Chapter 2
Sequences in $\mathbb{R}$

In this chapter we investigate the most basic concept of analysis, the limit of a sequence. The concepts of convergence and divergence will be the main focus of our discussion but we will also touch again on the notions of completeness and set structures in $\mathbb{R}$. It is also worth noting that each of the constructs comprising the remainder of the text, series, functional limits and continuity, the derivative, and the integral will rely heavily on our excursion here.

2.1 Sequences and Convergence

Like many mathematical definitions, a sequence can be defined rather intuitively. For instance, we could simply say that a sequence is a list of real numbers. This is likely how you have thought about sequences up to this point in your mathematical experience, however, as is often the case with definitions of this nature, there is a lack of precision in the wording which allows ambiguity to creep into the picture. Let's take a moment to think about these potential issues. For one, how long is this list? We will require that a sequence be a countably infinite list of real numbers, but our wording above would permit finite strings of numbers. Does the order of the terms in the sequence matter? Of course they do, but this is not reflected in our definition. These two problems are easily eliminated by restating our definition: A sequence is an ordered, countably infinite list of real numbers. This does not roll off the tongue quite as nicely and simply adding more adjectives is a rather imprecise use of language. As a formal definition, consider the following.

Definition 2.1.1. A *sequence* of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$.

This phrasing immediately gives some concrete meaning to our informal notion stated above. Specifically, if $f : \mathbb{N} \to \mathbb{R}$ is our sequence, then we can represent f as the collection of range values $(f(1), f(2), f(3), \ldots)$ of f, i.e. we have represented f as an ordered, countably infinite list of real numbers. This encompasses all of the intuition that we previously hoped for but we have only used well-defined

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mathematical terms to produce the result. In particular, we have an ordered list and the order of the *terms* (particular numbers appearing in the list) is important, e.g. $f(1)$ appears before $f(2)$, and so on. Repeated terms are allowed, but are not necessary. And, most importantly, the list has countably many terms.

The next step is to understand the several types of notation used to represent sequences. As it is impossible to write out an infinite string, we need a means of representing sequences which leaves no room for guesswork on the part of the reader. As a simple example, consider the list $(1, 1, 1, 1, \ldots)$. What is the fifth term? What is the ninth term? Are you sure that both of these are 1? Typically this is what the "..." should represent, but how do you know for certain that the pattern you've deciphered is in fact the correct one? Hence the ambiguity with our notion of lists. On the other hand, consider the function $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = 1$ for all $n \in \mathbb{N}$. What is the fifth term in this sequence? The function gives us a concrete means of identifying terms whereas the list leaves room for error. As such, we will prefer some form of function, or formulaic, notation for sequences.

Often we will abandon the formal use of f for our function and replace this with the simpler notation $f(n) = a_n$ representing the nth term of the sequence. We then represent the sequence by the symbol (a_n). For an explicit situation, consider the function $f(n) = 1/n$. We could represent this as $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$ or $(1/n)_{n=1}$ or $(a_n)_{n=1}$ where $a_n = 1/n$. Also, we will often want to begin our sequence at a position other than $n = 1$. When this is necessary we will use the notation $(a_n)_{n=k}$ if beginning with a natural number $k > 1$ or $k = 0$. This can be done in a more formal manner by expanding our definition of sequence to functions $f : K \to \mathbb{R}$ where $K \subset \mathbb{Z}$ has the form $\{n \in \mathbb{Z} : n \geq m\}$ for some fixed $m \in \mathbb{Z}$. As a last comment on notation, we use parenthesis in the notation for a sequence to preserve the distinction between the sequence as a newly defined object and not merely as a set of real numbers. However, each sequence is associated with its set of range values and we would use braces when representing this set. The set of range values for the sequence defined by the function $f(n) = 1$ is $\{1\}$ while the set of range values for the sequence $f(n) = 1/n$ is $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$.

Definition 2.1.2. Let (a_n) be a sequence in $\mathbb{R}$ and let $a \in \mathbb{R}$. We say that (a_n) *converges to* a if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for every $n \geq N$. In this case we call a the *limit* of the sequence and represent this symbolically by $(a_n) \to a$ or $\lim_{n\to\infty} a_n = a$.

As a first observation, the notational comment following the definition is phrased to imply that a sequence can have only one limit, signified by the phrase "the limit." This fact is not explicitly asserted in the definition of convergence, it requires proof which you will provide in Exercise 2.1.1. On occasion we will simplify the notation and write $\lim a_n = a$; this should cause no ambiguity for sequential limits as interesting phenomena only occur as we proceed further and further out into the sequence.

Before considering examples, it is beneficial to analyze what has just been set before us. We will see definitions of this sort on several occasions and an ever increasing level of comfort with such statements will be necessary. Suppose we

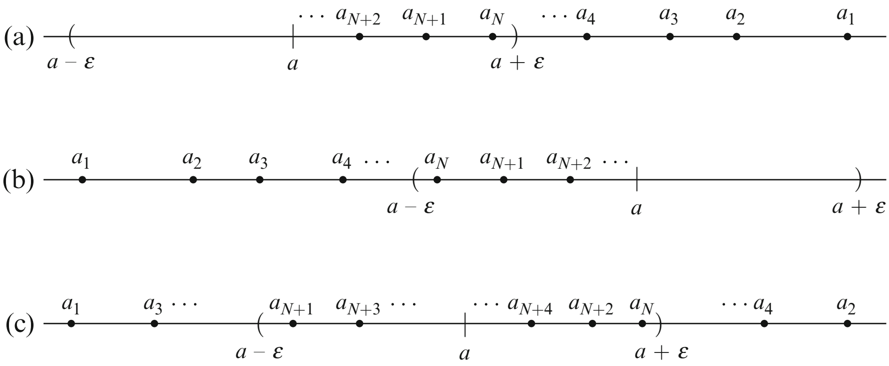

Fig. 2.1 Examples of convergent sequences: $(a_n) \to a$

are given a sequence (a_n). To show that it converges, we must first make a guess as to what its limit should be. In many of the exercises, you will be given the limit candidate; however, this is often not the case and we will also investigate elementary means of computing limits in the later sections. Suppose also then that we have a limit candidate, a.

The ε in the definition above is acting as an error bound, meaning that our goal is to only consider sequence terms which are within ε units of a. Moreover, we must do this according to a very specific rule. The N represents a position in the sequence, the 10th position, or the 104th position, and so on. The rule that we must abide by is as follows: *Given a positive number ε, we must find $N \in \mathbb{N}$ so that all terms at or beyond the Nth position in the sequence (signified by the statement $n \geq N$) are within ε units of a.*

In example (a) in Fig. 2.1, notice that the sequence terms approach a in a very specific manner while the opposite behavior is exhibited in example (b). The third example then demonstrates a more complicated case of sequence convergence. In each of the three examples it is clear that we have produced a natural number N so that each term at or beyond the Nth position satisfies $|a_n - a| < \varepsilon$.

The key to proving a convergence statement is then understanding how to make appropriate choices for N. And, as with many skills, this requires practice, a fact we will repeat over and over. People, for whatever reason, often respond positively to the idea of games and the challenge of making deliberate choices for N has been introduced as such on more than one occasion. Specifically, I choose an error bound ε and the challenge (by hook or by crook!) is then for you to come up with a choice for N which has the property that $|a_n - a| < \varepsilon$ for all $n \geq N$. Often this is done by algebraically manipulating the expression $|a_n - a| < \varepsilon$ and by using elementary estimating techniques which we will demonstrate below. Moreover, it is usually not the case that we choose N to be a specific number, but rather we will use facts such as the Archimedean Property to assert the existence of a natural number with a specific property. Our first example demonstrates this technique and we then move on to more involved examples.

Example 2.1.3. First, we use the definition to show that the sequence $(1/n)$ converges to 0. This is actually a restatement of the second part of the Archimedean Property. Suppose momentarily that we were to begin by considering specific values for ε. If $\varepsilon = 1$, how far out into the sequence must we travel in order to guarantee that the inequality $|1/n - 0| < 1$ is satisfied? This question is exactly what we must answer if we hope to find a valid choice for N. Using some basic facts, the previous inequality is equivalent to $1/n < 1$, which is then equivalent to $n > 1$. A bit of critical thinking now assures us that if we draw a line at the second position of our sequence, then all the terms thereafter will satisfy the inequality $|1/n - 0| < 1$, i.e. if $N = 2$, then $|1/n - 0| < \varepsilon$ for all $n \geq N$.

