

Chapter 2

Electromagnetic Theory

Abstract This chapter contains a summary of the main ingredients of the theory of electrostatics and the theory of magnetostatics required for the subsequent formulation of the theory governing electromechanical interactions in electro-sensitive and magneto-sensitive materials. In particular, expressions for the force and couple on an electric dipole in an electric field and on a magnetic dipole in a magnetic field are derived, which are important for examining polarization and magnetization in material media. The full system of four Maxwell equations that govern electromagnetic phenomena in material media is then obtained and the notions of polarization and magnetization that distinguish polarizable and magnetizable materials, respectively, from vacuum or from materials that are not sensitive to electric or magnetic fields are introduced. Maxwell's equations are then cast in their standard form in terms of free charge density and free current density. Based on Maxwell's equations, the final section of the chapter provides a derivation of the conditions that must be satisfied by the electric field, electric displacement, magnetic field and magnetic induction vectors on material boundaries.

2.1 Electrostatics

2.1.1 Preliminary Remarks

The equations that describe the forces generated by charged particles are well established and can be found in any textbook on electrostatics. In this section we begin with a short review of these equations and the associated field quantities and the Lorentz Law which gives the force acting on a charged particle in an electromagnetic field as a prelude to the introduction, in the next three sections, of magnetic fields and the full set of Maxwell's equations that completely describe the interaction between electric and magnetic fields in the nonrelativistic setting. Historically, electromagnetic theory has been developed with reference to observable

macroscopic events occurring in vacuum or in condensed matter, but the theory is first based on microscopic quantities such as point charges, dipoles and small current-carrying circuits and their distributions, which together with their forces of interaction are built into a continuum theory that describes experimentally observed phenomena. In this section we are concerned primarily with the equations of electrostatics. Of course, the notions of point charge, dipole, etc., are convenient mathematical idealizations used to build the basic theory and are associated with singularities. For example, the field of a point charge is infinite when evaluated at the point where the charge is located. Continuum theory avoids such singularities.

For detailed background covering the material in this section and Sects. 2.2–2.4, we refer to [Becker and Sauter \(1964\)](#), [Landau and Lifshitz \(1960\)](#), [Jackson \(1999\)](#) and [Stratton \(2007\)](#) and the recent book by [Kovetz \(2000\)](#).

2.1.2 The Electric Field

Consider a time-independent spatial distribution of charged particles that interact with one another by generating electrostatic forces. These interacting forces enable the electric field, denoted \mathbf{E} , to be defined at an arbitrary location \mathbf{x} . Consider the resultant force \mathbf{f} of the considered distribution of particles acting on a test particle with point charge e placed at position \mathbf{x} . The point charge must be small enough not to alter the original arrangement of the particles. As the magnitude of e approaches zero, it is clear that the measured force must approach zero as well. However, in the limit the ratio of force \mathbf{f} to the charge e remains finite and identifies the *electric field vector* \mathbf{E} at the point \mathbf{x} , i.e.

$$\mathbf{E}(\mathbf{x}) = \lim_{e \rightarrow 0} \frac{\mathbf{f}}{e}, \quad (2.1)$$

from which it follows that the electric field has dimensions of force per unit charge.

2.1.3 The Lorentz Law of Force

The Lorentz Law of force (Hendrik Antoon Lorentz, 1853–1928) is one of the fundamental elements of the classical theory of electromagnetism and defines the force exerted by an electromagnetic field on a charged particle. We first consider the case of a stationary point charge at rest at location \mathbf{x} subject to an electric field \mathbf{E} . This field exerts a force \mathbf{f} on the particle given by

$$\mathbf{f}(\mathbf{x}) = e\mathbf{E}(\mathbf{x}), \quad (2.2)$$

where e is the charge of the particle, which is small enough not to disturb the sources of the electric field and so not alter the electric field. This is a special case of the Lorentz Law in which the particle is stationary. For completeness, we now consider the general case in which the particle is moving, which requires the introduction of a magnetic field. Suppose that, in addition to the electric field \mathbf{E} , there is a time-independent magnetic field, described in terms of the magnetic induction vector \mathbf{B} , which will be discussed in detail in Sect. 2.2, and consider the particle, instead of being at rest, to be moving with velocity \mathbf{v} and instantaneously located at the point \mathbf{x} . The particle experiences an additional force perpendicular to its direction of motion and proportional to the magnitude of \mathbf{v} . The additional force is maximal when the motion of the particle is perpendicular to the direction of the magnetic induction \mathbf{B} and vanishes when the orientations of \mathbf{v} and \mathbf{B} coincide, and the total force on the particle, consisting of electric and magnetic components, is then

$$\mathbf{f}(\mathbf{x}) = e[\mathbf{E}(\mathbf{x}) + \mathbf{v} \times \mathbf{B}(\mathbf{x})]. \quad (2.3)$$

This is known as the *Lorentz force*.

2.1.4 Coulomb's Law

The laws of electrostatics have their origin in the experimental work performed by Coulomb (Charles Augustin de Coulomb, 1736–1806), who investigated the forces of interaction generated by a distribution of charged particles at rest. In particular, Coulomb was able to quantify the force of interaction between two charged particles. If the two particles have charges e_1 and e_2 and are placed at locations \mathbf{x}_1 and \mathbf{x}_2 , respectively, the interaction force is given by *Coulomb's Law*

$$\mathbf{f} = k e_1 e_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}, \quad (2.4)$$

where k is a constant of proportionality that depends on the units used. Coulomb's Law shows that the force depends linearly on the magnitude of each charge, is inversely proportional to the square of the distance between the two particles and is directed along the line connecting the two charges. It is attractive if one charge is positive and one negative, as illustrated in Fig. 2.1, and repulsive if they are both positively charged or both negatively charged.

Coulomb's Law can be generalized to the case of N interacting particles. The resultant net force acting on a test charge e_1 due to all other charged particles is obtained by using the principle of linear superposition and given by

$$\mathbf{f} = k e_1 \sum_{i=2}^N e_i \frac{\mathbf{x}_1 - \mathbf{x}_i}{|\mathbf{x}_1 - \mathbf{x}_i|^3}. \quad (2.5)$$

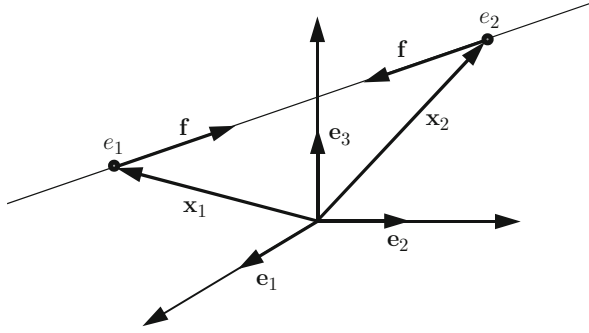


Fig. 2.1 The (attractive) force \mathbf{f} of interaction between two charged particles e_1 and e_2 for the case in which one charge is positive and one negative. Cartesian basis vectors \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 are also shown together with the position vectors \mathbf{x}_1 and \mathbf{x}_2 of e_1 and e_2

Coulomb also showed that the electric field \mathbf{E} generated by an isolated and stationary particle is proportional to its charge e and to the inverse square of the distance from the charge. The field at the point \mathbf{x} due to a point charge e placed at the origin is therefore given by

$$\mathbf{E}(\mathbf{x}) = k e \frac{\mathbf{x}}{r^3} = k e \frac{\hat{\mathbf{x}}}{r^2}, \quad (2.6)$$

where $r = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/r$ is a unit vector. If the charged particle is located at the fixed point \mathbf{x}' instead of the origin, then (2.6) is replaced by

$$\mathbf{E}(\mathbf{x}) = k e \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (2.7)$$

This formula for a single particle is easily extended to a distribution of N charged particles located at \mathbf{x}_i , $i = 1, \dots, N$, by the principle of linear superposition to give the electric field at the point $\mathbf{x} \neq \mathbf{x}_i$, $i = 1, \dots, N$, as

$$\mathbf{E}(\mathbf{x}) = k \sum_{i=1}^N e_i \frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|^3}. \quad (2.8)$$

2.1.5 Charge Conservation

The definition of the electric field up to this point assumes the existence of a discrete spatial distribution of charged particles. We now generalize this concept within the continuum framework by considering a charge distributed over a small volume in the neighbourhood of a point \mathbf{x} . Consider an infinitesimal element of volume dV and

let $\rho_e dV$ be the total charge within this element. Then ρ_e is the *charge density*, which may be positive or negative and depends, in general, on the position \mathbf{x} and time t : $\rho_e = \rho_e(\mathbf{x}, t)$. Thus, the individual charged particles within dV are ‘smoothed out’ in the continuum theory to form a continuous density function.

The velocities of the individual particles are treated similarly, and we denote by \mathbf{v} the mean velocity of the individual charges in dV . Then, we define the *current density* \mathbf{J} at \mathbf{x} by

$$\mathbf{J} = \rho_e \mathbf{v}, \quad (2.9)$$

from which can be determined the rate at which charges cross a unit surface of any orientation and the current flowing across an arbitrary surface, as discussed below.

The Lorentz force for a point charge subject to electromagnetic fields \mathbf{E} and \mathbf{B} is given in (2.3). For a charge distribution with density ρ_e and current density \mathbf{J} , the Lorentz force per unit volume is given by

$$\mathbf{f} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}. \quad (2.10)$$

Consider an arbitrary volume V *fixed* in space and bounded by a closed surface S with unit outward normal \mathbf{n} . The charge density per unit volume within V is ρ_e , and the rate at which charge flows out of V across S is given by $\mathbf{J} \cdot \mathbf{n}$ per unit area. The rate of increase of charge within V must arise from the *influx* of charge across S . Thus,

$$\frac{d}{dt} \int_V \rho_e dV = - \int_S \mathbf{J} \cdot \mathbf{n} dS, \quad (2.11)$$

which says that any change in charge within a confined volume must be balanced by the charge flowing across the bounding surface. By using the divergence theorem to convert the surface integral to an integral over the volume V , we obtain

$$\int_V \left(\frac{\partial \rho_e}{\partial t} + \text{div} \mathbf{J} \right) dV = 0, \quad (2.12)$$

which must hold for arbitrary V . Provided the integrand in (2.12) is continuous, we may deduce the local form of the *charge conservation equation* as

$$\frac{\partial \rho_e}{\partial t} + \text{div} \mathbf{J} = 0. \quad (2.13)$$

In a steady state situation (no time dependence) we have $\partial \rho_e / \partial t = 0$ and (2.13) reduces to

$$\text{div} \mathbf{J} = 0, \quad (2.14)$$

and the corresponding integral form is

$$\int_S \mathbf{J} \cdot \mathbf{n} dS = 0, \quad (2.15)$$

where S is an arbitrary closed surface.

2.1.6 Units

In this book we use the SI system of units in which the basic units are length (metres, m), mass (kilograms, kg), time (seconds, s) and electric charge (Coulomb, C). The electric charge on an electron, for example, is $e = -1.602 \times 10^{-19}$ C. From the equations connecting the electromagnetic field variables, it is then possible to derive the dimensions of all other quantities in terms of these four basic units. Force, for example, has dimensions kg m s^{-2} and is expressed in Newtons (N): 1 Newton is equal to 1 kg m s^{-2} . From (2.9) we find the dimensions of the current density \mathbf{J} to be $\text{C m}^{-2} \text{ s}^{-1}$ or Ampères per square metre (A m^{-2}), where an Ampère (A) has dimensions C s^{-1} .

