2

DEFINITION OF THE STOCHASTIC INTEGRAL

2.1 Introduction

In this chapter, we shall define stochastic integrals of the form $\int_{[0,t]} X\,dM$ where M is a right continuous local L^2-martingale and X is a process satisfying certain measurability and integrability assumptions, such that the family of stochastic integrals $\{\int_{[0,t]} X\,dM, t \in \mathbb{R}_+\}$ is a right continuous local L^2-martingale. For certain M and X, the integral can be defined path-by-path. For instance, if M is a right continuous local L^2-martingale whose paths are locally of bounded variation, and X is a continuous adapted process, then $\int_{[0,t]} X_s(\omega)\,dM_s(\omega)$ is well-defined as a Riemann-Stieltjes integral for each t and ω, namely by the limit as $n \to \infty$ of

$$\sum_{k=0}^{[2^n t]} X_{k2^{-n}}(\omega)\left(M_{(k+1)2^{-n}}(\omega) - M_{k2^{-n}}(\omega)\right).$$

The standard example of this path-by-path integral is obtained by setting $M_t = N_t - \alpha t$ where N is a Poisson process with parameter $\alpha > 0$. In this case, for any continuous adapted process X we have

$$\int_{[0,t]} X_s(\omega)\,dM_s(\omega) = \sum_{k=1}^{\infty} 1_{\{\tau_k \le t\}} X_{\tau_k}(\omega) - \alpha \int_0^t X_s(\omega)\,ds,$$

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where τ_k is the time of the k^{th} jump of N, and a.s. for each fixed t the sum on the right is of finitely many non-zero terms because almost surely there are only finitely many jumps of N in $[0,t]$.

The stochastic integral defined in the sequel is valid even when M does not have paths which are locally of bounded variation. Any non-constant continuous local martingale is such an M; the canonical example is a Brownian motion B in $\mathbb{R}$. Even the simple integral $\int_{[0,t]} B\,dB$ cannot be defined path-by-path in the Stieltjes sense, because almost every path of a Brownian motion is of unbounded variation on each time interval (see Freedman [33, p. 49]). In fact, the stochastic integral developed here, known as the Itô integral when M is a Brownian motion, is not defined path-by-path but via an isometry between a space of processes X that are square integrable with respect to a measure induced by M, and a space of square integrable stochastic integrals $\int X\,dM$.

As a guide to the reader, we provide the following outline of the several stages in the definition of the stochastic integral.

The measurability conditions on X will be specified first. In doing this, we adopt the modern view of X as a function on $\mathbb{R}_+ \times \Omega$ and require it to be measurable with respect to a σ-field $\mathcal{P}$ generated by a simple class $\mathcal{R}$ of "predictable rectangles." Although this definition of the measurable integrands may not be the most obvious one, it is convenient for a streamlined development of the integral. Moreover, we shall prove in Theorem 3.1 that the class of $\mathcal{P}$-measurable functions includes all of the left continuous adapted processes.

After a discussion of the σ-field $\mathcal{P}$, we shall consider the case where M is a right continuous L^2-martingale. A measure μ_M associated with M will be defined on $\mathcal{P}$ and then we shall define the integral $\int_{[0,t]} X\,dM$ in the following three steps.

(i) $\int X\,dM$ will be defined for any $\mathcal{R}$-simple process X in such a way that the following isometry holds:

$$E\left\{\left(\int X\,dM\right)^2\right\} = \int_{\mathbb{R}_+\times\Omega} (X)^2\,d\mu_M.$$

(ii) This isometry will then be used to extend the definition of $\int X\,dM$ to any $X \in \mathcal{L}^2 \equiv L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \mu_M)$.

(iii) For any process X satisfying $1_{[0,t]}X \in \mathcal{L}^2$ for each $t \in \mathbb{R}_+$, it will be shown that there is a version of $\{\int 1_{[0,t]}X\,dM, t \in \mathbb{R}_+\}$ which is a right continuous L^2-martingale, to be denoted by $\{\int_{[0,t]} X\,dM, t \in \mathbb{R}_+\}$.

Finally, the extension to the case where M is a right continuous local L^2-martingale and X is "locally" in $\mathcal{L}^2$ will be achieved using a sequence of optional times tending to ∞. The above definition of the stochastic integral will apply to the processes obtained by stopping $M - M_0$ and X at any one of these times, and then the integral for M and X will be defined as the almost sure limit of these integrals, as the optional times tend to ∞.

We now begin the above program with the definition of the σ-field $\mathcal{P}$.

2.2 Predictable Sets and Processes

The family of subsets of $\mathbb{R}_+ \times \Omega$ containing all sets of the form $\{0\} \times F_0$ and $(s,t] \times F$, where $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$ for $s < t$ in $\mathbb{R}_+$, is called the *class of predictable rectangles* and we denote it by $\mathcal{R}$. The (Boolean) ring $\mathcal{A}$ generated by $\mathcal{R}$ is the smallest family of subsets of $\mathbb{R}_+ \times \Omega$ which contains $\mathcal{R}$ and is such that if A_1 and A_2 are in the ring, then so too are their union $A_1 \cup A_2$ and difference $A_1 \backslash A_2$. Then $A_1 \cap A_2$ is also in $\mathcal{A}$. Indeed, it can be verified that the ring $\mathcal{A}$ consists of the empty set $\emptyset$ and all finite unions of disjoint rectangles in $\mathcal{R}$. The σ-field $\mathcal{P}$ of subsets of $\mathbb{R}_+ \times \Omega$ generated by $\mathcal{R}$ is called the *predictable σ-field* and sets in $\mathcal{P}$ are called *predictable (sets)*. A function $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called *predictable* if X is $\mathcal{P}$-measurable. This is denoted by $X \in \mathcal{P}$. If A is a set in $\mathcal{R}$, then $1_A(t,\cdot)$ is $\mathcal{F}_t$-measurable for each t. Consequently, 1_A is an adapted process. It follows by forming finite linear combinations that the same is true for any A in $\mathcal{A}$. Then by a monotone class theorem (see Section 1.2), any real-valued $\mathcal{P}$-measurable function is adapted. A real-valued $\mathcal{P}$-measurable function will be referred to as a predictable *process.*

Remark. In systematic studies of the theory of processes, it seems more natural to consider the σ-field $\mathcal{P}$ and predictable processes as defined on $(0,\infty) \times \Omega$. However, we find it convenient to have all processes defined at time zero. The consequence, which is of more logical than substantial significance, is that time zero and sets like $\{0\} \times F_0$ sometimes require slightly different treatment.

It is shown below that for any optional time τ,

$$[0,\tau] = \{(t,\omega) \in \mathbb{R}_+ \times \Omega : 0 \le t \le \tau(\omega)\}$$

is a predictable set. Such "intervals" play an important role in the final extension phase of the definition of the stochastic integral.

2.3 Stochastic Intervals

For optional times η and τ, the set

$$[\eta,\tau] = \{(t,\omega) \in \mathbb{R}_+ \times \Omega : \eta(\omega) \le t \le \tau(\omega)\}$$

is called a stochastic interval. Three other stochastic intervals $(\eta,\tau], (\eta,\tau)$, and $[\eta,\tau)$, with left end-point η and right end-point τ are defined similarly. The term *stochastic interval* will refer to any of these four kinds of intervals where η and τ are any optional times. Note that stochastic intervals are subsets of $\mathbb{R}_+ \times \Omega$ not $\overline{\mathbb{R}}_+ \times \Omega$; consequently (∞,ω) is never a member of such a set, even if $\tau(\omega) = \infty$. Also, we have not specified that $\eta \le \tau$, but by definition the intersection of $[\eta,\tau]$ with $\mathbb{R}_+ \times \{\omega : \eta > \tau\}$ is the empty set. If s, $t \in \mathbb{R}_+$, then $[s, t]$, $(s, t]$, $[s, t)$ and (s, t), may be interpreted as real or stochastic intervals. It will usually be clear from the context which interpretation is meant. For example, in equation (2.10), $1_{[0,t]}$ means the indicator function of the stochastic interval $[0, t] \times \Omega$.

The σ-field of subsets of $\mathbb{R}_+ \times \Omega$ generated by the class of stochastic intervals is called the *optional* σ-field and is denoted by $\mathcal{O}$. The graph of an optional time τ, denoted by

$$[\tau] = [0, \tau]\backslash[0, \tau) = \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \tau(\omega) = t\},$$

is in $\mathcal{O}$. A function $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ will be called *optional* iff X is $\mathcal{O}$-measurable. If A is a stochastic interval, then $1_A(t, \cdot)$ is $\mathcal{F}_t$-measurable for each t, by the optionality of the end-points of A. Then it follows as for predictable functions that any optional function is an adapted process, and we shall refer to it as an optional process.

We now investigate the relationship between $\mathcal{P}$ and $\mathcal{O}$. Each predictable rectangle of the form $(s,t]\times F$ where $F \in \mathcal{F}_s$ and $s < t$ in $\mathbb{R}_+$, is a stochastic interval of the form $(\eta, \tau]$ with $\eta \equiv s$, $\tau = s$ on $\Omega\backslash F$ and $\tau = t$ on F. Also, for $F_0 \in \mathcal{F}_0$, $\{0\} \times F_0 = \bigcap_n [0, \tau_n)$ where

$$\tau_n = \begin{cases} \frac{1}{n} & \text{on } F_0 \\ 0 & \text{on } \Omega\backslash F_0 \end{cases}$$

is optional for each n. It follows that $\mathcal{R} \subset \mathcal{O}$ and hence, since $\mathcal{R}$ generates $\mathcal{P}$, we have $\mathcal{P} \subset \mathcal{O}$. In the following lemma we show that certain types of stochastic intervals are predictable.