Be aware, however, that this does not show that the sequence converges to 0. We have demonstrated only for the choice of $\varepsilon = 1$ that it is possible to find an appropriate choice of N though the definition says that we must do this *for every* positive number ε. It would be an endless task if we were to consider the positive numbers one by one. Thus we will use the parameter ε to represent any positive value which will allow us to take care of all possibilities at once. In practice however, it is often instructive to first consider some specific values before moving on to the most general setting. Continuing with this example, find a suitable choice for N corresponding to $\varepsilon = 1/2, 1/10$, and $1/\sqrt{2}$.

Moving on, what happens if we simply set $\varepsilon > 0$? The task remains the same and the expression $|1/n - 0| < \varepsilon$ is equivalent to the statement that $1/n < \varepsilon$. Keep in mind also that after making our choice for N, we will only consider sequence terms which satisfy $n \geq N$, so we can think this way from the outset. This inequality can be rearranged to read $1/n \leq 1/N$. Thus if we can find an N for which $1/N < \varepsilon$, then we will have that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

for all $n \geq N$. The Archimedean Property now supplies the final part of this process. Piecing these thoughts together, we now give a formal proof.

Proof. We begin with an arbitrary $\varepsilon > 0$. Then, by the Archimedean Property we can find an $N \in \mathbb{N}$ with $1/N < \varepsilon$. Now, if we assume that $n \geq N$, it follows that $1/n \leq 1/N$ and thus

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

for all $n \geq N$. Therefore we conclude that 0 is the limit of the sequence $(1/n)$. $\square$

Later we will encounter another notion related to sequences which reproduces the first part of the Archimedean Property. For the moment however, let's continue on with a few more examples. The strategy is basically the same as in the simple example above, but as our sequences become more complicated, the relationship that determines how N is chosen for a given value of $\varepsilon > 0$ becomes less obvious.

One basic fact to remember is that when showing convergence for a specific sequence, the relationship between ε and N is likely, though not always, dictated by the Archimedean Property as you will see in the examples below.

Example 2.1.4. For a second example, let's show that $(1/(3n^2+1)) \to 0$. For $\varepsilon > 0$, how do we choose N? The idea is to first try to relate $|a_n - a|$ to n and then to N while keeping in mind that in our proof we will always assume that $n \geq N$. And off we go! Notice first that

$$3n^2 + 1 \geq 3n^2 \geq n^2 \geq n \geq N$$

where we are using the order properties of $\mathbb{R}$ to generate this string of inequalities. Still assuming that $n \geq N$, the previous inequality is equivalent to

$$\left|\frac{1}{3n^2+1} - 0\right| = \frac{1}{3n^2+1} \leq \frac{1}{3n^2} \leq \frac{1}{n^2} \leq \frac{1}{n} \leq \frac{1}{N}.$$

With this estimate in hand, it is apparent that we will obtain the desired conclusion by choosing $N \in \mathbb{N}$ with $1/N < \varepsilon$.

Proof. Let $\varepsilon > 0$. By the Archimedean Property we can find $N \in \mathbb{N}$ so that $1/N < \varepsilon$. Now, if we assume that $n \geq N$, we have that

$$\left|\frac{1}{3n^2+1} - 0\right| = \frac{1}{3n^2+1} \leq \frac{1}{3n^2} \leq \frac{1}{n^2} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

and therefore we conclude that $(1/(3n^2+1)) \to 0$. □

Example 2.1.5. As a third example, let's show that the sequence $(3/\sqrt{n+2}) \to 0$. Here we will take for granted the fact that the root function is increasing on $(0, \infty)$. With this, we see that if $n \geq N$, then

$$\sqrt{n+2} \geq \sqrt{n} \geq \sqrt{N}$$

which implies that

$$\frac{3}{\sqrt{n+2}} \leq \frac{3}{\sqrt{n}} \leq \frac{3}{\sqrt{N}}.$$

Now, it is not the case that $\sqrt{N} \geq N$ and this leaves us seemingly one step short. However, we can make use of the universal quantifier in the statement of the Archimedean Property. To be precise, $\varepsilon^2/9$ is positive and thus we may choose $N \in \mathbb{N}$ so that $1/N < \varepsilon^2/9$; an algebraic manipulation then shows that this is equivalent to $3/\sqrt{N} < \varepsilon$.

Proof. Let $\varepsilon > 0$. By the Archimedean Property we can choose $N \in \mathbb{N}$ so that $1/N < \varepsilon^2/9$. Notice that this is equivalent to $3/\sqrt{N} < \varepsilon$. If we now consider $n \geq N$, we know that $\sqrt{n+2} \geq \sqrt{n} \geq \sqrt{N}$ and hence

$$\left|\frac{3}{\sqrt{n+2}} - 0\right| = \frac{3}{\sqrt{n+2}} \leq \frac{3}{\sqrt{n}} \leq \frac{3}{\sqrt{N}} < \varepsilon.$$

Therefore we conclude that $(3/\sqrt{n+2}) \to 0$ as desired. □

Why so many examples? The idea is that observation is the key to learning a proof technique. Take a moment now to compare each of the above proofs with the definition of convergence. What do you notice? Think about the flow of the proof compared to the flow of the definition. You should find that there are three important parts in the definition and each proof above exhibits these same three components in the same order. Each proof begins with the designation that ε is a positive real number. Then, a choice for N is exhibited; however, the reason for the choice is not obvious until later. Then the implication "$n \geq N$ implies $|a_n - a| < \varepsilon$" is verified. In the second and third examples above, there are some additional thoughts involving algebraic manipulations which are present to aid the reader. They add a seamlessness to the proof, a continuity of ideas if you will. This strategy is basic to most of the convergence proofs that we will encounter and should therefore be practiced. As a final word of wisdom, keep in mind that in each of the examples above, the discussion proceeding the proof was an equally important component of our solution as it provided us the means to make the choice for N, necessary for the second part of the proof, and also provided the third component of the proof. This is how math is done! Proofs do not often come easily; ideas are scratched out before hand, often several times, and then a proof is given. Keep this in mind as you work through the exercises.

We will encounter the idea of convergence over and over in several different settings and a thorough grasp of each will be critical. It will also be equally important to understand when we have a non-convergent situation. We will use the word *divergent* to signify this. Our formal definition will simply be that a sequence diverges if it does not converge, and while this sounds simple enough, it should be noted that writing down a rigorous definition of this concept is a bit more cumbersome than our definition of convergence; we will provide a more formal discussion of such a statement in Example 2.1.11. One reason for this is that there are a variety of ways for a sequence to diverge. The remainder of this section will provide two divergence criteria and we will encounter several more in Sect. 2.2.

Definition 2.1.6 (Divergence Criterion 1). Let (a_n) be a sequence of real numbers. We say that (a_n) *diverges to* ∞ if for every $M > 0$, there is an $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$. In this situation we use the notation $\lim_{n\to\infty} a_n = \infty$.

The definition above has the same form as our definition of convergence and the proof style is also similar in that some scratch work is often necessary in order to be able to make an adequate choice for N corresponding to a fixed value of M. We omit these details for the two examples considered below.

Example 2.1.7. Consider the sequences (n^2) and $(n^2+\frac{n}{3}+2)$. For the first sequence, given $M > 0$, let's choose $N \in \mathbb{N}$ so that $N > \sqrt{M}$, which is permissible by the Archimedean Property. This is equivalent to requiring that $N^2 > M$. Using this last estimate, if $n \geq N$ we see that

$$n^2 \geq N^2 > M.$$

Therefore we may conclude that (n^2) diverges to infinity. Notice here that we made our initial choice for capital N based on this last inequality.

For the second sequence we let $M > 0$ and again choose $N \in \mathbb{N}$ so that $N > \sqrt{M}$. Now, if $n \geq N$, we can use the fact that $\frac{n}{3} + 2$ is a positive quantity to obtain the estimate

$$n^2 + \frac{n}{3} + 2 > n^2 > N^2 > M$$

showing that the sequence $(n^2 + \frac{n}{3} + 2)$ also diverges to infinity.