In SI units the constant k in (2.4), which defines the interaction force between two charged particles, has dimensions of $\text{kg m}^3 \text{ s}^{-2} \text{ C}^{-2}$. From (2.6) we find that the dimension of the electric field \mathbf{E} is $\text{kg m s}^{-2} \text{ C}^{-1}$, alternatively N C^{-1} or volt per metre (V m^{-1}). In SI units the numerical value of the constant of proportionality k in (2.4) is given by $1/4\pi\epsilon_0$, and Coulomb's Law (2.4) becomes

$$\mathbf{f} = \frac{e_1 e_2}{4\pi\epsilon_0} \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}, \quad (2.16)$$

where $\epsilon_0 \approx 8.854 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$ is the *permittivity of free space*. Similarly, the electric field at location \mathbf{x} of a point charge e at \mathbf{x}' is given by (2.7), which, using the expression for k , gives

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (2.17)$$

2.1.7 The Field of a Static Charge Distribution

If we replace the constant of proportionality k by its value appropriate to the units adopted herein, then (2.6) takes on the form

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\epsilon_0} \frac{\hat{\mathbf{x}}}{r^2} = -\frac{e}{4\pi\epsilon_0} \text{grad} \left(\frac{1}{r} \right), \quad (2.18)$$

where again $r = |\mathbf{x}|$, while grad represents the gradient operator with respect to \mathbf{x} . When the charged particle is placed at the position \mathbf{x}' instead of at the origin, the electric field is given by (2.7) or, alternatively, by

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\epsilon_0} \frac{\mathbf{R}}{R^3} = -\frac{e}{4\pi\epsilon_0} \text{grad} \left(\frac{1}{R} \right), \quad (2.19)$$

where we have introduced the notations $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $R = |\mathbf{R}|$. As in (2.18), the gradient operator is with respect to \mathbf{x} , i.e. the point at which \mathbf{E} is determined.

These results can be generalized to a region in space containing a continuous distribution of charge. In particular, consider a continuous distribution of charge with density ρ_e within a volume V , with the point charge e replaced by the charge $\rho_e dV$ in the volume element dV . If $\rho_e = 0$ outside the specified volume V , then the electric field at location \mathbf{x} is the sum of the contributions from all elements of charge $\rho_e dV$ within V . It is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho_e(\mathbf{x}') \frac{\mathbf{R}}{R^3} dV(\mathbf{x}') = -\frac{1}{4\pi\epsilon_0} \int_V \rho_e(\mathbf{x}') \text{grad} \left(\frac{1}{R} \right) dV(\mathbf{x}'), \quad (2.20)$$

where the integration is with respect to the \mathbf{x}' variable. The grad operator is again with respect to \mathbf{x} and can therefore be taken outside the integral, leading to an alternative expression for the electric field at point \mathbf{x} , specifically

$$\mathbf{E}(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0} \text{grad} \int_V \frac{\rho_e(\mathbf{x}')}{R} dV(\mathbf{x}'). \quad (2.21)$$

The gradient operator in the above equation acts on a scalar function, and it is therefore convenient to introduce a notation, namely, φ to represent this function. It is known as the *electrostatic potential* and allows (2.21) to be written compactly as

$$\mathbf{E}(\mathbf{x}) = -\text{grad} \varphi(\mathbf{x}), \quad (2.22)$$

where φ is given by

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_e(\mathbf{x}')}{R} dV(\mathbf{x}'). \quad (2.23)$$

The gradient operator in (2.22) operates on a scalar quantity, yielding a vector field. The standard vector identity $\text{curl}(\text{grad} \varphi) \equiv \mathbf{0}$ for any scalar function φ applied in the present context gives the *first equation of electrostatics*

$$\text{curl} \mathbf{E} = \mathbf{0}. \quad (2.24)$$

Far from the specified volume V , the electric field is approximately that of a point charge situated at the origin with a charge equal to the total charge within the

distribution. In this case we have $1/R \approx 1/r$ and the electrostatic potential (2.23) can be approximated by

$$\varphi(\mathbf{x}) \approx \frac{e}{4\pi\epsilon_0 r}, \quad (2.25)$$

where

$$e = \int_V \rho_e(\mathbf{x}') dV(\mathbf{x}') \quad (2.26)$$

is the total charge within V .

2.1.8 Gauss's Theorem

Equation (2.24) on its own is not sufficient to determine the electric field. The set of governing equations is completed by means of Gauss's theorem (Carl Friedrich Gauss, 1777–1855), which is now derived.

Consider first a particle carrying charge e placed at a position \mathbf{x}' within a volume V bounded by a closed surface S . Equation (2.19) gives the associated electric field at location \mathbf{x} . Specifically, we need to determine the electric field \mathbf{E} at a point \mathbf{x} on the surface S where the unit outward pointing normal vector is \mathbf{n} , say. Let $d\mathbf{S}$ ($= \mathbf{n} dS$) be an infinitesimal vector area element on the surface S at \mathbf{x} , where $dS > 0$. Then, the flux of \mathbf{E} across $d\mathbf{S}$ is given by

$$\mathbf{E} \cdot d\mathbf{S} = \frac{e}{4\pi\epsilon_0} \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}, \quad (2.27)$$

and the total flux of \mathbf{E} across the closed surface S is

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{e}{4\pi\epsilon_0} \int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}. \quad (2.28)$$

The integrand on the right-hand side defines the *solid angle*, denoted $d\Omega$, subtended by $d\mathbf{S}$ at \mathbf{x}' , i.e.

$$d\Omega = \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}, \quad (2.29)$$

a purely geometrical quantity, and (2.28) may therefore be written as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{e}{4\pi\epsilon_0} \int_S d\Omega. \quad (2.30)$$

If \mathbf{x}' lies within the volume V , then the solid angle is equal to 4π . On the other hand, if \mathbf{x}' lies outside the bounding surface, then positive contributions of $\mathbf{R} \cdot d\mathbf{S}/R^3$ to the integral are balanced by negative contributions and the integral vanishes, as can be shown by a simple application of the divergence theorem, noting that $1/R$ is a fundamental solution of Laplace's equation, i.e. $\nabla^2(1/R) = 0$. Thus,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \begin{cases} e/\epsilon_0 & \text{if } e \text{ is within } V \\ 0 & \text{if } e \text{ is outside } V. \end{cases} \quad (2.31)$$

We now extend this result to the case of a charge distribution within a volume V' which intersects V . The electric field at point \mathbf{x} on the surface S is

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_e(\mathbf{x}') \mathbf{R}}{R^3} dV(\mathbf{x}'), \quad (2.32)$$

and the flux of \mathbf{E} across the closed surface S , the boundary of V , is

$$\int_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{S}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho_e(\mathbf{x}') dV(\mathbf{x}') \int_S \frac{\mathbf{R} \cdot d\mathbf{S}(\mathbf{x})}{R^3}. \quad (2.33)$$

Using again the properties of solid angle, we have

$$\int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3} = \begin{cases} 4\pi & \text{if } \mathbf{x}' \text{ is within } V \\ 0 & \text{if } \mathbf{x}' \text{ is outside } V, \end{cases} \quad (2.34)$$

and hence

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_{V'(V)} \rho_e(\mathbf{x}') dV(\mathbf{x}'), \quad (2.35)$$

where $V'(V)$ is that part of V' contained within V . If e denotes the total charge inside the volume V , the above equation can be written more compactly as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{e}{\epsilon_0}, \quad (2.36)$$

which is *Gauss's theorem*. It states simply that the resultant flux of the electric field \mathbf{E} across any closed surface S is proportional to the total charge e contained within S .

To derive the associated local equation, we rewrite Gauss's theorem as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho_e dV, \quad (2.37)$$

where the integration on the right-hand side is restricted to the volume bounded by the surface S . Then, by use of the divergence theorem, (2.37) becomes

$$\int_V \left(\operatorname{div} \mathbf{E} - \frac{\rho_e}{\varepsilon_0} \right) dV = 0, \quad (2.38)$$

which must hold for arbitrary V . Therefore, provided the integrand in (2.38) is continuous, we deduce that

$$\operatorname{div} \mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \quad (2.39)$$

which is the local form of Gauss's theorem and the *second equation of electrostatics*. The equations

$$\operatorname{curl} \mathbf{E} = \mathbf{0}, \quad \operatorname{div} \mathbf{E} = \frac{\rho_e}{\varepsilon_0} \quad (2.40)$$

together govern the electrostatic field \mathbf{E} . Since (2.40)₁ is equivalent to $\mathbf{E} = -\operatorname{grad} \varphi$, we may substitute this into (2.40)₂ to obtain *Poisson's equation*

$$\nabla^2 \varphi = -\frac{\rho_e}{\varepsilon_0} \quad (2.41)$$

for the scalar potential φ for a given charge distribution with density ρ_e . It is easy to verify that the scalar potential (2.23) satisfies this equation whether \mathbf{x} is within or outside V (see, e.g., Jackson (1999), Sect. 1.7). *A fortiori*, the integrals in (2.20) and (2.23) are finite when \mathbf{x} is within V . For regions where $\rho_e = 0$, (2.41) reduces to *Laplace's equation*

$$\nabla^2 \varphi = 0. \quad (2.42)$$

Of course, for any particular boundary-value problem, appropriate boundary conditions need to accompany the equations. These will be given in a general form in Sect. 2.5 for a fixed surface and in Sect. 9.1 for a moving surface.

2.1.9 The Field of a Dipole

The equations of electrostatics discussed up to this point are concerned with the interactions of time-independent charges and fields in free space. When an electric field is applied to a solid medium, the configuration of charges is altered, and this leads, in particular, to the production of dipoles within the medium at the microscopic level, and the material is said to be *polarized*. A dipole can be visualized as two localized concentrations of charge with the same magnitude and opposite

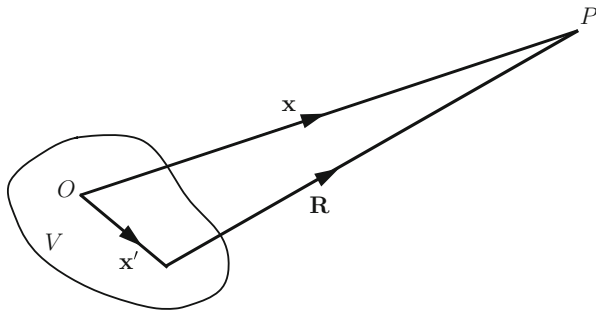


Fig. 2.2 Depiction of a volume V containing a charge distribution with density $\rho_e(\mathbf{x}')$ at point \mathbf{x}' relative to an origin O within V . There are no charges outside V . A field point P exterior to V has position vector \mathbf{x} relative to the origin O and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$

signs separated by a small distance. We shall consider polarization in Sect. 2.4, but here we discuss the field due to an idealized isolated dipole.

Equation (2.20) determines the electric field at location \mathbf{x} for a charge density contained within a volume V , and (2.23) gives the corresponding potential. To determine the field generated by an electric dipole, we again consider a distribution of charge with density $\rho_e(\mathbf{x}')$, but now confined to a *small* volume V , where \mathbf{x}' is the position vector of a typical point in V relative to an origin O located within V and $\rho_e = 0$ outside V . Let \mathbf{x} be the position vector of a point P far from V at which the electrostatic field is to be calculated, as depicted in Fig. 2.2.