Lemma 2.1. *Stochastic intervals of the form* $[0, \tau]$ *and* $(\eta, \tau]$ *are predictable.*

Proof. Since $(\eta, \tau] = [0, \tau]\backslash[0, \eta]$, it suffices to prove that a stochastic interval of the form $[0, \tau]$ is predictable. For this we use a standard approximation of τ by a decreasing sequence $\{\tau_n\}$ of countably valued optional times, defined by $\tau_n = 2^{-n}[2^n\tau + 1]$. Since $\tau_n \downarrow \tau$, we have $[0, \tau] = \bigcap_n [0, \tau_n]$. For each n,

$$[0, \tau_n] = (\{0\} \times \Omega) \cup \left(\bigcup_{k \in \mathbb{N}_0} (k2^{-n}, (k + 1)2^{-n}] \times \{\tau \geq k2^{-n}\} \right).$$

Here $\{\tau \geq k2^{-n}\} = \Omega\backslash\{\tau < k2^{-n}\} \in \mathcal{F}_{k2^{-n}}$, since τ is optional. It follows that $[0, \tau] \in \mathcal{P}$. ∎

Stochastic intervals, other than those mentioned in the preceding lemma, are not in general predictable without further restriction on the end-points. An $\mathcal{F}$-measurable function $\tau : \Omega \longrightarrow \overline{\mathbb{R}}_+$ is called a *predictable time* (or simply predictable) if there is a sequence of optional times $\{\tau_n\}$ which increases to τ such that each τ_n is *strictly* less than τ on $\{\tau \neq 0\}$. Such a sequence $\{\tau_n\}$ is called an *announcing sequence* for τ. It is easily verified that a predictable time is an optional time and as a partial converse, if τ is optional then $\tau + t$ is predictable for each constant $t > 0$. Intuitively speaking, if $\tau > 0$ is the first time some random event occurs, then τ is predictable if this event cannot take us by surprise because we are forewarned by a sequence of prior events, occurring at times τ_n. A very simple example of a predictable time is

$$0_{F_0} = \begin{cases} 0 & \text{on } F_0 \\ \infty & \text{on } F_0^c \,, \end{cases}$$

where $F_0 \in \mathcal{F}_0$. An announcing sequence for 0_{F_0} is $\{0_{F_0} \wedge n,\, n = 1, 2, \ldots\}$. An example of a non-predictable optional time is the time at which the first jump of a Poisson process occurs.

Parts (iii) and (iv) of the following lemma elucidate the reason for the names of the predictable and optional σ-fields.

Lemma 2.2.

(i) *If τ is a predictable time, then $[\tau, \infty)$ is predictable.*

(ii) *All stochastic intervals of the following forms are predictable: $(\eta, \tau]$ where η and τ are optional, $[\eta, \tau]$ and (τ, η) where η is predictable and τ is optional, $[\eta, \tau)$ where η and τ are both predictable.*

(iii) *The predictable σ-field is generated by the class of stochastic intervals of the form $[\tau, \infty)$ where τ is a predictable time.*

(iv) *The optional σ-field is generated by the class of stochastic intervals of the form $[\tau, \infty)$ where τ is an optional time.*

Proof. To prove (i), suppose τ is a predictable time and $\{\tau_n\}$ is an announcing sequence for τ. Since $\tau_n \uparrow \tau$ and $\tau_n < \tau$ on $\{\tau \neq 0\}$, we have

$$[\tau, \infty) = (\{0\} \times \{\tau = 0\}) \cup \left(\bigcap_n (\tau_n, \infty) \right).$$

Here $\{\tau = 0\} \in \mathcal{F}_0$ and $(\tau_n, \infty) = (I\!R_+ \times \Omega) \backslash [0, \tau_n]$ is predictable for each n, by Lemma 2.1. Hence $[\tau, \infty)$ is predictable, proving (i).

For an optional time τ, $[0, \tau]$ is predictable by Lemma 2.1, and if τ is predictable, then $[0, \tau)$—the complement of $[\tau, \infty)$—is predictable by part (i) above. Since each of the four kinds of stochastic intervals in (ii) can be written as a difference of two intervals of the above kind, with η in place of τ in one of them, the result (ii) follows.

For the proof of (iii), let $\mathcal{Q}$ denote the σ-field generated by the class of stochastic intervals of the form $[\tau, \infty)$ where τ is predictable. By part (i), $\mathcal{Q} \subset \mathcal{P}$ and to show $\mathcal{P} \subset \mathcal{Q}$, it suffices to prove $\mathcal{R} \subset \mathcal{Q}$. For any optional time τ we have $[0, \tau] = \bigcap_n [0, \tau + \frac{1}{n})$. Here $\tau + \frac{1}{n}$ is predictable and therefore, by complementation, $[0, \tau + \frac{1}{n}) \in \mathcal{Q}$. Consequently, $[0, \tau] \in \mathcal{Q}$. A predictable rectangle $(s, t] \times F$ for $F \in \mathcal{F}_s$ and $s < t$, is a stochastic interval of the form $(\eta, \tau] = [0, \tau] \backslash [0, \eta]$ and is therefore in $\mathcal{Q}$. If $F_0 \in \mathcal{F}_0$, then since 0_{F_0} is a predictable time, we have $\{0\} \times F_0 = [0_{F_0}, \infty) \backslash (0, \infty) \in \mathcal{Q}$. Thus, $\mathcal{R} \subset \mathcal{Q}$ and hence (iii) is proved.

Since $\mathcal{O}$ is generated by the stochastic intervals, to prove (iv) it suffices to show that all stochastic intervals are contained in the σ-field $\mathcal{S}$ generated by the class of stochastic intervals of the form $[\tau, \infty)$. If τ is optional, then $\tau + \frac{1}{n}$ is optional for each n and hence $(\tau, \infty) = \bigcup_n [\tau + \frac{1}{n}, \infty)$ is in $\mathcal{S}$. Since the class consisting of the stochastic intervals of the form $[\tau, \infty)$ and (τ, ∞) generates all stochastic intervals by combinations of the operations of complementation and differencing, it follows that all stochastic intervals are in $\mathcal{S}$, as required. ∎

For $\tau : \Omega \to \overline{I\!R}_+$, we have by the above lemma:

(i) if τ is predictable, then $[\tau, \infty)$ is predictable,

(ii) if τ is optional, then $[\tau, \infty)$ is optional.

The converses of these results are also true. The converse of (ii) follows from the result proved earlier that if $[\tau, \infty)$ is an optional set, then $1_{[\tau,\infty)}$ is an adapted process. For the more difficult proof of the converse of (i), we refer the reader to Dellacherie and Meyer [23, IV-76]. (*Warning*: Dellacherie and Meyer use the conclusion of (i) as their definition of a predictable time and derive the existence of an announcing sequence from it). Alternative characterizations of the predictable and optional σ-fields to those of Lemma 2.2 will be given in Chapter 3.

We conclude this section with the following result which is well known to experts, especially those interested in applications to mathematical economics. It is also referred to later in Section 9.4. The proof given here was told to us by Michael Sharpe. The argument for the "if" part is similar to that in Chung-Walsh [19]; the proof of the "only if" part is standard.

Proposition. *Every optional time is predictable if and only if every (local) martingale (adapted to $\{\mathcal{F}_t\}$) has a continuous version.*

Proof. For the "if" part, suppose every martingale has a continuous version. Let τ be an optional time. We may assume that τ is bounded because $\tau \wedge n \uparrow \tau$ and the limit of an increasing sequence of predictable times is predictable. Consider the supermartingale Y defined by

$$Y_t = E[(\tau - t)^+ \,|\, \mathcal{F}_t] = E[\tau \,|\, \mathcal{F}_t] - \tau \wedge t.$$

Since $\{E[\tau \,|\, \mathcal{F}_t], t \geq 0\}$ is a martingale, by assumption we may choose a continuous version of it. Then Y has continuous sample paths.

We first prove that P-a.s., $Y_t = 0$ for all $t \geq \tau$ and $Y_t > 0$ for all $t < \tau$. For $t \geq \tau$, this follows from the fact that Y has continuous paths and

$$\begin{aligned} Y_t 1_{\{t\geq\tau\}} &= E[(\tau - t)^+ 1_{\{t\geq\tau\}} \,|\, \mathcal{F}_t] \\ &= E[0 \,|\, \mathcal{F}_t] = 0 \quad P\text{-a.s.} \end{aligned}$$

To prove that $Y_t > 0$ for $t < \tau$, let $\sigma = \inf\{t \geq 0 : Y_t = 0\}$. Then by Doob's stopping theorem, P-a.s.,

$$Y_{\tau\wedge\sigma} = E[\tau \,|\, \mathcal{F}_{\tau\wedge\sigma}] - \tau \wedge \sigma = E[(\tau - \sigma)^+ \,|\, \mathcal{F}_{\tau\wedge\sigma}].$$

Now, by the result: $Y_\tau = 0$ P-a.s., the definition of σ implies that $Y_{\sigma\wedge\tau} = 0$ P-a.s. Thus, taking expectations in the above yields

$$0 = E[(\tau - \sigma)^+].$$

Hence $\tau \le \sigma$ P-a.s. and the desired property of Y follows. It then follows that $\tau_n \equiv \inf\{t \ge 0 : Y_t \le \frac{1}{n}\}$ is an announcing sequence for τ and hence τ is predictable.

For the "only if" part, suppose that every optional time is predictable. Let M be a local martingale. To prove M has a continuous version, it suffices by localization to consider the case where M is a uniformly integrable martingale. (Note that if M is a martingale, then $M_{\cdot\wedge n}$ is a uniformly integrable martingale for each positive integer n.) Then by Theorem 1.5, $M_\infty = \lim_{t\to\infty} M_t$ exists P-a.s. and $\{M_t, \mathcal{F}_t, t \in [0,\infty]\}$ is a martingale. Since every martingale has a version that is right continuous with finite left limits (see Chung [12, Section 1.4]), we may assume that M is such a version, and so M can only have jump discontinuities. For $\varepsilon > 0$, let $\tau \equiv \inf\{t \ge 0 : M_t - M_{t-} \ge \varepsilon\}$. Then τ is an optional time (see Exercise 2), and by assumption it is also predictable. Let $\{\tau_n\}$ be an announcing sequence for τ. By Doob's stopping theorem, for all positive integers n,

$$E[M_\tau \mid \mathcal{F}_{\tau_n}] = M_{\tau_n}.$$

Letting $n \to \infty$ in the above, we obtain

$$E\left[M_\tau \,\middle|\, \bigvee_{n=1}^{\infty} \mathcal{F}_{\tau_n}\right] = M_{\tau-} \le M_\tau - \varepsilon 1_{\{\tau<\infty\}}.$$

By taking expectations in the above, we obtain: $\varepsilon P(\tau < \infty) \le 0$ and hence $P(\tau < \infty) = 0$. Similarly, for $\sigma \equiv \inf\{t \ge 0 : M_t - M_{t-} \le -\varepsilon\}$ we have $P(\sigma < \infty) = 0$. Since $\varepsilon > 0$ was arbitrary, it follows that P-a.s., M has no jumps at all. ∎

Example. Suppose $\{\mathcal{F}_t\}$ is the filtration generated by a Hunt process (cf. [12, Chapter 3]) with continuous sample paths, where the filtration is augmented by the P-null sets in $\mathcal{F}$. It is known [19] that every (local) martingale adapted to $\{\mathcal{F}_t\}$ has a continuous version. Hence every optional time is predictable. In particular, these properties hold if $\{\mathcal{F}_t\}$ is the standard filtration associated with a d-dimensional Brownian motion.

