological subspace $\mathcal{M}_{co} \subset \mathcal{M}$ with the relative topology.

The spaces $\mathcal{M}$ and $\mathcal{M}_{co}$ are very important, for example, from the point of view of mathematical economics. In this area of research order-preserving mappings A are usually models of economic dynamics and the condition $A(0) = 0$ means that if we have no resources, then we produce nothing. Concavity means that the combination of resources allows one to produce at least the corresponding combination of outputs and even more than this combination. Monotonicity means that a larger input leads to a larger output. A particular class of concave operators are those operators which are positively homogeneous of degree $m \le 1$. Such operators were studied by many mathematical economists in the finite dimensional case (see [105] and the references mentioned there). For more information on ordered Banach spaces, order-preserving mappings and their applications see, for example, [3, 4].

We are now ready to state and prove the main result of this section. This result was established in [164].

Theorem 2.38 *There exist a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{M}$ and a set $\mathcal{F}_{co} \subset \mathcal{F} \cap \mathcal{M}_{co}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{M}_{co}$ such that for each $P \in \mathcal{F}$, there exists $x_P \in \langle 0, u_* \rangle$ for which the following two assertions hold*:

1. *The point x_P is an interior point of X_+ and $\lim_{t\to\infty} P^t x = x_P$ for each $x \in \langle 0, u_* \rangle$ which is an interior point of the cone X_+.*
2. *For each $\gamma, \varepsilon \in (0, 1)$, there exist an integer $N \geq 1$ and a neighborhood U of P in $\mathcal{M}$ such that for each $C \in U$, each $z \in \langle \gamma u_*, u_* \rangle$ and each integer $T \geq N$,*

$$\left\| C^T z - x_P \right\|_* \leq \varepsilon.$$

Proof of Theorem 2.38 For each $x, y \in X_+$ define

$$\lambda(x, y) = \sup\{r \in [0, \infty) : rx \leq y\}. \tag{2.269}$$

In the proof of Theorem 2.38 we will use several auxiliary results.

Lemma 2.39 *The function $y \to \lambda(u_*, y)$, $y \in X_+$, is continuous, concave and positively homogeneous.*

Proof All we need to show is that the function $y \to \lambda(u_*, y)$, $y \in X_+$, is continuous. To this end, assume that $y \in X_+$, $\{y_n\}_{n=1}^{\infty} \subset X_+$ and

$$\|y_n - y\|_* \to 0 \quad \text{as } n \to \infty. \tag{2.270}$$

We show that

$$\lambda(u_*, y_n) \to \lambda(u_*, y) \quad \text{as } n \to \infty. \tag{2.271}$$

It is well known that (2.271) is true if y is an interior point of X_+. Therefore we may assume that y is not an interior point of X_+.

Clearly,

$$\lambda(u_*, y) = 0. \tag{2.272}$$

We show that

$$\lim_{n\to\infty} \lambda(u_*, y_n) = 0. \tag{2.273}$$

Assume the contrary. Then there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ and a number $r > 0$ such that

$$y_{n_k} \geq r u_*, \quad k = 1, 2, \ldots. \tag{2.274}$$

Together with (2.270) this implies that

$$y \geq r u_* \quad \text{and} \quad \lambda(u_*, y) \geq r.$$

Since this contradicts (2.272), we see that (2.273) does hold. This completes the proof of Lemma 2.39. □

Define now an operator $\phi : \langle 0, u_* \rangle \to X_+$ by

$$\phi(x) = \lambda(u_*, x)^{1/2} u_*, \quad x \in \langle 0, u_* \rangle. \tag{2.275}$$

By using Lemma 2.39 one can easily check that

$$\phi \in \mathcal{M}_{co}. \tag{2.276}$$

Let $A \in \mathcal{M}$ and let $i \geq 1$ be an integer. Define an operator $A^{(i)} : \langle 0, u_* \rangle \to \langle 0, u_* \rangle$ by

$$A^{(i)} x = \left(1 - 2^{-i}\right) A x + 2^{-i} \phi(x), \quad x \in \langle 0, u_* \rangle. \tag{2.277}$$

Lemma 2.40 *Let $A \in \mathcal{M}$ and let $i \geq 1$ be an integer. Then $A^{(i)} \in \mathcal{M}$. Moreover, if $A \in \mathcal{M}_{co}$, then $A^{(i)} \in \mathcal{M}_{co}$.*

It is clear that for each $A \in \mathcal{M}$ and each integer $i \geq 1$,

$$\rho\left(A^{(i)}, A\right) \leq 2^{-i}. \tag{2.278}$$

Lemma 2.41 *Let $A \in \mathcal{M}$ and let $i \geq 1$ be an integer. Then*

$$A^{(i)}\left(16^{-i} u_*\right) \geq 8^{-i} u_*. \tag{2.279}$$

Proof By (2.277) and (2.275),

$$A^{(i)}\left(16^{-i} u_*\right) \geq 2^{-i} \phi\left(16^{-i} u_*\right) \geq 2^{-i} \left(16^{-i}\right)^{1/2} u_* \geq 8^{-i} u_*.$$ □

For each $A \in \mathcal{M}$ and each integer $i \geq 1$, we now define the operator $B^{(A,i)} : \langle 0, u_* \rangle \to \langle 0, u_* \rangle$ by

$$B^{(A,i)}(x) = \left(1 - 16^{-i}\right) A^{(i)} x + \min\left\{\lambda(u_*, x), 16^{-i}\right\} u_*, \quad x \in \langle 0, u_* \rangle. \tag{2.280}$$

Lemma 2.42 *Let $A \in \mathcal{M}$ and let $i \geq 1$ be an integer. Then*

$$B^{(A,i)}\left(16^{-i} u_*\right) \geq \left(8^{-i} + 2^{-1} \cdot 16^{-i}\right) u_* \tag{2.281}$$

and $B^{(A,i)} \in \mathcal{M}$. Moreover, if $A \in \mathcal{M}_{co}$, then $B^{(A,i)} \in \mathcal{M}_{co}$.

Proof It follows from (2.280) and (2.279) that

$$B^{(A,i)}\big(16^{-i}u_*\big) \geq \big(1-16^{-i}\big)\big(8^{-i}u_*\big)+16^{-i}u_* \geq \big(8^{-i}+2^{-1}\cdot 16^{-i}\big)u_*.$$

Therefore (2.271) is valid. By Lemma 2.40, $B^{(A,i)}(0)=0$ and the operator $B^{(A,i)}$ is monotone. Lemmas 2.39, 2.40 and (2.280) imply that $B^{(A,i)}$ is a continuous operator. It follows from Lemma 2.39 that the operator

$$x \to \min\big\{\lambda(u_*, x), 16^{-i}\big\}u_*, \quad x \in \langle 0, u_*\rangle,$$

is concave. When combined with (2.280), Lemma 2.40 and (2.264), this implies that

$$B^{(A,i)}(\alpha z) \geq \alpha B^{(A,i)}z \quad \text{for each } z \in \langle 0, u_*\rangle \text{ and each } \alpha \in [0,1],$$

and that if $A \in \mathcal{M}_{co}$, then $B^{(A,i)}$ is concave. This completes the proof of Lemma 2.42. □

It follows from (2.280), (2.278) and (2.267) that for each $A \in \mathcal{M}$ and each integer $i \geq 1$,

$$\rho\big(A, B^{(A,i)}\big) \leq 2^{-i}+16^{-i}. \tag{2.282}$$

Lemma 2.43 *Let $A \in \mathcal{M}$ and let $i \geq 1$ be an integer. Then*

$$\lim_{t\to\infty} \lambda\big(\big(B^{(A,i)}\big)^t(u_*), \big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big)\big)=1. \tag{2.283}$$

Proof Clearly,

$$\big(B^{(A,i)}\big)^{t+1}(u_*) \leq \big(B^{(A,i)}\big)^t(u_*), \quad t=1,2,\dots \tag{2.284}$$

and

$$\big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big) \leq \big(B^{(A,i)}\big)^t(u_*), \quad t=1,2,\dots.$$

Lemma 2.42 (see 2.361)) implies that for each integer $t \geq 1$,

$$\big(B^{(A,i)}\big)^{t+1}\big(16^{-i}u_*\big) \geq \big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big) \geq \big(8^{-i}+2^{-1}\cdot 16^{-i}\big)u_*. \tag{2.285}$$

For $t=0,1,\dots$ we set

$$\lambda_t = \lambda\big(\big(B^{(A,i)}\big)^t(u_*), \big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big)\big). \tag{2.286}$$

By (2.284),

$$\lambda_t \leq 1, \quad t=0,1,\dots. \tag{2.287}$$

Let $t \geq 0$ be an integer. It follows from (2.280), (2.286), (2.269), (2.285), Lemma 2.40 and (2.287) that

$$\begin{aligned}
\big(B^{(A,i)}\big)^{t+1}\big(16^{-i}u_*\big) &= B^{(A,i)}\big(\big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big)\big) \\
&= \big(1-16^{-i}\big)A^{(i)}\big(\big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big)\big) \\
&\quad + \min\big\{\lambda\big(u_*, \big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big)\big), 16^{-i}\big\}u_* \\
&\geq \big(1-16^{-i}\big)A^{(i)}\big(\lambda_t\big(B^{(A,i)}\big)^t(u_*)\big) + 16^{-i}u_* \\
&\geq \big(1-16^{-i}\big)\lambda_t A^{(i)}\big(\big(B^{(A,i)}\big)^t(u_*)\big) + 16^{-i}u_* \\
&= \lambda_t\big[\big(1-16^{-i}\big)A^{(i)}\big(\big(B^{(A,i)}\big)^t(u_*)\big) + 16^{-i}u_*\big] \\
&\quad + (1-\lambda_t)16^{-i}u_* \\
&= \lambda_t\big[\big(1-16^{-i}\big)A^{(i)}\big(B^{(A,i)}\big)^t(u_*) \\
&\quad + \min\big\{\lambda\big(u_*, \big(B^{(A,i)}\big)^t(u_*)\big), 16^{-i}\big\}u_*\big] + (1-\lambda_t)16^{-i}u_* \\
&= \lambda_t\big(B^{(A,i)}\big)^{t+1}(u_*) + (1-\lambda_t)16^{-i}u_* \\
&\geq \big(\lambda_t + (1-\lambda_t)16^{-i}\big)\big(B^{(A,i)}\big)^{t+1}(u_*).
\end{aligned}$$

This implies that

$$\lambda_{t+1} \geq \lambda_t + (1-\lambda_t)16^{-i}. \tag{2.288}$$

Combining (2.287) and (2.288), we see that

$$\Lambda = \lim_{t\to\infty} \lambda_t \tag{2.289}$$

exists. By (2.289) and (2.288), $\Lambda \geq \Lambda + (1-\Lambda)16^{-1}$. By (2.287) this implies that $\Lambda = 1$. Lemma 2.43 is proved. □

Lemma 2.44 *Let $A \in \mathcal{M}$ and let $i \geq 1$ be an integer. Then there exists $x^{(A,i)} \in \langle 0, u_*\rangle$ such that*

$$x^{(A,i)} \geq \big(8^{-i} + 2^{-1}\cdot 16^{-i}\big)u_* \tag{2.290}$$

and

$$\lim_{t\to\infty}\big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big) = \lim_{t\to\infty}\big(B^{(A,i)}\big)^t(u_*) = x^{(A,i)}. \tag{2.291}$$

Proof It is clear that inequalities (2.284) hold. Lemma 2.42 implies that for each integer $t \geq 1$, inequality (2.283) is also valid. By Lemma 2.43, (2.284) and (2.285),

$$\lim_{t\to\infty}\big[\big(B^{(A,i)}\big)^t u_* - \big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big)\big] = 0, \tag{2.292}$$

and $\{(B^{(A,i)})^t u_*\}_{t=1}^{\infty}$, as well as $\{(B^{(A,i)})^t(16^{-i}u_*)\}_{t=1}^{\infty}$, are Cauchy sequences.

Therefore there exist $x_1, x_2 \in \langle 0, u_*\rangle$ such that

$$x_1 = \lim_{t\to\infty}\big(B^{(A,i)}\big)^t\big(16^{-i}u_*\big) \quad \text{and} \quad x_2 = \lim_{t\to\infty}\big(B^{(A,i)}\big)^t u_*.$$

By (2.292) and (2.285), $x_1 = x_2 \geq (8^{-i} + 2^{-1} \cdot 16^{-i})u_*$. This completes the proof of Lemma 2.44. □

Lemma 2.45 *Let $A \in \mathcal{M}$, $\varepsilon > 0$, $z \in \langle 0, u_* \rangle$ and let $n \geq 1$ be an integer. Then there exists a neighborhood U of A in $\mathcal{M}$ such that for each $C \in U$,*

$$\|C^n z - A^n z\|_* < \varepsilon.$$

Proof We prove the lemma by induction. It is clear that the assertion of the lemma is valid for $n = 1$. Assume that it is valid for an integer $n \geq 1$. There exists

$$\delta \in (0, 8^{-1}\varepsilon) \tag{2.293}$$

such that

$$\|Ay - A(A^n z)\|_* \leq 8^{-1}\varepsilon \tag{2.294}$$

for each $y \in \langle 0, u_* \rangle$ satisfying $\|y - A^n z\|_* \leq \delta$. Since the assertion of the lemma is assumed to be valid for n, there exists a neighborhood U_0 of A in $\mathcal{M}$ such that for each $C \in U_0$,

$$\|C^n z - A^n z\|_* < \delta. \tag{2.295}$$

Set

$$U = \{C \in U_0 : \rho(C, A) < 8^{-1}\varepsilon\}, \tag{2.296}$$

and let $C \in U$. The definition of U implies that

$$\begin{aligned} \|A^{n+1}z - C^{n+1}z\|_* &\leq \|A^{n+1}z - AC^n z\|_* + \|AC^n z - C^{n+1}z\|_* \\ &\leq \|A^{n+1}z - AC^n z\|_* + 8^{-1}\varepsilon. \end{aligned} \tag{2.297}$$

By (2.295),

$$\|A^n z - C^n z\|_* < \delta.$$

It follows from this inequality and the choice of δ (see (2.293) and (2.294)) that

$$\|AC^n z - A(A^n z)\|_* \leq 8^{-1}\varepsilon.$$

Together with (2.297) this implies that

$$\|A^{n+1}z - C^{n+1}z\|_* \leq 4^{-1}\varepsilon.$$

This completes the proof of Lemma 2.45. □

Let $A \in \mathcal{M}$ and let $i \geq 1$ be an integer. By Lemma 2.44, there exists an integer $N(A, i) \geq 4$ such that

$$\|(B^{(A,i)})^{N(A,i)}(16^{-i}u_*) - (B^{(A,i)})^{N(A,i)}(u_*)\|_* \leq 16^{-i-1}. \tag{2.298}$$

By Lemma 2.45, there exists an open neighborhood $U(A,i)$ of $B^{(A,i)}$ in $\mathcal{M}$ such that

$$U(A,i) \subset \{C \in \mathcal{M} : \rho(C, B^{(A,i)}) \le 16^{-i-2}\}, \tag{2.299}$$

and for each $C \in U(A,i)$,

$$\begin{aligned} \|C^{N(A,i)}(16^{-i}u_*) - (B^{(A,i)})^{N(A,i)}(16^{-i}u_*)\|_* \le 16^{-i-2}, \\ \|C^{N(A,i)}(u_*) - (B^{(A,i)})^{N(A,i)}(u_*)\|_* \le 16^{-i-2}. \end{aligned} \tag{2.300}$$

Lemma 2.46 *Let $A \in \mathcal{M}$ and let $i \ge 1$ be an integer. Assume that $C \in U(A,i)$. Then*

$$C^t(16^{-i}u_*) \ge 8^{-i}u_*, \quad t = 1, 2, \ldots, \tag{2.301}$$

and for each $z \in \langle 16^{-i}u_*, u_*\rangle$ and each integer $T \ge N(A,i)$, the following inequality holds:

$$\|C^T z - x(A,i)\|_* \le 16^{-i-1} + 16^{-i-2}. \tag{2.302}$$

Proof By the definition of $U(A,i)$ (see (2.299)) and Lemma 2.42 (see (2.281)),

$$\|C(16^{-i}u_*) - B^{(A,i)}(16^{-i}u_*)\|_* \le 16^{-i-2}$$

and

$$\begin{aligned} C(16^{-i}u_*) &\ge B^{(A,i)}(16^{-i}u_*) - 16^{-i-2}u_* \\ &\ge (8^{-i} + 2^{-1} \cdot 16^{-i})u_* - 16^{-i-2}u_* \ge 8^{-i}u_*. \end{aligned} \tag{2.303}$$

Since the operator C is monotone, (2.303) implies that

$$C^{t+1}(16^{-i}u_*) \ge C^t(16^{-i}u_*), \quad t = 0, 1, \ldots. \tag{2.304}$$

Inequalities (2.304) and (2.303) imply (2.301), as claimed.