As another example, think about the sequence $(n^2 - \frac{n}{3} + 2)$. It looks very similar to the second sequence above, except for a sign change. How will this affect our proof? The details are requested of you in Exercise 2.1.5(b).

Definition 2.1.8. Let (a_n) be a sequence in $\mathbb{R}$. We say that (a_n) is a *Cauchy sequence* if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$.

This definition looks very similar to our definition of convergence and the proof strategy is also similar as we shall see below. The definition of limit dictates that the sequence terms get close and stay close to the limit while this definition asserts that the terms eventually get close and stay close to each other. Considering the fact that closeness is transitive in a certain sense (this is really what the triangle inequality says!), it seems likely that every convergent sequence must have this property.

Proposition 2.1.9. *Every convergent sequence is Cauchy.*

Before giving the proof, let us again consider what will be required. For a sequence (a_n) and $\varepsilon > 0$, we need to identify an $N \in \mathbb{N}$ so that $|a_n - a_m| < \varepsilon$ whenever $n, m \geq N$. However, we also have the fact that the sequence converges to some limit a. In other words, we can force the sequence terms to be as close to a as we desire. The measure of closeness that we will use in this case is $\varepsilon/2$. Also, we can modify the expression $|a_n - a_m|$ by (what some may call the mathematician's favorite trick) adding zero and using the triangle inequality. Consider the following:

$$\begin{aligned}|a_n - a_m| = |a_n - a + a - a_m| &= |(a_n - a) + (a - a_m)| \\ &\leq |a_n - a| + |a - a_m| \\ &= |a_n - a| + |a_m - a|.\end{aligned}$$

With this rather slick bit of algebra it follows that the quantity on the far left will be less than ε if each of the two terms on the right is less than $\varepsilon/2$.

Proof. Let (a_n) be a sequence of real numbers which converges to some limit a. In order to show that (a_n) is Cauchy, for each $\varepsilon > 0$ we must produce $N \in \mathbb{N}$ so that $|a_n - a_m| < \varepsilon$ whenever $n, m \geq N$. Let $\varepsilon > 0$. Since the sequence converges to a, we know that we can choose $N \in \mathbb{N}$ so that $|a_n - a| < \varepsilon/2$ for all $n \geq N$. With our choice of N, suppose $n, m \geq N$. Adding 0 and using the triangle inequality, we have

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$ and thus we have shown that our sequence is Cauchy. □

To close this section, we have another divergence criterion provided by the contrapositive of this last result.

Corollary 2.1.10 (Divergence Criterion 2). *If a sequence is not Cauchy, then it must diverge.*

Example 2.1.11. Consider the sequence $(1, -1, 1, -1, \ldots)$. It should be apparent that this sequence does not converge. How would we go about using the negation of the definition of convergence to provide a proof of this fact? Our definition centers around a limit candidate and thus we would need to show that there is no real number which acts as a limit for this sequence, i.e. for each real number we would have to show that the definition of convergence fails. To begin, let's show that the sequence does not converge to 1. Negating the definition of sequential limit we have:

> A sequence of real numbers (a_n) does not converge to a if there exist an $\varepsilon > 0$ so that for every $N \in \mathbb{N}$ there is an $n \geq N$ such that $|a_n - a| \geq \varepsilon$.

The statement says that no matter how far out in the sequence we traverse, we can always find terms that are more than some fixed distance away from our proposed limit. The game here lies in identifying an appropriate quantity for the fixed distance. Using the fact that the terms in the even positions in our sequence are 2 units away from 1, let's choose $\varepsilon = 2$. For $N \in \mathbb{N}$, choose n to be some even number greater than N. Then $|a_n - 1| = |-1 - 1| = 2 \geq \varepsilon$. Therefore we conclude that the sequence does not converge to 1.

Does this show that the sequence diverges? No, we have simply shown that the sequence does not converge to 1. Now we must argue that something similar happens for every other real number. There are some simplifying arguments that we could make, but it's beginning to sound like a lot of work! This is why Corollary 2.1.10 is so useful—it eliminates the need for working with a potential limit. To show that the sequence is not Cauchy we must find an $\varepsilon > 0$ so that for every $N \in \mathbb{N}$ there exist $n, m \geq N$ so that $|a_n - a_m| \geq \varepsilon$ (this is the formal negation of the definition of Cauchy). The game here is much the same except that in this case we are trying to show that no matter how far out in the sequence we look, we can always find terms that are more than a fixed distance away from each other, rather than from a potential limit. Considering the behavior of this specific sequence, we choose $\varepsilon = 2$ and let $N \in \mathbb{N}$. Now fix n to be some even number greater than N and m to be some odd number greater than N. It follows then that $a_n = -1$ and $a_m = 1$ and thus $|a_n - a_m| = |-1-1| = 2 \geq \varepsilon$. Thus we conclude that this given sequence is not Cauchy and hence not convergent.

Exercises

Exercise 2.1.1. Show that a convergent sequence has a unique limit.

Exercise 2.1.2. Verify each of the following limits using the definition of convergence.

(a) $\lim\limits_{n\to\infty} a = a$

(b) $\lim\limits_{n\to\infty} \dfrac{1}{6n^2+1} = 0$

(c) $\lim\limits_{n\to\infty} \dfrac{3n+1}{2n+5} = \dfrac{3}{2}$

(d) $\lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n+3}} = 0$

(e) $\lim\limits_{n\to\infty} \dfrac{n^2+6}{n^2} = 1$

(f) $\lim\limits_{n\to\infty} \dfrac{2n+3}{3n+1} = \dfrac{2}{3}$

Exercise 2.1.3. Let (a_n) be a sequence of positive numbers.

(a) If (a_n) converges to 0, show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty.$$

(b) If (a_n) diverges to ∞, show that

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Exercise 2.1.4. Let (a_n) and (b_n) be sequences with $a_n \leq b_n$ for all $n \in \mathbb{N}$. If $\lim a_n = \infty$, show that $\lim b_n = \infty$.

Exercise 2.1.5. (a) Suppose (a_n) and (b_n) are sequences of positive numbers with $\lim_{n\to\infty} a_n = a$ for $a > 0$, and $\lim_{n\to\infty} b_n = \infty$. Show that

$$\lim_{n\to\infty} a_n b_n = \infty.$$

(b) Use part (a) to show that the sequence $(n^2 - \frac{n}{3} + 2)$ from Example 2.1.7 diverges to infinity.

(c) Show that the sequence $(n^4/(n^2+1))$ diverges to infinity.
(d) Produce a definition similar to that of Definition 2.1.6 for the case that a sequence diverges to $-\infty$.
(e) Use your definition to show that the sequences $(-n^3)$ and $(n-n^3)$ diverge to $-\infty$.

Exercise 2.1.6. (a) Suppose that (a_n) and (b_n) are sequences with $(a_n) \to a$. Show that if $(a_n - b_n) \to 0$, then $(b_n) \to a$.
(b) Show that it is possible for two sequences (a_n) and (b_n) to both diverge even if $(a_n - b_n) \to 0$.

Exercise 2.1.7. Consider the statement of the Nested Interval Property (Theorem 1.4.2) with the additional hypothesis that $(b_n - a_n) \to 0$. Show that in this case the intersection $\cap_{n=1}^{\infty} I_n$ contains exactly one point.

Exercise 2.1.8. Let $r \in \mathbb{Q}$ and $t \in \mathbb{I}$. For each of the following, construct a sequence with the specified property and verify the convergence using only facts that we have proved.

(a) Construct a nonconstant sequence of rational numbers converging to r.
(b) Construct a sequence of irrational numbers converging to r.
(c) Construct a sequence of rational numbers converging to t.
(d) Construct a nonconstant sequence of irrational numbers converging to t.

2.2 Properties of Convergent Sequences

Convergent sequences have an abundance of useful properties. In this section we examine properties possessed by all convergent sequences as a means of identifying limits more easily and discuss several other divergence criteria.