Then $|\mathbf{x}'| \ll |\mathbf{x}|$ for all \mathbf{x}' in V , and we may use the Taylor expansion to obtain the approximation

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{r} - \mathbf{x}' \cdot \text{grad} \left(\frac{1}{r} \right), \quad (2.43)$$

recalling that $r = |\mathbf{x}|$. Hence, from (2.23), the electrostatic potential at \mathbf{x} is approximated as

$$\varphi(\mathbf{x}) \approx \frac{e}{4\pi\epsilon_0 r} - \frac{1}{4\pi\epsilon_0} \mathbf{p} \cdot \text{grad} \left(\frac{1}{r} \right), \quad (2.44)$$

where e is the total charge in V given by the formula (2.26) and \mathbf{p} is defined by

$$\mathbf{p} = \int_V \rho_e(\mathbf{x}') \mathbf{x}' dV(\mathbf{x}'). \quad (2.45)$$

If $e \neq 0$, then the origin can be translated to the centre of charge (analogous to the centre of mass in mechanics) so that $\mathbf{p} = \mathbf{0}$, in which case

$$\varphi(\mathbf{x}) \approx \frac{e}{4\pi\epsilon_0 r}, \quad (2.46)$$

which is the field of a point charge e located at the origin. Then, to a first approximation, the field at a large distance from a charge distribution is indistinguishable from that of a point charge, as already indicated in Sect. 2.1.7. On the other hand, if $e = 0$ and $\mathbf{p} \neq \mathbf{0}$, we have

$$\varphi(\mathbf{x}) \approx -\frac{1}{4\pi\epsilon_0} \mathbf{p} \cdot \text{grad} \left(\frac{1}{r} \right) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{x}}{r^3}. \quad (2.47)$$

This is the potential due to an *electric dipole* of strength \mathbf{p} situated at the origin. This is equivalent to having two charges of magnitude e and of equal and opposite signs very close together, say at distances $\pm \mathbf{d}/2$ from the origin, in which case $\mathbf{p} = e\mathbf{d}$. The above formula becomes exact in the limit in which \mathbf{d} approaches $\mathbf{0}$ as $e \rightarrow \infty$, while \mathbf{p} remains finite.

2.1.10 The Force and Couple on a Dipole in an Electric Field

We now calculate the total electric force on a distribution of charge contained within a volume V . Using that part of the Lorentz force density (2.3) due to the electric field \mathbf{E} , this is denoted \mathbf{F}_e , the subscript e signifying ‘electric’, and given by

$$\mathbf{F}_e = \int_V \rho_e(\mathbf{x}') \mathbf{E}(\mathbf{x}') dV(\mathbf{x}'). \quad (2.48)$$

Now take the origin to be within V and let V be a small volume. Then, by expanding $\mathbf{E}(\mathbf{x}')$ to the first order in \mathbf{x}' , we may approximate \mathbf{F}_e as

$$\mathbf{F}_e = \int_V \rho_e dV(\mathbf{x}') \mathbf{E}(\mathbf{0}) + \int_V \rho_e(\mathbf{x}') \mathbf{x}' dV(\mathbf{x}') \cdot [\text{grad} \mathbf{E}(\mathbf{0})], \quad (2.49)$$

where $\text{grad} \mathbf{E}(\mathbf{0})$ is $\text{grad} \mathbf{E}$ evaluated at the origin, and we are using the convention that the second-order tensor $\text{grad} \mathbf{E}$ is defined by $(\text{grad} \mathbf{E})\mathbf{a} = (\mathbf{a} \cdot \text{grad})\mathbf{E}$ for an arbitrary vector \mathbf{a} (see (A.22) in Appendix A for the component form of this definition). If the total charge in V vanishes then, by (2.45), this becomes

$$\mathbf{F}_e = (\mathbf{p} \cdot \text{grad})\mathbf{E}, \quad (2.50)$$

where $\text{grad} \mathbf{E}$ is evaluated at the origin. This is the force acting on a dipole of strength \mathbf{p} located at a point in an electric field, in this case the origin.

The total couple acting on the charge distribution, about the origin, is denoted \mathbf{G}_e and given by

$$\mathbf{G}_e = \int_V \rho_e(\mathbf{x}') \mathbf{x}' \times \mathbf{E}(\mathbf{x}') dV(\mathbf{x}'), \quad (2.51)$$

and when V is small, this is approximated, to the first order, as

$$\mathbf{G}_e = \mathbf{p} \times \mathbf{E}, \quad (2.52)$$

with \mathbf{E} evaluated at the origin. The formulas (2.50) and (2.52) for \mathbf{F}_e and \mathbf{G}_e are exact in the idealized limit of an isolated dipole.

We note that the equation $\text{curl} \mathbf{E} = \mathbf{0}$ is equivalent to the symmetry $(\text{grad} \mathbf{E})^T = \text{grad} \mathbf{E}$, where T denotes the transpose of a second-order tensor, and this fact can be used to rewrite the electric Lorentz force density, using (2.39), as

$$\rho_e \mathbf{E} = \varepsilon_0 (\text{div} \mathbf{E}) \mathbf{E} = \text{div} \boldsymbol{\tau}_e, \quad (2.53)$$

where $\boldsymbol{\tau}_e$ is the so-called *electrostatic Maxwell stress tensor*, which is defined by

$$\boldsymbol{\tau}_e = \varepsilon_0 [\mathbf{E} \otimes \mathbf{E} - \frac{1}{2} (\mathbf{E} \cdot \mathbf{E}) \mathbf{I}], \quad (2.54)$$

where \mathbf{I} is the identity tensor and \otimes indicates the tensor product of two vectors, which, in Cartesian component form, is defined by $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. The Maxwell stress tensor plays an important role in subsequent developments.

Finally in this section, it is also convenient as a precursor for later developments to introduce the so-called *electric displacement vector*, denoted \mathbf{D} , which for free space is related to \mathbf{E} simply by $\mathbf{D} = \varepsilon_0 \mathbf{E}$, and hence

$$\text{div} \mathbf{D} = \rho_e. \quad (2.55)$$

As we shall see in Sect. 2.4, (2.55) is one of Maxwell's equations, and it applies within material media, with ρ_e then replaced by the free charge density ρ_f (to be defined in Sect. 2.4), as well as in vacuum (where $\rho_e = 0$), and also when there is time dependence.

We also note that the force (2.50) can then be written as

$$\mathbf{F}_e = \varepsilon_0^{-1} (\mathbf{p} \cdot \text{grad}) \mathbf{D}. \quad (2.56)$$

For an isolated dipole, the formulas (2.50) and (2.56) are equivalent, but, as we shall see in Chap. 4, their counterparts are not equivalent in polarizable media.

2.2 Magnetostatics

2.2.1 Preliminary Remarks

In electrostatics the electric field is determined in terms of point charges according to (2.6) for a single point charge placed at the origin or in terms of a distribution of charges according to (2.21), for example. Magnetism is fundamentally different since the analogue of a point charge (a magnetic monopole) does not exist. The basic

unit of magnetism is the magnetic dipole, but magnetic fields are generated by moving charges and a magnetic dipole is equivalent to an idealized small current-carrying circuit. That currents generate magnetic fields is demonstrated by placing a small bar magnetic near a fixed wire carrying current. The magnet is deflected by the magnetic force generated by the current. Equally, when a current-carrying wire is placed in the vicinity of a fixed magnet, the wire is deflected. For example, a straight wire carrying current I (measured in Amperes) produces an azimuthal magnetic flux density (or magnetic induction) field of magnitude $\mu_0 I / 2\pi r$ at a perpendicular distance r from the wire, where the constant μ_0 is the *magnetic permeability of free space*, whose value is $4\pi \times 10^{-7} \text{ NA}^{-2}$. There is an important connection between μ_0 and the permittivity of free space ϵ_0 introduced in Sect. 2.1.6, namely that $1/\sqrt{\epsilon_0\mu_0} = c$, the speed of light, which is very slightly less than $3 \times 10^8 \text{ m s}^{-1}$.

In this section we are concerned with steady currents and magnetostatic fields, i.e. no time dependence is considered. For a distribution of current, the current density \mathbf{J} then satisfies (2.15), where S is a closed surface and the net flux of current out of the enclosed volume vanishes. Geometrically, we can think of *lines of current flow* within S having tangent in the direction of \mathbf{J} at each point. For example, a so-called *tube of current flow* is defined as the surface formed by all such lines that intersect a given closed curve, analogous to lines and tubes of flow in fluid dynamics. It follows that the flux of \mathbf{J} across a cross section of the tube is the same for all cross sections. *Steady current* therefore consists of closed tubes of current flow. The total current I passing across an *open surface* S is just the flux of \mathbf{J} across S and is given by

$$I = \int_S \mathbf{J} \cdot \mathbf{n} \, dS. \quad (2.57)$$

In practice, a thin conducting wire is a tube of flow of small cross section dS and current $I \approx \mathbf{J} \cdot dS$.

Each infinitesimal segment of a wire contributes to the magnetic field produced by the complete circuit. Let $d\mathbf{x}'$ be such a segment. Then, the (infinitesimal) contribution to the magnetic field induced (the magnetic induction) at the field point \mathbf{x} , say $d\mathbf{B}(\mathbf{x})$, is given by

$$d\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{x}' \times \mathbf{R}}{R^3}, \quad (2.58)$$

where again $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $R = |\mathbf{R}|$, \mathbf{x}' being the point at which $d\mathbf{x}'$ is situated. This is the counterpart for magnetostatics of Coulomb's Law in electrostatics and also has an inverse square character. This is the essence of the Biot–Savart Law, deduced on the basis of experiments of Biot and Savart and Ampère and the analysis of Ampère in the nineteenth century (Jean-Baptiste Biot, 1774–1862; Felix Savart, 1791–1841; André-Marie Ampère, 1775–1836).

2.2.2 The Biot–Savart Law and the Vector Potential

To obtain the total magnetic field $\mathbf{B}(\mathbf{x})$ generated by the entire length of the wire, we superpose linearly the contributions provided by all the segments $d\mathbf{x}'$ of the wire to obtain

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{x}' \times \mathbf{R}}{R^3}, \quad (2.59)$$

where C is the closed circuit of the wire. By replacing $I d\mathbf{x}'$ by $\mathbf{J}(\mathbf{x}')dV(\mathbf{x}')$, this formula is generalized to that for a current distribution of density $\mathbf{J}(\mathbf{x}')$ within a volume V , giving

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3} dV(\mathbf{x}'). \quad (2.60)$$

This important formula is known as the *Biot–Savart Law* for a volume current distribution. As for the formula (2.20), this applies for \mathbf{x} inside or outside V .

The integrand in the above formula can be written as

$$\frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3} = \text{grad} \left(\frac{1}{R} \right) \times \mathbf{J}(\mathbf{x}') = \text{curl} \left(\frac{\mathbf{J}(\mathbf{x}')}{R} \right), \quad (2.61)$$

where the operators grad and curl are with respect to \mathbf{x} , and hence, on taking the curl operation outside the integral, (2.60) can be rewritten as

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \text{curl} \left[\int_V \frac{\mathbf{J}(\mathbf{x}')}{R} dV(\mathbf{x}') \right]. \quad (2.62)$$

This prompts the introduction of a vector function \mathbf{A} defined by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{R} dV(\mathbf{x}'), \quad (2.63)$$

which is known as the *magnetostatic vector potential*. Equation (2.62) can now be written in the more concise form

$$\mathbf{B} = \text{curl} \mathbf{A}, \quad (2.64)$$

from which it follows that \mathbf{B} satisfies the equation

$$\text{div} \mathbf{B} = 0. \quad (2.65)$$

This is a fundamental equation of magnetostatics. Integration of this equation over a volume V and then use of the divergence theorem shows that the magnetic flux

through any closed surface within a magnetic field is always zero. It expresses the fact that magnetic poles cannot be isolated, i.e. there is no counterpart in magnetostatics of the electrostatic point charge. In fact, (2.65) is general and holds even when there is time dependence and electromagnetic coupling, both in free space and in material media.

2.2.3 Scalar Magnetic Potential

Consider again (2.59), which gives the magnetic field due to a thin closed current-carrying circuit C . We now write this in the alternative forms

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \text{grad} \left(\frac{1}{R} \right) \times d\mathbf{x}' = \frac{\mu_0 I}{4\pi} \text{curl} \int_C \frac{d\mathbf{x}'}{R}, \quad (2.66)$$

where again we recall that the derivatives are with respect to \mathbf{x} , not \mathbf{x}' .