Assume that $z \in \langle 16^{-i}u_*, u_*\rangle$ and let $T \ge N(A,i)$ be an integer. Since the operator C is monotone, it follows from (2.304) and the definition of $U(A,i)$ (see (2.300)) that

$$\begin{aligned} C^T z &\in \langle C^T(16^{-i}u_*), C^T(u_*)\rangle \subset \langle C^{N(A,i)}(16^{-i}u_*), C^{N(A,i)}u_*\rangle \\ &\subset \langle (B^{(A,i)})^{N(A,i)}(16^{-i}u_*) \\ &\quad - 16^{-i-2}u_*, (B^{(A,i)})^{N(A,i)}(u_*) + 16^{-i-2}u_*\rangle. \end{aligned} \tag{2.305}$$

By Lemma 2.44, (2.281), (2.305) and (2.298),

$$\begin{aligned} C^T z - x(A,i) \in \langle &(B^{(A,i)})^{N(A,i)}(16^{-i}u_*) - 16^{-i-2}u_* - x(A,i), \\ &(B^{(A,i)})^{N(A,i)}(u_*) + 16^{-i-2}u_* - x(A,i)\rangle \end{aligned}$$

and

$$\begin{aligned}
&x(A,i) - C^T z, -x(A,i) + C^T z \\
&\quad \le \big(B^{(A,i)}\big)^{N(A,i)}(u_*) - \big(B^{(A,i)}\big)^{N(A,i)}\big(16^{-i}u_*\big) + 16^{-i-2}u_* \\
&\quad \le \big(16^{-i-1} + 16^{-i-2}\big)u_*.
\end{aligned}$$

This implies (2.302) and completes the proof of Lemma 2.46. □

Completion of the proof of Theorem 2.38 Define

$$\mathcal{F} = \bigcap_{q=1}^{\infty} \bigcup \big\{U(A,i) : A \in \mathcal{M}, i = q, q+1, \dots\big\}$$

and

$$\mathcal{F}_{co} = \bigcap_{q=1}^{\infty} \bigcup \big\{U(A,i) \cap \mathcal{M}_{co} : A \in \mathcal{M}_{co}, i = q, q+1, \dots\big\}.$$

It is easy to see that $\mathcal{F}_{co} \subset \mathcal{F} \cap \mathcal{M}_{co}$, $\mathcal{F}$ is a countable intersection of open and everywhere dense sets in $\mathcal{M}$, and that $\mathcal{F}_{co}$ is a countable intersection of open and everywhere dense sets in $\mathcal{M}_{co}$. Assume that $P \in \mathcal{F}$ and $\varepsilon, \gamma \in (0,1)$. Choose a natural number q for which

$$64 \cdot 2^{-q} < 64^{-1} \min\{\varepsilon, \gamma\}. \tag{2.306}$$

There exist $A \in \mathcal{M}$ and a natural number $i \ge q$ such that

$$P \in U(A,i). \tag{2.307}$$

By Lemma 2.46,

$$C^t\big(16^{-i}u_*\big) \ge 8^{-i}u_* \quad \text{for all integers } t \ge 1 \text{ and all } C \in U(A,i), \tag{2.308}$$

and

$$\begin{aligned}
&\big\|C^T z - x(A,i)\big\|_* \le 16^{-i-1} + 16^{-i-2} \quad \text{for all } C \in U(A,i), \\
&\quad \text{each integer } T \ge N(A,i) \text{ and each } z \in \big\langle 16^{-i}u_*, u_*\big\rangle.
\end{aligned} \tag{2.309}$$

Now (2.309), (2.306) and (2.307) imply that

$$\begin{aligned}
&\big\|P^T z - x(A,i)\big\|_* \le \varepsilon \quad \text{for each integer } T \ge N(A,i) \\
&\quad \text{and each } z \in \langle \gamma u_*, u_*\rangle.
\end{aligned} \tag{2.310}$$

Since ε is an arbitrary number in the interval $(0, 1)$, we conclude that for each $z \in \langle \gamma u_*, u_* \rangle$, there exists $\lim_{t\to\infty} P^t z$. By (2.310),

$$\left\| \lim_{t\to\infty} P^T z - x(A,i) \right\|_* \le \varepsilon \quad \text{for each } z \in \langle \gamma u_*, u_* \rangle. \tag{2.311}$$

Hence

$$\lim_{t\to\infty} P^t z_1 = \lim_{t\to\infty} P^t z_2$$

for each $z_1, z_2 \in \langle \gamma u_*, u_* \rangle$.

Since $\gamma \in (0, 1)$ is also arbitrary, we conclude that

$$\lim_{t\to\infty} P^t z = x_P \tag{2.312}$$

for each $z \in \langle 0, u_* \rangle$ which is an interior point of X_+. By (2.308), x_P is an interior point of X_+. Now (2.309) implies that

$$\left\| x_P - x(A,i) \right\|_* \le 16^{-i-1} + 16^{-i-2}. \tag{2.313}$$

Assume that $C \in U(A,i)$, $z \in \langle \gamma u_*, u_* \rangle$, and let $T \ge N(A,i)$ be an integer. It follows from (2.309), (2.313) and (2.306) that

$$\begin{aligned} \left\| C^T z - x_P \right\|_* &\le \left\| x_P - x(A,i) \right\|_* + \left\| x(A,i) - C^T z \right\|_* \\ &\le 16^{-i-1} + 16^{-i-2} + \left\| x(A,i) - C^T z \right\|_* \\ &\le 2\left(16^{-i-1} + 16^{-i-2}\right) < \varepsilon. \end{aligned}$$

This completes the proof of Theorem 2.38. □

2.11 Positive Eigenvalues and Eigenvectors

In this section we consider a closed cone of positive operators on an ordered Banach space and prove that a generic element of this cone has a unique positive eigenvalue and a unique (up to a positive multiple) positive eigenvector. Moreover, the normalized iterations of such a generic element converge to its unique eigenvector. This section is based on [140].

Let $(X, \|\cdot\|)$ be a Banach space which is ordered by a closed convex cone X_+. For each $u, v \in X$ such that $u \le v$, we define $\langle u, v \rangle = \{z \in X : u \le z \le v\}$.

We assume that the cone X_+ has a nonempty interior and that for each $x, y \in X_+$ satisfying $x \le y$, the inequality $\|x\| \le \|y\|$ holds. We denote by $\mathrm{int}(X_+)$ the set of all interior points of X_+.

Fix an interior point η of the cone X_+ and define

$$\|x\|_\eta = \inf\left\{r \in [0, \infty) : -r\eta \le x \le r\eta\right\}, \quad x \in X. \tag{2.314}$$

Clearly, $\|\cdot\|_\eta$ is a norm on X which is equivalent to the original norm $\|\cdot\|$.

Let X' be the space of all linear continuous functionals $f : X \to R^1$ and let

$$X'_+ = \{ f \in X' : f(x) \geq 0 \text{ for all } x \in X_+ \}.$$

Denote by $\mathcal{A}$ the set of all linear operators $A : X \to X$ such that $A(X_+) \subset X_+$. Such operators are called positive. For the set $\mathcal{A}$ we define a metric $\rho(\cdot, \cdot)$ by

$$\rho(A, B) = \sup\{ \|Ax - Bx\|_\eta : x \in \langle 0, \eta \rangle \}, \quad A, B \in \mathcal{A}.$$

This metric ρ is equivalent to the metrics induced by the operator norms derived from $\|\cdot\|$ and $\|\cdot\|_\eta$. It is clear that the metric space $(\mathcal{A}, \rho)$ is complete. Since many linear operators between Banach spaces arising in classical and modern analysis are, in fact, positive operators, the theory of positive linear operators and its applications have drawn the attention of more and more mathematicians. See, for example, [3, 86, 170] and the references cited therein.

In this section we study the asymptotic behavior of powers of positive linear operators on the ordered Banach space X. We obtain generic convergence to an operator of the form $f(\cdot)\eta$, where f is a bounded linear functional and η is a unique (up to a positive multiple) eigenvector.

We denote by $\mathcal{A}_*$ the set of all $A \in \mathcal{A}$ such that $A\xi = \xi$ for some $\xi \in \text{int}(X_+)$ and by $\bar{\mathcal{A}}_*$ the closure of $\mathcal{A}_*$ in $(\mathcal{A}, \rho)$. We equip the subspace $\bar{\mathcal{A}}_* \subset \mathcal{A}$ with the same metric ρ.

In our paper [125] we established the following result.

Theorem 2.47 *There exists a set $\mathcal{F} \subset \bar{\mathcal{A}}_*$ which is a countable intersection of open and everywhere dense sets in $\bar{\mathcal{A}}*$ such that for each $B \in \mathcal{F}$, there exists an interior point ξ_B of X_+ satisfying $B\xi_B = \xi_B$, $\|\xi_B\|_\eta = 1$, and the following two assertions hold*:

1. *There exists $f_B \in X'_+$ such that* $\lim_{T\to\infty} B^T x = f_B(x)\xi_B$, $x \in X$.
2. *For each $\varepsilon > 0$, there exists a neighborhood $\mathcal{U}$ of B in $\bar{\mathcal{A}}_*$ and a natural number N such that for each $C \in \mathcal{U} \cap \mathcal{A}_*$, each integer $T \geq N$ and each $x \in \langle -\eta, \eta \rangle$,*

$$\left\| C^T x - f_B(x)\xi_B \right\| \leq \varepsilon.$$

Since the existence of fixed points and the convergence of iterates is of fundamental importance, it is of interest to look for a larger subset of $\mathcal{A}$ for which such a result continues to hold. To this end, we introduce the set $\mathcal{A}_{q*}$ of all $A \in \mathcal{A}$ for which there exist $c_0 \in (0, 1)$ and $c_1 > 1$ such that

$$c_0\eta \leq A^n\eta \leq c_1\eta \quad \text{for all integers } n \geq 1. \tag{2.315}$$

Note that our definition of $\mathcal{A}_{q*}$ does not depend on our choice of η. Since $\mathcal{A}_* \subset \mathcal{A}_{q*}$, it is natural to ask if there is also a generic result for the closure $\bar{\mathcal{A}}_{q*}$ of $\mathcal{A}_{q*}$. Note that in contrast with $\bar{\mathcal{A}}_*$, it is not clear *a priori* if $\mathcal{A}_*$ is dense in $\bar{\mathcal{A}}_{q*}$. However, as we show in our first result that this is indeed the case.

Theorem 2.48 $\bar{\mathcal{A}}_{q*} = \bar{\mathcal{A}}_*$.

Combining Theorems 2.47 and 2.48, we see that a generic element in $\bar{\mathcal{A}}_{q*}$ has a unique (up to a positive multiple) positive fixed point and all its iterations converge to some multiple of this fixed point.

Since the existence of positive eigenvectors which are not necessarily fixed points is even more important, we devote most of the section to this problem.

Known results about the existence of positive fixed points and eigenvectors include the classical Perron-Frobenius and Krein-Rutman theorems. For a survey of more recent results of the linear theory, see Sect. 2 in [106].

We begin with the following definition.

We say that an operator $A \in \mathcal{A}$ is regular if there exist $x_A \in \mathrm{int}(X_+)$ satisfying $\|x_A\|_\eta = 1$, $\alpha_A > 0$ and $f_A \in X'_+ \setminus \{0\}$ such that

$$Ax_A = \alpha_A x_A, \qquad \alpha_A^{-n} A^n x \to f_A(x) x_A \quad \text{as } n \to \infty,$$

uniformly for all $x \in \langle -\eta, \eta \rangle$.

Note that in the definition above, x_A, α_A and f_A are all uniquely defined and that if $x \in \mathrm{int}(X_+)$, then $\|A^n x\|_\eta^{-1} A^n x \to x_A$ as $n \to \infty$.

We denote by $\mathcal{A}_{reg}$ the set of all regular operators in $\mathcal{A}$ and by $\bar{\mathcal{A}}_{reg}$ its closure in the space $(\mathcal{A}, \rho)$. We endow the subspace $\bar{\mathcal{A}}_{reg} \subset \mathcal{A}$ with the same metric ρ.

We continue with two theorems on regular operators.

Theorem 2.49 *Let $A \in \mathcal{A}_{reg}$ and $\varepsilon > 0$. Then there exist an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{A}_{reg} \cap \mathcal{U}$,*

$$\|x_A - x_B\|_\eta \leq \varepsilon, \qquad |\alpha_A - \alpha_B| \leq \varepsilon$$

and for each $x \in \langle -\eta, \eta \rangle$ and each integer $n \geq N$,

$$\left\| \alpha_B^{-n} B^n x - f_A(x) x_A \right\|_\eta \leq \varepsilon.$$

Theorem 2.50 *Let $A \in \mathcal{A}_{reg}$, $\varepsilon > 0$ and $\Delta \in (0,1)$. Then there exist an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that the following assertion holds:*

Assume that $B \in \mathcal{U}$, $x_0 \in X_+$, $\alpha_0 > 0$, $\Delta\eta \leq x_0 \leq \eta$ and $\alpha_0 x_0 = Bx_0$. Then

$$\|x_A - x_0\|_\eta \leq \varepsilon, \qquad |\alpha_A - \alpha_0| \leq \varepsilon$$

and for each $x \in \langle -\eta, \eta \rangle$ and each integer $n \geq N$,

$$\left\| \alpha_0^{-n} B^n x - f_A(x) x_A \right\|_\eta \leq \varepsilon.$$

These theorems bring out the importance of regular operators. Such operators not only have a unique positive eigenvector but also enjoy certain convergence and stability properties. Therefore we would like to show that most operators in an appropriate space are indeed regular. Moreover, in analogy with the definition of $\mathcal{A}_{q*}$ we will also consider quasiregular operators.

We say that an operator $A \in \mathcal{A}$ is quasiregular if there exist $\alpha > 0$, $c_0 \in (0, 1)$ and $c_1 > 1$ such that

$$c_0\alpha^n \eta \leq A^n \eta \leq c_1\alpha^n \eta \quad \text{for all integers } n \geq 1.$$

Denote by $\mathcal{A}_{qreg}$ the set of all quasiregular $A \in \mathcal{A}$ and by $\bar{\mathcal{A}}_{qreg}$ the closure of $\mathcal{A}_{qreg}$ in $(\mathcal{A}, \rho)$. We endow the subspace $\bar{\mathcal{A}}_{qreg} \subset \mathcal{A}$ with the same metric ρ.

Theorem 2.51 *$\bar{\mathcal{A}}_{qreg} = \bar{\mathcal{A}}_{reg}$ and there exists a set $\mathcal{F} \subset \mathcal{A}_{reg}$ which is a countable intersection of open and everywhere dense subsets of $\bar{\mathcal{A}}_{reg}$.*

Theorems 2.48–2.51 were obtained in [140].

2.12 Proof of Theorem 2.48

In this section we are going to present the proof of Theorem 2.48. We precede this proof by a few preliminary results.

As usual, we set $A^0 = I$ (the identity) for each $A \in \mathcal{A}$. We denote by $g \cdot B$ the composition of $g \in X'$ and a linear operator $B : X \to X$.

Proposition 2.52 *Let $A \in \mathcal{A}$ and assume that there exist $c_0 \in (0, 1)$ and $c_1 > 1$ such that*

$$c_0\eta \leq A^n \eta \leq c_1\eta \quad \textit{for all integers } n \geq 1. \tag{2.316}$$

Then there exists $f_A \in X'_+$ such that

$$f_A(\eta) > 0 \quad \textit{and} \quad f_A \cdot A = f_A.$$

Proof There exists $g \in X'_+$ such that $g(\eta) = 1$. Denote by S the convex hull of the set $\{g \cdot A^n : n = 0, 1, \ldots\}$. Clearly for each $h \in S$,

$$c_0 \leq h(\eta) \leq c_1. \tag{2.317}$$

Denote by $\bar{S}$ the closure of S in the weak-star topology $\sigma(X', X)$. Clearly (2.317) holds for all $h \in \bar{S}$ and $\bar{S} \subset X'_+$. The set $\bar{S}$ is convex and by (2.317) compact in the weak-star topology. The operator $A' : f \to f \cdot A$, $f \in X'$, is weakly-star continuous and $A'(\bar{S}) \subset \bar{S}$. By Tychonoff's fixed point theorem, there exists $f_A \in \bar{S}$ for which $f_A \cdot A = f_A$. Since (2.317) holds for all $h \in \bar{S}$, $f_A(\eta) \geq c_0$. Proposition 2.52 is proved. □

Corollary 2.53 *Assume that $A \in \mathcal{A}$, $c_0 \in (0, 1)$, $c_1 > 1$, $\alpha > 0$ and*

$$\alpha^n c_0\eta \leq A^n \eta \leq \alpha^n c_1\eta \quad \textit{for all integers } n \geq 1. \tag{2.318}$$

Then there exists $f_A \in X'_+$ such that $f_A(\eta) > 0$ and $f_A \cdot A = \alpha f_A$.

Lemma 2.54 *Assume that $A \in \mathcal{A}$, there exist $c_1 > 1$ and $\alpha > 0$ such that*

$$A^n \eta \le \alpha^n c_1 \eta \quad \text{for all integers } n \ge 1, \tag{2.319}$$

and that there exists $f_A \in X'_+$ such that

$$f_A \cdot A = \alpha f_A \quad \text{and} \quad f_A(\eta) = 1. \tag{2.320}$$

Let $\gamma \in (0, 1)$. Define $A_\gamma \in \mathcal{A}$ by

$$A_\gamma x = (1-\gamma)Ax + \gamma\alpha f_A(x)\eta, \quad x \in X. \tag{2.321}$$

Then $f_A \cdot A_\gamma = \alpha f_A$ and for each integer $n \ge 1$, there exist positive constants $c_i^{(n)}$, $i = 0, \dots, n-1$, such that

$$\sum_{i=0}^{n-1} c_i^{(n)} = 1 - (1-\gamma)^n \tag{2.322}$$

and

$$(A_\gamma)^n x = (1-\gamma)^n A^n x + \alpha^n f_A(x) \sum_{i=0}^{n-1} \bigl(\alpha^{-i} c_i^{(n)} A^i \eta\bigr), \quad x \in X. \tag{2.323}$$

Proof We will prove this lemma by induction. Clearly $f_A \cdot A_\gamma = f_A$ and (2.322) and (2.323) hold for $n = 1$, $c_0 = \gamma$.

Assume that $k \ge 1$ is an integer and there exist positive constants $c_i^{(k)}$, $i = 0, \dots, k-1$, such that (2.322) and (2.323) hold with $n = k$. It then follows from (2.322) and (2.323) with $n = k$ and (2.321) that for each $x \in X$,

$$\begin{aligned}
(A_\gamma)^{k+1} x &= A_\gamma \bigl(A_\gamma^k x\bigr) \\
&= (1-\gamma)A\bigl[(A_\gamma)^k x\bigr] + \alpha\gamma f_A\bigl((A_\gamma)^k x\bigr)\eta \\
&= \alpha\gamma\alpha^k f_A(x)\eta \\
&\quad + (1-\gamma)A\left[(1-\gamma)^k A^k x + \alpha^k f_A(x)\left(\sum_{i=0}^{k-1} \alpha^{-i} c_i^{(k)} A^i \eta\right)\right] \\
&= \gamma\alpha^{k+1} f_A(x)\eta + (1-\gamma)^{k+1} A^{k+1} x \\
&\quad + \alpha^k f_A(x)(1-\gamma)\left(\sum_{i=0}^{k-1} \alpha^{-i} c_i^{(k)} A^{i+1} \eta\right) \\
&= \gamma\alpha^{k+1} f_A(x)\eta + (1-\gamma)^{k+1} A^{k+1} x
\end{aligned}$$

$$+\alpha^{k+1} f_A(x)(1-\gamma)\left(\sum_{i=1}^{k} \alpha^{-i} c_{i-1}^{(k)} A^i \eta\right)$$

$$= (1-\gamma)^{k+1} A^{k+1} x + \alpha^{k+1} f_A(x)\left(\gamma\eta + \sum_{i=1}^{k}\left((1-\gamma)\alpha^{-i} c_{i-1}^{(k)} A^i \eta\right)\right)$$

and

$$\gamma + \sum_{i=1}^{k}\left((1-\gamma)c_{i-1}^{(k)}\right) = \gamma + (1-\gamma)\left(1-(1-\gamma)^{(k)}\right) = 1-(1-\gamma)^{k+1}.$$

Therefore (2.322) and (2.321) are true for $n = k+1$ with $c_0^{(k+1)} = \gamma$ and $c_i^{(k+1)} = (1-\gamma)c_{i-1}^{(k)}$, $i = 1, \ldots, k$. This completes the proof of Lemma 2.54. □

Lemma 2.55 *Assume that $A \in \mathcal{A}$, there exist $c_0 \in (0,1)$, $c_1 > 1$ and $\alpha > 0$ such that*

$$\alpha^n c_0 \eta \le A^n \eta \le \alpha^n c_1 \eta \quad \text{for all integers } n \ge 1, \tag{2.324}$$

and that there exists $f_A \in X'_+$ such that (2.320) holds. Let $\gamma \in (0,1)$ and let $A_\gamma \in \mathcal{A}$ be defined by (2.321). Then there exists $x_A \in < c_0\eta, c_1\eta >$ such that

$$\alpha^{-n}(A_\gamma)^n x - f_A(x) x_A \to 0 \quad \text{as } n \to \infty,$$

uniformly for all $x \in \langle 0, \eta\rangle$. Moreover, $A_\gamma x_A = \alpha x_A$.