Next we define a measure on the predictable sets which is the key to the basic isometry used in defining the stochastic integral.

2.4 Measure on the Predictable Sets

Suppose that $Z = \{Z_t,\ t \in \mathbb{R}_+\}$ is a real-valued process adapted to the (standard) filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$, and $Z_t \in L^1$ for each $t \in \mathbb{R}_+$.

We define a set function λ_Z on $\mathcal{R}$ by

$$\begin{aligned} \lambda_Z\left((s,t] \times F\right) &= E\left(1_F(Z_t - Z_s)\right) \\ &\qquad \text{for } F \in \mathcal{F}_s \text{ and } s < t \text{ in } \mathbb{R}_+, \\ \lambda_Z\left(\{0\} \times F_0\right) &= 0 \quad \text{for } F_0 \in \mathcal{F}_0. \end{aligned} \tag{2.1}$$

We extend λ_Z to be a finitely additive set function on the ring $\mathcal{A}$ generated by $\mathcal{R}$ by defining

$$\lambda_Z(A) = \sum_{j=1}^{n} \lambda_Z(R_j)$$

for any $A = \bigcup_{j=1}^{n} R_j$, where $\{R_j, 1 \le j \le n\}$ is a finite collection of disjoint sets in $\mathcal{R}$. The value of $\lambda_Z(A)$ is the same for all representations of A as a finite disjoint union of sets in $\mathcal{R}$. We call λ_Z a *content* if $\lambda_Z \ge 0$ on $\mathcal{R}$ and hence on $\mathcal{A}$.

It is clear that if Z is a martingale then $\lambda_Z \equiv 0$, and if Z is a submartingale then $\lambda_Z \ge 0$. In particular, suppose $M = \{M_t, t \in \mathbb{R}_+\}$ is an L^2-martingale, then $(M)^2 = \{(M_t)^2, t \in \mathbb{R}_+\}$ is a submartingale and hence $\lambda_{(M)^2} \ge 0$. More explicitly, for $F \in \mathcal{F}_s$ and $s < t$,

$$\lambda_{(M)^2}\left((s,t] \times F\right) = E\left\{1_F(M_t - M_s)^2\right\}. \tag{2.2}$$

This is proved by setting $Y = 1_F$ in the following important identity. For

$s < t$ in $\mathbb{R}_+$ and any real-valued $Y \in b\mathcal{F}_s$,

$$
\begin{aligned}
E\left\{Y(M_t - M_s)^2\right\} &= E\left\{Y\left((M_t)^2 - 2M_tM_s + (M_s)^2\right)\right\} \\
&= E\left\{Y\left((M_t)^2 + (M_s)^2\right)\right\} - 2E\left\{YM_sE\left(M_t \,|\, \mathcal{F}_s\right)\right\} \\
&= E\left\{Y\left((M_t)^2 + (M_s)^2\right)\right\} - 2E\left\{Y(M_s)^2\right\} \\
&= E\left\{Y\left((M_t)^2 - (M_s)^2\right)\right\}.
\end{aligned}
\tag{2.3}
$$

The martingale property of M was used to obtain the third equality above.

We are interested in L^2-martingales M for which $\lambda_{(M)^2}$ can be extended to a measure on $\mathcal{P}$. It is shown in Section 2.8 that if Z is a right continuous positive submartingale, then the content λ_Z can be uniquely extended to a measure on $\mathcal{P}$, and this measure is σ-finite. Setting $Z = M^2$, we see that for a right continuous L^2-martingale M, there is a unique extension of $\lambda_{(M)^2}$ to a (σ-finite) measure on $\mathcal{P}$. An independent proof of this extendibility when M is a *continuous* L^2-martingale is given in Section 4.4.

Until stated otherwise, we suppose that $M = \{M_t, t \in \mathbb{R}_+\}$ is a right continuous L^2-martingale. We use μ_M to denote the unique measure on $\mathcal{P}$ which extends $\lambda_{(M)^2}$. This measure has been called the Doléans measure of M after C. Doléans-Dade who first made good use of it in a more general setting in [25]. We use $\mathcal{L}^2$ to denote $L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \mu_M)$, unless we need to emphasize the association with M in which case we use $\mathcal{L}^2(\mu_M)$.

Example. Consider a Brownian motion B in $\mathbb{R}$ with $B_0 \in L^2$ and let $\{\mathcal{F}_t\}$ denote its associated standard filtration. Then $\{B_t, \mathcal{F}_t, t \in \mathbb{R}_+\}$ is a continuous L^2-martingale. The following calculation shows that μ_B is the product measure $\lambda \times P$ on $\mathcal{P}$, where λ is the Lebesgue measure on $\mathbb{R}_+$. For $s < t$ and $F \in \mathcal{F}_s$ we have

$$
\begin{aligned}
\lambda_{(B)^2}\left((s,t] \times F\right) &= E\left(1_F(B_t - B_s)^2\right) \\
&= E\left\{1_F E\left((B_t - B_s)^2 \,|\, \mathcal{F}_s\right)\right\} \\
&= E\left\{(B_t - B_s)^2\right\} E\{1_F\} \\
&= (t - s)P(F) \\
&= (\lambda \times P)\left((s,t] \times F\right)
\end{aligned}
$$

The third equality above follows because $B_t - B_s$ is independent of $\mathcal{F}_s$, a consequence of the independence of the increments of B. The fourth equality follows because $B_t - B_s$ has mean zero and variance $t - s$. For $F_0 \in \mathcal{F}_0$,

$$\lambda_{(B)^2}(\{0\} \times F_0) = 0 = (\lambda \times P)(\{0\} \times F_0).$$

Thus, $\lambda_{(B)^2}$ agrees with $\lambda \times P$ on $\mathcal{R}$ and hence on $\mathcal{A}$. Since $\lambda \times P$ is a measure on $\mathcal{B} \times \mathcal{F} \supset \mathcal{P}$, we have $\mu_B = \lambda \times P$ on $\mathcal{P}$, by the uniqueness of the extension of $\lambda_{(B)^2}$ on $\mathcal{A}$ to μ_B on $\mathcal{P}$.

Example. Consider a Poisson process N with parameter $\alpha > 0$ and let $\{\mathcal{F}_t\}$ denote its associated standard filtration. Then $M \equiv \{N_t - \alpha t, \mathcal{F}_t, t \in \mathbb{R}_+\}$ is a right continuous L^2-martingale. In Exercise 4 you are asked to prove that $\alpha(\lambda \times P)$ is the Doléans measure for M. We shall not consider the Poisson process in detail in this text because stochastic integrals with respect to M can be defined using ordinary Lebesgue-Stieltjes integration (see Exercise 11). In addition, in our subsequent development of the stochastic calculus, from Chapter 4 onwards, we shall restrict ourselves to integrators that are *continuous* local martingales. By restricting to continuous integrators in this way, we are able to present the basic change of variable formula and ideas of stochastic calculus without the cumbersome notation and more elaborate considerations needed when one allows integrators with jumps.

2.5 Definition of the Stochastic Integral

First we define the stochastic integral $\int X\,dM$ when X is an $\mathcal{R}$-simple process and show that the map $X \to \int X\,dM$ is an isometry from a subspace of $\mathcal{L}^2$ into L^2. This isometry is the key to the extension of the definition to all X in $\mathcal{L}^2$.

When X is the indicator function of a predictable rectangle, the integral $\int X\,dM$ is defined as follows. For $s < t$ in $\mathbb{R}_+$ and $F \in \mathcal{F}_s$,

$$\int 1_{(s,t] \times F}\,dM \equiv 1_F(M_t - M_s) \tag{2.4}$$

and for $F_0 \in \mathcal{F}_0$,

$$\int 1_{\{0\} \times F_0}\,dM \equiv 0. \tag{2.5}$$

Let $\mathcal{E}$ denote the class of all functions $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ that are finite linear combinations of indicator functions of predictable rectangles. Such a function will be called an $\mathcal{R}$-simple process. Thus, $X \in \mathcal{E}$ can be expressed in the form

$$X = \sum_{j=1}^{n} c_j 1_{(s_j,t_j]\times F_j} + \sum_{k=1}^{m} d_k 1_{\{0\}\times F_{0k}} \tag{2.6}$$

where $c_j \in \mathbb{R}$, $F_j \in \mathcal{F}_{s_j}$, $s_j < t_j$ in $\mathbb{R}_+$ for $1 \le j \le n, n \in \mathbb{N}$, and $d_k \in \mathbb{R}$, $F_{0k} \in \mathcal{F}_0$ for $1 \le k \le m$, $m \in \mathbb{N}$. This representation, although not unique, can always be chosen such that the predictable rectangles $(s_j, t_j] \times F_j$ for $1 \le j \le n$ and $\{0\} \times F_{0k}$ for $1 \le k \le m$, are disjoint.

The integral $\int X \, dM$ for $X \in \mathcal{E}$ is defined by linearity. Thus, for X of the form (2.6) we have

$$\int X \, dM \equiv \sum_{j=1}^{n} c_j 1_{F_j} \left(M_{t_j} - M_{s_j}\right). \tag{2.7}$$

It can be easily verified that the value of the integral does not depend on the representation chosen for X.

Since $1_R \in \mathcal{L}^2$ for any predictable rectangle R, it follows that $\mathcal{E}$ is a subspace of $\mathcal{L}^2$; and since $M_t \in L^2$ for each t, $\int X \, dM$ is in L^2 for each $X \in \mathcal{E}$. The following theorem shows that the linear map $X \to \int X \, dM$ is an isometry from $\mathcal{E} \subset \mathcal{L}^2$ onto its image in L^2.