For points \mathbf{x} distant from C for which $|\mathbf{x}'| \ll |\mathbf{x}|$ for all \mathbf{x}' on C , we may use the Taylor expansion (2.43), i.e.

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{r} - \mathbf{x}' \cdot \text{grad} \left(\frac{1}{r} \right),$$

and since C is a closed curve, the first term in the integral vanishes and we obtain

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \text{curl} \left[\mathcal{M} \text{grad} \left(\frac{1}{r} \right) \right], \quad (2.67)$$

where the second-order tensor \mathcal{M} is defined by

$$\mathcal{M} = I \int_C d\mathbf{x}' \otimes \mathbf{x}' \quad (2.68)$$

and \otimes signifies the tensor product of two vectors. For any two vectors \mathbf{a} and \mathbf{b} , for example, this product is defined, in Cartesian components, by $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$, as noted in Sect. 2.1.10, while \mathcal{M} has components \mathcal{M}_{ij} and $(\mathcal{M}\mathbf{a})_i = \mathcal{M}_{ij} a_j$, $i, j \in \{1, 2, 3\}$, with summation over j from 1 to 3 (here we are using the summation convention for repeated indices). Moreover, \mathcal{M} is a skew-symmetric tensor, since, because C is a closed circuit,

$$\mathcal{M} + \mathcal{M}^T = I \int_C d(\mathbf{x}' \otimes \mathbf{x}') = \mathbf{O}, \quad (2.69)$$

the zero tensor, where T signifies the transpose of a second-order tensor.

The tensor \mathcal{M} is referred to as the *magnetic moment tensor*. Let \mathbf{m} denote the associated axial vector, defined by $\mathbf{m} = -\frac{1}{2}\epsilon\mathcal{M}$, where ϵ is the alternating tensor (see Appendix A.1). Expressed in components, this is written as $m_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk}$, with summation over indices j and k from 1 to 3. Then, for any vector \mathbf{a} , $\mathcal{M}\mathbf{a} = \mathbf{m} \times \mathbf{a}$, and, since \mathbf{m} is independent of \mathbf{x} , (2.67) becomes

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \text{curl} \left[\mathbf{m} \times \text{grad} \left(\frac{1}{r} \right) \right] = \frac{\mu_0}{4\pi} \text{curl} \text{curl} \left(\frac{\mathbf{m}}{r} \right). \quad (2.70)$$

Since $1/r$ satisfies Laplace's equation (provided $r \neq 0$), we may use the standard identity $\text{curl} \text{curl} = \text{grad} \text{div} - \nabla^2$ to rewrite the above as

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \text{grad} \text{div} \left(\frac{\mathbf{m}}{r} \right) = -\frac{\mu_0}{4\pi} \text{grad} \left(\frac{\mathbf{m} \cdot \mathbf{x}}{r^3} \right). \quad (2.71)$$

Thus, we may introduce a scalar potential function $\psi(\mathbf{x})$ defined by

$$\psi(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \cdot \mathbf{x}}{r^3} \quad (2.72)$$

such that

$$\mathbf{B}(\mathbf{x}) = -\text{grad} \psi(\mathbf{x}). \quad (2.73)$$

The potential (2.72) has the same structure as the potential associated with an electric dipole given in (2.47). Thus (2.72) is interpreted as the magnetostatic potential of a *magnetic dipole* of strength \mathbf{m} situated at the origin. Moreover, since

$$\mathbf{m} = -\frac{1}{2}\epsilon\mathcal{M} = \frac{1}{2}I \int_C \mathbf{x}' \times d\mathbf{x}', \quad (2.74)$$

the potential due to a magnetic dipole is equivalent to that due to a small current loop. More particularly, if C is a planar loop, then

$$\mathbf{m} = I \, d\mathbf{S} = I \, \mathbf{n} \, dS, \quad (2.75)$$

where dS is the plane area enclosed by the loop and \mathbf{n} is the unit normal to the plane of the loop, directed in the positive sense.

For a dipole situated at the point \mathbf{x}' , the potential in (2.72) is replaced by

$$\psi(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \cdot \mathbf{R}}{R^3}. \quad (2.76)$$

Now consider a circuit C of finite dimensions carrying current I , as depicted in Fig. 2.3. Let S be any regular surface that is bounded by C . Imagine that a fine network of curves is constructed on S such that each mesh is infinitesimal,

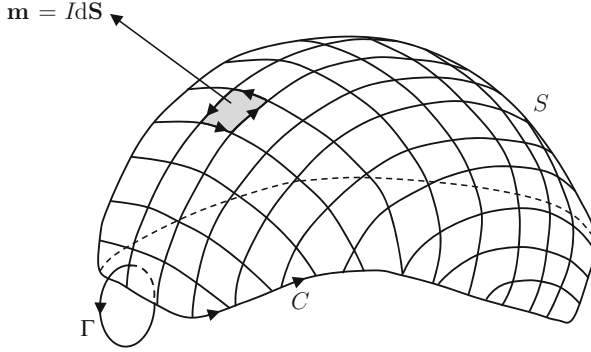


Fig. 2.3 An open surface S bounded by a closed circuit C carrying current I . On S is shown a network of curves made up of small current loops with current I corresponding to magnetic dipoles with magnetic moment $\mathbf{m} = I d\mathbf{S}$, where $d\mathbf{S}$ is the directed area element on S related to the direction of the current by the right-hand screw rule. The closed curve Γ encircles C once and hence cuts S

effectively plane and with vector area element $d\mathbf{S}$. We may regard the current I as flowing in each curve of the mesh because it cancels out on adjoining meshes. In effect, we have a surface S consisting of a distribution of magnetic dipoles $I d\mathbf{S}$. The potential at \mathbf{x} is due to contributions from all such dipoles. Inserting $\mathbf{m} = I d\mathbf{S}$ into (2.76) and integrating over S , we obtain the potential

$$\psi(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3}, \quad (2.77)$$

and with reference to Sect. 2.1.8, we see that

$$\int_S \frac{\mathbf{R} \cdot d\mathbf{S}}{R^3} = \Omega(\mathbf{x}) \quad (2.78)$$

is the solid angle subtended by S at \mathbf{x} . Thus,

$$\psi(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \Omega(\mathbf{x}). \quad (2.79)$$

The solid angle $\Omega(\mathbf{x})$ has the property that its value changes by 4π as the point \mathbf{x} crosses the surface S . This means the potential function ψ is multi-valued and changes in value by $\mu_0 I$ each time \mathbf{x} traverses a curve which cuts S once. Otherwise ψ is continuous.

2.2.4 Ampère's Circuital Law

Consider again an open surface S bounded by the circuit C carrying current I . Let the closed curve Γ encircle the circuit C just once, therefore cutting S , with the direction around Γ related to the direction of the current in C by the right-hand screw rule, as shown in Fig. 2.3. At any point of Γ , the magnetic induction is given by (2.73) with (2.79). The line integral of \mathbf{B} around Γ is

$$\int_{\Gamma} \mathbf{B} \cdot d\mathbf{x} = - \int_{\Gamma} \text{grad } \psi \cdot d\mathbf{x} = -[\psi]_{\Gamma}, \quad (2.80)$$

where $[\psi]_{\Gamma}$ is the change in ψ as Γ is traversed once. This is non-zero because ψ is multi-valued, and since Γ cuts S just once in the sense described above, Ω increases by -4π , and hence ψ by $-\mu_0 I$, for a single traversal of Γ . Therefore,

$$\int_{\Gamma} \mathbf{B} \cdot d\mathbf{x} = \mu_0 I, \quad (2.81)$$

where we note that the right-hand side is independent of the curve Γ . Now let Σ be an open surface bounded by Γ , and let I be the total current flowing through Σ . Then, the above argument can be applied to a current distribution \mathbf{J} such that $\int_{\Sigma} \mathbf{J} \cdot d\mathbf{S} = I$, leading to the formula

$$\int_{\Gamma} \mathbf{B} \cdot d\mathbf{x} = \mu_0 \int_{\Sigma} \mathbf{J} \cdot d\mathbf{S}. \quad (2.82)$$

This is a mathematical statement of *Ampère's Circuital Law*. By applying Stokes' theorem to (2.82), we obtain

$$\int_{\Sigma} (\text{curl } \mathbf{B} - \mu_0 \mathbf{J}) \cdot d\mathbf{S} = 0, \quad (2.83)$$

which holds for any open surface Σ associated with a Γ with the considered properties. Provided the integrand in (2.83) is continuous, we obtain the local form of one of the fundamental equations of magnetostatics, specifically

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J}. \quad (2.84)$$

We recall that in deriving this equation, it has been assumed that \mathbf{J} is time independent.

Returning to (2.64) we note that it is not affected by the addition of the gradient of an arbitrary scalar function (say φ) to the magnetic vector potential, i.e.

$$\mathbf{A} \rightarrow \mathbf{A} + \text{grad } \varphi, \quad (2.85)$$

which is known as a *gauge transformation*. This flexibility enables a restriction to be imposed on \mathbf{A} , a *gauge condition*, which is usually taken in the form

$$\operatorname{div} \mathbf{A} = 0. \quad (2.86)$$

Using (2.84) and (2.64), we have

$$\operatorname{curl}(\operatorname{curl} \mathbf{A}) = \mu_0 \mathbf{J}, \quad (2.87)$$

and, by using a standard vector identity, (2.87) can be written in the equivalent form

$$\operatorname{grad}(\operatorname{div} \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \quad (2.88)$$

Equation (2.86) is then used to reduce (2.88) to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad (2.89)$$

which, for given \mathbf{J} , is Poisson's equation for the magnetostatic vector potential. It can be verified that the expression for \mathbf{A} given in (2.63) is a solution of (2.89) whether \mathbf{x} is inside or outside V .

2.2.5 Force and Couple on a Dipole in a Magnetic Field

We now derive expressions for the (mechanical) force and couple on a magnetic dipole placed in a magnetic field. For this purpose we recall from Sect. 2.1.3 that the Lorentz force acting on a charged particle e moving with velocity \mathbf{v} in an electromagnetic field with electric field \mathbf{E} and magnetic induction \mathbf{B} is $e\mathbf{E} + e\mathbf{v} \times \mathbf{B}$. In the case of a continuous distribution of charge with density ρ_e and current with density \mathbf{J} , the Lorentz force per unit volume is given by (2.10) as $\rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}$. We now focus on the magnetic contribution $\mathbf{J} \times \mathbf{B}$ to the Lorentz force.

Consider a material volume V in which there is a current distribution with density \mathbf{J} , and let \mathbf{B} be the magnetic induction field permeating the material. Then, the magnetic contribution to the Lorentz force acting on V , which we denote by \mathbf{F}_m , is

$$\mathbf{F}_m = \int_V \mathbf{J} \times \mathbf{B} \, dV, \quad (2.90)$$

where the subscript m signifies 'magnetic'. Now suppose that V consists simply of a single current loop C carrying current I . Then, we may replace the volume integral by a line integral around C , and (2.90) becomes

$$\mathbf{F}_m = I \int_C d\mathbf{x} \times \mathbf{B} = I \int_S (d\mathbf{S} \times \operatorname{grad}) \times \mathbf{B}, \quad (2.91)$$

where S is a regular open surface bounded by C and the latter integral has been obtained by an application of Stokes' theorem.