Proof By Lemma 2.54 and (2.324), for each integer $n \ge 1$ there exists

$$z_n \in \langle c_0\eta, c_1\eta\rangle \tag{2.325}$$

such that

$$(A_\gamma)^n x = (1-\gamma)^n A^n x + \alpha^n\left(1-(1-\gamma)^n\right) f_A(x) z_n, \quad x \in X. \tag{2.326}$$

For each integer $n \ge 1$, by (2.320), (2.324) and (2.325),

$$\begin{aligned}(A_\gamma)^n \eta &= (1-\gamma)^n A^n \eta + \alpha^n\left(1-(1-\gamma)^n\right) z_n \\ &\in (1-\gamma)^n\left\langle \alpha^n c_0\eta, \alpha^n c_1\eta\right\rangle + \alpha^n\left(1-(1-\gamma)^n\right)\langle c_0\eta, c_1\eta\rangle \\ &\subset \alpha^n\langle c_0\eta, c_1\eta\rangle.\end{aligned} \tag{2.327}$$

Let $\varepsilon > 0$. By (2.326), there exists an integer $n(\varepsilon) \ge 1$ such that for each $x \in \langle c_0\eta, c_1\eta\rangle$ and each integer $n \ge n(\varepsilon)$,

$$\left\|\alpha^{-n}(A_\gamma)^n x - f_A(x) z_n\right\| \le \varepsilon.$$

Since $\{\alpha^{-i}(A_\gamma)^i\eta\}_{i=0}^\infty \subset \langle c_0\eta, c_1\eta\rangle$ and $f_A \cdot A_\gamma = \alpha f_A$, we conclude that for each integer $n \geq n(\varepsilon)$ and each integer $i \geq 0$,

$$\begin{aligned}\varepsilon &\geq \left\|\alpha^{-n}(A_\gamma)^n\left(\alpha^{-i}(A_\gamma)^i\eta\right) - f_A\left(\alpha^{-i}\left(A_\gamma^i\eta\right)\right)z_n\right\| \\ &= \left\|\alpha^{-n-i}(A_\gamma)^{n+i}\eta - z_n\right\|\end{aligned}$$

and therefore $\|z_n - z_{n+i}\| \leq 2\varepsilon$. This implies that $\{z_n\}_{n=1}^\infty$ is a Cauchy sequence. Hence there exists a vector $x_A \in \langle c_0\eta, c_1\eta\rangle$ such that $\lim_{i\to\infty}\|z_i - x_A\| = 0$. Let $\varepsilon > 0$. There exists an integer $n_0 \geq 1$ such that $\|z_i - x_A\| \leq \varepsilon/2$ for all integers $i \geq n_0$. By (2.326) and (2.324), there exists an integer $n_1 > n_0$ such that for each integer $n \geq n_1$ and each $x \in \langle 0, \eta\rangle$,

$$\left\|\alpha^{-n}(A_\gamma)^n x - f_A(x)z_n\right\| \leq 2^{-1}\varepsilon.$$

It follows from this last inequality and the definition of n_0 that for each $x \in \langle 0, \eta\rangle$ and each integer $n \geq n_1$,

$$\left\|\alpha^{-n}(A_\gamma)^n x - f_A(x)x_A\right\| \leq \varepsilon.$$

This completes the proof of Lemma 2.55. □

Proof of Theorem 2.48 It is, of course, sufficient to show that $\mathcal{A}_{q*} \subset \bar{\mathcal{A}}_*$. Towards this end, let $A \in \mathcal{A}_{q*}$. Then there exist $c_0 \in (0, 1)$ and $c_1 > 1$ such that

$$c_0\eta \leq A^n\eta \leq c_1\eta \quad \text{for all integers } n \geq 1.$$

By Proposition 2.52, there exists $f_A \in X'_+ \setminus \{0\}$ such that $f_A \cdot A = f_A$ and $f_A(\eta) = 1$.

For each $\gamma \in (0, 1)$, define $A_\gamma \in \mathcal{A}$ by

$$A_\gamma x = (1-\gamma)Ax + \gamma f_A(x)\eta, \quad x \in X.$$

By Lemma 2.55, A_γ belongs to $\mathcal{A}_*$. On the other hand, $\lim_{\gamma\to 0^+} A_\gamma = A$. Thus $\mathcal{A}_{q*} \subset \bar{\mathcal{A}}_*$ and Theorem 2.48 is proved. □

2.13 Auxiliary Results for Theorems 2.49–2.51

For each $x, y \in X_+$, define

$$\begin{aligned}\lambda(x, y) &= \sup\left\{\lambda \in [0, \infty) : \lambda x \leq y\right\}, \\ r(x, y) &= \inf\left\{r \in [0, \infty) : y \leq rx\right\}.\end{aligned} \tag{2.328}$$

Here we use the usual convention that the infimum of the empty set is ∞.

Lemma 2.56 *Assume that $A \in \mathcal{A}$, $n \geq 1$ is an integer and $\varepsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$ and each $x \in \langle -\eta, \eta \rangle$,*

$$\left\| A^n x - B^n x \right\|_\eta \leq \varepsilon.$$

Proof We prove the lemma by induction. Clearly for $n = 1$ the lemma is true. Assume that $k \geq 1$ is an integer and that the lemma holds for $n = k, \dots, 1$. There is a number $c_0 > 0$ such that $\|Ax\|_\eta \leq c_0$ for each $x \in \langle -\eta, \eta \rangle$. Since the lemma is true for $n = k$, there exists a neighborhood $\mathcal{U}_1$ of A in $\mathcal{A}$ such that $\|A^k x - B^k x\|_\eta \leq (4 + 4c_0)^{-1}\varepsilon$ for each $B \in \mathcal{U}_1$ and for each $x \in \langle -\eta, \eta \rangle$. It follows that there exists $c_1 > 1$ such that $\|B^k x\|_\eta \leq c_1$ for each $B \in \mathcal{U}_1$ and each $x \in \langle -\eta, \eta \rangle$. Since the lemma holds for $n = 1$, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$ and each $x \in \langle -\eta, \eta \rangle$, $\|Ax - Bx\|_\eta \leq (4c_1)^{-1}\varepsilon$.

Assume now that $B \in \mathcal{U}$ and $x \in \langle -\eta, \eta \rangle$. Then

$$\left\| A^{k+1} x - B^{k+1} x \right\|_\eta \leq \left\| A^{k+1} x - A B^k x \right\|_\eta + \left\| A B^k x - B^{k+1} x \right\|_\eta. \tag{2.329}$$

It follows from the definition of c_0 and $\mathcal{U}_1$ that

$$\left\| A^{k+1} x - A B^k x \right\|_\eta \leq \varepsilon/4. \tag{2.330}$$

By the definition of $\mathcal{U}$ and c_1, $\|AB^k x - B^{k+1}x\|_\eta \leq \varepsilon/4$. Together with (2.329) and (2.330), this implies that $\|A^{k+1}x - B^{k+1}x\|_\eta \leq \varepsilon$. In other words, the lemma also holds for $n = k + 1$. This completes the proof of Lemma 2.56. □

Let $A \in \mathcal{A}$ be regular,

$$\begin{gathered}
x_A \in \operatorname{int}(X_+), \qquad \|x_A\|_\eta = 1, \qquad \alpha_A > 0, \\
f_A \in X'_+ \setminus \{0\}, \qquad A x_A = \alpha_A x_A, \\
\alpha_A^{-n} A^n x \to f_A(x) x_A \quad \text{as } n \to \infty, \text{ uniformly on } \langle -\eta, \eta \rangle.
\end{gathered} \tag{2.331}$$

Assumptions (2.331) and Lemma 2.56 imply the following result.

Lemma 2.57 *Let $\varepsilon > 0$. Then there exists an integer $N(\varepsilon) \geq 1$ such that for each integer $N > N(\varepsilon)$, there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$ and each $x \in \langle -\eta, \eta \rangle$,*

$$\left\| \alpha_A^{-n} B^n x - f_A x_A \right\|_\eta \leq \varepsilon, \quad n = N(\varepsilon), \dots, N.$$

Corollary 2.58 *Assume that $0 < \Delta_1 < 1 < \Delta_2$ and $\theta > 1$. Then there exists an integer $N_0 \geq 1$ such that for each integer $N > N_0$, there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $x \in \langle \Delta_1 \eta, \Delta_2 \eta \rangle$, each $B \in \mathcal{U}$ and each integer $n \in [N_0, N]$,*

$$B^n x \in \left\langle \theta^{-1} \alpha_A^n f_A(x) x_A, \theta \alpha_A^n f_A(x) x_A \right\rangle.$$

Lemma 2.59 *Assume that* $0 < \Delta_1 < 1 < \Delta_2$ *and* $\theta > 1$. *Then there exist an integer* $N_0 \geq 1$ *and a neighborhood* $\mathcal{U}$ *of* A *in* $\mathcal{A}$ *such that for each* $B \in \mathcal{U}$, $x \in \langle \Delta_1 \eta, \Delta_2 \eta \rangle$ *and each integer* $n \geq N_0$,

$$r\big(x_A, B^n x\big) \leq \theta \lambda\big(x_A, B^n x\big). \tag{2.332}$$

Proof We may assume that

$$\Delta_2 > \theta \quad \text{and} \quad \theta \Delta_1 < \lambda(\eta, x_A). \tag{2.333}$$

Choose $\theta_0 > 1$ such that

$$\theta_0^2 < \theta. \tag{2.334}$$

By Corollary 2.58, there exist an integer $N_0 \geq 1$ and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $x \in \langle \Delta_1 \eta, \Delta_2 \eta \rangle$, each $B \in \mathcal{U}$ and each integer $n \in [N_0, 8N_0 + 8]$,

$$B^n x \in \big\langle \theta_0^{-1} \alpha_A^n f_A(x) x_A, \theta_0 \alpha_A^n f_A(x) x_A \big\rangle. \tag{2.335}$$

Assume that $B \in \mathcal{U}$ and $x \in \langle \Delta_1 \eta, \Delta_2 \eta \rangle$. By the definition of $\mathcal{U}$ and N_0, the inclusion (2.335) is valid for each integer $n \in [N_0, 8N_0 + 8]$. The relations (2.335) and (2.334) imply that for each integer $n \in [N_0, 8N_0 + 8]$,

$$r\big(x_A, B^n x\big) \leq \theta_0 \alpha_A^n f_A(x), \qquad \lambda\big(x_A, B^n x\big) \geq \theta_0^{-1} \alpha_A^n f_A(x)$$

and

$$r\big(x_A, B^n x\big) \leq \theta_0^2 \lambda\big(x_A, B^n x\big) \leq \theta \lambda\big(x_A, B^n x\big).$$

It remains to be shown that (2.332) is valid for all integers $n > 8N_0 + 8$.

Assume the contrary. Then there exists an integer

$$N_1 > 8N_0 + 8 \tag{2.336}$$

such that

$$r\big(x_A, B^n x\big) \leq \theta \lambda\big(x_A, B^n x\big) \quad \text{for all integers } n \in [N_0, N_1 - 1] \tag{2.337}$$

and

$$r\big(x_A, B^{N_1} x\big) > \theta \lambda\big(x_A, B^{N_1} x\big). \tag{2.338}$$

Consider the vector $B^{N_1 - N_0} x$. By (2.336) and (2.337), we see that

$$r\big(x_A, B^{N_1 - N_0} x\big) \leq \theta \lambda\big(x_A, B^{N_1 - N_0} x\big) \tag{2.339}$$

and

$$\theta^{-1} r\big(x_A, B^{N_1 - N_0} x\big) x_A \leq B^{N_1 - N_0} x \leq r\big(x_A, B^{N_1 - N_0} x\big) x_A.$$

By (2.338),

$$r\big(x_A, B^{N_1-N_0}x\big) > 0. \tag{2.340}$$

It follows from (2.339), (2.340), (2.331) and (2.333) that

$$r\big(x_A, B^{N_1-N_0}x\big)^{-1}B^{N_1-N_0}x \in \big\langle\theta^{-1}x_A, x_A\big\rangle \subset \big\langle\theta^{-1}\lambda(\eta, x_A)\eta, \eta\big\rangle \subset \langle\Delta_1\eta, \Delta_2\eta\rangle.$$

It follows from this relation and the definition of $\mathcal{U}$ and N_0 (see (2.335)) that

$$r\big(x_A, B^{N_1-N_0}x\big)^{-1}B^{N_1}x \in \big\langle\theta_0^{-1}\alpha_A^{N_0}f_A(x)x_A, \theta_0\alpha_A^{N_0}f_a(x)x_A\big\rangle,$$

$$r\big(x_A, B^{N_1}x\big) \le \theta_0\alpha_A^{N_0}f_A(x)r\big(x_A, B^{N_1-N_0}x\big),$$

$$\lambda\big(x_A, B^{N_1}x\big) \ge \theta_0^{-1}\alpha_A^{N_0}f_A(x)r\big(x_A, B^{N_1-N_0}x\big),$$

and by (2.333),

$$r\big(x_A, B^{N_1}x\big) \le \theta\lambda\big(x_A, B^{N_1}x\big),$$

an inequality which contradicts (2.338). Thus (2.332) is indeed valid for all $n \ge N_0$ and Lemma 2.59 is proved. □

Lemma 2.60 *Let $\gamma > 1$. Then there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{A}_{reg} \cap \mathcal{U}$, the inequalities $\gamma^{-1}x_A \le x_B \le \gamma x_A$ hold.*

Proof Choose a positive number $\theta > 1$ such that

$$\theta^2 < \gamma. \tag{2.341}$$

By Lemma 2.59, there exists an integer $N_0 \ge 1$ and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$ and each integer $n \ge N_0$,

$$r\big(x_A, B^n\eta\big) \le \theta\lambda\big(x_A, B^n\eta\big). \tag{2.342}$$

Assume that $B \in \mathcal{A}_{reg} \cap \mathcal{U}$. Then

$$\lim_{n\to\infty} \alpha_B^{-n}B^n\eta = f_B(\eta)x_B. \tag{2.343}$$

By the definition of $\mathcal{U}$ and N_0, (2.342) is valid for each integer $n \ge N_0$. This implies that for each integer $n \ge N_0$,

$$\alpha_B^{-n}\lambda\big(x_A, B^n\eta\big)x_A \le \alpha_B^{-n}B^n\eta \le \alpha_B^{-n}r\big(x_A, B^n\eta\big)x_A$$

and

$$r\big(x_A, \alpha_B^{-n}B^n\eta\big) \le \theta\lambda\big(x_A, \alpha_B^{-n}B^n\eta\big).$$

When combined with (2.343), this implies that

$$r\big(x_A, f_B(\eta)x_B\big) \le \theta^2\lambda\big(f_B(\eta)x_B, x_A\big) \quad \text{and} \quad r(x_A, x_B) \le \theta^2\lambda(x_A, x_B). \tag{2.344}$$

It follows from (2.331), (2.334) and (2.341) that

$$\lambda(x_A, x_B)x_A \le x_B \le r(x_A, x_B)x_A \le r(x_A, x_B)\eta,$$
$$x_A \le \lambda(x_A, x_B)^{-1}x_B \le \lambda(x_A, x_B)^{-1}\eta, \qquad r(x_A, x_B) \ge 1, \qquad \lambda(x_A, x_B)^{-1} \ge 1,$$
$$r(x_A, x_B) \le \theta^2, \qquad \lambda(x_A, x_B) \ge \theta^{-2}$$

and finally, that

$$\gamma^{-1}x_A \le \theta^{-2}x_A \le x_B \le \theta^2 x_A \le \gamma x_A.$$

Lemma 2.60 is proved. □

Lemma 2.61 *Let $\theta > 1$ and $\Delta \in (0,1)$. Then there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$, $z \in X_+$ and $\alpha > 0$ satisfying*

$$\|z\|_\eta = 1, \qquad z \ge \Delta\eta \quad \text{and} \quad Bz = \alpha z, \tag{2.345}$$

the following inequalities hold: $\theta^{-1}x_A \le z \le \theta x_A$.

Proof By Lemma 2.59, there exists an integer $N_0 \ge 1$ and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$, each integer $n \ge N_0$ and for each $x \in \langle 4^{-1}\Delta\eta, 4\eta\rangle$,

$$r\big(x_A, B^n x\big) \le \theta\lambda\big(x_A, B^n x\big). \tag{2.346}$$

Assume that $B \in \mathcal{U}$, $z \in X_+$, $\alpha > 0$ and that (2.345) is valid. By (2.345) and the definition of $\mathcal{U}$ and N_0 (see (2.346)), for each integer $n \ge N_0$,

$$\begin{aligned} \alpha^n r(x_A, z) &= r\big(x_A, \alpha^n z\big) = r\big(x_A, B^n z\big) \le \theta\lambda\big(x_A, B^n z\big) \\ &= \theta\lambda\big(x_A, \alpha^n z\big) = \alpha^n\theta\lambda(x_A, z) \quad \text{and} \quad r(x_A, z) \le \theta\lambda(x_A, z). \end{aligned} \tag{2.347}$$

It follows from (2.345), (2.331) and (2.347) that

$$\lambda(x_A, z)x_A \le z \le r(x_A, z)x_A \le r(x_A, z)\eta, \qquad r(x_A, z) \ge 1,$$
$$x_A \le \lambda(x_A, z)^{-1}z \le \lambda(x_A, z)^{-1}\eta, \qquad \lambda(x_A, z) \le 1,$$
$$r(x_A, z) \le \theta, \qquad \lambda(x_A, z) \ge \theta^{-1}$$

and finally, that $\theta^{-1}x_A \le z \le \theta x_A$. This completes the proof of Lemma 2.61. □

Lemma 2.62 *Let $\varepsilon \in (0,1)$ and $\Delta \in (0,1)$. Then there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$, $z \in X_+$ and $\alpha > 0$ satisfying*

$$\|z\|_\eta = 1, \qquad z \ge \Delta\eta \quad \text{and} \quad Bz = \alpha z, \tag{2.348}$$

we have $|\alpha - \alpha_A| \le \varepsilon$.