Theorem 2.3. *For $X \in \mathcal{E}$ we have the isometry*

$$E\left\{\left(\int X \, dM\right)^2\right\} = \int_{\mathbb{R}_+\times\Omega} (X)^2 \, d\mu_M. \tag{2.8}$$

Proof. Let $X \in \mathcal{E}$ be expressed in the form (2.6) where the predictable rectangles $R_j \equiv (s_j, t_j] \times F_j$ for $1 \le j \le n$ and $\{0\} \times F_{0k}$ for $1 \le k \le m$ are disjoint. Then by (2.7) we have

$$
\begin{aligned}
\left(\int X\,dM\right)^2 &= \sum_{j=1}^{n} c_j^2 1_{F_j}\left(M_{t_j} - M_{s_j}\right)^2 \\
&\quad + 2\sum_{j=1}^{n}\sum_{k=j+1}^{n} c_j c_k 1_{F_j \cap F_k}\left(M_{t_j} - M_{s_j}\right)\left(M_{t_k} - M_{s_k}\right).
\end{aligned}
\tag{2.9}
$$

For $1 \le j < k \le n$, since $R_j \cap R_k = \emptyset$, either

(i) $F_j \cap F_k = \emptyset$, or

(ii) $(s_j, t_j] \cap (s_k, t_k] = \emptyset$.

If (i) holds, the term indexed by j and k in the double sum above is zero. If (ii) holds, we may assume without loss of generality that $t_j \le s_k$. By the martingale property we have $E\left(M_{t_k} - M_{s_k} \mid \mathcal{F}_{s_k}\right) = 0$. This implies the basic "orthogonality property" that in the Hilbert space L^2, the increment $M_{t_k} - M_{s_k}$ of M is orthogonal to the subspace $L^2(\Omega, \mathcal{F}_{s_k}, P)$, i.e., for any $Y \in L^2(\Omega, \mathcal{F}_{s_k}, P)$,

$$
E\left\{Y\left(M_{t_k} - M_{s_k}\right)\right\} = E\left\{Y E\left(M_{t_k} - M_{s_k} \mid \mathcal{F}_{s_k}\right)\right\} = 0.
$$

Since $1_{F_j \cap F_k}\left(M_{t_j} - M_{s_j}\right) \in L^2(\Omega, \mathcal{F}_{s_k}, P)$, it follows that the expected value of the term indexed by j and k in the double sum in (2.9) is also zero if (ii) holds. Thus, by taking expectations in (2.9) and using (2.1)–(2.2), we obtain

$$
\begin{aligned}
E\left\{\left(\int X\,dM\right)^2\right\} &= \sum_{j=1}^{n} c_j^2 E\left\{1_{F_j}\left(M_{t_j} - M_{s_j}\right)^2\right\} \\
&= \sum_{j=1}^{n} c_j^2 \mu_M\left((s_j, t_j] \times F_j\right) + \sum_{k=1}^{m} d_k^2 \mu_M\left(\{0\} \times F_{0k}\right) \\
&= \int_{\mathbb{R}_+ \times \Omega} (X)^2\, d\mu_M. \quad \blacksquare
\end{aligned}
$$

The extension of the definition of $\int X\,dM$ from integrands X in $\mathcal{E}$ to those in $\mathcal{L}^2$ is based on the isometry (2.8) and the fact that $\mathcal{E}$ is dense in the Hilbert space $\mathcal{L}^2$. A proof of the latter statement is given below.

Lemma 2.4. *The set of $\mathcal{R}$-simple processes $\mathcal{E}$ is dense in the Hilbert space $\mathcal{L}^2$.*

Proof. Since $\mathcal{P}$ is generated by the ring $\mathcal{A}$ and μ_M is σ-finite, then for each $\varepsilon > 0$, and $A \in \mathcal{P}$ such that $\mu_M(A) < \infty$, there is $A_1 \in \mathcal{A}$ such that $\mu_M(A \Delta A_1) < \varepsilon$ where $A \Delta A_1$ is the symmetric difference of A and A_1 (see Halmos [37; p. 42, 49]). It follows that any $\mathcal{P}$-simple function in $\mathcal{L}^2$ can be approximated arbitrarily closely in the $\mathcal{L}^2$-norm by functions in $\mathcal{E}$. The proof is completed by invoking the standard result that the set of $\mathcal{P}$-simple functions is dense in $\mathcal{L}^2$. ∎

If we regard $\mathcal{L}^2$ and L^2 as Hilbert spaces, then the map $X \to \int X\,dM$ is a linear isometry from the dense subspace $\mathcal{E}$ of $\mathcal{L}^2$ into L^2, and hence can be uniquely extended to a linear isometry from $\mathcal{L}^2$ into L^2 (see Taylor [74, p. 99]). For $X \in \mathcal{L}^2$, we define $\int X\,dM$ as the image of X under this isometry. Then (2.8) holds for all X in $\mathcal{L}^2$ and we refer to it simply as "the isometry" since it is the only one we use.

Notation. Let $\Lambda^2(\mathcal{P}, M)$ denote the space of all $X \in \mathcal{P}$ such that $1_{[0,t]}X \in \mathcal{L}^2$ for each $t \in \mathbb{R}_+$. Here $1_{[0,t]}X$ denotes the process defined by

$$(1_{[0,t]}X)(s,\omega) = 1_{[0,t]}(s)X(s,\omega) \quad \text{for all } (s,\omega) \in \mathbb{R}_+ \times \Omega.$$

Let $X \in \Lambda^2(\mathcal{P}, M)$. For each t, $\int 1_{[0,t]}X\,dM$ is well-defined and has the isometry property:

$$E\left\{\left(\int 1_{[0,t]}X\,dM\right)^2\right\} = \int_{[0,t]\times\Omega} (X)^2\,d\mu_M. \tag{2.10}$$

By definition, $\mu_M(\{0\} \times \Omega) = 0$, hence by (2.10) we have

$$\int 1_{\{0\}\times\Omega}X\,dM = 0 \quad \text{a.s.} \tag{2.11}$$

If $X \in \mathcal{E}$ and (2.6) is a representation for X, then for each t, $1_{[0,t]}X$ is in $\mathcal{E}$ and

$$\int 1_{[0,t]} X \, dM = \sum_{j=1}^{n} c_j 1_{F_j} \left(M_{t_j \wedge t} - M_{s_j \wedge t} \right). \tag{2.12}$$

Here the right member of (2.12) is a right continuous L^2-martingale indexed by t. By using the isometry, we shall extend this to prove for $X \in \Lambda^2(\mathcal{P}, M)$ that $\left\{ \int 1_{[0,t]} X \, dM, t \in I\!R_+ \right\}$ is an L^2-martingale which has a right continuous version; thus showing that these properties of M are preserved by the integration.

Theorem 2.5. *Let $X \in \Lambda^2(\mathcal{P}, M)$ and for each t let $Y_t = \int 1_{[0,t]} X \, dM$. Then $Y = \{Y_t, t \in I\!R_+\}$ is a zero-mean L^2-martingale and there is a version of Y with all paths right continuous.*

Proof. Let $n \in I\!N$. Then $1_{[0,n]} X \in \mathcal{L}^2$ and by Lemma 2.4 there is a sequence $\{X^k, k \in I\!N\}$ in $\mathcal{E}$ which converges to $1_{[0,n]} X$ in $\mathcal{L}^2$. It follows that for each $t \in [0, n]$, $1_{[0,t]} X^k$ converges to $1_{[0,t]} X$ in $\mathcal{L}^2$ as $k \to \infty$, and hence by the isometry, $Y_t^k \equiv \int 1_{[0,t]} X^k \, dM$ converges to $Y_t = \int 1_{[0,t]} X \, dM$ in L^2. For each k, by the remarks following equation (2.12), $Y^k = \left\{ Y_t^k, t \in I\!R_+ \right\}$ is a right continuous L^2-martingale. Since the martingale property is preserved by L^2-limits (see Proposition 1.3), it follows that $\{Y_t, t \in [0, n]\}$ is an L^2-martingale. Since n was arbitrary, we conclude that $\{Y_t, t \in I\!R_+\}$ is an L^2-martingale. By (2.11), $Y_0 = 0$ a.s. and hence $E(Y_t) = E(Y_0) = 0$ for all t.

Since $\{Y_t, \mathcal{F}_t, t \in I\!R_+\}$ is a martingale and $\{\mathcal{F}_t\}$ is a standard filtration, by [12, p. 29], there is a version of $\{Y_t, t \in I\!R_+\}$ with all paths right continuous. Another proof of this last property of Y can be obtained by replacing "continuous" with "right continuous" in the proof of Theorem 2.6 below. ∎

Theorem 2.6. *Suppose the hypotheses of Theorem 2.5 hold and M has continuous paths. Then there is a version of Y with continuous paths.*

Proof. We first show that for each $n \in I\!N$ there is a continuous version Z^n of $\{Y_t, t \in [0, n]\}$. For $j < k$ and Y^j, Y^k as in the above proof, $Y^k - Y^j$ is a continuous L^2-martingale and thus by the basic inequality (1.3) of Theorem 1.4 we have

$$P\left(\sup_{0\le t\le n}\left|Y_t^k - Y_t^j\right| \ge \frac{1}{2^m}\right) \le 2^{2m} E\left(\left|Y_n^k - Y_n^j\right|^2\right) \tag{2.13}$$

for each $m \in I\!N$. Since Y_n^k converges to Y_n in L^2 as $k \to \infty$, there is a subsequence $\{Y_n^{k_m}, m \in I\!N\}$ such that

$$E\left(\left|Y_n^{k_{m+1}} - Y_n^{k_m}\right|^2\right) \le \frac{1}{2^{3m}}. \tag{2.14}$$

By combining (2.13) and (2.14), we obtain

$$\sum_m P\left(\sup_{0\le t\le n}\left|Y_t^{k_{m+1}} - Y_t^{k_m}\right| \ge \frac{1}{2^m}\right) \le \sum_m \frac{1}{2^m} < \infty.$$