Next, we take C and S to be infinitesimal so that the derivatives of \mathbf{B} are approximately uniform over S . Then (2.91) is approximated as $\mathbf{F}_m \approx I(\mathbf{dS} \times \text{grad}) \times \mathbf{B}$, and by setting $I \mathbf{dS} = \mathbf{m}$ to be the equivalent magnetic dipole and taking the limit $I \rightarrow \infty$ as $\mathbf{dS} \rightarrow 0$ while keeping \mathbf{m} finite, we obtain the exact result $\mathbf{F}_m = (\mathbf{m} \times \text{grad}) \times \mathbf{B}$, which is evaluated at the location of the dipole. By standard vector identities and the fact that $\text{div} \mathbf{B} = 0$, this *force on a dipole \mathbf{m} in a magnetic induction field \mathbf{B}* may be written as

$$\mathbf{F}_m = (\text{grad} \mathbf{B})^T \mathbf{m}. \quad (2.92)$$

In (2.92) and henceforth, similarly to Sect. 2.1.10, we adopt the following conventions: for two vector fields \mathbf{u} and \mathbf{v} , we define the products $(\text{grad} \mathbf{u})^T \mathbf{v}$ and $(\text{grad} \mathbf{u}) \mathbf{v} \equiv (\mathbf{v} \cdot \text{grad}) \mathbf{u}$ via their index notation representations $u_{j,i} v_j$ and $u_{i,j} v_j$, respectively, where $_{,j} = \partial/\partial x_j$ and $(\text{grad} \mathbf{u})_{ij} = u_{i,j}$.

The (magnetic) couple on V , denoted \mathbf{G}_m , about a fixed origin due to the magnetic Lorentz force is given by

$$\mathbf{G}_m = \int_V \mathbf{x} \times (\mathbf{J} \times \mathbf{B}) \, dV, \quad (2.93)$$

where \mathbf{x} is the position vector relative to the chosen origin. When V consists of just a current loop C , this becomes

$$\mathbf{G}_m = I \int_C \mathbf{x} \times (\mathbf{dx} \times \mathbf{B}) = I \int_C (\mathbf{dx} \otimes \mathbf{x}) \mathbf{B} - I \int_C (\mathbf{x} \cdot \mathbf{dx}) \mathbf{B}. \quad (2.94)$$

Once more we take C to be infinitesimal, but now it suffices, as a first approximation, to take \mathbf{B} to be uniform over C so that it can be taken outside the integrals. Then, since C is a closed circuit, the final integral in (2.94) vanishes, and on use of (2.68) \mathbf{G}_m can be written compactly as

$$\mathbf{G}_m = \mathcal{M} \mathbf{B} = \mathbf{m} \times \mathbf{B}, \quad (2.95)$$

again with \mathbf{B} evaluated at the location of the dipole, and this is exact in the limit described above. This is the *couple on a dipole \mathbf{m} in a magnetic induction field \mathbf{B}* .

Thus far the development has been based entirely on the use of the *magnetic induction* vector \mathbf{B} , but at this point it is instructive to introduce the so-called *magnetic field* vector, which is denoted by \mathbf{H} . For the field due to an isolated dipole placed in a vacuum, for example, \mathbf{B} and \mathbf{H} are simply related by $\mathbf{B} = \mu_0 \mathbf{H}$, where the constant μ_0 is again the permeability of free space. This relationship applies at any point in free space or in non-magnetizable materials, whatever the source of the magnetic field, in which case \mathbf{B} and \mathbf{H} satisfy the same equations. In particular, $\text{curl} \mathbf{H} = \mathbf{0}$, or equivalently $(\text{grad} \mathbf{H})^T = \text{grad} \mathbf{H}$, and (2.92) and (2.95) can be written in the alternative forms

$$\mathbf{F}_m = \mu_0 (\mathbf{m} \cdot \text{grad}) \mathbf{H}, \quad \mathbf{G}_m = \mu_0 \mathbf{m} \times \mathbf{H}. \quad (2.96)$$

We emphasize that while the two expressions for \mathbf{F}_m are equivalent in the present context, their counterparts are not equivalent in magnetizable media, and the distinction will be recognized as important, in particular when dealing with deformable media.

Similarly to the electric Maxwell stress tensor introduced in Sect. 2.1.10, we derive a magnetic Maxwell stress tensor. The magnetic contribution to the Lorentz force density may be rewritten, using (2.84), as

$$\mathbf{J} \times \mathbf{B} = \mu_0^{-1}(\text{curl} \mathbf{B}) \times \mathbf{B} = \mu_0^{-1}[(\text{grad} \mathbf{B})\mathbf{B} - (\text{grad} \mathbf{B})^T \mathbf{B}] = \text{div} \boldsymbol{\tau}_m, \quad (2.97)$$

where $\boldsymbol{\tau}_m$ is the *magnetic Maxwell stress tensor*, defined by

$$\boldsymbol{\tau}_m = \mu_0^{-1}[\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}]. \quad (2.98)$$

Here, the subscript m indicates ‘magnetic’, not ‘Maxwell’. As with its electric counterpart, this tensor has an important role to play subsequently.

2.3 Faraday’s Law of Induction

2.3.1 Preliminary Remarks

Having summarized the basic equations for both electrostatics and magnetostatics, we now consider time-dependent fields, in which case there is in general a strong coupling between the electric and magnetic fields. In experiments in 1819, Oersted (Hans Christian Ørsted, 1777–1851) showed that a steady current produces a steady magnetic field and established a connection between the electric current and the magnetic field. Faraday (Michael Faraday, 1791–1867) in his initial experiments investigated the possibility of a steady magnetic field producing a steady electric current, which, as we now know, is not possible. In the process, however, Faraday made the transformational discovery that a *time-varying* magnetic field will induce the flow of an electric current in a closed circuit and therefore an electric field. This phenomenon is known as *electromagnetic induction* and requires the additional information that is embodied in *Faraday’s Law of Induction*, which we examine in detail in this section.

Before discussing Faraday’s Law, we note that a time-varying electric field always generates a magnetic field. Equation (2.39) shows that a changing electric field is necessarily associated with a charge density ρ_e that depends on time. The equation of charge conservation (2.13) connects a time-varying ρ_e to the current density \mathbf{J} , which then generates a magnetic field, as quantified by the Biot–Savart Law (2.60).

2.3.2 Electromotive Force

Consider the uniform flow of an electric current in a closed circuit, which is equivalent to the average motion of charges (conduction electrons) along the wire. This motion suffers resistance analogous to dynamic friction, and therefore some force is required to drive the electrons along the wire and to maintain the current flow. This driving force is known as the *electromotive force*, abbreviated as emf. A uniform current in a closed loop of wire can therefore only be achieved if the component of the driving force tangential to the wire does *net work* in driving the electrons once round the loop in the direction of the current.

Let \mathbf{f} denote the driving force per unit charge. Then the net work done around the closed circuit C is

$$\int_C \mathbf{f} \cdot d\mathbf{x} \neq 0, \quad (2.99)$$

which implies that \mathbf{f} is non-conservative. In the case of a battery, for example, this is the voltage (which drives the current). The use of an *electrostatic* field as the driving force can be excluded because the field \mathbf{E} is the gradient of the electrostatic potential, so that $\mathbf{f} = \mathbf{E} = -\text{grad}\varphi$ and \mathbf{E} is conservative provided φ is single valued.

2.3.3 Flux of a Magnetic Field Through a Moving Circuit

For the development that follows, it is important to remember that Faraday has shown experimentally that the same electromotive force can be induced in a closed circuit whether the closed circuit is moved while keeping the applied magnetic field stationary or whether the closed circuit is fixed and the applied magnetic field varies in time. Faraday also observed that the emf generated is proportional to the change of the magnetic field per unit time, faster change inducing a larger emf. He also noticed that the induced emf is proportional to the area bounded by the closed circuit.

The objective now is to determine an expression for the electromotive force in a small closed circuit C that is moving with velocity \mathbf{v} without rotation in a *magnetostatic field* \mathbf{B} . Let \mathbf{u} denote the velocity of the charges relative to the wire so that $\mathbf{v} + \mathbf{u}$ is their resultant velocity. The magnetic force per unit charge is then given by

$$\mathbf{f} = (\mathbf{v} + \mathbf{u}) \times \mathbf{B}, \quad (2.100)$$

and the electromotive force is the integral of this around the circuit C :

$$\int_C \mathbf{f} \cdot d\mathbf{x} = \int_C [(\mathbf{v} + \mathbf{u}) \times \mathbf{B}] \cdot d\mathbf{x} = - \int_C \mathbf{B} \cdot [(\mathbf{v} + \mathbf{u}) \times d\mathbf{x}]. \quad (2.101)$$

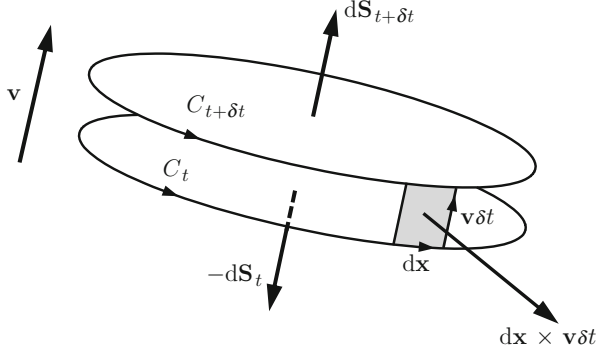


Fig. 2.4 A small circuit moves with uniform velocity \mathbf{v} without rotation. At time t it is at C_t and the approximately plane surface bounded by C_t is denoted by S_t , with vector area element $d\mathbf{S}_t$ related to the direction of C_t by the right-hand screw rule. At an infinitesimal time δt later, the curve is at $C_{t+\delta t}$ and S_t has moved to $S_{t+\delta t}$ with vector area element $d\mathbf{S}_{t+\delta t}$. The normal vector to the ribbon-like surface which completes the boundary of the volume formed by the motion of the curve is also shown, pointing out of the volume

The magnetic force generated by the motion of the charges along the wire is $\mathbf{u} \times \mathbf{B}$, which is perpendicular to the current flow and therefore does no work around the circuit. The expression for the electromotive force therefore reduces to

$$\int_C \mathbf{f} \cdot d\mathbf{x} = - \int_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{x}). \quad (2.102)$$

In the time interval from time t to $t + \delta t$, the closed circuit C moves from the location C_t to location $C_{t+\delta t}$, where δt is infinitesimal. Let S_t denote an approximately plane surface bounded by C_t and $S_{t+\delta t}$ the corresponding surface bounded by $C_{t+\delta t}$ with surface normal vectors related to the direction of C by the right-hand screw rule. The surfaces consisting of the ribbon-like surface swept out by the motion of C and the surfaces S_t and $S_{t+\delta t}$ form the boundary of a closed volume. The outward normals to the three surfaces are in the directions $d\mathbf{x} \times \mathbf{v}$, $-d\mathbf{S}_t$ and $d\mathbf{S}_{t+\delta t}$, respectively, as illustrated in Fig. 2.4. Clearly,

$$\int_C \mathbf{B} \cdot (d\mathbf{x} \times \mathbf{v}) \delta t, \quad - \int_{S_t} \mathbf{B} \cdot d\mathbf{S}, \quad \int_{S_{t+\delta t}} \mathbf{B} \cdot d\mathbf{S} \quad (2.103)$$

are the fluxes of \mathbf{B} across the respective surfaces *out* of the enclosed volume.