Proof Choose a number $\gamma > 1$ for which

$$(\alpha_A + 1)(\gamma - 1) \le \varepsilon/8.$$

By Lemma 2.61, there exists a neighborhood $\mathcal{U}_1$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}_1$, $z \in X_+$ and $\alpha > 0$ satisfying (2.348), the following inequalities hold:

$$\gamma^{-1} x_A \le z \le \gamma x_A. \tag{2.349}$$

There exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}$,

$$\|Ay - By\|_\eta \le \varepsilon/8 \quad \text{for all } y \in \gamma\langle -\eta, \eta\rangle. \tag{2.350}$$

Assume that $B \in \mathcal{U}$, $z \in X_+$, $\alpha > 0$ and that (2.348) is true. Then by the definition of $\mathcal{U}_1$, (2.349) holds.

It follows from (2.348) and (2.331) that

$$\begin{aligned} |\alpha - \alpha_A| = \big|\|\alpha z\|_\eta - \|\alpha_A x_A\|_\eta\big| &\le \|\alpha z - \alpha_A x_A\|_\eta = \|Bz - Ax_A\|_\eta \\ &\le \|Ax_A - Az\|_\eta + \|Az - Bz\|_\eta. \end{aligned} \tag{2.351}$$

By our choice of γ, (2.349) and (2.331),

$$\begin{aligned} (1-\gamma)\alpha_A\eta \le (1-\gamma)\alpha_A x_A = A(1-\gamma)x_A &\le Ax_A - Az \\ \le \big(1-\gamma^{-1}\big)Ax_A &\le (\gamma - 1)\alpha_A \eta \end{aligned}$$

and

$$\|Ax_A - Az\|_\eta \le \varepsilon/8. \tag{2.352}$$

It follows from (2.349) and (2.350) that

$$z \le \gamma x_A \le \gamma\eta \quad \text{and} \quad \|Az - Bz\|_\eta \le 8^{-1}\varepsilon.$$

When combined with (2.351) and (2.352), this implies that $|\alpha_A - \alpha| \le \varepsilon$. Lemma 2.62 is proved. □

Lemmas 2.62 and 2.60 imply the following result.

Lemma 2.63 *Let $\varepsilon \in (0, 1)$. Then there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{A}_{reg} \cap \mathcal{U}$ we have $|\alpha_B - \alpha_A| \le \varepsilon$.*

2.14 Proofs of Theorems 2.49 and 2.50

In this section we prove Lemma 2.64. Theorem 2.50 follows when this lemma is combined with Lemmas 2.61 and 2.62. Theorem 2.49 is a consequence of Lemmas 2.60, 2.63 and 2.64.

Lemma 2.64 *Let $A \in \mathcal{A}$ be regular and let ε and Δ belong to the interval* $(0, 1)$. *Then there exist an integer $N \geq 1$ and a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ such that the following assertion holds*:

If

$$\begin{aligned} B \in \mathcal{U}, \qquad & x_0 \in \operatorname{int}(X_+), \\ \Delta\eta \leq x_0 \leq \eta, \qquad & \alpha_0 > 0 \quad \textit{and} \quad \alpha_0 x_0 = B x_0, \end{aligned} \tag{2.353}$$

then for each $x \in \langle -\eta, \eta \rangle$ and each integer $n \geq N$,

$$\left\| \alpha_0^{-n} B^n x - f_A(x) x_A \right\|_\eta \leq \varepsilon. \tag{2.354}$$

Proof Choose a positive number ε_0 for which

$$8\varepsilon_0 < 4^{-1} \varepsilon \Delta.$$

By Lemma 2.56, there exist a neighborhood $\mathcal{U}_1$ of A in $\mathcal{A}$ and an integer $N \geq 1$ such that for each $B \in \mathcal{U}_1$,

$$\left\| \alpha_A^{-N} B^N x - f_A(x) x_A \right\|_\eta \leq 16^{-1} \varepsilon_0 \quad \text{for all } x \in \langle -\eta, \eta \rangle. \tag{2.355}$$

There exists a number $c_1 > 1$ such that

$$\left\| B^N x \right\|_\eta \leq c_1 \quad \text{for } x \in \langle -\eta, \eta \rangle \quad \text{and} \quad B \in \mathcal{U}_1, \quad \text{and} \quad f_A(\eta) \leq c_1. \tag{2.356}$$

There exists a number $\delta_1 \in (0, \min\{1, \alpha_A/8\})$ such that

$$\left| \alpha^{-N} - \alpha_A^{-N} \right| c_1 \leq 16^{-1} \varepsilon_0 \quad \text{for each } \alpha \text{ satisfying } |\alpha - \alpha_A| \leq \delta_1. \tag{2.357}$$

By Lemmas 2.62 and 2.61 there exists a neighborhood $\mathcal{U}_2$ of A in $\mathcal{A}$ such that for each $B \in \mathcal{U}_2$, $z \in X_+$ and $\alpha > 0$ satisfying $\Delta\eta \leq z \leq \eta$ and $Bz = \alpha z$, the following inequalities are true:

$$|\alpha - \alpha_N| \leq \delta_1 \quad \text{and} \quad \|z - x_A\|_\eta \leq 16^{-1} \varepsilon_0 c_1^{-1}. \tag{2.358}$$

Set

$$\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2. \tag{2.359}$$

Assume that $B \in \mathcal{U}$, $x_0 \in X_+$, $\alpha_0 > 0$ and that (2.353) holds. By the definition of $\mathcal{U}_1$ and N, (2.355) holds. It follows from the definition of $\mathcal{U}_2$ (see (2.358)) and (2.353) that $|\alpha_0 - \alpha_N| \leq \delta_1$. By the latter inequality, (2.357), (2.356) and (2.355),

$$\left\| \alpha_0^{-N} B^N x - f_A(x) x_A \right\|_\eta \leq 8^{-1} \varepsilon_0 \quad \text{for all } x \in \langle -\eta, \eta \rangle. \tag{2.360}$$

By the definition of $\mathcal{U}_2$ (see (2.358)) and (2.353),

$$\|x_0 - x_A\|_\eta \leq 16^{-1} \varepsilon_0 c_1^{-1}. \tag{2.361}$$

This inequality, when combined with (2.360) and (2.356), implies that

$$\left\|\alpha_0^{-N} B^N x - f_A(x)x_0\right\|_\eta \le 8^{-1}\varepsilon_0 + 16^{-1}\varepsilon_0 \quad \text{for all } x \in \langle -\eta, \eta\rangle. \tag{2.362}$$

By (2.362) and (2.353), we have

$$\alpha_0^{-N} B^N x - f_A(x)x_0 \in \varepsilon_0\big(8^{-1} + 16^{-1}\big)\langle -\eta, \eta\rangle \subset \varepsilon_0\big(8^{-1} + 16^{-1}\big)\Delta^{-1}\langle -x_0, x_0\rangle$$

for all $x \in \langle -\eta, \eta\rangle$.

It follows from this relation and (2.353) that for each $x \in \langle -\eta, \eta\rangle$ and each integer $n \ge N$,

$$\begin{aligned}\alpha_0^{-n} B^n x - f_A(x)x_0 &= \alpha_0^{-n+N} B^{n-N}\big[\alpha_0^{-N} B^N x - f_A(x)x_0\big]\\ &\subset \varepsilon_0\big(8^{-1} + 16^{-1}\big)\Delta^{-1}\alpha_0^{N-n} B^{n-N}\langle -x_0, x_0\rangle\\ &\subset \varepsilon_0\big(8^{-1} + 16^{-1}\big)\Delta^{-1}\langle -x_0, x_0\rangle\\ &\subset \varepsilon_0\big(8^{-1} + 16^{-1}\big)\Delta^{-1}\langle -\eta, \eta\rangle\end{aligned}$$

and

$$\left\|\alpha_0^{-n} B^n x - f_A(x)x_0\right\|_\eta \le \Delta^{-1}\varepsilon_0/4.$$

When combined with (2.361), (2.356) and (2.354), this implies that for each $x \in \langle -\eta, \eta\rangle$ and each integer $n \ge N$,

$$\left\|\alpha_0^{-n} B^n x - f_A(x)x_A\right\|_\eta \le \Delta^{-1}\varepsilon_0/4 + 16^{-1}\varepsilon_0 < \varepsilon.$$

Lemma 2.64 is proved. □

2.15 Proof of Theorem 2.51

It follows from Lemma 2.55 and Corollary 2.53 that $\mathcal{A}_{qreg} \subset \bar{\mathcal{A}}_{reg}$. This clearly implies that $\bar{\mathcal{A}}_{reg} = \bar{\mathcal{A}}_{qreg}$.

To construct the set $\mathcal{F}$ we let $A \in \mathcal{A}_{reg}$,

$$\begin{gathered} x_A \in \operatorname{int}(X_+), \qquad f_A \in X'_+ \setminus \{0\}, \qquad \alpha_A > 0,\\ A x_A = \alpha_A x_A, \qquad f_A \cdot A = \alpha_A \cdot f_A, \\ \alpha_A^{-n} A^n x \to f_A(x)x_A \quad \text{as } n \to \infty, \text{ uniformly on } \langle \eta, \eta\rangle. \end{gathered} \tag{2.363}$$

Let $i \ge$ be an integer. By Lemmas 2.60 and 2.63, Theorem 2.49, Lemmas 2.61 and 2.62, and Theorem 2.50, there exist a number $r(A, i) \in (0, 4^{-i})$ and an integer $N(A, i) \ge 1$ such that the following two assertions hold:

1. Assume that $B \in \mathcal{A}_{reg}$ and $\rho(A,B) < r(A,i)$. Then

$$\left(1-4^{-i}\right)x_A \le x_B \le \left(1+4^{-i}\right)x_A, \qquad |\alpha_A - \alpha_B| \le 4^{-i}\min\{1,\alpha_A\}$$

and

$$\left\|\alpha_B^{-n}B^n x - f_A(x)x_A\right\|_\eta \le 4^{-i} \quad \text{for all } x \in \langle -\eta, \eta\rangle \text{ and each integer } n \ge N.$$

2. Assume that $B \in \mathcal{A}$, $\rho(A,B) < r(A,i)$, $x_0 \in X_+$, $\alpha_0 > 0$, $\alpha_0 x_0 = Bx_0$ and $4^{-1}x_A \le x_0 \le \eta$. Then

$$\left(1-4^{-i}\right)x_A \le x_0 \le \left(1+4^{-i}\right)x_A, \qquad |\alpha_A - \alpha_0| \le 4^{-i}\min\{1,\alpha_A\}$$

and

$$\left\|\alpha_0^{-n}B^n x - f_A(x)x_A\right\|_\eta \le 4^{-i}$$

$$\text{for all } x \in \langle -\eta, \eta\rangle \text{ and each integer } n \ge N(A,i).$$

Now set

$$\mathcal{U}(A,i) = \left\{B \in \mathcal{A} : \rho(B,A) < r(A,i)\right\} \tag{2.364}$$

and define

$$\mathcal{F} = \left[\bigcap_{i=1}^{\infty}\bigcup\left\{\mathcal{U}(A,i) : A \in \mathcal{A}_{reg}\right\}\right] \cap \bar{\mathcal{A}}_{reg}. \tag{2.365}$$

Evidently, $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of $\bar{\mathcal{A}}_{reg}$.

It remains to be shown that $\mathcal{F} \subset \mathcal{A}_{reg}$. To this end, assume that $B \in \mathcal{F}$. There exist $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}_{reg}$ and a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that

$$B \in \mathcal{U}(A_k, i_k) \quad \text{and} \quad \mathcal{U}(A_{k+1}, i_{k+1}) \subset \mathcal{U}(A_k, i_k), \quad k = 1, 2, \ldots. \tag{2.366}$$

Let $k \ge 1$. It follows from assertion 1 and (2.366) that for each integer $j \ge 1$,

$$\left(1-4^{-i_k}\right)x_{A_k} \le x_{A_{k+j}} \le \left(1+4^{-i_k}\right)x_{A_k} \tag{2.367}$$

and

$$|\alpha_{A_k} - \alpha_{A_{k+j}}| \le 4^{-i_k}\min\{1, \alpha_{A_k}\}.$$

It is clear that both $\{x_{A_p}\}_{p=1}^{\infty}$ and $\{\alpha_{A_p}\}_{p=1}^{\infty}$ are Cauchy sequences. Therefore there exist the limits

$$x_* = \lim_{s\to\infty} x_{A_s}, \qquad \alpha_* = \lim_{s\to\infty} \alpha_{A_s}. \tag{2.368}$$

Set

$$\lambda_* = \inf\left\{\lambda(x_{A_k}, \eta) : k = 1, 2, \ldots\right\}. \tag{2.369}$$

By (2.367), λ_* is positive. By (2.367) and (2.368),

$$\begin{aligned} &\left(1-4^{-i_k}\right)x_{A_k} \le x_* \le \left(1+4^{-i_k}\right)x_{A_k}, \\ &|\alpha_{A_k}-\alpha_*| \le 4^{-i_k}\min\{1,\alpha_{A_k}\}, \quad x_* \le \eta. \end{aligned} \tag{2.370}$$

By (2.368) and (2.366),

$$Bx_* = B\Big(\lim_{k\to\infty} x_{A_k}\Big) = \lim_{k\to\infty} A_k x_{A_k} = \lim_{k\to\infty} \alpha_{A_k} x_{A_k} = \alpha_* x_*. \tag{2.371}$$

Let $k \ge 1$ be an integer. It follows from assertion 2, (2.366), (2.370) and (2.371) that

$$\begin{aligned} &\left\|\alpha_*^{-n}B^n x - f_{A_k}(x)x_{A_k}\right\|_\eta \le 4^{-i_k} \quad \text{for all } x \in \langle -\eta, \eta\rangle \\ &\text{and each integer } n \ge N(A_k, i_k). \end{aligned} \tag{2.372}$$

Note that (see (2.363) and (2.369))

$$x_{A_k} = f_{A_k}(x_{A_k})x_{A_k}, \qquad f_{A_k}(x_{A_k}) = 1$$

and

$$f_{A_k}(\eta) \le f_{A_k}(x_{A_k}) \cdot \lambda_*^{-1} = \lambda_*^{-1}.$$

When combined with (2.372) and (2.370), this implies that

$$\left\|\alpha_*^{-n}B^n x - f_{A_k}(x)x_A\right\|_\eta \le 4^{-i_k} + 4^{-i_k}\lambda_*^{-1} \tag{2.373}$$

for all $x \in \langle -\eta, \eta\rangle$ and each integer $n \ge N(A_k, i_k)$. Since k is an arbitrary natural number, we obtain that for each $x \in X$, there exists

$$\lim_{n\to\infty} \alpha_*^{-n}B^n x = f_B(x)x_*, \tag{2.374}$$

where $f_B \in X'_+$. It follows from (2.373) and (2.374) that for each integer $k \ge 1$, each integer $n \ge N(A_k, i_k)$ and each $x \in \langle -\eta, \eta\rangle$,

$$\left\| f_B(x)x_* - f_{A_k}(x)x_*\right\|_\eta \le 4^{-i_k} + 4^{-i_k}\lambda_*^{-1}$$

and

$$\left\|\alpha_*^{-n}B^n x - f_B(x)x_*\right\|_\eta \le 2\left(4^{-i_k} + 4^{-i_k}\lambda_*^{-1}\right).$$

Therefore $B \in \mathcal{A}_{reg}$ and Theorem 2.51 is established.

2.16 Convergence of Inexact Orbits for a Class of Operators

In this section we exhibit a class of nonlinear operators with the property that their iterates converge to their unique fixed points even when computational errors are present. We also show that most (in the sense of Baire category) elements in an appropriate complete metric space of operators do, in fact, possess this property.

Assume that (X, ρ) is a complete metric space and let the operator $A : X \to X$ have the following properties:

(A1) there exists a unique $x_A \in X$ such that $Ax_A = x_A$;
(A2) $A^n x \to x_A$ as $n \to \infty$, uniformly on all bounded subsets of X;
(A3) A is uniformly continuous on bounded subsets of X;
(A4) A is bounded on bounded subsets of X.

Many operators with these properties can be found, for example, in [23, 33, 50, 85, 108, 114, 126, 127, 137]. We mention, in particular, the classes of operators introduced by Rakotch [114] and Browder [23]. Note that if X is either a closed and convex subset of a Banach space or a closed and ρ-convex subset of a complete hyperbolic metric space [124], then (A4) follows from (A3).

In view of (A2), it is natural to ask if the convergence of the orbits of A will be preserved even in the presence of computational errors. In this section we provide affirmative answers to this question. More precisely, we have the following results which were obtained in [35].

Theorem 2.65 *Let K be a nonempty, bounded subset of X and let $\varepsilon > 0$ be given. Then there exist $\delta = \delta(\varepsilon, K) > 0$ and a natural number N such that for each natural number $n \geq N$, and each sequence $\{x_i\}_{i=0}^n \subset X$ which satisfies*

$$x_0 \in K \quad \textit{and} \quad \rho(Ax_i, x_{i+1}) \leq \delta, \quad i = 0, \ldots, n-1,$$

the following inequality holds:

$$\rho(x_i, x_A) \leq \varepsilon, \quad i = N, \ldots, n.$$

Corollary 2.66 *Assume that $\{x_i\}_{i=0}^\infty \subset X$, $\{x_i\}_{i=0}^\infty$ is bounded, and that*

$$\lim_{i \to \infty} \rho(Ax_i, x_{i+1}) = 0.$$

Then $\rho(x_i, x_A) \to 0$ as $i \to \infty$.

Theorem 2.67 *Let $\varepsilon > 0$ be given. Then there exists $\delta = \delta(\varepsilon) > 0$ such that for each sequence $\{x_i\}_{i=0}^\infty \subset X$ which satisfies*

$$\rho(x_0, x_A) \leq \delta \quad \textit{and} \quad \rho(x_{i+1}, Ax_i) \leq \delta, \quad i = 0, 1, \ldots,$$

the following inequality holds:

$$\rho(x_i, x_A) \leq \varepsilon, \quad i = 0, 1, \ldots.$$

These results show that, roughly speaking, in order to achieve an ε-approximation of x_A, it suffices to compute *inexact orbits* of A, that is, sequences $\{x_i\}_{i=0}^\infty$ such that

$$x_0 \in X \quad \text{and} \quad \rho(x_{i+1}, Ax_i) \le \delta \quad \text{for any } i \ge 0,$$

where δ is a sufficiently small positive number.