Definition 2.2.1. Let (a_n) be a sequence in $\mathbb{R}$. We say (a_n) is *bounded* if there is an $M > 0$ so that $|a_n| \leq M$ for every $n \in \mathbb{N}$.

For example, the sequences $(1, -1, 1, -1, \ldots)$, $(1/n)$, and $(\sin(n))$ are bounded while the sequences (n) and $(1, 1, 2, 1/2, 3, 1/3, 4, 1/4, \ldots)$ are not bounded. A sequence that is not bounded is typically said to be *unbounded*.

Proposition 2.2.2. *If (a_n) is a convergent sequence, then (a_n) is bounded.*

Proof. Let (a_n) be a sequence with limit a. We begin by first finding a bound for the tail of the sequence and then extend this to a bound for the entire sequence. Since the sequence converges, choose $N \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq N$. The choice of $\varepsilon = 1$ here is arbitrary. The triangle inequality now implies that

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a|$$

for all $n \geq N$ and thus $1 + |a|$ is a bound for all terms at or beyond position N. To complete the proof, consider the terms at the front end of the sequence. There are at most $N - 1$ distinct terms and we can simply choose the largest in absolute value to act as a bound for this set. Combining this idea with the previous bound for the tail, we set $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a|\}$. It then follows immediately that $|a_n| \leq M$ for all $n \in \mathbb{N}$. □

Corollary 2.2.3 (Divergence Criterion 3). *A sequence that is not bounded must diverge.*

How does this relate to our first divergence criterion? An example of an unbounded sequence which does not diverge to infinity is given by

$$(1, 1, 2, 1/2, 3, 1/3, 4, 1/4, \ldots).$$

On the other hand, any sequence which does diverge to infinity is unbounded. Thus this last result is a generalization of sorts of its predecessor. The concept of bounded can be broken down into two categories, bounded above and below. These ideas are explored in Exercise 2.2.2. We now examine the algebraic properties of convergent sequences.

Theorem 2.2.4 (Algebraic Limit Theorem). *Let (a_n) and (b_n) be sequences converging to a and b, respectively. The following algebraic properties then hold:*

(a) (ka_n) *converges to* ka *for every* $k \in \mathbb{R}$;
(b) $(a_n + b_n)$ *converges to* $a + b$;
(c) $(a_n b_n)$ *converges to* ab;
(d) if $b_n \neq 0$ *for every* $n \in \mathbb{N}$ *and* $b \neq 0$, *then* (a_n/b_n) *converges to* a/b.

The proof below will verify the statements (b) and (c) while the proof of the remaining parts will be considered in the exercises. In order to verify the statement in part (b), for $\varepsilon > 0$, we need to find $N \in \mathbb{N}$ so that $|(a_n + b_n) - (a + b)| < \varepsilon$ for all $n \geq N$. Though the proof here is abstract in nature, the strategy is very much the same as we encountered when showing convergence for a specific sequence. We know what is required, but we need to do some scratch work before giving a formal proof in order to understand how to make a suitable choice for N. In this situation, we know that $(a_n) \to a$, $(b_n) \to b$ and, by the triangle inequality,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

Notice that this inequality relates the quantity we wish to make small to the quantities we know we can control. To be specific, in order to guarantee that the desired quantity is less than ε, it suffices to consider terms from both of the given sequences which are within $\varepsilon/2$ units of their respective limits. The question now is whether or not we can make both of these things happen simultaneously, and the technique we employ will be quite common as we proceed.

By the definition of convergence, there exists $N_1 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ for all $n \geq N_1$, and $N_2 \in \mathbb{N}$ so that $|b_n - b| < \varepsilon/2$ for all $n \geq N_2$. We must now choose an N which will guarantee that $|(a_n + b_n) - (a + b)| < \varepsilon$ for all $n \geq N$. The key to making this choice is being aware that we need the assumption $n \geq N$ to imply that $n \geq N_1$ and $n \geq N_2$. Thus we set $N = \max\{N_1, N_2\}$. To verify that our logic is in order, if $n \geq N$, then $n \geq N \geq N_1$ and $n \geq N \geq N_2$ which forces both quantities $|a_n - a|$ and $|b_n - b|$ to be less than $\varepsilon/2$, which in turn implies that $|(a_n + b_n) - (a + b)| < \varepsilon$. At this point, we have essentially given a proof, but a formal write-up is always necessary.

Proof. Let (a_n) and (b_n) be sequences converging to a and b, respectively, and let $\varepsilon > 0$. By the definition of convergence, choose $N_1 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ for all $n \geq N_1$, and $N_2 \in \mathbb{N}$ so that $|b_n - b| < \varepsilon/2$ for all $n \geq N_2$. Also, set $N = \max\{N_1, N_2\}$. If we now consider $n \geq N$, it follows that $n \geq N_1$ and $n \geq N_2$. With this fact and the triangle inequality, we see that

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whenever $n \geq N$. Therefore we conclude that $(a_n + b_n) \to a + b$. $\square$

The proof of (c) is more involved due to the fact that multiplication is a more complicated operation than addition, but the basic approach is the same. For $\varepsilon > 0$, we must find an $N \in \mathbb{N}$ such that $|a_n b_n - ab| < \varepsilon$ for all $n \geq N$, and our first goal is to relate the quantity $|a_n b_n - ab|$ to $|a_n - a|$ and $|b_n - b|$. This is where the complication arises. It is unfortunately *not true* that $(a_n b_n - ab) = (a_n - a)(b_n - b)$ and hence we will need a bit of cleverness to find a viable relationship. Observe what happens if we add and subtract the term $a_n b$,

$$\begin{aligned} |a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a|. \end{aligned}$$

The introduction of this term allows us to factor, resulting in an expression consisting of terms which we can control. Thus, for $\varepsilon > 0$, we can guarantee that $|a_n b_n - ab| < \varepsilon$ if we can force the terms $|a_n||b_n - b|$ and $|b||a_n - a|$ to be less than $\varepsilon/2$.

Considering the first term, the fact that (a_n) converges guarantees us that $|a_n|$ is bounded by Proposition 2.2.2, i.e. there is an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. If we then choose $N_1 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/2M$ for all $n \geq N_1$, we have

$$|a_n||b_n - b| \leq M|b_n - b| < M\left(\frac{\varepsilon}{2M}\right) = \varepsilon/2$$

whenever $n \geq N_1$.

For the second of these terms, we use a similar technique, but we also have to be cautious as it may be the case that $b = 0$; in short, we must avoid division by 0.

The proof can be split into cases $b = 0$ and $b \neq 0$, but we can avoid this since $|b| < |b| + 1$ and $|b| + 1 > 0$. With this in mind, choose $N_2 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2(|b| + 1)$ for all $n \geq N_2$. Then,

$$|b||a_n - a| < (|b| + 1)|a_n - a| < (|b| + 1)\left(\frac{\varepsilon}{2(|b| + 1)}\right) = \varepsilon/2$$

for all $n \geq N_2$. At this point we have all the pieces needed to give a formal proof. We will again choose N to be the maximum of N_1 and N_2 and then verify that our work provides the desired conclusion.

Proof. Let $(a_n) \to a$ and $(b_n) \to b$, and let $\varepsilon > 0$. By Proposition 2.2.2, there is an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. By the definition of convergence, we can choose $N_1 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/2M$ for all $n \geq N_1$. We also choose $N_2 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2(|b| + 1)$ for all $n \geq N_2$. Now set $N = \max\{N_1, N_2\}$. If $n \geq N$, our assumptions and the triangle inequality show that

$$\begin{aligned}
|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\
&= |a_n||b_n - b| + |b||a_n - a| \\
&< M|b_n - b| + (|b| + 1)|a_n - a| \\
&< M\left(\frac{\varepsilon}{2M}\right) + (|b| + 1)\left(\frac{\varepsilon}{2(|b| + 1)}\right) \\
&= \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Thus $(a_n b_n) \to ab$. □

The theorem above will be a powerful tool in theoretical situations but can also be used to calculate limits when the given function can be broken down algebraically into simpler pieces. For more complicated sequences, identifying a candidate for N is often difficult and the theorem provides a way around this.