An application of the Borel-Cantelli lemma then yields

$$P\left(\sup_{0\le t\le n}\left|Y_t^{k_{m+1}} - Y_t^{k_m}\right| \ge \frac{1}{2^m} \quad \text{i.o.}\right) = 0,$$

where i.o. is our abbreviation for "infinitely often". It follows that there is a set Ω_n of probability one such that for each $\omega \in \Omega_n$, $\{Y^{k_m}(t,\omega), m \in I\!N\}$ converges uniformly for $t \in [0,n]$ to some limit $Z^n(t,\omega)$. Since $Y^{k_m}(\,\cdot\,,\omega)$ is continuous on $[0,n]$, so is $Z^n(\,\cdot\,,\omega)$, by the uniformity of the convergence. Moreover, for each $t \in [0,n]$, $Y_t^{k_m}$ converges a.s. to Z_t^n, and in L^2 to Y_t, as $m \to \infty$; hence $Z_t^n = Y_t$ a.s. Thus, $Z^n = \{Z_t^n, t \in [0,n]\}$ is a continuous version of $\{Y_t, t \in [0,n]\}$ on Ω_n. For $n_1 < n_2$, $\{Z_t^n, t \in [0,n_1]\}$ and $\{Z_t^{n_2}, t \in [0,n_1]\}$ are both continuous versions of $\{Y_t, t \in [0,n_1]\}$ on $\Omega_{n_1} \cap \Omega_{n_2}$, and are therefore indistinguishable there. It follows that there is a set $\Omega_0 \subset \bigcap_n \Omega_n$ of probability one such that for each $\omega \in \Omega_0$, $\lim_{n\to\infty} Z^n(t,\omega)$ exists and is finite for each $t \in I\!R_+$, and for each $n \in I\!N$ this limit equals $Z^n(t,\omega)$ for each $t \in [0,n]$. If we denote this limit by $Z(t,\omega)$, then Z is a continuous version of Y on Ω_0. It can easily be extended to a continuous version on Ω. ∎

Notation. We shall use the notation $\{\int_{[0,t]} X\,dM, t \in I\!R_+\}$ to denote a right continuous version of $\{\int 1_{[0,t]} X\,dM, t \in I\!R_+\}$ and $\int_{(s,t]} X\,dM$ to denote $\int_{[0,t]} X\,dM - \int_{[0,s]} X\,dM$ for $s < t$ in $I\!R_+$. If M is known to be continuous, we shall use $\{\int_0^t X\,dM, t \in I\!R_+\}$ to denote a continuous version of $\{\int 1_{[0,t]} X\,dM, t \in I\!R_+\}$ and $\int_s^t X\,dM$ to denote $\int_0^t X\,dM - \int_0^s X\,dM$ for $s < t$.

In the following theorem, we list some properties of the stochastic integral $\int_{[0,t]} X\,dM$.

Theorem 2.7. *Let $X \in \Lambda^2(\mathcal{P}, M)$ and let Y denote the right continuous stochastic integral process $\{\int_{[0,t]} X\,dM, t \in I\!R_+\}$. Then the following properties hold.*

(i) *For $s < t$ in $I\!R_+$ and any r.v. $Z \in b\mathcal{F}_s$, we have $1_{(s,t]}Z \in \mathcal{P}$, $1_{(s,t]}ZX \in \Lambda^2(\mathcal{P}, M)$, and a.s.*

$$\int 1_{(s,t]} ZX\,dM = Z \int_{(s,t]} X\,dM. \tag{2.15}$$

(ii) *The measure μ_Y associated with the right continuous L^2-martingale Y has density $(X)^2$ with respect to μ_M, i.e., for any $A \in \mathcal{P}$,*

$$\mu_Y(A) = \int_A (X)^2\,d\mu_M. \tag{2.16}$$

(iii) *For any bounded optional time τ,*

$$Y_\tau \equiv \int_{[0,\tau]} X\,dM = \int 1_{[0,\tau]} X\,dM \quad \text{a.s.} \tag{2.17}$$

Remark. The first equality in (2.17) is by definition, where for each $\omega, Y_\tau(\omega)$ is the value of $Y_t(\omega)$ at $t = \tau(\omega)$; whereas the integral on the far right of (2.17) is a random variable defined via the L^2-isometry. Their a.s. equality must therefore be proved.

Proof. For $s < t$ in $I\!R_+$ and $Z \in \mathcal{F}_s$, $1_{(s,t]}Z \in \mathcal{P}$ follows by linearity and a monotone class argument from the fact that $1_{(s,t]\times G} \in \mathcal{P}$ for $G \in \mathcal{F}_s$.

Then, since $X \in \mathcal{P}$, $1_{(s,t]}ZX \in \mathcal{P}$. (For a partial converse see Exercise 8.) Furthermore, if Z is bounded, then since $X \in \Lambda^2(\mathcal{P}, M)$, we have $1_{(s,t]}ZX \in \Lambda^2(\mathcal{P}, M)$. Now that the measurability and integrability properties in part (i) have been established, we focus on the proof of (2.15). Note that (2.15) is easily verified if $Z = 1_G$ for some $G \in \mathcal{F}_s$ and $X = 1_{(u,v] \times F}$ for some $u < v$ in $\mathbb{R}_+$ and $F \in \mathcal{F}_u$. It then follows by linearity that (2.15) holds when Z is an $\mathcal{F}_s$-simple function and X is in $\mathcal{E}$. For general Z and X, there is a bounded sequence $\{Z^k\}$ of $\mathcal{F}_s$-simple functions converging to Z pointwise on Ω, and a sequence $\{X^k\}$ of functions in $\mathcal{E}$ such that $\lim_{k \to \infty} 1_{(s,t]}X^k = 1_{(s,t]}X$ in $\mathcal{L}^2$. Since $\{Z^k\}$ is bounded, it follows that $\lim_{k \to \infty} 1_{(s,t]}Z^k X^k = 1_{(s,t]}ZX$ in $\mathcal{L}^2$ also. Now,

$$
\begin{aligned}
&\int 1_{(s,t]}ZX\, dM - Z \int_{(s,t]} X\, dM \\
&= \int 1_{(s,t]} \left(ZX - Z^k X^k\right) dM \\
&\quad + \left\{ \int 1_{(s,t]}Z^k X^k\, dM - Z^k \int_{(s,t]} X^k\, dM \right\} \\
&\quad + Z^k \int_{(s,t]} (X^k - X)\, dM + (Z^k - Z) \int_{(s,t]} X\, dM.
\end{aligned}
\tag{2.18}
$$

We claim that the terms following the equals sign above converge to zero in L^1 as $k \to \infty$. By the simple function case discussed above, the second term (in braces) is zero. The first and third terms converge to zero in L^2, by the isometry. The last term tends to zero in L^1, by Schwarz's inequality and bounded convergence. Since the expression in (2.18) preceding the equals sign is independent of k, it follows that it is zero a.s., proving (i).

For the proof of part (ii), it suffices to prove (2.16) for $A \in \mathcal{R}$, since the measures μ_Y and $(X)^2 d\mu_M$ on $\mathcal{P}$ are uniquely determined by their values on $\mathcal{R}$. If $A = \{0\} \times F_0$ for $F_0 \in \mathcal{F}_0$, both sides of (2.16) are zero. On the other hand, if $A = 1_{(s,t] \times F}$ for some $s < t$ and $F \in \mathcal{F}_s$, then

$$
\mu_Y(A) = E\left\{1_F (Y_t - Y_s)^2\right\} = E\left\{ (1_F \int_{(s,t]} X\, dM)^2 \right\}
$$

which by part (i) equals

$$E\left\{\left(\int 1_{(s,t]\times F} X\, dM\right)^2\right\} = \int 1_{(s,t]\times F}(X)^2\, d\mu_M = \int_A (X)^2\, d\mu_M .$$

The first equality above follows by the isometry. Thus (2.16) holds for all A in $\mathcal{R}$ and hence for all A in $\mathcal{P}$.

For the proof of part (iii), let τ be an optional time, bounded by C, say. We approximate τ in the standard way by a sequence $\{\tau_n,\, n \in \mathbb{N}\}$ of optional times such that for each n, τ_n takes only finitely many values and (2.17) holds with τ_n in place of τ.

As in the proof of Lemma 2.1, for each n let $\tau_n = 2^{-n}\,[2^n\tau + 1]$. Also let $N_n = [2^n C]$. Then

$$[0,\tau_n] = (\{0\}\times\Omega) \cup \bigcup_{k=0}^{N_n} (k2^{-n},(k+1)2^{-n}] \times \{\tau \geq k2^{-n}\} \tag{2.19}$$

is in $\mathcal{A}$ and by the boundedness of τ_n, $1_{[0,\tau_n]}X \in \mathcal{L}^2$. Now, for each n,

$$\begin{aligned} Y_{\tau_n} &= \sum_{k=0}^{N_n} 1_{\{k2^{-n}\leq\tau<(k+1)2^{-n}\}} Y_{(k+1)2^{-n}} \\ &= \sum_{k=0}^{N_n} 1_{\{\tau\geq k2^{-n}\}} \left(Y_{(k+1)2^{-n}} - Y_{k2^{-n}}\right). \end{aligned}$$

Here the second equality is obtained by partial summation using $Y_0 = 0$ and $0 \leq \tau < (N_n+1)2^{-n}$. Thus by the definition of Y_t and part (i) we have a.s.

$$Y_{\tau_n} = \sum_{k=0}^{N_n} \int 1_{(k2^{-n},(k+1)2^{-n}]\times\{\tau\geq k2^{-n}\}} X\, dM.$$

By linearity, (2.11), and (2.19), it follows that a.s.

$$Y_{\tau_n} = \int 1_{[0,\tau_n]} X\, dM. \tag{2.20}$$

Since $\tau_n \downarrow \tau$ and Y is right continuous, the left side of (2.20) converges pointwise on Ω to Y_τ as $n \to \infty$; and since τ_n is bounded by $C+1$, it follows by dominated convergence and the isometry that the right side converges to $\int 1_{[0,\tau]} X\, dM$ in L^2. Hence (2.17) holds. ∎

The following corollary will be needed in the next section.

Corollary 2.8. *Let $s < t$ in $\mathbb{R}_+$, $F \in \mathcal{F}_s$, and τ be an optional time. Then we have a.s.:*

$$\int 1_{[0,\tau]} 1_{(s,t]\times F}\, dM = 1_F(M_{t\wedge\tau} - M_{s\wedge\tau}). \tag{2.21}$$

Proof. Let $X = 1_{(s,t]\times F}$. Then,

$$\int 1_{[0,u]} X\, dM = 1_F(M_{t\wedge u} - M_{s\wedge u}).$$

The right side of the above equality is right continuous in u and therefore may be used as the right continuous version $\int_{[0,u]} X\, dM$ of the left side. By replacing u by $\tau \wedge t$, we obtain

$$\int_{[0,\tau\wedge t]} X\, dM = 1_F(M_{t\wedge\tau} - M_{s\wedge\tau}).$$

It follows from (2.17) with $\tau \wedge t$ in place of τ there, that the left side of the above is equal a.s. to the left side of (2.21). Hence (2.21) holds a.s. ∎

Our definitions of the measure μ_M and the stochastic integral $\int_{[0,t]} X\, dM$ only involved the increments of M. Hence the values of these quantities would remain unchanged if we replaced M by $M - M_0$ in their definitions. Indeed, the following depends on this.