However, sometimes the operator A is not given explicitly and only some approximation of it, B_i, is available at each step i of the inexact orbit computing procedure. The next result shows that for certain operators A, the procedure of approximating x_A by inexact orbits is stable in the sense that, even in this case, the orbits determined by the sequence of operators B_i approach x_A provided that each B_i is a sufficiently accurate approximation of A in the topology of uniform convergence on bounded subsets of X. To be precise, we set, for each $x \in X$ and $E \subset X$,

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}.$$

Denote by $\mathcal{A}$ the set of all self-mappings $A : X \to X$ which have properties (A3) and (A4). Fix $\theta \in X$. For each natural number n, set

$$E_n = \{(A, B) \in \mathcal{A} \times \mathcal{A} : \rho(Ax, Bx) \le 1/n \text{ for all } x \in B(\theta, n)\}. \tag{2.375}$$

We equip the set $\mathcal{A}$ with the uniformity determined by the base E_n, $n = 1, 2, \ldots$. This uniformity is metrizable by a complete metric.

Denote by $\mathcal{A}_{reg}$ the set of all mappings $A \in \mathcal{A}$ which satisfy (A1) and (A2), and by $\bar{\mathcal{A}}_{reg}$ the closure of $\mathcal{A}_{reg}$ in $\mathcal{A}$.

Theorem 2.68 *Assume that $A \in \mathcal{A}_{reg}$ and x_A is a fixed point of A. Let $m, \varepsilon > 0$ be given. Then there exist a neighborhood $\mathcal{U}$ of A in $\mathcal{A}$ and a natural number N such that for each $x \in B(\theta, m)$, each integer $n \ge N$, and each sequence $\{B_i\}_{i=1}^n \subset \mathcal{U}$,*

$$\rho(B_i \cdots B_1 x, x_A) \le \varepsilon \quad \textit{for } i = N, \ldots, n.$$

As a matter of fact, it turns out that the stability property established in this theorem is generic. That is, it holds for most (in the sense of Baire category) operators in the closure of $\mathcal{A}_{reg}$.

Theorem 2.69 *The set $\mathcal{A}_{reg}$ contains an everywhere dense G_δ subset of $\bar{\mathcal{A}}_{reg}$.*

2.17 Proofs of Theorem 2.65 and Corollary 2.66

We first prove Theorem 2.65. To this end, set, for $x \in X$ and $r > 0$,

$$B(x, r) = \{y \in X : \rho(x, y) \le r\}.$$

We may assume without loss of generality that

$$\varepsilon \le 1 \quad \text{and} \quad B(x_A, 4) \subset K. \tag{2.376}$$

By (A2), there exists a natural number $N \geq 4$ such that

$$\rho\big(A^n x, x_A\big) \leq \varepsilon/4 \quad \text{for all integers } n \geq N \text{ and all } x \in K. \tag{2.377}$$

By (A4), the set $A^m(K)$ is bounded for all natural numbers m. Hence there exists a positive number $S > 0$ such that

$$A^i(K) \subset B(x_A, S), \quad i = 0, \dots, 2N. \tag{2.378}$$

(Here we use the convention that A^0 is the identity operator.) By induction and (A3), we define a finite sequence of positive numbers $\{\gamma_i\}_{i=0}^{2N}$ so that

$$\gamma_{2N} = \varepsilon/4$$

and, for each $i = 0, 1, \dots, 2N-1$,

$$\gamma_i \leq \gamma_{i+1} \tag{2.379}$$

and

$$\rho(Ax, Ay) \leq 2^{-1}\gamma_{i+1} \quad \text{for all } x, y \in B(x_A, S+4) \quad \text{with} \quad \rho(x, y) \leq \gamma_i. \tag{2.380}$$

Set

$$\delta = \gamma_0/2. \tag{2.381}$$

First, we prove the following auxiliary result.

Lemma 2.70 *Suppose that* $\{z_i\}_{i=0}^{2N} \subset X$ *satisfies*

$$z_0 \in K \quad \textit{and} \quad \rho(z_{i+1}, Az_i) \leq \delta, \quad i = 0, \dots, 2N-1. \tag{2.382}$$

Then

$$\rho(z_i, x_A) \leq \varepsilon, \quad i = N, \dots, 2N.$$

Proof We will show that for $i = 1, \dots, 2N$,

$$\rho\big(z_i, A^i z_0\big) \leq \gamma_i. \tag{2.383}$$

Clearly, (2.383) holds for $i = 1$ by (2.382) and (2.381).

Assume that $i \in \{2, \dots, 2N\}$ and

$$\rho\big(z_{i-1}, A^{i-1} z_0\big) \leq \gamma_{i-1}. \tag{2.384}$$

Then (2.382) implies that

$$\begin{aligned} \rho\big(z_i, A^i z_0\big) &\leq \rho(z_i, Az_{i-1}) + \rho\big(Az_{i-1}, A\big(A^{i-1} z_0\big)\big) \\ &\leq \delta + \rho\big(Az_{i-1}, A\big(A^{i-1} z_0\big)\big). \end{aligned} \tag{2.385}$$

It follows from the definition of γ_{i-1} (see (2.379)), (2.384), (2.382) and (2.378) that

$$A^{i-1}z_0, z_{i-1} \in B(x_A, S+1).$$

By these inclusions, the definition of γ_{i-1} (see (2.380) with $j = i-1$) and (2.384),

$$\rho\big(A\big(A^{i-1}z_0\big), Az_{i-1}\big) \le \gamma_i/2.$$

When combined with (2.385) and (2.381), this inequality implies that

$$\rho\big(z_i, A^i z_0\big) \le \delta + \gamma_i/2 \le \gamma_i.$$

Therefore (2.383) is valid for all $i \in \{1, \dots, 2N\}$. Together with (2.377), (2.379), (2.382) and (2.383), this last inequality implies that for all $i \in \{N, \dots, 2N\}$, we have

$$\rho(z_i, x_A) \le \rho\big(z_i, A^i z_0\big) + \rho\big(A^i z_0, x_A\big) \le \gamma_i + \varepsilon/4 \le \varepsilon/2.$$

Lemma 2.70 is proved. □

Now we are ready to complete the proof of Theorem 2.65.

To this end, assume that $n \ge N$ is a natural number and that the sequence $\{x_i\}_{i=0}^n \subset X$ satisfies

$$x_0 \in K \quad \text{and} \quad \rho(Ax_i, x_{i+1}) \le \delta, \quad i = 0, \dots, n-1.$$

We will show that

$$\rho(x_i, x_A) \le \varepsilon, \quad i = N, \dots, n. \tag{2.386}$$

If $n \le 2N$, then (2.386) follows from Lemma 2.70. Therefore we may confine our attention to the case where $n > 2N$. Again by Lemma 2.70,

$$\rho(x_i, x_A) \le \varepsilon, \quad i = N, \dots, 2N. \tag{2.387}$$

Assume by way of contradiction that there exists an integer $q \in (2N, n]$ such that

$$\rho(x_q, x_A) > \varepsilon. \tag{2.388}$$

In view of (2.387), we may assume without loss of generality that

$$\rho(x_i, x_A) \le \varepsilon, \quad i \in \{2N, \dots, q-1\}. \tag{2.389}$$

Define $\{z_i\}_{i=0}^{2N} \subset X$ by

$$z_i = x_{i+q-N}, \quad i = 0, \dots, N, \qquad z_{i+1} = Az_i, \quad i = N, \dots, 2N-1. \tag{2.390}$$

We will show that the sequence $\{z_i\}_{i=0}^{2N}$ satisfies (2.382). To meet this goal, we only need to show that $z_0 \in K$. By (2.390), (2.389) and (2.387),

$$z_0 = x_{q-N} \quad \text{and} \quad \rho(z_0, x_A) \le \varepsilon.$$

The last inequality and (2.376) imply that $z_0 \in K$. Therefore (2.382) holds. It now follows from Lemma 2.70 and (2.390) that

$$\rho(x_A, x_q) = \rho(x_A, z_N) \leq \varepsilon.$$

This, however, contradicts (2.388). The contradiction we have reached proves (2.386) and this completes the proof of Theorem 2.65.

Finally, we are going to prove Corollary 2.66.

Set $K = \{x_n : n = 0, 1, \ldots\}$ and let $\varepsilon > 0$ we given. Let $\delta > 0$ and a natural number N be as guaranteed by Theorem 2.65. There exists a natural number j such that for each integer $i \geq j$, we have $\rho(Ax_i, x_{i+1}) \leq \delta$. It follows from the last inequality and the choice of δ that $\rho(x_i, x_A) \leq \varepsilon$ for all integers $i \geq j + N$. Since ε is an arbitrary positive number, this implies that $\lim_{i\to\infty} x_i = x_A$. The proof of Corollary 2.66 is complete.

Corollary 2.66 provides a partial answer to a question raised in [77] in the wake of Theorem 1 of [75], which is also concerned with the stability of iterations.

2.18 Proof of Theorem 2.67

We may assume without loss of generality that $\varepsilon \leq 1$. By Theorem 2.65, there exist a natural number N and a real number $\delta_0 \in (0, \varepsilon)$ such that the following property holds.

(P1) For each natural number $n \geq N$ and each sequence $\{y_i\}_{i=0}^n \subset X$ which satisfies

$$y_0 \in B(x_A, 4) \quad \text{and} \quad \rho(y_{i+1}, Ay_i) \leq \delta_0, \quad i = 0, \ldots, n-1, \tag{2.391}$$

the following inequality holds:

$$\rho(y_i, x_A) \leq \varepsilon, \quad i = N, \ldots, n. \tag{2.392}$$

By property (A4), the set $A^i(B(x_A, 4))$ is bounded for any integer $i \geq 1$. Choose a number $s > 1$ such that

$$\bigcup_{i=0}^{N} A^i\big(B(x_A, 4)\big) \subset B(x_A, s). \tag{2.393}$$

By induction and (A3), we define a finite sequence of positive numbers $\{\gamma_i\}_{i=0}^N$ so that

$$\begin{aligned} &\gamma_i \leq 1, \quad i = 0, \ldots, N, \\ &\gamma_N \leq \delta_0/4, \qquad \gamma_i \leq \gamma_{i+1}, \quad i = 0, \ldots, N-1, \end{aligned} \tag{2.394}$$

and for each $j \in \{0, \ldots, N-1\}$,

$$\begin{aligned} &\rho(Ax, Ay) \leq 2^{-1}\gamma_{j+1} \quad \text{for all } x, y \in B(x_A, s+4) \\ &\quad \text{with} \quad \rho(x, y) \leq \gamma_j. \end{aligned} \tag{2.395}$$

Set

$$\delta = \gamma_0/4. \tag{2.396}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$,

$$\rho(x_0, x_A) \le \delta \quad \text{and} \quad \rho(x_{i+1}, Ax_i) \le \delta, \quad i = 0, 1, \dots. \tag{2.397}$$

We will show that

$$\rho(x_i, x_A) \le \varepsilon \tag{2.398}$$

for all integers $i \ge 0$. By (2.397), (2.396) and (P1), inequality (2.398) holds for all integers $i \ge N$. Therefore we only need to prove (2.398) for $i < N$. Clearly, (2.398) holds for $i = 0$.

We will show that for $i = 0, \dots, N$, we have

$$\rho(x_i, x_A) = \rho\big(x_i, A^i x_A\big) \le \gamma_i. \tag{2.399}$$

By (2.397) and (2.396), this is true for $i = 0$. Assume that $i \in \{1, \dots, N\}$ and

$$\rho\big(x_{i-1}, A^{i-1} x_A\big) = \rho(x_{i-1}, x_A) \le \gamma_{i-1}. \tag{2.400}$$

Then (2.397) implies that

$$\rho(x_i, x_A) \le \rho(x_i, Ax_{i-1}) + \rho(Ax_{i-1}, x_A) \le \delta + \rho(Ax_{i-1}, x_A). \tag{2.401}$$

It follows from (2.400) and (2.394) that

$$x_{i-1} \in B(x_A, s). \tag{2.402}$$

By (2.402), (2.400) and the definition of γ_{i-1} (see (2.395) with $j = i - 1$),

$$\rho(Ax_{i-1}, x_A) \le 2^{-1}\gamma_i. \tag{2.403}$$

Using (2.401), (2.403), (2.396) and (2.394), we obtain

$$\rho(x_i, x_A) \le \delta + 2^{-1}\gamma_i \le \gamma_i.$$

Thus (2.399) indeed holds for all $i \in \{0, \dots, N\}$. This fact, when combined with (2.394), implies that (2.398) is true for all $i \in \{0, \dots, N\}$. This completes the proof of Theorem 2.67.

2.19 Proof of Theorem 2.68

We may assume, without any loss of generality, that $\varepsilon < 1$ and that $m \ge 1$ is an integer such that

$$m \ge \rho(x_A, \theta) + 4. \tag{2.404}$$

By Theorem 2.65, there exist $\delta \in (0, \varepsilon)$ and a natural number N such that the following property holds.

(P2) For each natural number $n \geq N$ and each sequence $\{x_i\}_{i=0}^n \subset X$ which satisfies

$$x_0 \in B(\theta, m) \quad \text{and} \quad \rho(Ax_i, x_{i+1}) \leq \delta, \quad i = 0, \ldots, n-1, \tag{2.405}$$

the following inequality holds:

$$\rho(x_i, x_A) \leq \varepsilon, \quad i = N, \ldots, n. \tag{2.406}$$

Set

$$K_0 = B(\theta, m) \quad \text{and} \quad K_{i+1} = \{z \in X : \rho(z, A(K_i)) \leq 1\},$$
$$i = 0, 1, \ldots. \tag{2.407}$$

Clearly, the set K_i is bounded for any integer $i \geq 0$. Choose a natural number $q \geq 8$ such that

$$\bigcup_{i=0}^{2N} K_i \subset B(\theta, q) \quad \text{and} \quad 1/q < \delta/8. \tag{2.408}$$

We are going to use the following technical result.

Lemma 2.71 *Assume that*

$$z \in B(\theta, m) \quad \textit{and} \quad \{B_i\}_{i=1}^{2N} \subset \{C \in \mathcal{A} : (C, A) \in E_q\}, \tag{2.409}$$

where E_q is given by (2.375). *Then*

$$\rho(B_i \cdots B_1 z, x_A) \leq \varepsilon, \quad i = N, \ldots, 2N. \tag{2.410}$$

Proof Set

$$z_0 = z \quad \text{and} \quad z_i = B_i z_{i-1}, \quad i = 1, \ldots, 2N. \tag{2.411}$$

We will show that

$$z_i \in K_i \tag{2.412}$$

for $i = 0, \ldots, 2N$. Clearly, (2.412) holds for $i = 0$. Assume that $i \in \{0, \ldots, 2N-1\}$ and (2.412) is valid. Inclusions (2.412) and (2.408) imply that

$$z_i \in K_i \subset B(\theta, q). \tag{2.413}$$

When combined with (2.409), (2.375) and (2.411), this last inclusion implies that

$$\rho(Az_i, z_{i+1}) = \rho(Az_i, B_{i+1} z_i) \leq 1/q. \tag{2.414}$$

Consequently, (2.414), (2.413) and (2.407) imply that $z_{i+1} \in K_{i+1}$. Therefore (2.412) is true for all $i = 0, \dots, 2N$. This implies (see (2.408)) that

$$\{z_i\}_{i=0}^{2N} \subset B(\theta, q).$$

It follows from this inclusion, (2.408), (2.409) and (2.411) that for $i = 0, \dots, 2N-1$,

$$\rho(z_{i+1}, Az_i) = \rho(B_{i+1}z_i, Az_i) \le 1/q < \delta.$$

By (P2), we see that

$$\rho(B_i \cdots B_1 z, x_A) = \rho(z_i, x_A) \le \varepsilon, \quad i = N, \dots, 2N.$$

Lemma 2.71 is proved. □

Now we are ready to complete the proof of Theorem 2.68. To this end, set

$$\mathcal{U} = \{C \in \mathcal{A} : (C, A) \in E_q\}. \tag{2.415}$$

Let $n \ge N$ be an integer, $x \in B(\theta, m)$, and $\{B_i\}_{i=1}^n \subset \mathcal{U}$. We will show that

$$\rho(B_i \cdots B_1 x, x_A) \le \varepsilon \quad \text{for } i = N, \dots, n. \tag{2.416}$$

If $n \le 2N$, then (2.416) follows from Lemma 2.71. Therefore we may restrict our attention to the case $n > 2N$. By Lemma 2.71,

$$\rho(B_i \cdots B_1 x, x_A) \le \varepsilon, \quad i = N, \dots, 2N. \tag{2.417}$$

Suppose now that there exists an integer $p > 2N$, $p \le n$, such that

$$\rho(B_p \cdots B_1 x, x_A) > \varepsilon. \tag{2.418}$$

According to (2.417), we may assume, without loss of generality, that

$$\rho(B_i \cdots B_1 x, x_A) \le \varepsilon, \quad i = 2N, \dots, p-1. \tag{2.419}$$

Define $\{D_i\}_{i=0}^{2N} \subset \mathcal{A}$ by

$$D_i = B_{i+p-N}, \quad i = 0, \dots, N, \qquad D_i = A, \quad i = N+1, \dots, 2N, \tag{2.420}$$

and let

$$z = B_{p-N} \cdots B_1 x.$$

It follows from (2.417), (2.419), (2.420) and (2.404) that

$$\rho(z, x_A) \le \varepsilon \quad \text{and} \quad z \in B(\theta, m).$$

Applying now Lemma 2.71 to the mappings $\{D_i\}_{i=0}^{2N}$ defined by (2.420), we deduce that

$$\varepsilon \geq \rho(D_N \cdots D_1 z, x_A) = \rho(x_A, B_p \cdots B_{p-N+1} z) = \rho(x_A, B_p \cdots B_1 x),$$

which contradicts (2.418). Hence (2.416) is true and Theorem 2.68 is established.

2.20 Proof of Theorem 2.69

Let $A \in \mathcal{A}_{reg}$ and let $k \geq 1$ be an integer. There is $x_A \in K$ such that

$$Ax_A = x_A. \tag{2.421}$$

According to Theorem 2.68, there exist a natural number $N(A,k)$ and an open neighborhood $\mathcal{U}(A,k)$ of A in $\mathcal{A}$ such that the following property holds.

(P3) For each $x \in B(\theta, k)$, each natural number $n \geq N(A,k)$ and each $B \in \mathcal{U}(A,k)$, we have $\rho(B^n, x_A) \leq 1/k$.

Define

$$\mathcal{F} = \left[\bigcap_{q=1}^{\infty} \bigcup \{\mathcal{U}(A,k) : A \in \mathcal{A}_{reg}, k \geq q \text{ an integer}\}\right] \cap \bar{\mathcal{A}}_{reg}. \tag{2.422}$$

Clearly, $\mathcal{F}$ is an everywhere dense G_δ subset of $\bar{\mathcal{A}}_{reg}$.