Example 2.2.5. Consider the sequence $(3/(3n^2 + 4))$. We can factor this and write

$$\frac{3}{3n^2 + 4} = \frac{1}{n^2} \cdot \frac{3}{(3 + (4/n^2))}.$$

For the second factor here, we know that the sequence $(1/n^2)$ converges to zero and thus the denominator $(3 + (4/n^2))$ converges to 3 by parts (a) and (b) of the Algebraic Limit Theorem. This forces the entire second term to converge to 1 by part (d). Applying part (c), we see that the original sequence then converges to zero since the first factor above converges to zero.

The previous theorem allows us to understand the algebra of sequences, that is, the theorem demonstrates that convergent sequences respect the field operations in $\mathbb{R}$. Our next result will show that similar properties hold with respect to the order on $\mathbb{R}$.

Theorem 2.2.6 (Order Limit Theorem). *Let (a_n) and (b_n) be sequences converging to a and b, respectively.*

(a) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$. On the other hand, if $a_n \leq 0$ for all $n \in \mathbb{N}$, then $a \leq 0$.
(b) If $a_n \leq b_n$ for every $n \in \mathbb{N}$, then $a \leq b$.

Proof. For the first statement in (a), we suppose there is a sequence $(a_n) \to a$ with $a_n \geq 0$ for all $n \in \mathbb{N}$. To show that $a \geq 0$, we assume to the contrary that $a < 0$. With this additional hypothesis, we will show that there must be a term in the sequence which is negative, thus providing our contradiction. To use the given convergence, we should progress far enough out in our sequence to force the terms into the negative realm, but what ε will accomplish this goal? Consider the distance between a and 0. This is represented most easily by $|a|$. It should also be apparent that any term which satisfies $|a_n - a| < |a|$ will be closer to a than a is to 0, that is, any such term must be negative.

With this reasoning, let $\varepsilon = |a|$. Since the sequence converges to a, it must then be the case that there is an $N \in \mathbb{N}$ with $|a_n - a| < \varepsilon$ for all $n \geq N$. In particular, using our choice for ε, we have that $|a_N - a| < |a|$ which implies that $-|a| < a_N - a < |a|$. Again our goal is to show that $a_N < 0$. Working with the right-hand inequality we obtain $a_N < a + |a|$. Also, since $a < 0$ by hypothesis, we have

$$a_N < a + |a| = a + (-a) = 0.$$

Thus we have shown that $a_N < 0$, contradicting our hypothesis that $a_n \geq 0$ for all $n \in \mathbb{N}$. Therefore it must be the case that $a \geq 0$.

For the second statement in (a), suppose we are given a sequence $(a_n) \to a$ with $a_n \leq 0$ for all $n \in \mathbb{N}$. The sequence $(-a_n)$ then satisfies $-a_n \geq 0$ for all n and by Theorem 2.2.4(a), the sequence converges to $-a$. By the previous case, we have that $-a \geq 0$ which immediately implies that $a \leq 0$.

Part (b) now follows immediately from (a) by considering the sequence $(a_n - b_n)$. The fact that $a_n \leq b_n$ implies that $a_n - b_n \leq 0$ and Exercise 2.2.4(b) indicates that $(a_n - b_n) \to a - b$. Applying part (a), we see that $a - b \leq 0$ which allows us to conclude that $a \leq b$. □

Our next result is in the same vein as the Order Limit Theorem, but it has a subtlety which we will exploit several times in the coming chapters.

Theorem 2.2.7 (Squeeze Theorem). *Let $(a_n), (b_n), and (c_n)$ be sequences with $a_n \leq b_n \leq c_n$ for every $n \in \mathbb{N}$. If $(a_n) \to a$ and $(c_n) \to a$, then $(b_n) \to a$.*

Before talking about the proof, consider this statement in light of the Order Limit Theorem. Naively, we could take the given inequality, $a_n \leq b_n \leq c_n$, and apply the limit process to arrive at the statement $a \leq \lim b_n \leq a$. From this it follows immediately that $\lim b_n = a$. Is this a valid argument? In actuality it is not as our hypotheses do not specify convergence of the sequence (b_n), which would be necessary in order to apply the order statements previously presented. This is the beauty of the Squeeze Theorem—it asserts that the sequence (b_n) converges *and* specifies the value of its limit.

Seeking a rigorous argument, we discuss some important points and leave the presentation of a formal proof as Exercise 2.2.13. First, observe that

$$|b_n - a| = |b_n - a_n + a_n - a| \leq |b_n - a_n| + |a_n - a|. \tag{2.1}$$

Thus we can show that $(b_n) \to a$ if we can demonstrate control over both terms on the right. Next, notice that the inequality $a_n \leq c_n$ combined with the fact that both sequences converge to a shows that the sequence $(c_n - a_n)$ consists only of positive terms and converges to 0. The compound inequality $a_n \leq b_n \leq c_n$ can also be rewritten as $0 \leq b_n - a_n \leq c_n - a_n$. Thus we can control the size of the quantity $|b_n - a_n|$ by controlling $|c_n - a_n|$, since we know that $|c_n - a_n| = c_n - a_n$ and that this sequence converges to 0. This is half the battle. The second term on the right hand side of Eq. (2.1) can then be made small by appealing to the hypothesis that $(a_n) \to a$. The task of formalizing a rigorous argument is now left to you.

Subsequences

Definition 2.2.8. Let (a_n) be a sequence in $\mathbb{R}$ and let $n_1 < n_2 < n_3 < \ldots$ be a sequence of natural numbers. The function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(j) = a_{n_j}$ is called a *subsequence* of the given sequence (a_n). We use the notation $(a_{n_j})_{j=1}$ or simply (a_{n_j}) to denote the subsequence.

The multiple subscripts can be confusing at first attempt, but your intuition concerning the common usage of the prefix "sub" is likely correct. A subsequence is simply a sequence chosen from the terms of a given sequence in which we have not changed the order in which the terms appear. The function notation simply allows us to define this in a rigorous fashion and the subscripts enable us to keep track of the position of a term both in the original sequence and in the subsequence. In particular, the term a_{n_j} appears as the jth term of the subsequence and as the n_jth term of the original sequence. The fact that the sequence $n_1 < n_2 < n_3 < \ldots$ is chosen as a strictly increasing sequence of natural numbers guarantees that the order of the terms is preserved.

Example 2.2.9. Just as sets can have many subsets, the same is true of sequences and subsequences. Consider, for example, the sequence

$$(1/n) = (1, 1/2, 1/3, 1/4, \ldots, 1/n, \ldots).$$

The most obvious subsequences are those in which we take terms appearing in the original sequence in related positions, e.g. the even terms, the odd terms, every third term, and so on. To write these out explicitly is a simple matter. For the three subsequences just mentioned we have

$$\begin{aligned}(a_{n_j})_{j=1} &= (1/2, 1/4, 1/6, 1/8, \ldots),\\ (a_{n_k})_{k=1} &= (1, 1/3, 1/5, 1/7, \ldots),\\ (a_{n_l})_{l=1} &= (1/3, 1/6, 1/9, 1/12, \ldots).\end{aligned}$$

It will also be necessary on occasion to explicitly identify the function $j \mapsto n_j$ which defines the relationship between position in the subsequence to position in the original. The three maps needed for this example are $n_j = 2j$, $n_k = 2k - 1$, and $n_l = 3l$.

Example 2.2.10. As a more particular class of examples, let $(a_n)_{n=1}$ be a sequence and let $k \in \mathbb{N}$. We define the *k–tail* of (a_n) by the sequence $(a_n)_{n=k}$; in essence, we simply remove the first $k - 1$ terms of the sequence. In this situation, $n_1 = k$, $n_2 = k + 1$, and in general, $n_j = k + (j - 1)$ producing the subsequence

$$(a_{n_j})_{j=1} = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots) = (a_k, a_{k+1}, a_{k+2}, \ldots) = (a_n)_{n=k}.$$

Exercise 2.2.18 explores this type of subsequence in more detail.

The following proposition demonstrates that the convergence behavior of a sequence carries over to every subsequence.