Since $\text{div } \mathbf{B} = 0$, it follows from the divergence theorem that

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad (2.104)$$

for any closed surface S . Hence, the sum of the fluxes in (2.103) must vanish:

$$\int_C \mathbf{B} \cdot (d\mathbf{x} \times \mathbf{v}) \delta t + \int_{S_{t+\delta t}} \mathbf{B} \cdot d\mathbf{S} - \int_{S_t} \mathbf{B} \cdot d\mathbf{S} = 0. \quad (2.105)$$

Thus, from (2.102) and (2.105), in the limit as $\delta t \rightarrow 0$, we obtain

$$\int_C \mathbf{f} \cdot d\mathbf{x} = \lim_{\delta t \rightarrow 0} \left\{ -\frac{1}{\delta t} \left[\int_{S_{t+\delta t}} \mathbf{B} \cdot d\mathbf{S} - \int_{S_t} \mathbf{B} \cdot d\mathbf{S} \right] \right\} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}. \quad (2.106)$$

This shows the important property that the electromotive force is equal to the change of the magnetic flux across any surface S bounded by C . The minus sign on the right-hand side of (2.106) indicates that the induced electric current will produce a magnetic field that always opposes the change of the magnetic flux. The latter connection is known as Lenz's Law (Heinrich Friedrich Emil Lenz, 1804–1865).

If the circuit C is moved through a magnetic field in such a way that the flux through C changes, then an electromotive force is induced and a current will flow. For a rigid closed circuit moving with uniform translation (no rotation), a current will flow provided \mathbf{B} changes with position. If \mathbf{B} is uniform, then the flux will not change unless C rotates.

Since the circuit C is moving, we may write, with reference to the integral on the right-hand side of (2.106),

$$\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \mathbf{B}_{,t} \cdot d\mathbf{S}, \quad (2.107)$$

where $\mathbf{B}_{,t}$ denotes the *material time derivative* (or total time derivative), which accounts for the motion of C (and hence of S) and is given by

$$\mathbf{B}_{,t} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \text{grad})\mathbf{B}, \quad (2.108)$$

where $\partial \mathbf{B} / \partial t$ is the time derivative at fixed \mathbf{x} . For a definition of the material time derivative for a moving and deforming material, we refer to Sect. 3.1.4.1. Since \mathbf{v} is independent of position on S , it is easy to show from the identity (A.19) in Appendix A.2 that $(\mathbf{v} \cdot \text{grad})\mathbf{B} = -\text{curl}(\mathbf{v} \times \mathbf{B})$. Using this in the above equation together with (2.107) and then applying Stokes' theorem, we obtain from (2.106) the formula

$$\int_C \mathbf{f} \cdot d\mathbf{x} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{x}. \quad (2.109)$$

Equation (2.109) gives an expression for the electromotive force alternative to (and equivalent to) that in (2.106). The right-hand side in each case is the rate of change of the flux of \mathbf{B} through S . For an observer in a frame of reference moving with C , S appears fixed and (2.109) admits the possibility that \mathbf{B} can vary with t as well as \mathbf{x} . We recall that in deriving (2.106), we assumed that \mathbf{B} was time independent. The derivation carries over to the case when \mathbf{B} depends on time. Thus, the flux through the circuit C may be changed by either a time-dependent magnetic field or by the motion of C , or both simultaneously.

2.3.4 Faraday's Law

In the previous subsection we quantified the electromotive force generated by moving a closed circuit in a magnetostatic field. We have shown that a magnetic force is the driving agency that generates the flow of charges. Here, on the other hand, we consider an observer rigidly attached to the moving circuit, i.e. the circuit is held fixed in the moving frame of reference. Let \mathbf{E}' and \mathbf{B}' be the electric and magnetic induction fields as measured in this frame of reference, which we take to have *constant* velocity \mathbf{v} . We now relate these to the corresponding fields in the fixed frame.

Consider the force on a unit point charge moving with velocity \mathbf{u} in an electromagnetic field. In a fixed frame of reference the magnetic force (2.1.3) is

$$\mathbf{f} = \mathbf{E} + (\mathbf{u} + \mathbf{v}) \times \mathbf{B}. \quad (2.110)$$

The force measured in the moving frame is

$$\mathbf{f}' = \mathbf{E}' + \mathbf{u} \times \mathbf{B}' \quad (2.111)$$

since the point charge has velocity \mathbf{u} relative to this frame of reference. According to Newton's Second Law, the force must be the same in both frames. Therefore,

$$\mathbf{E} + (\mathbf{u} + \mathbf{v}) \times \mathbf{B} = \mathbf{E}' + \mathbf{u} \times \mathbf{B}', \quad (2.112)$$

which must hold for arbitrary \mathbf{u} . By taking $\mathbf{u} = \mathbf{0}$, it follows that

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad (2.113)$$

and hence on substituting back in (2.112), we obtain

$$\mathbf{B}' = \mathbf{B}. \quad (2.114)$$

As already noted, the magnetic force $\mathbf{u} \times \mathbf{B}$ in (2.100) does not contribute to the electromotive force given by (2.101) and (2.102). According to the moving observer the driving force cannot be magnetic since the stationary circuit does not experience magnetic forces in a magnetostatic field. However, the moving observer perceives a non-conservative electric field so that electric force becomes the driving agency that moves the charges. The electric force per unit charge measured in the moving frame is $\mathbf{f} = \mathbf{E}'$, and the electromotive force on the complete circuit C is then given by

$$\int_C \mathbf{E}' \cdot d\mathbf{x} = - \int_S \mathbf{B}'_{,t} \cdot d\mathbf{S}, \quad (2.115)$$

which shows that, contrary to the electrostatic case, the electric field inside the conducting circuit is non-conservative when there is a change in the magnetic flux. Note that not only is $\mathbf{B}' = \mathbf{B}$ but also $\mathbf{B}'_{,t} = \mathbf{B}_{,t}$ since the latter is the time derivative of \mathbf{B} following the motion of the circuit. By Stokes' theorem, (2.115) can be written as

$$\int_S (\text{curl } \mathbf{E}' + \mathbf{B}'_{,t}) \cdot d\mathbf{S} = 0, \quad (2.116)$$

and since S is arbitrary, it follows that

$$\text{curl } \mathbf{E}' + \mathbf{B}'_{,t} = \mathbf{0}, \quad (2.117)$$

as measured in the frame of reference of the moving circuit.

Substitution of (2.113) into (2.117) gives

$$\text{curl } (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{B}_{,t} = \mathbf{0}, \quad (2.118)$$

and hence, using (2.108) and the formula $\text{curl } (\mathbf{v} \times \mathbf{B}) = -(\mathbf{v} \cdot \text{grad})\mathbf{B}$, we obtain the important equation

$$\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad (2.119)$$

connecting the fields \mathbf{E} and \mathbf{B} measured in a fixed frame of reference. This shows that the structure of (2.117) is invariant under changes of uniformly moving frames of reference.

Equation (2.106) is known as *Faraday's Law of Induction*, which we repeat here compactly as

$$\int_C \mathbf{f} \cdot d\mathbf{x} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}. \quad (2.120)$$

Faraday's Law is very general and allows for both time and space variation of \mathbf{E} and \mathbf{B} and is independent of observer. Different observers measure different voltages (emfs), the same flux, but different rates of change of that flux. Faraday's Law is based on experiments in which the magnetic flux through a thin wire circuit is made to vary in a variety of ways. It is the basis of the dynamo and the electric motor (which involve rotations of wire loops in magnetic fields).

Thus, the electric and magnetic fields are in general intimately connected when there is time variation or motion.

Equation (2.119) replaces the electrostatic equation $\text{curl } \mathbf{E} = \mathbf{0}$ for time-varying situations. By taking the divergence of (2.119) we obtain

$$\text{div} \left(\frac{\partial \mathbf{B}}{\partial t} \right) = \frac{\partial}{\partial t} (\text{div } \mathbf{B}) = 0. \quad (2.121)$$

If $\text{div } \mathbf{B} = 0$ at some initial time, then it follows from (2.121) that $\text{div } \mathbf{B} = 0$ for all time. Thus, the magnetostatic equation $\text{div } \mathbf{B} = 0$ still holds in the time-varying situation.

2.4 Maxwell's Equations

2.4.1 The Full Set of Maxwell's Equations

We begin this section by first recalling the fundamental equations governing time-independent electric and magnetic fields that were discussed, respectively, in Sects. 2.1 and 2.2. The equations of electrostatics were derived in Sects. 2.1.7 and 2.1.8 and are

$$\text{curl } \mathbf{E} = \mathbf{0}, \quad \text{div } \mathbf{E} = \frac{\rho_e}{\epsilon_0}. \quad (2.122)$$

Similarly, for the magnetostatic field we have the two equations

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J}, \quad \text{div } \mathbf{B} = 0, \quad (2.123)$$

from Sects. 2.2.3 and 2.2.2, respectively, where we emphasize that \mathbf{J} is a *steady* current density. Applying the divergence operator to both sides of (2.123)₁ shows that

$$\text{div } \mathbf{J} = 0. \quad (2.124)$$

For time-dependent fields, the equation of charge conservation (2.124) is no longer valid and must be replaced by (2.13), which we write here as

$$\text{div } \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0. \quad (2.125)$$

In Sect. 2.3 we have seen that a steady current produces a magnetic field and that a time-varying magnetic field will induce a flow of electric charges and therefore produce an electric field. Clearly, the fundamental equations describing static fields need to be modified to reflect these experimental facts. Equations (2.122)₂ and (2.123)₂ remain unchanged, while (2.122)₁ is replaced by (2.119), i.e.

$$\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad (2.126)$$

which is the local form of Faraday's Law.

This leaves (2.123)₁, which no longer holds since it implies (2.124) and not (2.125). To compensate for this difference we write, instead of (2.123)₁,

$$\text{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mathbf{G}, \quad (2.127)$$

where \mathbf{G} is an unknown vector function that has to be determined. On taking the divergence of this equation and using (2.122)₂ and (2.125) we obtain

$$\text{div} \mathbf{G} = -\mu_0 \text{div} \mathbf{J} = \mu_0 \frac{\partial \rho_e}{\partial t} = \mu_0 \varepsilon_0 \text{div} \left(\frac{\partial \mathbf{E}}{\partial t} \right). \quad (2.128)$$

The equations are now self-consistent if we set

$$\mathbf{G} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (2.129)$$

so that

$$\mu_0^{-1} \text{curl} \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (2.130)$$

A divergence-free vector can be added to \mathbf{G} , but this is inessential.

We now collect together the four fundamental differential equations for time-dependent fields as

$$\text{div} \mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \quad \text{div} \mathbf{B} = 0, \quad (2.131)$$

$$\text{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (2.132)$$

These are the four *Maxwell equations* (James Clerk Maxwell, 1831–1879) that govern the fields \mathbf{E} and \mathbf{B} everywhere when the charge density ρ_e and current density \mathbf{J} are known. When coupled with the Lorentz Law of force, they constitute an exact and complete description of classical (non-relativistic) electromagnetic phenomena.

On taking the curl of (2.132)₂ and making use of (2.132)₁, we obtain

$$\text{curl}(\text{curl} \mathbf{E}) = -\frac{\partial}{\partial t}(\text{curl} \mathbf{B}) = -\frac{\partial}{\partial t} \left(\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \quad (2.133)$$

Combining this with the identity $\text{curl}(\text{curl} \mathbf{E}) = \text{grad}(\text{div} \mathbf{E}) - \nabla^2 \mathbf{E}$ and (2.131)₁, we arrive at the equation

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \text{grad} \left(\frac{\rho_e}{\varepsilon_0} \right) + \mu_0 \frac{\partial \mathbf{J}}{\partial t}. \quad (2.134)$$

This is the *inhomogeneous wave equation* for \mathbf{E} , where $c = (\mu_0 \varepsilon_0)^{-1/2}$ is the speed of light anticipated at the beginning of Sect. 2.2 (i.e. the speed of electromagnetic effects in free space). The right-hand side of (2.134) is the *source* term. Similarly, taking the curl of (2.132)₁ and using (2.131)₂ and (2.132)₂, the corresponding wave equation for the magnetic induction \mathbf{B} is obtained as

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \text{curl} \mathbf{J}. \quad (2.135)$$

In free space, where $\rho_e = 0$ and $\mathbf{J} = \mathbf{0}$, we obtain the *homogeneous wave equations*

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}. \quad (2.136)$$

Now, from (2.131)₂, we may write $\mathbf{B} = \text{curl} \mathbf{A}$, \mathbf{A} being a time-dependent vector potential. Substitution of this into (2.132)₂ yields

$$\text{curl} \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0}, \quad (2.137)$$

and hence we may introduce a scalar field φ such that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad} \varphi. \quad (2.138)$$

On substituting this into (2.131)₁ we may rearrange it as a wave equation for φ , specifically

$$\nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho_e}{\varepsilon_0} - \frac{\partial}{\partial t} \left(\text{div} \mathbf{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right). \quad (2.139)$$

Similarly, substitution into (2.132)₁ leads to a wave equation for \mathbf{A} , i.e.