Let $B \in \mathcal{F}$. We claim that $B \in \mathcal{A}_{reg}$. Indeed, let q be a natural number. There exists a mapping $A_q \in \mathcal{A}_{reg}$ with a fixed point x_{A_q} and a natural number $k_q \geq q$ such that

$$B \in \mathcal{U}(A_q, k_q). \tag{2.423}$$

This inclusion together with (P3) imply that the following property holds.

(P4) For each point $x \in B(\theta, q) \subset B(\theta, k_q)$ and each natural number $n \geq N(A_q, k_q)$,

$$\rho\big(B^n x, x_{A_q}\big) \leq k_q^{-1} \leq 1/q.$$

Since q is an arbitrary natural number, we obtain that for any $x \in X$, the sequence $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence and its limit is the unique fixed point x_B of B. Thus

$$\lim_{n\to\infty} B^N z = x_B \quad \text{for any } z \in X.$$

Property (P4) implies that

$$\rho(x_{A_q}, x_B) \leq 1/q. \tag{2.424}$$

Finally, it follows from property (P4) and (2.424) that for any $x \in B(\theta, q)$ and any $n \geq N(A_q, k_q)$,

$$\rho\big(B^n x, x_B\big) \leq 2/q.$$

This implies that $B^n x \to x_B$ as $n \to \infty$, uniformly on any bounded subset of X. This completes the proof of Theorem 2.69.

2.21 Inexact Orbits of Nonexpansive Operators

Let (X, ρ) be a complete metric space, $A : X \to X$ be a continuous mapping, and let $F(A)$ be the set of all fixed points of A. We assume that $F(A) \neq \emptyset$ and that for each $x, y \in X$,

$$\rho(Ax, Ay) \leq \rho(x, y). \tag{2.425}$$

By A^0 we denote the identity self-mapping of A. We assume that for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in (X, ρ). (Clearly, its limit belongs to $F(A)$.)

The following result was obtained in [34].

Theorem 2.72 *Let $x_0 \in X$, $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\sum_{n=0}^{\infty} r_n < \infty$,*

$$\{x_n\}_{n=0}^{\infty} \subset X, \quad \rho(x_{n+1}, Ax_n) \leq r_n, \quad n = 0, 1, \ldots. \tag{2.426}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of A in (X, ρ).

Proof Fix a natural number k and consider the sequence $\{A^n x_k\}_{n=0}^{\infty}$. This sequence converges to $y_k \in F(A)$. By induction we will show that for each integer $i \geq 0$,

$$\rho\big(A^i x_k, x_{k+1}\big) \leq \sum_{j=k-1}^{i+k-1} r_j - r_{k-1}. \tag{2.427}$$

Clearly, for $i = 0$ (2.427) is valid. Assume that (2.427) is valid for an integer $i \geq 0$. By (2.426), (2.425) and (2.427),

$$\begin{aligned}\rho\big(x_{k+i+1}, A^{i+1} x_k\big) &\leq \rho(x_{k+i+1}, Ax_{k+i}) + \rho\big(Ax_{k+i}, A\big(A^i x_k\big)\big)\\ &\leq r_{k+i} + \rho\big(x_{k+i}, A^i x_k\big) \leq \sum_{j=k-1}^{i+k} r_i - r_{k-1}.\end{aligned}$$

Therefore (2.427) holds for all integers $i \geq 0$.

By (2.427), we have for each integer $i \geq 0$,

$$\rho(x_{k+i}, y_k) \leq \rho\big(x_{k+i}, A^i x_k\big) + \rho\big(A^i x_k, y_k\big) \leq \sum_{j=k}^{\infty} r_j + \rho\big(A^i x_k, y_k\big). \tag{2.428}$$

Since $A^i x_k$ converges to y_k in (X, ρ), there is an integer $i_0 \geq 1$ such that for each integer $i \geq i_0$,

$$\rho\big(A^i x_k, y_k\big) \leq \sum_{j=k}^{\infty} r_j/4. \tag{2.429}$$

By (2.429) and (2.428), for each pair of integers $i_1, i_2 \geq i_0$,

$$\rho(x_{k+i_1}, x_{k+i_2}) \leq \rho(x_{k+i_1}, y_k) + \rho(y_k, x_{k+i_2}) \leq 3\sum_{j=k}^{\infty} r_j.$$

Thus we have shown that for each natural number k, there is an integer $i_0 \geq 1$ such that for each pair of integers $i_1, i_2 \geq i_0$,

$$\rho(x_{k+i_1}, x_{k+i_2}) \leq 3\sum_{j=k}^{\infty} r_j.$$

Since $\sum_{j=1}^{\infty} r_j < \infty$, we see that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists $\bar{x} = \lim_{n\to\infty} x_n$. Together with (2.428), this equality implies that

$$\rho(\bar{x}, y_k) \leq \sum_{j=k}^{\infty} r_j.$$

Since $\sum_{j=1}^{\infty} r_j < \infty$, this inequality implies that

$$\bar{x} = \lim_{k\to\infty} y_k$$

and $A\bar{x} = \bar{x}$. Theorem 2.72 is proved. □

Now we present another result which was obtained in [34].

Let X be a nonempty closed subset of a Banach space $(E, \|\cdot\|)$ with a dual space $(E^*, \|\cdot\|_*)$ and let $A : X \to X$ satisfy

$$\|Ax - Ay\| \leq \|x - y\| \quad \text{for each } x, y \in X. \tag{2.430}$$

As usual, we denote by A^0 the identity self-mapping of X. Consider the following assumptions.

(A1) For each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges weakly in X.

(A2) For each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges weakly in X to a fixed point of A.

Theorem 2.73 *Assume that* (A1) *holds. Let* $x_0 \in X$,

$$\{r_n\}_{n=0}^{\infty} \subset (0, \infty), \sum_{n=0}^{\infty} r_n < \infty, \tag{2.431}$$

$$\{x_n\}_{n=0}^{\infty} \subset X, \qquad \|x_{n+1} - Ax_n\| \le r_n, \quad n = 0, 1, \ldots. \tag{2.432}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges weakly in* X. *Moreover, if* (A2) *holds, then its limit is a fixed point of* A.

Proof Fix a natural number k and consider a sequence $\{A^n x_k\}_{n=0}^{\infty}$. This sequence converges weakly to $y_k \in X$. (Note that if (A2) holds, then $Ay_k = y_k$.) By induction we will show that for each integer $i \ge 0$,

$$\left\|A^i x_k - x_{k+i}\right\| \le \sum_{j=k-1}^{i+k-1} r_j - r_{k-1}. \tag{2.433}$$

It is clear that (2.433) is valid for $i = 0$. Assume that $i \ge 0$ is an integer and that (2.433) is valid. By (2.432) and (2.430),

$$\begin{aligned}
\left\|x_{k+i+1} - A^{i+1}x_k\right\| &\le \|x_{k+i+1} - Ax_{k+i}\| + \left\|Ax_{k+i} - A\left(A^i x_k\right)\right\| \\
&\le r_{k+i} + \left\|x_{k+i} - A^i x_k\right\| \\
&\le r_{k+i} + \sum_{j=k-1}^{i+k-1} r_j - r_{k-1} = \sum_{j=k-1}^{i+k} r_j - r_{k-1}.
\end{aligned}$$

Therefore (2.433) holds for all integers $i \ge 0$. Fix an integer $q \ge 1$. By (2.433), we have

$$\left\|A^q x_k - x_{k+q}\right\| \le \sum_{j=k}^{\infty} r_j. \tag{2.434}$$

By (2.430) and (2.434), we have for each integer $i \ge 0$,

$$\left\|A^{q+i} x_k - A^i x_{k+q}\right\| \le \left\|A^q x_k - x_{k+q}\right\| \le \sum_{j=k}^{\infty} r_j. \tag{2.435}$$

In view of (2.435) and the definition of y_k and y_{k+q},

$$\|y_k - y_{k+q}\| \le \sum_{j=k}^{\infty} r_j. \tag{2.436}$$

Since the above inequality holds for each pair of natural numbers q and k and since $\sum_{j=0}^{\infty} r_j < \infty$, we conclude that $\{y_k\}_{k=1}^{\infty}$ is a Cauchy sequence and there exists

$$y_* = \lim_{k\to\infty} y_k \tag{2.437}$$

in the norm topology of E. (Note that if (A2) holds, then $Ay_* = y_*$.) By (2.437) and (2.436),

$$\|y_k - y_*\| \le \sum_{j=k}^{\infty} r_j \quad \text{for all integers } k \ge 1. \tag{2.438}$$

In order to complete the proof it is sufficient to show that $\lim_{k\to\infty} x_k = y_*$ in the weak topology.

Let $f \in E^*$ be a continuous linear functional on E such that $\|f\|_* \le 1$ and let $\varepsilon > 0$ be given. It is sufficient to show that $|f(y_* - x_i)| \le \varepsilon$ for all large enough integers i.

There is an integer $k \ge 1$ such that

$$\sum_{j=k}^{\infty} r_j < \varepsilon/4. \tag{2.439}$$

By (2.438) and (2.434), for each integer $i \ge 1$,

$$\begin{aligned} \left|f(y_* - x_{k+i})\right| &\le \left|f(y_* - y_k)\right| + \left|f\left(y_k - A^i x_k\right)\right| + \left|f\left(A^i x_k - x_{k+i}\right)\right| \\ &\le \|y_* - y_k\| + \left|f\left(y_k - A^i x_k\right)\right| + \left\|A^i x_k - x_{k+i}\right\| \\ &\le \sum_{j=k}^{\infty} r_j + \left|f\left(y_k - A^i x_k\right)\right| + \sum_{j=k}^{\infty} r_j. \end{aligned} \tag{2.440}$$

Since $y_k = \lim_{i\to\infty} A^i x_k$ in the weak topology of X, there is a natural number i_0 such that

$$\left|f\left(y_k - A^i x_k\right)\right| \le \varepsilon/4 \quad \text{for all natural numbers } i \ge i_0. \tag{2.441}$$

By (2.440), (2.439), (2.441), we have for each integer $i \ge i_0$,

$$\left|f(y_* - x_{k+i})\right| \le \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4.$$

Theorem 2.73 is proved. □

2.22 Convergence to Attracting Sets

In this section we continue to study the influence of errors on the convergence of orbits of nonexpansive mappings in either metric or Banach spaces.

Let (X, ρ) be a metric space. For each $x \in X$ and each closed nonempty subset $A \subset X$, put

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\}.$$

Theorem 2.74 *Let $T : X \to X$ satisfy*

$$\rho(Tx, Ty) \le \rho(x, y) \quad \textit{for all } x, y \in X. \tag{2.442}$$

Suppose that F is a nonempty closed subset of X such that for each $x \in X$,

$$\lim_{i \to \infty} \rho\big(T^i x, F\big) = 0.$$

Assume that $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$,

$$\{x_n\}_{n=0}^{\infty} \subset X \quad \textit{and} \quad \rho(x_{n+1}, Tx_n) \le \gamma_n, \quad n = 0, 1, \dots. \tag{2.443}$$

Then

$$\lim_{n \to \infty} \rho(x_n, F) = 0.$$

Proof Let $\varepsilon > 0$. Then there is an integer $k \ge 1$ such that

$$\sum_{i=k}^{\infty} \gamma_i < \varepsilon. \tag{2.444}$$

Define a sequence $\{y_i\}_{i=k}^{\infty}$ by

$$\begin{aligned} y_k &= x_k, \\ y_{i+1} &= Ty_i \quad \text{for all integers } i \ge k. \end{aligned} \tag{2.445}$$

By (2.443) and (2.445),

$$\rho(x_{k+1}, y_{k+1}) \le \gamma_k. \tag{2.446}$$

Assume that $q \ge k + 1$ is an integer and that for $i = k + 1, \dots, q$,

$$\rho(x_i, y_i) \le \sum_{j=k}^{i-1} \gamma_j. \tag{2.447}$$

(Note that in view of (2.446), inequality (2.447) is valid when $q = k + 1$.)

By (2.442) and (2.447),

$$\rho(Ty_q, Tx_q) \le \rho(y_q, x_q) \le \sum_{j=k}^{q-1} \gamma_j.$$

When combined with (2.445) and (2.443), this implies that

$$\rho(x_{q+1}, y_{q+1}) \le \rho(x_{q+1}, Tx_q) + \rho(Tx_q, Ty_q) \le \gamma_q + \sum_{j=k}^{q-1} \gamma_j = \sum_{j=k}^{q} \gamma_j,$$

so that (2.447) also holds for $i = q + 1$. Thus we have shown that for all integers $q \geq k + 1$,

$$\rho(y_q, x_q) \leq \sum_{j=k}^{q-1} \gamma_j < \sum_{j=k}^{\infty} \gamma_j < \varepsilon, \tag{2.448}$$

by (2.444). In view of (2.445) and the hypotheses of the theorem we note that

$$\lim_{i \to \infty} \rho(y_i, F) = 0. \tag{2.449}$$

By (2.448) and (2.449),

$$\limsup_{i \to \infty} \rho(x_i, F) \leq \varepsilon.$$

Since ε is an arbitrary positive number, we conclude that

$$\lim_{i \to \infty} \rho(x_i, F) = 0,$$

as asserted. □

Theorem 2.75 *Let X be a nonempty and closed subset of a reflexive Banach space $(E, \|\cdot\|)$ and let $T : X \to X$ be such that*

$$\|Tx - Ty\| \leq \|x - y\| \quad \textit{for all } x, y \in X. \tag{2.450}$$

Let F be a nonempty and closed subset of X such that for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ is bounded and all its weak limit points belong to F.

Assume that $\{\gamma_i\}_{i=0}^{\infty} \subset (0, \infty)$, $\sum_{i=0}^{\infty} \gamma_i < \infty$, $\{x_i\}_{i=0}^{\infty} \subset X$ and

$$\|x_{i+1} - Tx_i\| \leq \gamma_i \quad \textit{for all integers } i \geq 0. \tag{2.451}$$

Then the sequence $\{x_i\}_{i=0}^{\infty} \subset X$ is bounded and all its weak limit points also belong to F.

Proof Let $\varepsilon > 0$ be given. There is an integer $k \geq 1$ such that

$$\sum_{i=k}^{\infty} \gamma_i < \varepsilon. \tag{2.452}$$

Define a sequence $\{y_i\}_{i=k}^{\infty}$ by

$$y_k = x_k, \qquad y_{i+1} = Ty_i \quad \text{for all integers } i \geq k. \tag{2.453}$$

Arguing as in the proof of Theorem 2.74, we can show that for all integers $q \geq k+1$,

$$\|y_q - x_q\| \leq \sum_{j=k}^{q-1} \gamma_j < \varepsilon. \tag{2.454}$$

Obviously, (2.454) implies that the sequence $\{x_k\}_{k=0}^{\infty}$ is bounded.

Assume now that z is a weak limit point of the sequence $\{x_k\}_{k=0}^{\infty}$. There exists a subsequence $\{x_{i_p}\}_{p=1}^{\infty}$ which weakly converges to z. We may assume without loss of generality that $\{y_{i_p}\}_{p=1}^{\infty}$ weakly converges to $\tilde{z} \in F$. By (2.454) and the weak lower semicontinuity of the norm,

$$\|\tilde{z} - z\| \le \varepsilon.$$

Since ε is an arbitrary positive number, we conclude that

$$z \in F.$$

Theorem 2.75 is proved. □

Both Theorems 2.74 and 2.75 were obtained in [111].

2.23 Nonconvergence to Attracting Sets

In this section, which is based on [111], we show that both Theorems 2.72 and 2.74 cannot, in general, be improved. We begin with Theorem 2.72.

Proposition 2.76 *For any normed space X, there exists an operator $T : X \to X$ such that $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges for each $x \in X$ and, for any sequence of positive numbers $\{\gamma_n\}_{n=0}^{\infty}$, there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ with $\|x_{n+1} - Tx_n\| \le \gamma_n$ for all nonnegative integers n, which converges if and only if the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is summable, i.e., $\sum_{n=0}^{\infty} \gamma_n < \infty$.*

Proof This is a simple fact because we may take T to be the identity operator: $Tx = x$, $\forall x$. Then we may take x_0 to be an arbitrary element of X with $\|x_0\| = 1$, and define by induction

$$x_{n+1} = Tx_n + \gamma_n x_0, \quad n = 0, 1, 2, \ldots.$$

Evidently, $\|x_{n+1} - Tx_n\| = \gamma_n$ and $x_{n+1} = x_0(1 + \sum_{i=0}^{n} \gamma_i)$ for all integers $n \ge 0$, so that the convergence of $\{x_n\}_{n=0}^{\infty}$ is equivalent to the summability of the sequence $\{\gamma_n\}_{n=0}^{\infty}$. □

Counterexamples to possible improvements of Theorem 2.74 are more difficult to construct because this theorem deals with convergence to attracting sets. For simplicity, we assume that the non-summable sequence $\{\gamma_n\}_{n=0}^{\infty}$ decreases to 0 and that $\gamma_1 \le 1$.

Proposition 2.77 *Let X be an arbitrary (but not one-dimensional) normed space and let a non-summable sequence of positive numbers $\{\gamma_n\}_{n=0}^{\infty}$ decrease to 0. Then there exist a subspace $F \subset X$ and a nonexpansive (with respect to an equivalent norm on X) operator $T : X \to X$ such that $\rho(T^n u, F) \to 0$ as $n \to \infty$ for any*

$u \in X$, and there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset X$ such that $\|u_{n+1} - Tu_n\| \le \gamma_n$ for all integers $n \ge 0$, but $\rho(u_n, F)$ does not tend to 0 as $n \to \infty$.

Proof We take any 2-dimensional subspace of X, identify it with R^2 (with coordinates (x, y)), and perform all constructions and proofs only in this subspace, taking as F the one-dimensional space $L := \{(x, y) \in R^2 : y = 0\}$. The same counterexample may be then applied to the whole space X if we take F to be an algebraic complement of the one-dimensional space $\{(x, y) \in R^2 : x = 0\}$ which contains L.

So, consider a plane with orthogonal axes x, y and the norm $\|u\| = \|(x, y)\| = \max(|x|, |y|)$ (recall that in a finite dimensional space all norms are equivalent). At the first stage, we only consider the case where $\gamma_{n+1}/\gamma_n \ge 1/2$ for all n and we define a decreasing function $y = \gamma(x)$ which equals γ_n at $x = 2n$, $n = 1, 2, \dots,$ and is linear on the intermediate segments. Finally, we define the operator T as the superposition $T = T_4T_3T_2T_1$ of the following four mappings: (a) $T_1 : (x, y) \mapsto (|x|, |y|)$; (b) $T_2 : (x, y) \mapsto (x, \min(1, y))$; (c) $T_3 : (x, y) \mapsto (x + 2, y)$; (d) $T_4 : (x, y) \mapsto (x, [1 - \gamma(x)]y)$.