Proposition 2.2.11. *If (a_n) is a sequence converging to a, then every subsequence of (a_n) also converges to a.*

Before giving the proof, let us again sketch out a few ideas. We know that (a_n) converges to a and thus for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ with $|a_n - a| < \varepsilon$ for all $n \geq N$. To show that (a_{n_j}) converges to a, our focus will be on identifying $J \in \mathbb{N}$ such that $|a_{n_j} - a| < \varepsilon$ whenever $j \geq J$. In short, we need to choose J large enough so that all the terms in the subsequence appearing after the J position appear in the original sequence past the N position in order to guarantee the desired degree of closeness to a. The fact that a subsequence is defined by a strictly increasing sequence of natural numbers implies that if we choose $J \in \mathbb{N}$ such that $n_J \geq N$, which we can do since such a sequence is unbounded in $\mathbb{N}$, it follows that $n_j \geq n_J \geq N$ whenever $j \geq J$. In other words, the position of the term a_{n_j} in the original sequence is at or beyond position N.

Proof. Let (a_n) be a sequence with limit a, and let (a_{n_j}) be any subsequence. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $|a_n - a| < \varepsilon$ for all $n \geq N$. Next choose $J \in \mathbb{N}$ such that $n_J \geq N$. Then if $j \geq J$, it is apparent that $n_j \geq n_J \geq N$ and thus $|a_{n_j} - a| < \varepsilon$. Therefore we conclude that (a_{n_j}) converges to a. □

As with our previous divergence criteria, the contrapositive of Proposition 2.2.11 provides us with yet another means of detecting divergence, but we first have a definition to help with terminology.

Definition 2.2.12. Let (a_n) be a sequence and let $a \in \mathbb{R}$. We call a a *subsequential limit of* (a_n) if there is a subsequence (a_{n_j}) of (a_n) which converges to a.

Example 2.2.13. Consider again the sequence $(1, -1, 1, -1, \ldots)$. This sequence has exactly two subsequential limits since the subsequence consisting of all the odd terms converges to 1 while the subsequence consisting of all the even terms converges to -1. To be explicit, the subsequence defined by $n_j = 2j - 1$ gives

$$(a_{n_j}) = (a_1, a_3, a_5, \ldots) = (1, 1, 1, \ldots)$$

and the rule $n_j = 2j$ yields the subsequence

$$(a_{n_j}) = (a_2, a_4, a_6, \ldots) = (-1, -1, -1, \ldots).$$

Corollary 2.2.14 (Divergence Criterion 4). *If a sequence (a_n) has two distinct subsequential limits, then the sequence (a_n) diverges.*

Example 2.2.15. Consider the sequence $(1, 1, 1, 1/2, 1, 1/3, 1, 1/4, \ldots)$. You should see two subsequences embedded here, one which converges to 1 and one which converges to 0. Take a moment to write out the defining functions for these two subsequences, after which we can conclude that this sequence diverges. What would happen if we changed the sequence to $(0, 1, 0, 1/2, 0, 1/3, 0, 1/4, \ldots)$?

Examining the various divergence criteria, it seems that a sequence can diverge in several distinct ways. While this is true to some degree, we have purposefully overplayed some of these scenarios. There are actually only two distinct cases. An unbounded sequence must have a subsequence that diverges to ∞ or $-\infty$ (Exercise 2.2.2), while a bounded sequence that diverges must have two subsequences with distinct limits. This statement encompasses divergence Criteria 1, 3, and 4, and is restated for the sake of convenience below. Divergence Criterion 2 can potentially be applied to any divergence situation. Its true role in the behavior of convergent sequences will be examined in the next section.

Proposition 2.2.16 (Divergence Criterion for Sequential Limits). *A sequence (a_n) diverges provided one of the following occur:*

(a) the sequence is unbounded, i.e. the sequence has a subsequence which diverges to either ∞ or $-\infty$;
(b) the sequence has two subsequences with distinct limits.

Exercises

Exercise 2.2.1. Let (a_n) be a Cauchy sequence. Show that (a_n) is bounded.

Exercise 2.2.2. We say a sequence is *bounded below* if there exist $m \in \mathbb{R}$ such that $m \leq a_n$ for all $n \in \mathbb{N}$ and *bounded above* if there exist $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$.

(a) Show that a sequence is bounded if and only if it is bounded above and below.
(b) Show that a sequence which is not bounded above has a subsequence which diverges to ∞.
(c) Show that a sequence which is not bounded below has a subsequence which diverges to $-\infty$.

Exercise 2.2.3. Verify each of the following limits.

(a) $\lim_{n\to\infty} \dfrac{4n+7}{2n-9} = 2$

(b) $\lim_{n\to\infty} \sqrt{\dfrac{1}{6n^2+1}} = 0$

(c) $\lim_{n\to\infty} \dfrac{\sqrt{9n^2+n}}{n+1} = 3$

(d) $\lim_{n\to\infty} \dfrac{n^3+6n-1}{n^4+n+2} = 0$

(e) $\lim_{n\to\infty} \dfrac{n^2+-3n+6}{n^2} = 1$

(f) $\lim_{n\to\infty} \sqrt{9n^2+n} - 3n = \dfrac{1}{6}$

Exercise 2.2.4. (a) Prove part (a) of Theorem 2.2.4.
(b) Suppose that $(a_n) \to a$ and $(b_n) \to b$. Show that $(a_n - b_n) \to a - b$.
(c) In the proof of part (c) of Theorem 2.2.4, we relied on the fact that a convergent sequence is bounded in order to show that the term $|a_n||b_n - b| < \varepsilon/2$. Explain why this was necessary. In other words, why couldn't we simply choose $N_1 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/2|a_n|$ for all $n \geq N$?
(d) Suppose (b_n) is a sequence satisfying $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$. Show then that there is a number $K > 0$ such that $|b_n| > K$ for all $n \in \mathbb{N}$.
(e) Use part (d) and techniques similar to those used to prove Theorem 2.2.4(c) to prove Theorem 2.2.4(d).
(f) Extend part (d) of Theorem 2.2.4 by showing that the statement still holds if we remove the requirement that $b_n \neq 0$ for all $n \in \mathbb{N}$. First try extending part (d) to this more general situation.

Exercise 2.2.5. Let (a_n) be a sequence with $(a_n) \to a$. Show that $(a_n^k) \to a^k$ for every $k \in \mathbb{N}$.

Exercise 2.2.6. Let (a_n) be a sequence with $(a_n) \to a$ and $a_n \geq 0$ for every $n \in \mathbb{N}$. Show that $(\sqrt{a_n}) \to \sqrt{a}$. To do this, consider two cases: $a = 0$ and $a > 0$.

Exercise 2.2.7. Let (a_n) be a sequence with $(a_n) \to a$.

(a) Show that $(|a_n|)$ converges to $|a|$. It may be helpful to consider Exercise 1.3.2.
(b) On the other hand, provide an example of a sequence (a_n) such that $(|a_n|) \to 1$, but (a_n) does not converge to 1 or -1.
(c) However, if $(|a_n|) \to 0$, show that $(a_n) \to 0$.

Exercise 2.2.8. Let (a_n) and (b_n) be sequences.

(a) If (a_n) converges but (b_n) diverges, show that $(a_n + b_n)$ must diverge.
(b) If (a_n) and (b_n) both diverge, must it be the case that $(a_n + b_n)$ diverges? Explain in detail.
(c) If (a_n) diverges and $c \in \mathbb{R}$ with $c \neq 0$, show that (ca_n) diverges.

Exercise 2.2.9. Let (a_n) and (b_n) be sequences.

(a) If (a_n) is a bounded sequence and (b_n) converges to zero, show that the product sequence $(a_n b_n)$ converges to zero. Note: You cannot use the Algebraic Limit Theorem to prove this. Why not?
(b) Does a similar result hold if (b_n) converges to a nonzero limit?
(c) Does the result hold if (a_n) is not bounded and $(b_n) \to 0$.

Exercise 2.2.10. Let (a_n) and (b_n) be sequences which diverge to infinity.

(a) If $c > 0$, show that (ca_n) diverges to infinity.
(b) Show that $(a_n + b_n)$ diverges to infinity.
(c) Show that $(a_n b_n)$ diverges to infinity.
(d) What conclusions can we draw about the behavior of (a_n / b_n)?

Exercise 2.2.11. Provide an example for each of the following statements.