2.6 Extension to Local Integrators and Integrands

So far we have considered stochastic integrals $\int_{[0,t]} X\, dM$ where the integrator is a right continuous L^2-martingale and the integrand is in $\Lambda^2(\mathcal{P}, M)$. As a final extension we shall define the stochastic integral for integrators and integrands which only possess these properties in a local sense. Consequently, we shall no longer assume that M is a right continuous L^2-martingale. Instead, for the rest of this chapter, we suppose that M is a right continuous *local* L^2-martingale (see Section 1.10 for the definition). If $\{\tau_k\}$ is a localizing sequence for M, we use M^k to denote the right continuous L^2-martingale $\{M_{t\wedge\tau_k} - M_0, t \in \mathbb{R}_+\}$ for each k.

Next we define the class of integrands associated with M.

Definition. Let $\Lambda(\mathcal{P}, M)$ denote the class of all processes X for which there is a localizing sequence $\{\tau_k\}$ for M such that M^k is an L^2-martingale and

$$1_{[0,\tau_k]}X \in \Lambda^2(\mathcal{P}, M^k) \text{ for each } k. \tag{2.22}$$

Such a sequence will be called a localizing sequence for (X, M).

Example. Suppose M has continuous paths and X is a continuous adapted process. We claim $X \in \Lambda(\mathcal{P}, M)$ and

$$\tau_k = \inf\{t > 0 : |M_t - M_0| \vee |X_t| > k\}$$

defines a localizing sequence for (X, M).

For the proof of this claim we note that by results in Chapter 3, X is predictable. By the definition of τ_k, $(X)^2 \le k^2$ on $(0, t\wedge\tau_k]$. Moreover, by the isometry and Theorem 2.7(iii) we have

$$\int_{\mathbb{R}_+\times\Omega} 1_{(0,t\wedge\tau_k]}\, d\mu_{M^k} = E\left\{(M_{t\wedge\tau_k} - M_0)^2\right\} \le k^2.$$

Thus, by combining the above with the fact that $\mu_{M^k}(\{0\}\times\Omega) = 0$, we obtain

$$\int_{\mathbb{R}_+\times\Omega} 1_{[0,t\wedge\tau_k]}(X)^2\, d\mu_{M^k} \le k^4,$$

which proves the assertion. ∎

An important special case of the above example is obtained by setting $X = M$.

Let $X \in \Lambda(\mathcal{P}, M)$ and $\{\tau_k\}$ be a localizing sequence for (X, M). Then $Y^k \equiv \{\int_{[0,t]} 1_{[0,\tau_k]}X\, dM^k, t \in \mathbb{R}_+\}$ is a right continuous L^2-martingale for each k, by the notational convention following Theorem 2.6. We shall define $Y = \{\int_{[0,t]} X\,dM, t \in I\!R_+\}$ as the a.s. limit of the Y^k's, just as Z was defined from the Z^n's in the proof of Theorem 2.6. The difference being that here we use random truncation times τ_k whereas constant times n were used before. To validate this procedure, we need to verify that the following consistency condition holds:

(i) for each k, for almost every ω:

$$Y_t^m(\omega) = Y_t^k(\omega) \text{ for all } t \in [0, \tau_k] \text{ and } m \geq k, \tag{2.23}$$

and to show that

(ii) the definition of Y is independent (up to indistinguishability) of the choice of a localizing sequence for (X, M).

These assertions are formally obvious, but their proofs are long in details. They follow from the two lemmas below which are spelled out for the meticulous reader.

Lemma 2.9. *Let τ and η be optional times such that $M^\tau = \{M_{t\wedge\tau} - M_0, t \in I\!R_+\}$ and $M^\eta = \{M_{t\wedge\eta} - M_0, t \in I\!R_+\}$ are right continuous L^2-martingales. Let μ^τ and μ^η denote the measures μ_{M^τ} and μ_{M^η} on $\mathcal{P}$ associated respectively with M^τ and M^η. Then μ^τ and μ^η induce the same measure on the stochastic interval $[0, \tau\wedge\eta]$, i.e., for each $A \in \mathcal{P}$:*

$$\mu^\tau(A \cap [0, \tau\wedge\eta]) = \mu^\eta(A \cap [0, \tau\wedge\eta]). \tag{2.24}$$

Proof. Since the predictable rectangles generate $\mathcal{P}$, it suffices to prove (2.24) when A is a predictable rectangle. Clearly both sides of (2.24) are zero when $A = \{0\} \times F_0$ for some $F_0 \in \mathcal{F}_0$. On the other hand, if $A = (s,t] \times F$ for some $s < t$ and $F \in \mathcal{F}_s$, then by the isometry

$$\mu^\tau(A \cap [0, \tau\wedge\eta]) = E\left\{\left(\int 1_{[0,\tau\wedge\eta]} 1_{(s,t]\times F}\, dM^\tau\right)^2\right\}.$$

By Corollary 2.8 and since $M_u^\tau = M_u - M_0$ for $0 \le u \le \tau$, the right side above equals

$$E\{1_F(M_{t\wedge\tau\wedge\eta} - M_{s\wedge\tau\wedge\eta})^2\}.$$

Since the last expression is symmetric in τ and η, (2.24) follows for $A = (s,t] \times F$, and hence for all A in $\mathcal{P}$. ∎

Lemma 2.10. *Let τ and η be optional times such that $M^\tau = \{M_{t\wedge\tau} - M_0, t \in \mathbb{R}_+\}$ and $M^\eta = \{M_{t\wedge\eta} - M_0, t \in \mathbb{R}_+\}$ are right continuous L^2-martingales, and $1_{[0,\tau]}X \in \Lambda^2(\mathcal{P}, M^\tau)$ and $1_{[0,\eta]}X \in \Lambda^2(\mathcal{P}, M^\eta)$. Let Y^τ and Y^η respectively denote the right continuous L^2-martingales $\{\int_{[0,t]} 1_{[0,\tau]}X\, dM^\tau, t \in \mathbb{R}_+\}$ and $\{\int_{[0,t]} 1_{[0,\eta]}X\, dM^\eta, t \in \mathbb{R}_+\}$. Then*

$$P\{Y_t^\tau = Y_t^\eta \text{ for } 0 \le t \le \tau\wedge\eta\} = 1. \tag{2.25}$$

Proof. To prove (2.25), it is equivalent to prove the processes $\{Y^\tau_{t\wedge\tau\wedge\eta}, t \ge 0\}$ and $\{Y^\eta_{t\wedge\tau\wedge\eta}, t \ge 0\}$ are indistinguishable, and since these are right continuous, it suffices to prove

$$Y^\tau_{t\wedge\tau\wedge\eta} = Y^\eta_{t\wedge\tau\wedge\eta} \quad \text{a.s.} \tag{2.26}$$

for each t. It is easily verified using (2.17) and (2.21) that (2.26) holds if X is the indicator function of a predictable rectangle, and hence by linearity if X is in $\mathcal{E}$. For the general case, using the same notation as in Lemma 2.9, we have $1_{[0,t\wedge\tau\wedge\eta]}X \in \mathcal{L}^2(\mu^\tau)$. Hence there is a sequence $\{X^n\}$ in $\mathcal{E}$ which converges to $1_{[0,t\wedge\tau\wedge\eta]}X$ in $\mathcal{L}^2(\mu^\tau)$ and therefore $1_{[0,t\wedge\tau\wedge\eta]}(X^n - X) \to 0$ in $\mathcal{L}^2(\mu^\tau)$ as $n \to \infty$. The latter convergence is also in $\mathcal{L}^2(\mu^\eta)$, since μ^τ and μ^η induce the same measure on $[0, t\wedge\tau\wedge\eta]$ by Lemma 2.9. We have already verified that (2.26) holds if X is replaced by X^n. By letting $n \to \infty$ and using (2.17) and the isometries (for τ and η), it follows that (2.26) holds for X. ∎

By setting $\tau = \tau_m$ and $\eta = \tau_k$ in Lemma 2.10, we obtain (2.23). Consequently, there is a set Ω_0 of probability one such that for each $\omega \in \Omega_0, \lim_{m\to\infty} Y^m(t,\omega)$ exists and is finite for each t, and for each k and $t \in [0, \tau_k]$ this limit equals $Y^k(t,\omega)$. We denote this limit by $Y(t,\omega)$. Then $Y(\cdot,\omega)$ is right continuous for each $\omega \in \Omega_0$ and can easily be defined so that it is right continuous for $\omega \in \Omega \backslash \Omega_0$. Then for each k, almost surely, $Y_{t \wedge \tau_k} = Y_t^k$ for all t. Hence Y is a right continuous local L^2-martingale with localizing sequence $\{\tau_k\}$.

We shall denote Y_t by $\int_{[0,t]} X\, dM$ and $Y_t - Y_s$ by $\int_{(s,t]} X\, dM$. If M is actually continuous, then so is Y, and Y_t will be denoted by $\int_0^t X\, dM$ and $Y_t - Y_s$ by $\int_s^t X\, dM$.

The fact that the definition of Y is independent (up to indistinguishability) of the choice of a localizing sequence for (X, M) is an easy consequence of Lemma 2.10. The formal proof is left to the reader.

The following theorem is an immediate consequence of the above discussion and the example which follows (2.22). Recall that a continuous local martingale is automatically a local L^2-martingale.

Theorem 2.11. *Let M be a continuous local martingale and X be a continuous adapted process. Then $X \in \Lambda(\mathcal{P}, M)$ and $\{\int_0^t X\, dM, t \in \mathbb{R}_+\}$ is a continuous local martingale.*

If M is a right continuous L^2-martingale and $X \in \Lambda^2(\mathcal{P}, M)$, the above definition of $\int_{[0,t]} X\, dM$ is consistent with that given in the previous section because the integrals are unchanged if M is replaced by $M - M_0$. This replacement is also used to simplify some later proofs by reducing to the case $M_0 = 0$ so that $M_t^k = M_{t \wedge \tau_k}$. In connection with this, we emphasize that only the integrator can be replaced by $M - M_0$. In particular if M is a continuous local martingale we have

$$
\begin{aligned}
\int_0^t M\, dM &= \int_0^t (M - M_0)\, dM + M_0(M_t - M_0) \\
&= \int_0^t (M - M_0)\, d(M - M_0) + M_0(M_t - M_0).
\end{aligned}
\tag{2.27}
$$

2.7 Substitution Formula

Theorem 2.12. *Let M be a right continuous local martingale, $X \in \Lambda(\mathcal{P}, M)$ and $Y_t = \int_{[0,t]} X dM$ for all $t \geq 0$. Suppose $Z \in \Lambda(\mathcal{P}, Y)$. Then $XZ \in \Lambda(\mathcal{P}, M)$ and a.s. for all $t \geq 0$,*

$$\int_{[0,t]} Z dY = \int_{[0,t]} XZ dM. \tag{2.28}$$

Proof. By taking the minimum of a localizing sequence $\{\sigma_n\}$ for (X, M) (see (2.22)) and a localizing sequence $\{\rho_n\}$ for (Z, Y), we obtain a sequence $\{\tau_n \equiv \sigma_n \wedge \rho_n\}$ that is simultaneously localizing for (X, M) and (Z, Y). By stopping M and Y with this sequence and multiplying X and Z by $1_{[0,\tau_n]}$, we see that it suffices to prove the theorem for the case in which M and Y are L^2-martingales, $X \in \Lambda^2(\mathcal{P}, M)$ and $Z \in \Lambda^2(\mathcal{P}, Y)$. The proof for this case is divided into three parts.