The principal point of the proof is to show that the operator T is nonexpansive.

This is obviously true for the first three mappings T_1, T_2 and T_3, so we need only consider the fourth operator T_4. For simplicity, we may assume from the very beginning that $T = T_4$.

For arbitrary $x_1 < x_2$, let $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$. Then $Tu_1 = (x_1, [1 - \gamma(x_1)]y_1)$ and $Tu_2 = (x_2, [1 - \gamma(x_2)]y_2)$. Our aim is to show that $\|Tu_1 - Tu_2\| \le \|u_1 - u_2\|$, where $\|u_1 - u_2\| = \max(x_2 - x_1, |y_2 - y_1|)$ and $\|Tu_1 - Tu_2\| = \max(x_2 - x_1, |[1 - \gamma(x_2)]y_2 - [1 - \gamma(x_1)]y_1|)$. Since after the application of the first two mappings T_1 and T_2, the second coordinate y already belongs to $[0, 1]$, the case where $x_2 - x_1 \ge 1$ is trivial, because then $\|Tu_1 - Tu_2\| = \|u_1 - u_2\| = x_2 - x_1$. Hence we may assume in what follows that $x_2 - x_1 < 1$ and thus we need only consider one of the following two possibilities: either both x_1 and x_2 belong to the same interval $[2n, 2(n + 1)]$ or they belong to two adjoining intervals $[2n, 2(n + 1)]$ and $[2(n + 1), 2(n + 2)]$ for some $n = 1, 2, \dots$. We claim that in both cases,

$$\gamma(x_1) - \gamma(x_2) \le (x_2 - x_1)\gamma(x_1). \tag{2.455}$$

If $2n \le x_1 < x_2 \le 2(n + 1)$, then the points u_1 and u_2 lie on the straight line connecting the points $(2n, 1 - \gamma_n)$ and $(2(n + 1), 1 - \gamma_{n+1})$, so that the ratio $(\gamma(x_1) - \gamma(x_2))/(x_2 - x_1)$ coincides with the slope of this line:

$$k_n = (\gamma_n - \gamma_{n+1})/2 \le \gamma_n/2 \le \gamma_{n+1} \le \gamma(x_1).$$

In the second case the same ratio is less than or equal to $\max(k_n, k_{n+1})$, where

$$k_{n+1} = (\gamma_{n+1} - \gamma_{n+2})/2 \le \gamma_{n+1} \le \gamma(x_1),$$

and therefore inequality (2.455) is proved in both cases.

Note that in order to compare the distances between u_1 and u_2, and between Tu_1 and Tu_2, it is enough to show that

$$\bigl|y_2\bigl[1 - \gamma(x_2)\bigr] - y_1\bigl[1 - \gamma(x_1)\bigr]\bigr| \le \max\bigl(x_2 - x_1, |y_2 - y_1|\bigr). \tag{2.456}$$

If $y_1 \geq y_2$, then

$$y_1\big[1-\gamma(x_1)\big]-y_2\big[1-\gamma(x_2)\big]=(y_1-y_2)-\big[y_1\gamma(x_1)-y_2\gamma(x_2)\big]\leq y_1-y_2,$$

because $\gamma(x_1) \geq \gamma(x_2)$. On the other hand,

$$\begin{aligned}y_1\big[1-\gamma(x_1)\big]-y_2\big[1-\gamma(x_2)\big]&=(y_1-y_2)\big[1-\gamma(x_2)\big]+y_1\big[\gamma(x_2)-\gamma(x_1)\big]\\&\geq -(x_2-x_1)\gamma(x_1)y_1\end{aligned}$$

by (2.455). Now inequality (2.456) follows because $\gamma(x_1)y_1 < 1$.

If $y_2 - y_1 \geq 0$, then also $y_2[1-\gamma(x_2)] - y_1[1-\gamma(x_1)] \geq 0$ and it suffices to estimate this difference only from above. Bearing in mind that all $y \leq 1$, we obtain by (2.455) that

$$\begin{aligned}&y_2\big[1-\gamma(x_2)\big]-y_1\big[1-\gamma(x_1)\big]\\&\quad=(y_2-y_1)\big[1-\gamma(x_1)\big]+y_2\big[\gamma(x_1)-\gamma(x_2)\big]\\&\quad\leq(y_2-y_1)\big[1-\gamma(x_1)\big]+\gamma(x_1)(x_2-x_1)\leq\max(x_2-x_1,y_2-y_1),\end{aligned}$$

as needed.

Let $u = (x, y)$ be an arbitrary point in R^2. Then $T_2T_1u \in \{(x, y) : x \geq 0, 0 \leq y \leq 1\}$ and thereafter the operators T_1 and T_2 coincide with the identity mapping. Defining the integer k by $2k \leq x < 2(k+1)$, we see that

$$\rho\big(T^n u, F\big)=y\prod_{i=1}^{n}\big[1-\gamma(x+2i)\big]\leq y\prod_{i=k+1}^{k+n}(1-\gamma_i)\longrightarrow 0$$

as $n \to \infty$, because the series $\sum_{i=1}^{\infty}\gamma_i$ is divergent.

To finish the proof for the case where $\gamma_{n+1}/\gamma_n \geq 1/2$ for all natural numbers n, we define $u_n = (2(n-1), 1)$ for $n = 1, 2, \ldots$. Then $Tu_n = T_4T_3u_n = (2n, 1-\gamma_n)$ and $\|u_{n+1} - Tu_n\| = \gamma_n$. At the same time, $\rho(u_n, F) = 1$ for all n and does not tend to 0.

We now proceed to the general case where the given sequence $\{\gamma_n\}_{n=0}^{\infty}$ does not satisfy the condition $\gamma_{n+1}/\gamma_n \geq 1/2$ for all $n \geq 0$. We then define by induction a new sequence:

$$\gamma_1' = \gamma_1, \qquad \gamma_{n+1}' = \max\big\{\gamma_{n+1}, \gamma_n'/2\big\}, \quad n = 1, 2, \ldots,$$

so that $\gamma_{n+1}'/\gamma_n' \geq 1/2$. Using the new sequence $\{\gamma_n'\}_{n=0}^{\infty}$, we construct the operator T as before, replacing each γ_n by γ_n'. The sequence $\{u_n\}_{n=0}^{\infty}$ will be defined by induction. Let $u_1 = (0, 1)$. If the point $u_n = (x_n, y_n)$ has already been defined, then to obtain the next point $u_{n+1} = (x_{n+1}, y_{n+1})$, we put $x_{n+1} = x_n + 2$, $y_{n+1} = y_n$ if $\gamma_n' = \gamma_n$, and $y_{n+1} = y_n[1-\gamma_n']$ if $\gamma_n' > \gamma_n$. Since $Tu_n = (x_{n+1}, y_n[1-\gamma_n'])$ for each n, we find that $\|u_{n+1} - Tu_n\| \leq \gamma_n$ for all n, as needed.

It is easy to see that

$$y_{n+1} = \prod_{k=1}^{n} \left(1 - \sigma_k \gamma_k'\right),$$

where $\sigma_k = 1$ when $\gamma_n' > \gamma_n$ and $\sigma_k = 0$ otherwise. But the series $\sum_{k=1}^{\infty} \sigma_k \gamma_k'$ converges, since the ratio of any two consecutive nonzero terms here is not greater than $1/2$. Therefore

$$\rho(u_n, F) \geq \prod_{k=1}^{\infty} \left(1 - \sigma_k \gamma_k'\right) > 0.$$

That is, the sequence $\{\rho(u_n, F)\}$ again does not tend to zero, as claimed. □

2.24 Convergence and Nonconvergence to Fixed Points

In Sect. 2.23 we have shown that Theorems 2.72 and 2.74 cannot be, in general, improved. However in Proposition 2.76 every point of the space is a fixed point of the operator T and the inexact orbits tend to infinity. In Proposition 2.77 the attracting set F is unbounded and the operator T depends on the sequence of errors. In this section we construct an operator T on a complete metric space X such that all of its orbits converge to its unique fixed point, and for any nonsummable sequence of errors and any initial point, there exists a divergent inexact orbit with a convergent subsequence. On the other hand, we emphasize that while the example of the present section is for a particular subset of an infinite-dimensional Banach space, the examples in Sect. 2.23 apply to general normed spaces, even finite-dimensional ones.

Let X be the set of all sequences $x = \{x_i\}_{i=1}^{\infty}$ of nonnegative numbers such that $\sum_{i=1}^{\infty} x_i \leq 1$. For $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty} \in X$, set

$$\rho\left(\{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} |x_i - y_i|. \tag{2.457}$$

Clearly, (X, ρ) is a complete metric space.

Define a mapping $T : X \to X$ as follows:

$$T\left(\{x_i\}_{i=1}^{\infty}\right) = (x_2, x_3, \ldots, x_i, \ldots), \quad \{x_i\}_{i=1}^{\infty} \in X. \tag{2.458}$$

In other words, for any $\{x_i\}_{i=1}^{\infty} \in X$,

$$T\left(\{x_i\}_{i=1}^{\infty}\right) = \{y_i\}_{i=1}^{\infty}, \quad \text{where } y_i = x_{i+1} \text{ for all integers } i \geq 1. \tag{2.459}$$

Set $T^0 x = x$ for all $x \in X$. Clearly,

$$\rho(Tx, Ty) \leq \rho(x, y) \quad \text{for all } x, y \in X \tag{2.460}$$

and

$$T^n x \text{ converges to } (0,0,\dots,\dots) \quad \text{as } n \to \infty \tag{2.461}$$

for all $x \in X$.

The following result was obtained in [111].

Theorem 2.78 *Let* $\{r_i\}_{i=0}^{\infty} \subset [0,\infty)$,

$$\sum_{i=0}^{\infty} r_i = \infty, \tag{2.462}$$

and $x = \{x_i\}_{i=1}^{\infty} \in X$. *Then there exists a sequence* $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ *such that*

$$y^{(0)} = x, \qquad \rho\big(Ty^{(i)}, y^{(i+1)}\big) \le r_i, \quad i = 0,1,\dots,$$

the sequence $\{y^{(i)}\}_{i=0}^{\infty}$ *does not converge in* (X,ρ), *but* $(0,0,\dots)$ *is a limit point of* $\{y^{(i)}\}_{i=0}^{\infty}$.

In the proof of this theorem we may assume without loss of generality that

$$r_i \le 16^{-1} \quad \text{for all integers } i \ge 0. \tag{2.463}$$

We precede the proof of Theorem 2.78 with the following lemma.

Lemma 2.79 *Let* $z^{(0)} = \{z_i^{(0)}\}_{i=1}^{\infty} \in X$ *and let* $k \ge 0$ *be an integer. Then there exist an integer* $n \ge 4$ *and a sequence* $\{z^{(i)}\}_{i=0}^{n} \subset X$ *such that*

$$\rho\big(z^{(i+1)}, Tz^{(i)}\big) \le r_{k+i}, \quad i = 0,\dots,n-1,$$

and

$$\rho\big(z^{(n)}, (0,0,0,\dots)\big) \ge 4^{-1}.$$

Proof There is a natural number $m > 4$ such that

$$\sum_{i=m}^{\infty} z_i^{(0)} < 16^{-1}. \tag{2.464}$$

Set

$$z^{(i+1)} = Tz^{(i)}, \quad i = 0,\dots,m-1. \tag{2.465}$$

Clearly,

$$z^{(m)} = \big(z_{m+1}^{(0)}, z_{m+2}^{(0)}, \dots, z_i^{(0)}, \dots\big). \tag{2.466}$$

By (2.462), there is a natural number $n > m$ such that

$$\sum_{j=k+m}^{k+n} r_j \geq 2^{-1}. \tag{2.467}$$

By (2.467) and (2.463), $n \geq m + 7$ and we may assume without loss of generality that

$$\sum_{j=k+m}^{k+n-1} r_j < 1/2. \tag{2.468}$$

In view of (2.457) and (2.463)

$$\sum_{j=k+m}^{k+n-1} r_j = \sum_{j=k+m}^{k+n} r_j - r_{k+n} \geq 2^{-1} - 16^{-1}. \tag{2.469}$$

For $i = m + 1, \ldots, n$, define $z^{(i)} = \{z_j^{(i)}\}_{j=1}^{\infty}$ as follows:

$$\begin{aligned} z_j^{(i)} &= z_{j+i}^{(0)}, \quad j \in \{1, 2, \ldots\} \setminus \{n + 1 - i\}, \\ z_{n+1-i}^{(i)} &= z_{n+1}^{(0)} + \sum_{j=k+m}^{k+i-1} r_j. \end{aligned} \tag{2.470}$$

Clearly, for $i = m + 1, \ldots, n$, $z^{(i)}$ is well-defined and by (2.470), (2.464) and (2.468),

$$\sum_{j=1}^{\infty} z_j^{(i)} = \sum_{j=i+1}^{\infty} z_j^{(0)} + \sum_{j=k+m}^{k+i-1} r_j \leq \sum_{j=m}^{\infty} z_j^{(0)} + \sum_{j=k+m}^{k+n-1} r_j \leq 16^{-1} + 2^{-1} < 1.$$

Thus $z^{(i)} \in X$, $i = m + 1, \ldots, n$.

Let $i \in \{m, \ldots, n - 1\}$. In order to estimate $\rho(z^{(i+1)}, Tz^{(i)})$, we first set

$$\{\tilde{z}_j\}_{j=1}^{\infty} = Tz^{(i)}. \tag{2.471}$$

In view of (2.471), (2.458) and (2.459), $\tilde{z}_j = z_{j+1}^{(i)}$ for all integers $j \geq 1$. When combined with (2.470), this implies that

$$\tilde{z}_j = z_{j+1+i}^{(0)} \quad \text{for all } j \in \{1, 2, \ldots\} \setminus \{n - i\} \tag{2.472}$$

and

$$\tilde{z}_{n-i} = z_{n+1-i}^{(i)} = z_{n+1}^{(0)} + \sum_{j=k+m}^{k+i-1} r_j.$$

By (2.472), $\tilde{z}_j = z_j^{(i+1)}$ for all $j \in \{1, 2, \ldots\} \setminus \{n-i\}$. Together with (2.473), (2.457), (2.472) and (2.470), this equality implies that

$$\rho\big(z^{(i+1)}, Tz^{(i)}\big) = \rho\big(z^{(i+1)}, \{\tilde{z}_j\}_{j=1}^{\infty}\big) = \big|z_{n-i}^{(i+1)} - \tilde{z}_{n-i}\big| = r_{k+i}.$$

It follows from this relation, which holds for all $i \in \{m, \ldots, n-1\}$, and from (2.465) that

$$\rho\big(z^{(i+1)}, Tz^{(i)}\big) \le r_{k+i}, \quad i = 0, \ldots, n-1.$$

By (2.457), (2.470) and (2.469),

$$\rho\big(z^{(n)}, (0,0,0,\ldots)\big) \ge z_1^{(n)} = z_{n+1}^{(0)} + \sum_{j=k+m}^{k+n-1} r_j \ge 2^{-1} - 16^{-1}.$$

This completes the proof of Lemma 2.79. □

Proof of Theorem 2.78 In order to prove the theorem, we construct by induction, using Lemma 2.79, sequences of nonnegative integers $\{t_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=0}^{\infty}$, and a sequence $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ such that

$$y^{(0)} = x, \tag{2.473}$$

$$\rho\big(y^{(i+1)}, Ty^{(i)}\big) \le r_i \quad \text{for all integers } i \ge 0, \tag{2.474}$$

$$t_0 = s_0 = 0, \qquad s_k < s_{k+1} < t_{k+1} \quad \text{for all integers } k \ge 0, \tag{2.475}$$

and for all integers $k \ge 1$,

$$\rho\big(y^{(s_k)}, (0,0,0\ldots)\big) \le 1/k \quad \text{and} \quad \rho\big(y^{(t_k)}, (0,0,0\ldots)\big) \ge 1/4. \tag{2.476}$$

In the sequel we use the notation $y^{(i)} = \{y_j^{(i)}\}_{j=1}^{\infty}$, $i = 0, 1, \ldots$.

Set

$$y^{(0)} = x \quad \text{and} \quad t_0, s_0 = 0. \tag{2.477}$$

Assume that $q \ge 0$ is an integer and that we have already defined two sequences of nonnegative numbers $\{t_k\}_{k=0}^{q}$ and $\{s_k\}_{k=0}^{q}$, and a sequence $\{y^{(i)}\}_{i=0}^{t_q} \subset X$ such that (2.474) holds for all integers i satisfying $0 \le i < s_q$, (2.477) holds,

$$t_k < s_{k+1} < t_{k+1} \quad \text{for all integers } k \text{ satisfying } 0 \le k < q,$$

and (2.476) holds for all integers k satisfying $0 < k \le q$. (Note that for $q = 0$ this assumption does hold.)

Now we show that this assumption also holds for $q + 1$.

Indeed, there is a natural number $s_{q+1} > t_q + 1$ such that

$$\sum_{j=s_{q+1}-1-t_q}^{\infty} y_j^{(t_q)} < (q+1)^{-1}. \tag{2.478}$$

Set

$$y^{(i+1)} = Ty^{(i)}, \quad i = t_q, \ldots, s_{q+1} - 1. \tag{2.479}$$

By (2.479), (2.457), (2.458), (2.459) and (2.478),

$$\rho\big(y^{(s_{q+1})}, (0, 0, \ldots)\big) = \sum_{j=1}^{\infty} y_j^{(s_{q+1})} = \sum_{j=s_{q+1}-t_q+1}^{\infty} y_j^{(t_q)} < (q+1)^{-1}. \tag{2.480}$$

Applying Lemma 2.79 with

$$z^{(0)} = y^{(s_{q+1})} \quad \text{and} \quad k = s_{q+1}, \tag{2.481}$$

we obtain that there exist an integer $n \geq 4$ and a sequence $\{y^{(i)}\}_{i=s_{q+1}}^{s_{q+1}+n} \subset X$ such that

$$\rho\big(y^{(i+1)}, Ty^{(i)}\big) \leq r_i, \quad i = s_{q+1}, \ldots, s_{q+1} + n - 1, \tag{2.482}$$

and

$$\rho\big(y^{(s_{q+1}+n)}, (0, 0, 0 \ldots)\big) \geq 1/4. \tag{2.483}$$

Put

$$t_{q+1} = s_{q+1} + n.$$

In this way we have constructed a sequence $\{y^{(i)}\}_{i=0}^{t_{q+1}} \subset X$ and two sequences of nonnegative integers $\{t_k\}_{k=0}^{q+1}$ and $\{s_k\}_{k=0}^{q+1}$ such that (2.477) holds, (2.474) holds for all integers i satisfying $0 \leq i < t_{q+1}$ (see (2.479) and (2.482)), $t_k < s_{k+1} < t_{k+1}$ for all integers k satisfying $0 \leq k < q + 1$, and (2.476) holds for all integers k satisfying $0 < k \leq q + 1$ (see (2.480), (2.482) and (2.483)).