(a) A sequence (a_n) which diverges and

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1.$$

(b) A sequence (a_n) which converges to 0 and a sequence (b_n) which diverges to infinity with

$$\lim_{n \to \infty} a_n b_n = 1.$$

Exercise 2.2.12. (a) Extend Theorem 2.2.6(a) by showing that the result holds when we replace 0 by any other real number c.
(b) Do the results of Theorem 2.2.6 hold if we replace $\leq$ and $\geq$ with $<$ and $>$? Explain.

Exercise 2.2.13. Using the discussion from the text, supply a formal proof of Theorem 2.2.7.

Exercise 2.2.14. Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Show that there exist sequences (x_n) and (y_n) in A which converge to $\sup(A)$ and $\inf(A)$, respectively.

Exercise 2.2.15. Suppose (a_n) is a sequence with positive terms. If $\lim n a_n$ exists, show that $(a_n) \to 0$.

Exercise 2.2.16. For each of the following, give an example of a sequence with the specified property.

(a) A sequence with a subsequence that diverges to ∞ and a subsequence which diverges to $-\infty$.
(b) A sequence for which $0, 1, -1$ are subsequential limits. Write out the function that defines the sequence and the three subsequences.
(c) A sequence for which each natural number is a subsequential limit.
(d) A sequence for which every real number is a subsequential limit.

Exercise 2.2.17. Suppose that (a_n) is a sequence with the property that the subsequences (a_{n_j}) and (a_{n_k}), where $n_j = 2j$ and $n_k = 2k - 1$, both converge to a. Show that (a_n) also converges to a.

Exercise 2.2.18. Let $(a_n)_{n=1}$ be a sequence. Show that the following are equivalent.

(a) The sequence (a_n) converges.
(b) There is a $k \in \mathbb{N}$ so that the k–tail of (a_n) converges.
(c) The k–tail of (a_n) converges for every $k \in \mathbb{N}$.

2.3 Completeness in $\mathbb{R}$ Revisited

Completeness is essentially the defining property of the real number system and enriches the system with an abundance of properties not shared by its subsystems—the natural numbers, the integers, and the rational numbers. Here we present three theorems which are crucial to the study of sequences all of which follow from the fact that $\mathbb{R}$ is complete. After discussing these theorems we will investigate their true connection to our axiom of completeness.

Definition 2.3.1. A sequence (a_n) is said to be *increasing* if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$. A *monotone sequence* is one that is either increasing or decreasing.

Example 2.3.2. The sequences $(n/(n+1))$ and (n) are both increasing and the sequences $(1/n)$ and $(n - n^2)$ are both decreasing; we could also say that all four sequences are monotone sequences; however, it's often best to be as specific about the behavior as possible. For a non-example, the sequence $((-1)^n)$ is not monotone since it is neither increasing nor decreasing.

Consider the first two sequences above. The first converges to 1 while the second diverges to infinity even though they have the same monotonic behavior. The next theorem provides the rationale for this distinction in behavior.

Theorem 2.3.3 (Monotone Convergence Theorem). *Every bounded and monotone sequence must converge. In particular, if a sequence is bounded above and increasing then it converges and likewise, if it is bounded below and decreasing then it converges.*

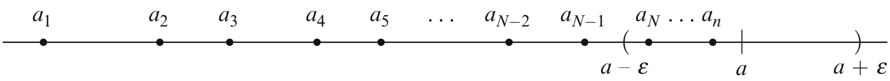

Fig. 2.2 Monotone Convergence Theorem

Proof. Without loss of generality, let us assume that (a_n) is an increasing sequence which is bounded above and consider the set of range values of the sequence, $A = \{a_n : n \in \mathbb{N}\}$. Since the sequence in question is bounded above, the set A must also be bounded above; it is also nonempty. Therefore the supremum exists and we set $a = \sup(A)$.

We claim now that $(a_n) \to a$. To verify this, let $\varepsilon > 0$. By Lemma 1.2.10, there is an element of A, call it a_N, such that $a - \varepsilon < a_N$. If we now consider $n \in \mathbb{N}$ with $n \geq N$, the fact that our sequence is increasing together with the fact that a is the supremum of A guarantees us that

$$a - \varepsilon < a_N \leq a_n \leq a < a + \varepsilon$$

(see Fig. 2.2); we can rephrase this by saying that $|a_n - a| < \varepsilon$ whenever $n \geq N$ and therefore conclude that $(a_n) \to a$. □

Notice that the proof provides a bit more information than is explicitly stated in the theorem—the sequence must converge and the supremum of the set of terms is the limit. We now encourage the reader to supply a proof for the alternate situation involving decreasing sequences. In the exercises you will also show that any increasing sequence that is not bounded above must diverge to infinity and the analogous statement for decreasing sequences which are not bounded below.

Example 2.3.4. The Monotone Convergence Theorem is a very practical tool, particularly for sequences defined recursively. Take the sequence (a_n) defined by the recursive relationship $a_1 = 3$ and $a_{n+1} = (2a_n + 1)/3$. In order to apply the theorem, we must first show that the sequence is monotone and bounded. To do this we write out the first few terms in an effort to observe behavior and then work to prove our claims. Computing, we see that the first three terms are 3, 7/3, and 17/9. It appears that the sequence is decreasing with an upper bound of 3 (obvious since the terms are decreasing). Coming up with a lower bound is a more complicated process, but since all the operations involve positive values, it seems likely that 0 will serve as a lower bound. We must now work to show that the inequality

$$0 \leq a_{n+1} \leq a_n \leq 3$$

holds for each $n \in \mathbb{N}$. Notice we have summed up both the decreasing nature of the sequence and the proposed bounds with this single compound inequality. Since the statement must be shown for each natural number, we proceed by mathematical induction.

For our base case, we take $n = 1$ and it is certainly true that

$$0 \leq 7/3 \leq 3 \leq 3.$$

For the inductive hypothesis, suppose that for some $j \in \mathbb{N}$ we know that the inequality

$$0 \leq a_{j+1} \leq a_j \leq 3$$

holds true. We now show that the corresponding inequality holds for $j + 1$. The strategy here will be to produce the desired inequality by algebraically manipulating the known inequality using the algebraic processes given in the recursive definition of the sequence, with care given to the order of the operations. Moving forward, we multiply across the inequality

$$0 \leq a_{j+1} \leq a_j \leq 3$$

by 2, add 1, and then divide by 3 to obtain the inequality

$$1/3 \leq (2a_{j+1} + 1)/3 \leq (2a_j + 1)/3 \leq 7/3.$$

The inner most expressions take a simpler form as a_{j+2} and a_{j+1}, respectively,

$$1/3 \leq a_{j+2} \leq a_{j+1} \leq 7/3.$$

We had hoped for an upper bound of 3 and a lower bound of 0 and the previous inequality certainly implies that

$$0 \leq a_{j+2} \leq a_{j+1} \leq 3.$$

By induction, we conclude that our claim is true for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem, the sequence converges to some value L.

To find the limit, we apply the limit process to the both sides of the recursive equation $a_{n+1} = (2a_n + 1)/3$. By the Algebraic Limit Theorem, the right-hand side converges to $(2L + 1)/3$. The sequence (a_{n+1}) is a subsequence of the original sequence and therefore also converges to L. Hence we arrive at the equation $L = (2L + 1)/3$, which we can solve to find that $L = 1$.

Example 2.3.5. Let $x > 1$ and consider the sequence (x^n). In order to understand the behavior of such a sequence, let's examine the case $x = 2$ for a moment. The resulting sequence is $(2, 4, 8, 16, \ldots, 2^n, \ldots)$. It appears as if we have an increasing sequence which diverges to infinity. Returning to our general situation, we will show that this behavior is common to all sequences of the form above. First, to show that the sequence is increasing, we use the order properties of $\mathbb{R}$. Let $n \in \mathbb{N}$. Multiplying the inequality $x > 1$ by x^n yields $x^n x > x^n$ which implies that $x^{n+1} > x^n$. We thus conclude that the sequence is increasing since this relationship holds for all $n \in \mathbb{N}$.