(i) *Suppose Z is an $\mathcal{R}$-simple integrand. Then, for each $t \in \mathbb{R}_+$, (2.28) holds a.s.*

Proof of (i). Note that $1_{[0,t]} XZ \in \mathcal{L}^2(\mu_M)$, since this holds with X in place of XZ and Z is bounded. It follows from (2.5) with Y in place of M, and (2.10) with XZ in place of X and $\mu_M(\{0\} \times \Omega) = 0$, that (2.28) holds when $Z = 1_{\{0\} \times F_0}$ where $F_0 \in \mathcal{F}_0$. By Theorem 2.7, (2.28) also holds when $Z = 1_{(r,s] \times F_r}$ for $F_r \in \mathcal{F}_r$, $0 \leq r < s < \infty$. It then follows by linearity that (2.28) holds for any $\mathcal{R}$-simple Z.

(ii) *For each $t \in \mathbb{R}_+$,*

$$\int 1_{[0,t]} Z^2 d\mu_Y = \int 1_{[0,t]} X^2 Z^2 d\mu_M, \tag{2.29}$$

and hence $XZ \in \Lambda^2(\mathcal{P}, M)$.

Proof of (ii). If Z is $\mathcal{R}$-simple, then by the isometry (2.8) and part (i) above we have

$$
\begin{aligned}
\int 1_{[0,t]} Z^2 d\mu_Y &= E\left[\left(\int_{[0,t]} Z dY\right)^2\right] \\
&= E\left[\left(\int_{[0,t]} XZ dM\right)^2\right] \\
&= \int 1_{[0,t]} X^2 Z^2 d\mu_M .
\end{aligned}
\tag{2.30}
$$

The result for $Z \in \Lambda^2(\mathcal{P}, Y)$ then follows by applying a monotone class theorem.

(iii) *Almost surely, (2.28) holds for all $t \in I\!R_+$.*

Proof of (iii). Since both sides of (2.28) are right continuous processes, it suffices to show that (2.28) holds P-a.s for fixed t. By part (i) above, this holds for all $\mathcal{R}$-simple functions Z. For a general $Z \in \Lambda^2(\mathcal{P}, Y)$, there is a sequence $\{Z^{(m)}\}$ of $\mathcal{R}$-simple integrands such that $1_{[0,t]} Z^{(m)}$ converges to $1_{[0,t]} Z$ in $\mathcal{L}^2(\mu_Y)$ as $m \to \infty$. It follows by the L^2-isometry that

$$
\int_{[0,t]} Z^{(m)} dY \to \int_{[0,t]} Z dY \quad \text{in } L^2. \tag{2.31}
$$

Hence, using (2.29) with Z replaced by $Z^{(m)} - Z$, we see that as $m \to \infty$, $1_{[0,t]} X (Z^{(m)} - Z)$ converges to zero in $\mathcal{L}^2(\mu_M)$, and so by the isometry (2.8),

$$
\int_{[0,t]} X Z^{(m)} dM \to \int_{[0,t]} X Z dM \quad \text{in } L^2. \tag{2.32}
$$

Since the $Z^{(m)}$'s are $\mathcal{R}$-simple, the left members of (2.31) and (2.32) are equal for each m, by part (i), and hence the right members are equal P-a.s. ∎

2.8 A Sufficient Condition for Extendibility of λ_Z

For the proofs below, we have benefited from the presentation given in Letta [53].

For any set $A \subset I\!R_+ \times \Omega$ and $\omega \in \Omega$, let A^ω denote the ω-section of A:

$$A^\omega = \{t \in I\!R_+ : (t, \omega) \in A\}.$$

The *début* D_A of A is defined by

$$D_A(\omega) = \inf A^\omega, \quad \text{for all} \quad \omega \in A, \tag{2.33}$$

where $\inf \emptyset = \infty$.

Lemma 2.13.

(i) *If $B_j \subset I\!R_+$ for $1 \le j \le k \le \infty$, then*

$$\inf_{1 \le j \le k} (\inf B_j) = \inf \left(\bigcup_{j=1}^{k} B_j \right).$$

(ii) *If for each $n \ge 1$, C_n is a compact subset of $I\!R_+$ and $C_n \supset C_{n+1}$ for all n, then*

$$\lim_{n \to \infty} \uparrow (\inf C_n) = \inf \left(\bigcap_{n=1}^{\infty} C_n \right).$$

Remark. The empty set $\emptyset$ is a compact set.

Proof. We prove part (ii) only. If there is an n such that $C_n = \emptyset$, then $\inf C_n = \infty$ and the result reduces to $\infty = \infty$. On the other hand, suppose none of the C_n is empty, let $C = \bigcap_{n=1}^{\infty} C_n$ and $t_n = \inf C_n$ for each n. Then, $t_n \in C_n$ and $t^* \equiv \lim_{m\to\infty} \uparrow t_m \in C_n$ for all $n \ge 1$. Thus, $t^* \in C$, $t^* \ge \inf C \ge t_n$ for all n. Hence $t^* = \inf C$. ∎

Example. If the sets in Lemma 2.13(ii) are not compact, the result can fail, as the following example illustrates. If $C_n = (1 - \frac{1}{n}, 1) \cup \{2\}$, then $C_n \downarrow \{2\}$, but $\inf C_n = 1 - \frac{1}{n} \uparrow 1 \neq \inf \bigcap_n C_n = 2$.

A subset A of $\mathbb{R}_+ \times \Omega$ is called a *stochastic compact* if for each $\omega \in \Omega$, A^ω is a compact subset of $\mathbb{R}_+$.

Lemma 2.14.

(i) *If $A_j \subset \mathbb{R}_+ \times \Omega$ for $1 \le j \le k \le \infty$, then*

$$\inf_{1 \le j \le k} D_{A_j} = D_{\bigcup_{j=1}^k A_j}.$$

(ii) *If for each $n \ge 1$, A_n is a stochastic compact and $A_n \supset A_{n+1}$ for all n, then*

$$\lim_{n\to\infty} \uparrow D_{A_n} = D_{\bigcap_{n=1}^\infty A_n}.$$

Proof. This is an immediate consequence of Lemma 2.13. ∎

Definition. A random variable X is said to have *finite range* if $\{X(\omega) : \omega \in \Omega\}$ is a finite set.

Lemma 2.15. *If $A \in \mathcal{A}$, then D_A is optional and has finite range.*

Proof. Since $A \in \mathcal{A}$, there are finitely many disjoint predictable rectangles $R_1, \ldots, R_n$, such that $A = \bigcup_{j=1}^n R_j$. By Lemma 2.14(i),

$$D_A = \min_{1 \le j \le n} D_{R_j}.$$

If $R = (s,t] \times F_s$ for $F_s \in \mathcal{F}_s$, then $D_R = s1_{F_s} + \infty \cdot 1_{F_s^c}$ is an optional time. If $R = \{0\} \times F_0$ for $F_0 \in \mathcal{F}_0$, then $D_R = 0 \cdot 1_{F_0} + \infty \cdot 1_{F_0^c}$ is an optional time. Since the minimum of a finite number of optional times, each of which has finite range, is optional and has finite range, the result follows. ∎

Remark. More generally, the début of any optional (in fact, of any progressively measurable) set is optional (see Chung [12, Theorem 3, Section 1.5]).

Theorem 2.16. *Let Z be a right continuous positive submartingale. Then the content λ_Z defined in Section 2.4 can be uniquely extended to a measure on $\mathcal{P}$, and this measure is σ-finite.*

Proof. We first note for later reference that for each $t \geq 0$, from the non-negativity and submartingale property of Z,

$$0 \leq Z_s \leq E[Z_t \,|\, \mathcal{F}_s] \quad \text{for all} \quad 0 \leq s \leq t,$$

and hence $\{Z_s, 0 \leq s \leq t\}$ is uniformly integrable.

Since

$$\lambda_Z([0, t] \times \Omega) = E[Z_t - Z_0] < \infty \qquad \text{for all} \;\; t \geq 0,$$

any extension of λ_Z to a measure on $\mathcal{P}$ must be σ-finite.