In other words, the assumption made concerning q also holds for $q + 1$. It follows that we have indeed constructed two sequences of nonnegative integers $\{t_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=0}^{\infty}$, and a sequence $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ which satisfy (2.473)–(2.476). This completes the proof of Theorem 2.78. □

2.25 Convergence to Compact Sets

In this section, we study the influence of computational errors on the convergence to compact sets of orbits of nonexpansive mappings in Banach and metric spaces.

Let (X, ρ) be a complete metric space. For each $x \in X$ and each nonempty closed subset $A \subset X$, put

$$\rho(x, A) = \inf\big\{\rho(x, y) : y \in A\big\}.$$

For each mapping $T : X \to X$, set $T^0 x = x$ for all $x \in X$.

The following result was obtained in [112].

Theorem 2.80 *Let $T : X \to X$ satisfy*

$$\rho(Tx, Ty) \le \rho(x, y) \quad \textit{for all } x, y \in X. \tag{2.484}$$

Suppose that for each $x \in X$, there exists a nonempty compact set $E(x) \subset X$ such that

$$\lim_{i\to\infty} \rho\big(T^i x, E(x)\big) = 0. \tag{2.485}$$

Assume that $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$,

$$\{x_n\}_{n=0}^{\infty} \subset X \quad \textit{and} \quad \rho(x_{n+1}, Tx_n) \le \gamma_n, \quad n = 0, 1, \ldots. \tag{2.486}$$

Then there exists a nonempty compact subset F of X such that

$$\lim_{n\to\infty} \rho(x_n, F) = 0.$$

Proof In order to prove the theorem it is sufficient to show that any subsequence of $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence.

To see this, it is sufficient to show that for any $\varepsilon > 0$, the following assertion holds:

(P1) Any subsequence of $\{x_n\}_{n=0}^{\infty}$ possesses a subsequence which is contained in a ball with radius ε.

Indeed, there is an integer $k \ge 1$ such that

$$\sum_{i=k}^{\infty} \gamma_i < \varepsilon/8. \tag{2.487}$$

Define a sequence $\{y_i\}_{i=k}^{\infty}$ by

$$\begin{aligned} y_k &= x_k, \\ y_{i+1} &= Ty_i \quad \text{for all integers } i \ge k. \end{aligned} \tag{2.488}$$

There exists a nonempty compact set $E \subset X$ such that

$$\lim_{i\to\infty} \rho(y_i, E) = 0. \tag{2.489}$$

By (2.486) and (2.488),

$$\rho(x_{k+1}, y_{k+1}) \le \gamma_k. \tag{2.490}$$

Assume that $q \ge k + 1$ is an integer and that for $i = k + 1, \ldots, q$,

$$\rho(x_i, y_i) \le \sum_{j=k}^{i-1} \gamma_j. \tag{2.491}$$

(Note that in view of (2.490), inequality (2.491) is valid when $q = k + 1$.)

By (2.484) and (2.491),

$$\rho(Ty_q, Tx_q) \le \rho(y_q, x_q) \le \sum_{j=k}^{q-1} \gamma_j.$$

When combined with (2.486), this implies that

$$\rho(x_{q+1}, y_{q+1}) \le \rho(x_{q+1}, Tx_q) + \rho(Tx_q, Ty_q) \le \gamma_q + \sum_{j=k}^{q-1} \gamma_j = \sum_{j=k}^{q} \gamma_j,$$

so that (2.491) also holds for $i = q + 1$. Thus we have shown that for all integers $q \ge k + 1$,

$$\rho(y_q, x_q) \le \sum_{j=k}^{q-1} \gamma_j < \sum_{j=k}^{\infty} \gamma_j < \varepsilon/8 \tag{2.492}$$

by (2.487). In view of (2.489), for all large enough natural numbers q, we have

$$\rho(x_q, E) < \varepsilon/4. \tag{2.493}$$

By (2.493), there exist an integer $q_0 > k$ and a sequence $\{z_i\}_{i=q_0}^{\infty} \subset K$ such that

$$\rho(x_i, z_i) < \varepsilon/3 \quad \text{for all integers } i \ge q_0. \tag{2.494}$$

Consider any subsequence $\{x_{q_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$. Since the set E is compact, the sequence $\{z_{q_i}\}_{i=1}^{\infty}$ possesses a convergent subsequence $\{z_{q_{i_j}}\}_{j=1}^{\infty}$.

We may assume without loss of generality that all elements of this convergent subsequence belong to $B(u, \varepsilon/16)$ for some $u \in X$.

In view of (2.494),

$$x_{q_{i_j}} \in B(u, \varepsilon/2) \quad \text{for all sufficiently large natural numbers } j.$$

Thus (P1) holds and this completes the proof of the theorem. □

Note that Theorem 2.80 is an extension of Theorem 2.72.

The following result, which was obtained in [112], shows that both Theorems 2.72 and 2.80 cannot, in general, be improved (cf. Proposition 2.77).

Proposition 2.81 *For any normed space X, there exists an operator $T : X \to X$ such that $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges for each $x \in X$ and, for any sequence of positive numbers $\{\gamma_n\}_{n=0}^{\infty}$, there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ with $\|x_{n+1} - Tx_n\| \le \gamma_n$ for all nonnegative integers n, which converges to a compact set if and only if the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is summable, i.e., $\sum_{n=0}^{\infty} \gamma_n < \infty$.*

Proof This is a simple fact because we may take T to be the identity operator: $Tx = x$, $\forall x$. Then we may take as x_0 to be an arbitrary element of X with $\|x_0\| = 1$ and define by induction

$$x_{n+1} = Tx_n + \gamma_n x_0, \quad n = 0, 1, 2, \ldots.$$

Evidently, $\|x_{n+1} - Tx_n\| = \gamma_n$ and $x_{n+1} = x_0(1 + \sum_{i=0}^{n} \gamma_i)$ for all integers $n \geq 0$, so that the convergence of $\{x_n\}_{n=0}^{\infty}$ to a compact set is equivalent to the summability of the sequence $\{\gamma_n\}_{n=0}^{\infty}$. Proposition 2.81 is proved. □

2.26 An Example of Nonconvergence to Compact Sets

In the previous section, we have shown that Theorems 2.72 and 2.80 cannot, in general, be improved. However, in Proposition 2.81 every point of the space is a fixed point of the operator T and the inexact orbits tend to infinity. In this section, we construct an operator T on a certain complete metric space X (a bounded, closed and convex subset of a Banach space) such that all of its orbits converge to its unique fixed point, and for any nonsummable sequence of errors and any initial point, there exists an inexact orbit which does not converge to any compact set. This example is based on [112].

Let X be the set of all sequences $x = \{x_i\}_{i=1}^{\infty}$ of nonnegative numbers such that $\sum_{i=1}^{\infty} x_i \leq 1$. For $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$ in X, set

$$\rho\big(\{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty}\big) = \sum_{i=1}^{\infty} |x_i - y_i|. \tag{2.495}$$

Clearly, (X, ρ) is a complete metric space.

Define a mapping $T : X \to X$ as follows:

$$T\big(\{x_i\}_{i=1}^{\infty}\big) = (x_2, x_3, \ldots, x_i, \ldots), \quad \{x_i\}_{i=1}^{\infty} \in X. \tag{2.496}$$

In other words, for any $\{x_i\}_{i=1}^{\infty} \in X$,

$$T\big(\{x_i\}_{i=1}^{\infty}\big) = \{y_i\}_{i=1}^{\infty}, \quad \text{where } y_i = x_{i+1} \text{ for all integers } i \geq 1. \tag{2.497}$$

Set $T^0 x = x$ for all $x \in X$. Clearly,

$$\rho(Tx, Ty) \leq \rho(x, y) \quad \text{for all } x, y \in X \tag{2.498}$$

and

$$T^n x \text{ converges to } (0, 0, \ldots, \ldots) \quad \text{as } n \to \infty \tag{2.499}$$

for all $x \in X$.

Theorem 2.82 *Let* $\{r_i\}_{i=0}^{\infty} \subset [0, \infty)$,

$$\sum_{i=0}^{\infty} r_i = \infty, \tag{2.500}$$

and $x = \{x_i\}_{i=1}^{\infty} \in X$. *Then there exists a sequence* $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ *such that*

$$y^{(0)} = x, \qquad \rho\big(Ty^{(i)}, y^{(i+1)}\big) \le r_i, \quad i = 0, 1, \ldots, \tag{2.501}$$

and that the following property holds:

there is no nonempty compact set $E \subset X$ *such that*

$$\lim_{i \to \infty} \rho\big(y^{(i)}, E\big) = \emptyset.$$

In the proof of this theorem, we may assume without any loss of generality that

$$r_i \le 16^{-1} \quad \text{for all integers } i \ge 0. \tag{2.502}$$

We precede the proof of Theorem 2.82 with the following lemma.

Lemma 2.83 *Let* $z^{(0)} = \{z_i^{(0)}\}_{i=1}^{\infty} \in X$, *let* $k \ge 0$ *be an integer and let* j_0 *be a natural number. Then there exist an integer* $n \ge 4$ *and a sequence* $\{z^{(i)}\}_{i=0}^{n} \subset X$ *such that*

$$\rho\big(z^{(i+1)}, Tz^{(i)}\big) \le r_{k+i}, \quad i = 0, \ldots, n-1,$$

and

$$z^{(n)} = \big(z_1^{(n)}, \ldots, z_i^{(n)}, \ldots\big) = \big\{z_i^{(n)}\big\}_{i=1}^{\infty}$$

with $z_{j_0+1}^{(n)} \ge 4^{-1}$.

Proof There is a natural number $m > 4$ such that

$$\begin{gathered} m > j_0 + 4, \\ \sum_{i=m}^{\infty} z_i^{(0)} < 16^{-1}. \end{gathered} \tag{2.503}$$

Set

$$z^{(i+1)} = Tz^{(i)}, \quad i = 0, \ldots, m-1. \tag{2.504}$$

Then

$$z^{(m)} = \big(z_{m+1}^{(0)}, z_{m+2}^{(0)}, \ldots, z_i^{(0)}, \ldots\big). \tag{2.505}$$

By (2.500), there is a natural number $n > m$ such that

$$\sum_{j=k+m}^{k+n} r_j \geq 2^{-1}. \tag{2.506}$$

By (2.506) and (2.502),

$$n \geq m + 7 \tag{2.507}$$

and we may assume without loss of generality that

$$\sum_{j=k+m}^{k+n-1} r_j < 1/2. \tag{2.508}$$

In view of (2.506) and (2.502),

$$\sum_{j=k+m}^{k+n-1} r_j = \sum_{j=k+m}^{k+n} r_j - r_{k+n} \geq 2^{-1} - 16^{-1}. \tag{2.509}$$

For $i = m+1, \ldots, n$, define $z^{(i)} = \{z_j^{(i)}\}_{j=1}^{\infty}$ as follows:

$$\begin{aligned} z_j^{(i)} &= z_{j+i}^{(0)}, \quad j \in \{1, 2, \ldots\} \setminus \{n + 1 + j_0 - i\}, \\ z_{n+1+j_0-i}^{(i)} &= z_{n+1+j_0}^{(0)} + \sum_{j=k+m}^{k+i-1} r_j. \end{aligned} \tag{2.510}$$

Clearly, for $i = m+1, \ldots, n$, $z^{(i)}$ is well-defined and by (2.510), (2.503) and (2.508),

$$\sum_{j=1}^{\infty} z_j^{(i)} = \sum_{j=i+1}^{\infty} z_j^{(0)} + \sum_{j=k+m}^{k+i-1} r_j \leq \sum_{j=m}^{\infty} z_j^{(0)} + \sum_{j=k+m}^{k+n-1} r_j \leq 16^{-1} + 2^{-1} < 1.$$

Thus $z^{(i)} \in X$, $i = m+1, \ldots, n$.

Let $i \in \{m, \ldots, n-1\}$. We now estimate $\rho(z^{(i+1)}, Tz^{(i)})$. If $i = m$, then by (2.496), (2.497), (2.505) and (2.514),

$$\rho\left(z^{(i+1)}, Tz^{(i)}\right) \leq r_{k+i}. \tag{2.511}$$

Let $i > m$. We first set

$$\{\tilde{z}_j\}_{j=1}^{\infty} = Tz^{(i)}. \tag{2.512}$$

In view of (2.506), (2.496) and (2.497), $\tilde{z}_j = z^{(i)}_{j+1}$ for all integers $j \geq 1$. When combined with (2.510), this implies that

$$\begin{aligned}\tilde{z}_j &= z^{(0)}_{j+1+i} \quad \text{for all } j \in \{1, 2, \dots\} \setminus \{n - i + j_0\},\\ \tilde{z}_{n+j_0-i} &= z^{(i)}_{n+1+j_0-i} = z^{(0)}_{n+1+j_0} + \sum_{j=k+m}^{k+i-1} r_j.\end{aligned} \tag{2.513}$$

By (2.510) and (2.513),

$$\tilde{z}_j = z^{(i+1)}_j \tag{2.514}$$

for all $j \in \{1, 2, \dots\} \setminus \{n + j_0 - i\}$. It now follows from (2.512), (2.514), (2.510) and (2.513) that

$$\begin{aligned}\rho\big(z^{(i+1)}, Tz^{(i)}\big) &= \rho\big(z^{(i+1)}, \{\tilde{z}_j\}_{j=1}^{\infty}\big) = \big|z^{(i+1)}_{n+j_0-i} - \tilde{z}_{n+j_0-i}\big|\\ &= \left|z^{(0)}_{n+1+j_0} + \sum_{j=k+m}^{k+i} r_j - \left(z^{(0)}_{n+1+j_0} + \sum_{j=k+m}^{k+i-1} r_j\right)\right| < r_{k+i}.\end{aligned}$$

When combined with (2.504), this implies that

$$\rho\big(z^{(i+1)}, Tz^{(i)}\big) \leq r_{k+i}, \quad i = 0, \dots, n-1.$$

By (2.509) and (2.510),

$$z^{(n)}_{j_0+1} = z^{(n)}_{n+1+j_0-n} \geq \sum_{j=k+m}^{k+n-1} r_j \geq 4^{-1}.$$

This completes the proof of Lemma 2.83. □

Proof of Theorem 2.82 In order to prove the theorem, we construct by induction, using Lemma 2.83, a sequence of nonnegative integers $\{s_k\}_{k=0}^{\infty}$ and a sequence $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ such that

$$y^{(0)} = x,$$

$$\rho\big(y^{(i+1)}, Ty^{(i)}\big) \leq r_i \quad \text{for all integers } i \geq 0, \tag{2.515}$$

$$s_0 = 0, \qquad s_k < s_{k+1} \quad \text{for all integers } k \geq 0, \tag{2.516}$$

and for all integers $k \geq 1$,

$$y^{(s_k)}_{k+1} \geq 1/4. \tag{2.517}$$

In the sequel we use the notation $y^{(i)} = \{y^{(i)}_j\}_{j=1}^{\infty}$, $i = 0, 1, \dots$.

Set

$$y^{(0)} = x, \qquad s_0 = 0. \tag{2.518}$$

Assume that $q \geq 0$ is an integer and we have already defined a (finite) sequence of nonnegative integers $\{s_k\}_{k=0}^{q}$ and a (finite) sequence $\{y^{(i)}\}_{i=0}^{s_q} \subset X$ such that (2.518) is valid, (2.515) holds for all integers i satisfying $0 \leq i < s_q$,

$$s_i < s_{i+1} \quad \text{for all integers } i \text{ satisfying } 0 \leq i < q,$$

and that (2.517) holds for all integers k satisfying $0 < k \leq q$. (Note that for $q = 0$ this assumption does hold.)

Now we show that this assumption also holds for $q + 1$.

Indeed, applying Lemma 2.83 with

$$z^{(0)} = y^{(s_q)} \quad \text{and} \quad j_0 = q + 1, \qquad k = s_q,$$

we obtain that there exist an integer $s_{q+1} \geq 4 + s_q$ and a sequence $\{y^{(i)}\}_{i=s_q}^{s_{q+1}} \subset X$ such that

$$\rho\big(y^{(i+1)}, Ty^{(i)}\big) \leq r_i, \quad i = s_q, \ldots, s_{q+1} - 1,$$

and

$$y_{q+2}^{(s_{q+1})} \geq 1/4.$$

Thus the assumption made for q also holds for $q + 1$. Therefore we have constructed by induction a sequence $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ and a sequence of nonnegative integers $\{s_k\}_{k=0}^{\infty}$ which satisfy (2.515) and (2.516) for all integers $i, k \geq 0$, respectively, and (2.517) for all integers $k \geq 1$.

Finally, we show that there is no nonempty compact set $E \subset X$ such that

$$\lim_{i \to \infty} \rho\big(y^{(i)}, E\big) = 0.$$

Assume the contrary. Then there does exist a nonempty compact set $E \subset X$ such that

$$\lim_{i \to \infty} \rho\big(y^{i)}, E\big) = 0.$$

This implies that any subsequence of $\{y^{(k)}\}_{k=0}^{\infty}$ possesses a convergent subsequence.

Consider such a subsequence $\{y^{(s_q)}\}_{q=1}^{\infty}$. This subsequence has a convergent subsequence $\{y^{s_{q_p}}\}_{p=1}^{\infty}$. There are, therefore, a point $z = \{z_i\}_{i=0}^{\infty} \in X$ such that

$$z = \lim_{p \to \infty} y^{(s_{q_p})}$$

and a natural number p_0 such that

$$\rho\big(z, y^{(s_{q_p})}\big) \leq 16^{-1} \quad \text{for all integers } p \geq p_0. \tag{2.519}$$

By (2.518) and (2.519), we have for all integers $p \ge p_0$,

$$\left|z_{q_p+1} - y_{q_p+1}^{(s_{q_p})}\right| \le \rho\big(z, y^{(s_{q_p})}\big) \le 16^{-1}$$

and

$$z_{q_p+1} \ge y_{q_p+1}^{(s_{q_p})} - 16^{-1} \ge 8^{-1}.$$

This, of course, contradicts the inequality $\sum_{i=1}^{\infty} z_i \le 1$. The contradiction we have reached completes the proof of Theorem 2.82. □

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