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} - \text{grad} \left(\text{div} \mathbf{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right). \quad (2.140)$$

Since there is flexibility in the definition of \mathbf{A} (as noted in Sect. 2.2.4, the gradient of an arbitrary scalar function may be added to \mathbf{A}), these equations suggest that the additional condition

$$\text{div} \mathbf{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0 \quad (2.141)$$

should be adopted. This is a gauge condition, extending that in (2.86) to the time-varying situation and known as the *Lorenz condition* (Ludwig Valentin Lorenz, 1829–1891). Note that Lorentz and Lorenz are different. The wave equations then become

$$\nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho_e}{\varepsilon_0}, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \quad (2.142)$$

These have solutions analogous to those in the static case given by (2.23) and (2.63), respectively, namely

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_e(\mathbf{x}', t')}{R} dV(\mathbf{x}'), \quad \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}', t')}{R} dV(\mathbf{x}'), \quad (2.143)$$

where t' is the retarded time $t - R/c$. Thus, the structure of the potentials carries over to the dynamic situation.

2.4.2 Polarization and Magnetization in Materials

In Sects. 2.1.10 and 2.2.5, respectively, we introduced the *electric displacement vector* \mathbf{D} and the *magnetic field vector* \mathbf{H} in free space. These are simply related to \mathbf{E} and \mathbf{B} , respectively, by a constant factor in each case. Thus,

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad (2.144)$$

where ε_0 is the electric permittivity and μ_0 the magnetic permeability of free space introduced earlier. In material media these relations do not hold in general, and to aid the description of the electric and magnetic properties of materials, we introduce two additional vectors, defined by

$$\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E}, \quad \mathbf{M} = \mu_0^{-1} \mathbf{B} - \mathbf{H}. \quad (2.145)$$

The vector \mathbf{P} is called the *polarization density* and \mathbf{M} the *magnetization density*. We now provide physical interpretations for these quantities.

We recall from (2.44) that the electrostatic potential at \mathbf{x} due to a point charge e and dipole \mathbf{p} at the origin is given by

$$\varphi(\mathbf{x}) = \frac{e}{4\pi\varepsilon_0 r} - \frac{1}{4\pi\varepsilon_0} \mathbf{p} \cdot \text{grad} \left(\frac{1}{r} \right), \quad (2.146)$$

where we have replaced the approximation by an equality by neglecting higher-order terms. Thus, we are considering an isolated point charge and an isolated dipole situated at the origin. There is no net charge on a dipole since it consists of equal amounts of positive and negative charge (these are said to be *bound charges*), whereas e is regarded as a *free charge*. We now generalize these notions and consider a continuous distribution of free charges and dipoles in a volume V with densities $\rho_f(\mathbf{x}')$ and $\mathbf{P}(\mathbf{x}')$ at the point \mathbf{x}' , where $\rho_f(\mathbf{x}')$ represents the *free charge density*. Then, the potential at \mathbf{x} due to this distribution is

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \left[\frac{\rho_f(\mathbf{x}')}{R} + \mathbf{P}(\mathbf{x}') \cdot \text{grad}' \left(\frac{1}{R} \right) \right] dV(\mathbf{x}'), \quad (2.147)$$

where $R = |\mathbf{x} - \mathbf{x}'|$ and grad' is the gradient with respect to \mathbf{x}' . We also note that

$$\text{grad}' \left(\frac{1}{R} \right) = -\text{grad} \left(\frac{1}{R} \right). \quad (2.148)$$

By the divergence theorem, (2.147) can be rewritten as

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_f(\mathbf{x}') - \text{div}' \mathbf{P}(\mathbf{x}')}{R} dV(\mathbf{x}') + \int_S \frac{\mathbf{P}(\mathbf{x}') \cdot d\mathbf{S}(\mathbf{x}')}{R}, \quad (2.149)$$

where div' is the divergence operator with respect to \mathbf{x}' and S is the bounding surface of V .

From the first integral in (2.149) it can be seen that the term $-\text{div}' \mathbf{P}(\mathbf{x}')$ acts like an additional charge density. It is referred to as the *bound charge density* and denoted ρ_b , i.e.

$$\rho_b(\mathbf{x}) = -\text{div} \mathbf{P}(\mathbf{x}) \quad (2.150)$$

at any point \mathbf{x} in V . Thus, the total charge density consists of free charge and bound charge, and we write, for any point \mathbf{x} in V ,

$$\rho_e(\mathbf{x}) = \rho_f(\mathbf{x}) + \rho_b(\mathbf{x}). \quad (2.151)$$

This provides the interpretation of \mathbf{P} . The first term in the formula (2.149) can then be recognized as the same as that in (2.23), which did not account for the surface term included here. The complete expression (2.149) is a solution of Poisson's equation (2.41).

It follows from (2.131)₁ and (2.145)₁ that

$$\text{div} \mathbf{D} = \rho_f. \quad (2.152)$$

This is the equation that replaces (2.131)₁ in the case of polarizable materials, and it applies for both static and time-dependent fields.

Turning now to (2.132)₁, we may use (2.145), (2.150) and (2.151) to rewrite it as

$$\text{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} - \text{curl} \mathbf{M} - \frac{\partial \mathbf{P}}{\partial t}. \quad (2.153)$$

We now interpret the different terms on the right-hand side of this equation. The application of a magnetic field to a material generates a flow of electrons and an alignment of intrinsic magnetic dipoles known as magnetization and quantified by the magnetization density (or magnetic moment per unit volume) introduced

in (2.145)₂. The effect of the magnetization is to induce a *bound current density* resulting from the motion of bound charges in atoms. We denote this by \mathbf{J}_b . Moreover, when the polarization changes in time, it generates an additional current, characterized by the *polarization current density*, which we denote by \mathbf{J}_p , and the difference $\mathbf{J} - \mathbf{J}_b - \mathbf{J}_p$ is the *free current density*, which we denote by \mathbf{J}_f . Thus,

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p. \quad (2.154)$$

The connections

$$\mathbf{J}_b = \text{curl} \mathbf{M}, \quad \mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \quad (2.155)$$

then follow.

To see why $\text{curl} \mathbf{M}$ can be interpreted as a current density, consider the following. From Sect. 2.2.3 the vector potential at the point \mathbf{x} associated with an isolated magnetic dipole situated at the origin is

$$\mathbf{A}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \mathcal{M} \text{grad} \left(\frac{1}{r} \right), \quad (2.156)$$

where \mathcal{M} is the magnetization tensor given by (2.68), which is skew-symmetric and is related to the magnetic moment vector \mathbf{m} (the axial vector of \mathcal{M}) by $\mathbf{m} = -\frac{1}{2}\epsilon \mathcal{M}$. Conversely, \mathcal{M} is given in terms of \mathbf{m} by $\mathcal{M} = -\epsilon \mathbf{m}$, or, in components, $\mathcal{M}_{ij} = -\epsilon_{ijk} m_k$.

Suppose now there is a distribution of dipoles with density $\mathbf{M}(\mathbf{x}')$ and tensor density $\mathcal{M}(\mathbf{x}')$ within a volume V , vanishing outside V . Then, the vector potential at \mathbf{x} is given by the integral

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \mathcal{M}(\mathbf{x}') \text{grad}' \left(\frac{1}{R} \right) dV(\mathbf{x}'), \quad (2.157)$$

where (2.148) has again been used. By applying the divergence theorem, we obtain

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\text{div}' \mathcal{M}(\mathbf{x}')}{R} dV(\mathbf{x}') + \frac{\mu_0}{4\pi} \int_S \frac{\mathcal{M}(\mathbf{x}') d\mathbf{S}(\mathbf{x}')}{R}, \quad (2.158)$$

where the skew-symmetry of \mathcal{M} has been used. But it is easy to show that $\text{div}' \mathcal{M}(\mathbf{x}') = \text{curl}' \mathbf{M}(\mathbf{x}')$ and hence

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\text{curl}' \mathbf{M}(\mathbf{x}')}{R} dV(\mathbf{x}') + \frac{\mu_0}{4\pi} \int_S \frac{\mathcal{M}(\mathbf{x}') d\mathbf{S}(\mathbf{x}')}{R}. \quad (2.159)$$

Comparison of the first term in the above with (2.63) shows that, for a distribution of dipoles, $\text{curl} \mathbf{M}$ behaves like a current density and therefore has the interpretation

indicated above. The expression (2.159) is a solution of Poisson's equation (2.89) for $\mathbf{J} = \mathbf{J}_b$.

Clearly $\text{div} \mathbf{J}_b = 0$ and, by definition, \mathbf{J}_p satisfies the charge conservation equation

$$\text{div} \mathbf{J}_p = \frac{\partial}{\partial t} (\text{div} \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}, \quad (2.160)$$

from which it may be deduced that the free charge satisfies separately the charge conservation equation

$$\text{div} \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}. \quad (2.161)$$

It follows from (2.132)₁ and (2.145)₂ that

$$\text{curl} \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad (2.162)$$

which shows that only the free electric current density \mathbf{J}_f remains in Maxwell's equation. The term $\partial \mathbf{D} / \partial t$, the time derivative of the electric displacement, plays a role similar to a current density and is known as the *displacement current*.

To summarize, the four Maxwell equations in material matter may be written as

$$\text{div} \mathbf{D} = \rho_f, \quad \text{div} \mathbf{B} = 0, \quad (2.163)$$

$$\text{curl} \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad \text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.164)$$

which are equivalent to (2.131) and (2.132). For a detailed treatment of Maxwell's equations, see, for example, the classic texts by Jackson (1999), Landau and Lifshitz (1960) and Stratton (2007), and, for an interesting historical overview, we refer to the book by Maugin (1988).

In (2.145) there are three vector fields associated with electric effects and three vector fields associated with magnetic effects. In each case there is one connection between the three vectors. These apply to all polarizable or magnetizable materials. To distinguish between different materials an additional connection is needed in each case. Such a connection is known as a *constitutive equation*. For polarizable materials this may take the form of an explicit expression for \mathbf{P} in terms of either \mathbf{D} or \mathbf{E} , with either \mathbf{E} or \mathbf{D} as the independent electric variable, or of either \mathbf{E} or \mathbf{D} in terms of one of the other variables. Similarly, for magnetizable materials any one of \mathbf{H} , \mathbf{B} or \mathbf{M} may be adopted as the independent variable and the constitutive law specified accordingly.