By the Caratheodory extension theorem [37, p. 54], since the σ-ring generated by $\mathcal{A}$ is $\mathcal{P}$, to prove that λ_Z is uniquely extendible to a measure on $\mathcal{P}$, it suffices to prove λ_Z is countably additive on $\mathcal{A}$. For this it is enough to show that for any sequence $\{G_n\} \subset \mathcal{A}$ such that $G_n \downarrow \emptyset$, $\lambda_Z(G_n) \to 0$. Now for any $G \in \mathcal{A}$,

$$\begin{aligned}\lambda_Z(G) &= \lambda_Z\left(G \cap (\{0\} \times \Omega)\right) + \lambda_Z\left(G \cap ((0, \infty) \times \Omega)\right)\\ &= \lambda_Z\left(G \cap ((0, \infty) \times \Omega)\right),\end{aligned}$$

where $G \cap ((0, \infty) \times \Omega) \in \mathcal{A}$. Thus, it suffices to consider $\{G_n\} \subset \mathcal{A} \cap ((0, \infty) \times \Omega)$. Note that if $G = (s, t] \times F_s$ for $F_s \in \mathcal{F}_s$, then for any $m \in \mathbb{N}$ such that $s + \frac{1}{m} < t$, $H \equiv (s + \frac{1}{m}, t] \times F_s \in \mathcal{R}$, $K \equiv [s + \frac{1}{m}, t] \times F_s$ is a stochastic compact, and $H \subset K \subset G$. Furthermore,

$$\lambda_Z(G \backslash H) = \lambda_Z(G) - \lambda_Z(H) = E\left(1_{F_s}(Z_{s+\frac{1}{m}} - Z_s)\right) \to 0$$

as $m \to \infty$, by the right continuity of Z and the uniform integrability of $\{Z_u : s \leq u \leq s+1\}$. Since each set in $\mathcal{A} \cap ((0, \infty) \times \Omega)$ is a finite union of disjoint sets in $\mathcal{R} \cap ((0, \infty) \times \Omega)$, it follows that given any $\varepsilon > 0$, for each $j \geq 1$, there is $H_j \in \mathcal{A}$ and a stochastic compact K_j such that

$H_j \subset K_j \subset G_j$ and $\lambda_Z(G_j) - \lambda_Z(H_j) < \varepsilon 2^{-j}$. Since $\{G_n\}$ is decreasing, we have for each n,

$$\hat{H}_n \equiv \bigcap_{j=1}^{n} H_j \subset \hat{K}_n \equiv \bigcap_{j=1}^{n} K_j \subset G_n \equiv \bigcap_{j=1}^{n} G_j,$$

where $\hat{H}_n \in \mathcal{A}$, $\hat{K}_n$ is a stochastic compact, and

$$\lambda_Z(G_n \backslash \hat{H}_n) = \lambda_Z(G_n) - \lambda_Z(\hat{H}_n) < \varepsilon. \tag{2.34}$$

By hypothesis $G_n \downarrow \emptyset$, hence $\hat{K}_n \downarrow \emptyset$, and then by Lemma 2.14(ii),

$$D_{\hat{K}_n} \uparrow D_\emptyset = \infty.$$

Since $T_n \equiv D_{\hat{H}_n} \geq D_{\hat{K}_n}$, we have $T_n \uparrow \infty$ as $n \to \infty$. Thus, for any $t \geq 0$ and $\omega \in \Omega$, there is $N(\omega) \in \mathbb{N}$ such that $T_n(\omega) \wedge t = t$ for all $n \geq N(\omega)$. Thus, $Z_{T_n \wedge t} \to Z_t$ pointwise on Ω as $n \to \infty$. Now, by Lemma 2.15, T_n is optional and has finite range. Hence $T_n \wedge t$ also has these properties, and then by the submartingale property of Z,

$$0 \leq Z_{T_n \wedge t} \leq E[Z_t \,|\, \mathcal{F}_{T_n \wedge t}] \quad \text{for all} \quad n \geq 1.$$

(Note we did not need to use the right continuity of Z for this.) Consequently, $\{Z_{T_n \wedge t},\, n \geq 1\}$ is uniformly integrable and it follows that

$$\lim_{n \to \infty} E[Z_{T_n \wedge t}] = E[Z_t].$$

By the definition of λ_Z,

$$\lambda_Z\left((T_n \wedge t, t]\right) = E[Z_t - Z_{T_n \wedge t}] \to 0 \quad \text{as} \quad n \to \infty. \tag{2.35}$$

Since $G_1 \in \mathcal{A} \cap ((0,\infty) \times \Omega)$, there is $t > 0$ such that $G_1 \subset [0,t]$ and so $\hat{H}_n \subset [0,t]$. On the other hand, by the form of the sets in $\mathcal{A} \cap ((0,\infty) \times \Omega)$, the graph $\{(D_{\hat{H}_n}(\omega), \omega) : \omega \in \Omega\} \subset \overline{\mathbb{R}}_+ \times \Omega$ of the début of the set $\hat{H}_n$ does not meet $\hat{H}_n$, and so $\hat{H}_n \subset (T_n, \infty)$. Hence, $\hat{H}_n \subset (T_n \wedge t, t]$. It follows from (2.35) that $\lambda_Z(\hat{H}_n) \to 0$ as $n \to \infty$, and hence by (2.34), $\lambda_Z(G_n) \to 0$ as $n \to \infty$, as desired. ∎

Corollary 2.17. *If M is a right continuous L^2-martingale, then $\lambda_{(M)^2}$ has a unique extension to a measure on $\mathcal{P}$, and this measure is σ-finite.*

2.9 Exercises

1. Let $\mathcal{G}_1$ denote the σ-field generated by all stochastic intervals of the form $[\eta, \tau]$ where η and τ are optional times. Similarly, let $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ denote the σ-fields generated by the stochastic intervals of the form $[\eta, \tau)$, $(\eta, \tau]$ and (η, τ), respectively. Can you identify $\mathcal{G}_i$ ($i \in \{1, 2, 3, 4\}$) with $\mathcal{P}$, $\mathcal{O}$ or neither?
(*Answer:* $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{O}$, $\mathcal{G}_3 \subset \mathcal{P}$ but $\mathcal{G}_3 \neq \mathcal{P}$, and $\mathcal{G}_4 \subset \mathcal{O}$ but $\mathcal{G}_4 \neq \mathcal{O}$.)

2. Suppose $X = \{X_t, t \geq 0\}$ is a right continuous process that has finite left limits on $(0, \infty)$ and is adapted to $\{\mathcal{F}_t, t \geq 0\}$. Prove that $\tau \equiv \inf\{t \geq 0 : X_t - X_{t-} \geq \varepsilon\}$ is an optional time.
(This result can be proved from first principles, but it is also a special case of the optionality of the début of a progressively measurable set, cf. Chung [12, Section 1.5].)

In the next two exercises, let $\{N_t, t \in \mathbb{R}_+\}$ be a Poisson process with parameter $\alpha > 0$ and let $\{\mathcal{F}_t, t \geq 0\}$ be the associated standard filtration as defined in Section 1.8.

3. Let T be the time of the first jump of N. Show the following.
(a) T is an optional time.
(b) T is *not* a predictable time.
Hint for (b): For a proof by contradiction, suppose that $\{T_n\}$ is an announcing sequence for T. Then use the memoryless property of the Poisson process at the optional times T_n to conclude that for any $\varepsilon > 0$:

$$P(T - T_n > \varepsilon) = P(T > \varepsilon).$$

Since $T_n \uparrow T$ P-a.s., the left member above tends to zero as $n \to \infty$. On the other hand, the right member is independent of n and strictly positive.

4. Show that $M_t = N_t - \alpha t$ defines a (right continuous) L^2-martingale and that its Doléans measure is $\mu_M = \alpha(\lambda \times P)$ (cf. Exercise 2 of Chapter 1).

5. Let T be an optional time relative to $\{\mathcal{F}_t\}$. Suppose the following two properties hold:

(i) for each $t > 0$, $P(0 < T \leq t) > 0$, and
(ii) for $A \in \mathcal{F}_t$, $P(A \cap \{T > t\})$ is either equal to 0 or to $P(T > t)$.

Prove that if S is also an optional time relative to $\{\mathcal{F}_t\}$, then

$$P(0 < S < T) < P(T > 0).$$

This implies that T is not predictable. Since the first jump time of a Poisson process satisfies (i)-(ii), this gives an alternative proof of 3(b) above.

6. Let $(W, \mathcal{G}, \nu)$ be a σ-finite measure space where the σ-field $\mathcal{G}$ is generated by a ring $\mathcal{H}$ such that $\nu(A) < \infty$ for each $A \in \mathcal{H}$. Prove that if $A_1 \in \mathcal{G}$ and $\nu(A_1) < \infty$, then for each $\varepsilon > 0$ there is $A_2 \in \mathcal{H}$ such that $\nu(A_1 \Delta A_2) < \varepsilon$.

7. Use the monotone class theorem for functions (see Section 1.2) to give an alternative proof of Lemma 2.4.
Hint: First prove this result for bounded functions in $L^2([0,t] \times \Omega, \mathcal{P}, \mu_M)$.

8. Suppose X is a predictable process and $X > 0$. Prove that for any $0 \leq s < t < \infty$ and random variable $Z \in \mathcal{F}$, $1_{(s,t]}ZX$ is predictable if and only if Z is $\mathcal{F}_s$-measurable.

9. Prove that (2.17) is also true if τ is a finite optional time and $X \in \mathcal{L}^2$.

10. Let $\{\tau_k\}$ be a localizing sequence for a right continuous local L^2-martingale M. For fixed k, define

$$Z_t = M_{t \wedge \tau_k \wedge k} - M_0 \quad \text{for all} \quad t \geq 0.$$

Show that Z is an L^2-martingale and that its Doléans measure μ_Z is finite on $(\mathbb{R}_+ \times \Omega, \mathcal{P})$.

11. Let N be a Poisson process with parameter $\alpha > 0$. Define $M_t = N_t - \alpha t$. Note from Exercise 4 that $\mu_M = \alpha(\lambda \times P)$. Suppose $X \in \Lambda^2(\mathcal{P}, M)$. Then the stochastic integral $\int_{[0,t]} X dM$ is well defined for all $t \in I\!R_+$. Prove that for each t, $\int_{[0,t]} X_s dN_s$ and $\int_{[0,t]} X_s ds$ are almost surely well defined as Lebesgue-Stieltjes integrals and that they define random variables satisfying

$$\int_{[0,t]} X_s dN_s = \int_{[0,t]} X dM + \alpha \int_{[0,t]} X_s ds, \tag{2.36}$$

where the integral with respect to dM is a stochastic integral defined by the L^2-isometry.

Hint: First prove (2.36) for an $\mathcal{R}$-simple X and then use a monotone class argument to extend to $X \in \Lambda^2(\mathcal{P}, M)$.

In Exercises 12 and 13 below, B is a Brownian motion in $I\!R$ with $B_0 \in L^2$. You may use the result from the first example in Section 4.2 in solving these exercises.

12. Working from the fundamental definition, evaluate the stochastic integral $\int_0^t B_s\, dB_s$. If you are ambitious, try also $\int_0^t B_s^k\, dB_s$ for $k = 2, 3, \ldots,$ where B_s^k denotes the k^{th} power of B_s.

13. Show that for each fixed $t \geq 0$, the approximating sums:

$$\sum_{k=0}^{2^n-1} B\left(\left(k + \frac{1}{2}\right) t2^{-n}\right) \{B((k+1)t2^{-n}) - B(kt2^{-n})\}$$

converge in L^2 to $\int_0^t B_s\, dB_s + t/2$ as $n \to \infty$. Note that this limit does *not* define a martingale. It defines what is usually called the *Stratonovitch* integral of B with respect to B.

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