Basic examples of constitutive laws include those for *linear isotropic media*, for which the equations in (2.144) are replaced by

$$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_r \mu_0 \mathbf{H}, \quad (2.165)$$

where ε_r and μ_r are the *relative dielectric permittivity* and *relative magnetic permeability*, respectively. From (2.145) and (2.165), the polarization and magnetization are given by

$$\mathbf{P} = \frac{\varepsilon_r - 1}{\varepsilon_r} \mathbf{D}, \quad \mathbf{M} = \frac{\mu_r - 1}{\mu_0 \mu_r} \mathbf{B}, \quad (2.166)$$

so that \mathbf{P} and \mathbf{M} , respectively, are parallel to the electric displacement \mathbf{D} and the magnetic induction \mathbf{B} . Also, the units of the electric polarization \mathbf{P} and the electric displacement \mathbf{D} as well as the units of the magnetization vector \mathbf{M} and the magnetic field vector \mathbf{H} coincide; see (2.165) and (2.166). In vacuo or in non-polarizable material, $\varepsilon_r = 1$, while in vacuo or in non-magnetizable media, $\mu_r = 1$. In polarizable materials $\varepsilon_r > 1$ and \mathbf{P} is in the same direction as \mathbf{D} . For most materials $\mu_r > 1$; however, there are some magnetizable materials for which $\mu_r < 1$ and \mathbf{M} is therefore opposite in direction to \mathbf{B} . For details of the permittivity and permeability constants of dielectric and magnetic materials, a convenient source of information is [Wikipedia \(2013\)](#), which contains references to multiple sources.

2.5 Boundary Conditions

Maxwell's equations (2.163) and (2.164) are valid for any material medium provided \mathbf{D} and \mathbf{H} are given by appropriate constitutive laws. To these equations we need to append boundary conditions in order to formulate and solve boundary-value problems. In general the field vectors \mathbf{E} , \mathbf{D} , \mathbf{B} and \mathbf{H} are discontinuous across surfaces between different media or across a surface bounding the material. In this section we derive, using (2.163) and (2.164) in integral form together with the divergence and Stokes' theorems, as appropriate, the equations satisfied by the discontinuities. We consider only stationary surfaces. The results will be generalized to moving surfaces in Chap. 9.

2.5.1 Boundary Conditions for \mathbf{E} and \mathbf{D}

Let S be a *stationary* surface which carries free surface charge σ_f per unit area. The two sides of S are distinguished as side 1 and side 2, and field vectors on the two sides of S are identified with subscripts 1 and 2. Let \mathbf{n} be the unit normal to S pointing from side 1 to side 2. The 'jump' in a vector on S is the difference between its values on side 2 and side 1, evaluated on S . Thus \mathbf{E} , for example, has jump $\mathbf{E}_2 - \mathbf{E}_1$, which is denoted $\llbracket \mathbf{E} \rrbracket$, and similarly for the other vectors. The jump conditions satisfied by \mathbf{E} and \mathbf{D} are summarized as

$$\mathbf{n} \times \llbracket \mathbf{E} \rrbracket = \mathbf{0}, \quad \mathbf{n} \cdot \llbracket \mathbf{D} \rrbracket = \sigma_f. \quad (2.167)$$

We now establish these results.

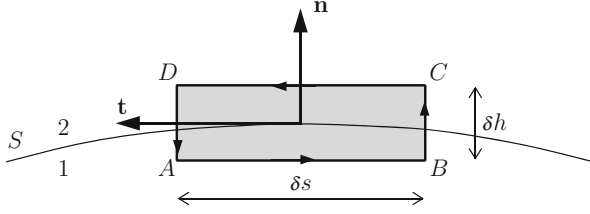


Fig. 2.5 A small plane area intersecting the surface S with corners A, B, C, D in the plane of the unit normal \mathbf{n} to the surface and a unit tangent vector \mathbf{t} . The unit normal points from side 1 to side 2 of the surface. The bounding curve is traversed in the direction of the arrows along the path $ABCD A$

Consider the Maxwell equation (2.164)₂ integrated over an open surface Σ with bounding curve Γ . After application of Stokes' theorem, it becomes

$$\int_{\Gamma} \mathbf{E} \cdot d\mathbf{x} = - \int_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \quad (2.168)$$

Let Σ be an infinitesimal plane rectangular surface with Γ identified by its corner points $ABCD$ lying in the plane of the unit normal \mathbf{n} to a surface S and a unit tangent vector \mathbf{t} to the surface and intersecting S , as shown in Fig. 2.5. The sides AB and CD of Γ are parallel to \mathbf{t} and have lengths δs . The sides BC and DA are parallel to \mathbf{n} and have lengths δh . Then, application of (2.168) to Σ and Γ yields the approximate result

$$- \int_{AB} \mathbf{E} \cdot \mathbf{t} ds + \int_{BC} \mathbf{E} \cdot \mathbf{n} dh + \int_{CD} \mathbf{E} \cdot \mathbf{t} ds - \int_{DA} \mathbf{E} \cdot \mathbf{n} dh \approx - \frac{\partial \mathbf{B}}{\partial t} \cdot (\mathbf{n} \times \mathbf{t}) \delta h \delta s. \quad (2.169)$$

Taking the limit as $\delta h \rightarrow 0$ and then dividing by δs and letting $\delta s \rightarrow 0$, we obtain $\mathbf{E}_2 \cdot \mathbf{t} - \mathbf{E}_1 \cdot \mathbf{t} = 0$, i.e. $\mathbf{t} \cdot [\mathbf{E}] = 0$. This holds for an arbitrary \mathbf{t} normal to \mathbf{n} , and hence the result (2.167)₁ follows.

Now consider a cylinder (or 'pill box') of infinitesimal height δh and cross-sectional area $\delta S = \mathbf{n} \delta S$ straddling the surface S , as depicted in Fig. 2.6. Equation (2.163)₁ is integrated over the volume V of the cylinder and the divergence theorem then applied to give

$$\int_{\Sigma} \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f dV, \quad (2.170)$$

where Σ is the bounding surface of the cylinder.

Since δh is infinitesimal and the flux of \mathbf{D} across the lateral surface of the cylinder becomes negligible as $\delta h \rightarrow 0$, the only contributions to the surface integral come from the top and bottom surfaces of the cylinder. The right-hand side of (2.170) is the total free charge in V , which consists of the surface charge $\sigma_f \delta S$.

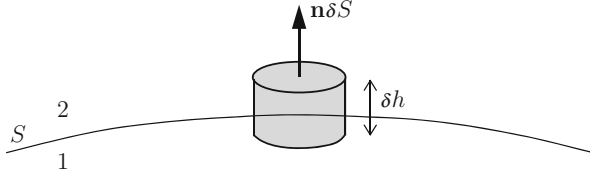


Fig. 2.6 A ‘pill-box’ of height δh and cross-sectional area δS straddling the surface S with unit normal \mathbf{n} pointing from side 1 to side 2 of S

Equation (2.170) is therefore approximated simply as $\mathbf{D}_2 \cdot \mathbf{n} \delta S - \mathbf{D}_1 \cdot \mathbf{n} \delta S \approx \sigma_f \delta S$, which, after dividing by δS and taking the limit $\delta S \rightarrow 0$, yields $\mathbf{n} \cdot \llbracket \mathbf{D} \rrbracket = \sigma_f$, and hence (2.167)₂ is established. Clearly, if the surface S is free of distributed charge σ_f , then the normal component of \mathbf{D} is continuous.

If the material medium is surrounded by a non-polarizable medium or a vacuum (where $\mathbf{P} = \mathbf{0}$), the boundary conditions (2.167) can be written in the alternative forms

$$\llbracket \mathbf{D} \rrbracket = \sigma_f \mathbf{n} + (\mathbf{n} \cdot \mathbf{P}) \mathbf{n} - \mathbf{P}, \quad \varepsilon_0 \llbracket \mathbf{E} \rrbracket = \sigma_f \mathbf{n} + (\mathbf{n} \cdot \mathbf{P}) \mathbf{n}, \quad (2.171)$$

where use has been made of the connection (2.145)₁.

2.5.2 Boundary Conditions for \mathbf{B} and \mathbf{H}

The counterparts of the boundary conditions (2.167) for the magnetic vectors are

$$\mathbf{n} \times \llbracket \mathbf{H} \rrbracket = \mathbf{K}_f, \quad \mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0, \quad (2.172)$$

where \mathbf{K}_f is the free current surface density on the surface S per unit area. The proof of (2.172) follows the same pattern as for (2.167) and is given below.

Consider again the cylinder of infinitesimal height δh and cross-sectional area $\delta S = \mathbf{n} \delta S$ straddling the surface S in Fig. 2.6. Equation (2.163)₂, when integrated over the volume V of the cylinder followed by an application of the divergence theorem, yields

$$\int_{\Sigma} \mathbf{B} \cdot d\mathbf{S} = 0. \quad (2.173)$$

Again, since δh is infinitesimal and the flux of \mathbf{B} across the lateral surface of the cylinder becomes negligible as $\delta h \rightarrow 0$ and only the integrals over the top and bottom surfaces of the cylinder contribute non-negligible values. Equation (2.173) is therefore approximated simply as $\mathbf{B}_2 \cdot \mathbf{n} \delta S - \mathbf{B}_1 \cdot \mathbf{n} \delta S \approx 0$, which, after dividing by δS and taking the limit $\delta S \rightarrow 0$, yields $\mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0$, and hence (2.172)₂ is established.

Next, consider (2.164)₂ integrated over the open surface Σ with bounding curve Γ shown in Fig. 2.5. On application of Stokes' theorem, it becomes

$$\int_{\Gamma} \mathbf{H} \cdot d\mathbf{x} = \int_{\Sigma} \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}, \quad (2.174)$$

which yields the approximate result

$$\begin{aligned} & - \int_{AB} \mathbf{H} \cdot \mathbf{t} ds + \int_{BC} \mathbf{H} \cdot \mathbf{n} dh + \int_{CD} \mathbf{H} \cdot \mathbf{t} ds - \int_{DA} \mathbf{H} \cdot \mathbf{n} dh \\ & \approx \left[\left(\int_{BC} \mathbf{J}_f dh + \frac{\partial \mathbf{D}}{\partial t} dh \right) \times \mathbf{n} \right] \cdot \mathbf{t} \delta s. \end{aligned} \quad (2.175)$$

In the limit as $\delta h \rightarrow 0$ the term in $\partial \mathbf{D} / \partial t$ in the integral on the right-hand side becomes negligible, as do the integrals along BC and DA on the left-hand side, while the term in \mathbf{J}_f becomes the surface current density \mathbf{K}_f with $\mathbf{n} \cdot \mathbf{K}_f = 0$. Then dividing by δs and letting $\delta s \rightarrow 0$, we obtain $\mathbf{H}_2 \cdot \mathbf{t} - \mathbf{H}_1 \cdot \mathbf{t} = (\mathbf{K}_f \times \mathbf{n}) \cdot \mathbf{t}$. Setting $\mathbf{n} \times \mathbf{t} = \mathbf{k}$ and noting that $\mathbf{k} \times \mathbf{n} = \mathbf{t}$, it follows that $\{\mathbf{n} \times [\mathbf{H}]\} \cdot \mathbf{k} = \mathbf{K}_f \cdot \mathbf{k}$. Since \mathbf{t} is an arbitrary tangent, then so is \mathbf{k} . This holds for arbitrary \mathbf{k} normal to \mathbf{n} , and hence the result (2.172)₂ follows.

Note that if outside the material is a vacuum or a non-magnetizable material $\mathbf{M} = \mathbf{0}$ outside the material, in which case, by combining the two boundary conditions (2.172) and using the connection (2.145)₂, we obtain

$$[\mathbf{H}] = (\mathbf{n} \cdot \mathbf{M})\mathbf{n} - \mathbf{n} \times \mathbf{K}_f, \quad [\mathbf{B}] = \mu_0 \mathbf{n} \times (\mathbf{n} \times \mathbf{M}) - \mu_0 \mathbf{n} \times \mathbf{K}_f. \quad (2.176)$$

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