If the sequence is unbounded, then by Exercise 2.3.6 we can conclude that it diverges to infinity, providing the desired conclusion. At the present, we are not capable of dealing with powers in an efficient manner as we have not discussed logarithms. To alleviate this issue, we will use a proof by contradiction. Assume to the contrary then that the sequence (x^n) is bounded. By the Monotone Convergence Theorem, it must be the case that $(x^n) \to L$ for some $L \in \mathbb{R}$. Moreover, we also know that $L = \sup\{x^n : n \in \mathbb{N}\}$. To obtain our contradiction, we will work to produce a term in the sequence greater than L. To accomplish this, we will use the convergence of the sequence to force the terms close enough to L so that when we move to a subsequent term, the increasing behavior of the sequence forces a term to bypass L. The key in all of this is the fact that the sequence is strictly increasing and the distance between successive terms is completely determined by the distance between x and 1. Let $\varepsilon = |x-1|$ and choose $N \in \mathbb{N}$ such that $|x^n - L| < \varepsilon$ for all $n \geq N$. In particular, the term x^N satisfies $|x^N - L| < |x-1|$. We claim that $x^{N+1} > L$. Using the fact that $x > 1$ and the increasing nature of the sequence shows that

$$|x^{N+1} - x^N| = x^N|x-1| > |x-1| > |x^N - L|.$$

The inequality indicates that the distance between x^{N+1} and x^N is greater than the distance between x^N and L. Combining this fact with the increasing behavior of the sequence and the fact that L is the supremum of the sequence terms allows us to conclude that $x^N \leq L < x^{N+1}$. However this is a contradiction since $x^n \leq L$ for all $n \in \mathbb{N}$. Therefore it must be the case that the sequence (x^n) is unbounded and hence diverges to ∞.

Example 2.3.6. Exercise 2.3.7 shows that for a real number $0 < x < 1$, the sequence $(x^n) \to 0$. With this we can show that the Cantor set (Example 1.4.15) contains no intervals. In our construction of C, we first constructed sets C_n which had the property that each is a union of 2^n closed intervals of length $1/3^n$. If the Cantor set contained an interval, say of length $\varepsilon > 0$, then this interval would have to be contained in C_n for every $n \in \mathbb{N}$; in particular, for any fixed $n \in \mathbb{N}$, this interval would have to be entirely contained in *one* of the 2^n intervals of length $1/3^n$ which comprise C_n. However, since $(1/3^n) \to 0$, we can choose $N \in \mathbb{N}$ such that $0 < 1/3^N < \varepsilon$. From this we conclude that the Cantor Set contains no intervals.

The second major theorem of this section concerns subsequences of bounded sequences and is one of the most notable theorems in the study of real analysis. The proof here is somewhat more involved than those we have encountered up to this point and we will discuss some of the ideas before presenting the proof.

Theorem 2.3.7 (Bolzano–Weierstrass Theorem). *Every bounded sequence has a convergent subsequence.*

The proof technique we use is not as obvious as some of the others we've encountered and the basic outline is as follows. Using the fact that the sequence is bounded, we can identify an $M > 0$ so that $|a_n| \leq M$ for all $n \in \mathbb{N}$. We can

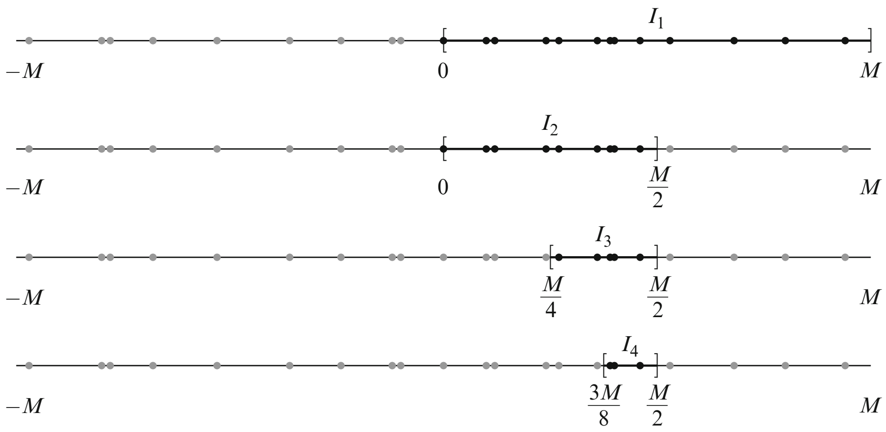

Fig. 2.3 Construction of $\{I_j\}$ within $[-M, M]$

rephrase this in terms of set inclusion, $a_n \in [-M, M]$ for all n. The idea then is to construct a collection of closed, bounded, nested intervals $\{I_j : j \in \mathbb{N}\}$ based on the behavior of our sequence and such that the length of these intervals decreases to zero. Exercise 2.1.7 then guarantees us that the intersection of these sets contains a unique element a. We then construct a subsequence (a_{n_j}) with the term a_{n_j} coming from the interval I_j and claim that this subsequence must converge to a. One of the most beautiful aspects of this theorem is that it is constructive in its approach, meaning that it constructs both the subsequence and the limit in an explicit manner, rather than simply using a reduction technique and applying an abstract result. The computer scientist will recognize this process as a binary search algorithm. We also mention that the use of the word countable in the proof will mean a countably infinite collection.

Proof. Let (a_n) be a bounded sequence of real numbers. Using this hypothesis, we can find $M > 0$ such that $a_n \in [-M, M]$ for all $n \in \mathbb{N}$. To construct a sequence of closed, bounded, nested intervals with lengths decreasing to zero, we will successively cut the interval $[-M, M]$ into halves. For I_1, we bisect $[-M, M]$ into the closed subintervals $[-M, 0]$ and $[0, M]$. The fact that sequences must have countably many terms guarantees us that at least one of these two subintervals must contain countably many terms of the sequence (a_n). Choose I_1 to be one such subinterval (if both $[-M, 0]$ and $[0, M]$ contain infinitely many terms of the sequence, then the choice is completely arbitrary). Note that the length of I_1 is M.

To select I_2, we bisect I_1 into closed subintervals of length $M/2 = M/2^1$. Again, at least one of these subintervals must contain countably many terms of the sequence (a_n) and we choose I_2 to be one such closed interval. Notice also that $I_2 \subseteq I_1$. Continuing inductively, we can construct a collection of closed, bounded, nested intervals $\{I_j\}$ (Fig. 2.3) with the property that each interval contains countably many terms of the sequence (a_n) and the length of I_j is $M/2^{j-1}$. From Exercise 2.3.7, we see that the sequence $M/2^{j-1}$ converges to zero as $j \to \infty$. By Exercise 2.1.7, it follows that the intersection $\cap_{j=1}^{\infty} I_j$ contains a unique element which we shall denote by a. The convergence of the sequence $M/2^{j-1}$ will also be key in the final component of the proof.

At this point we are ready to define a subsequence; this is also done in an inductive manner. Choose $n_1 \in \mathbb{N}$ so that $a_{n_1} \in I_1$. Next, choose $n_2 \in \mathbb{N}$ so that $n_2 > n_1$ and $a_{n_2} \in I_2$, which is permissible since I_2 contains countably many terms of the sequence (a_n). Continuing, for each $j \in \mathbb{N}$, we choose $n_j \in \mathbb{N}$ so that $n_j > n_{j-1}$ and $a_{n_j} \in I_j$.

We claim now that the subsequence (a_{n_j}) converges to a. Let $\varepsilon > 0$ and choose $J \in \mathbb{N}$ such that $M/2^J < \varepsilon$. Then if $j \geq J + 1$, we have that $j - 1 \geq J$ and $M/2^{j-1} \leq M/2^J < \varepsilon$. In other words, for $j \geq J + 1$, the length of the interval I_j is less than ε. Finally, if we consider a term a_{n_j} with $j \geq J + 1$, then both a_{n_j} and a are in I_j and, using the restriction on the length of this interval, it is clear that

$$|a_{n_j} - a| \leq \frac{M}{2^{j-1}} \leq \frac{M}{2^J} < \varepsilon.$$

Thus we conclude that $(a_{n_j}) \to a$ as desired. □

Notice that boundedness is a hypothesis in both of the previous two statements. We then are able to draw conclusions regarding the convergence behavior of the sequence in question. The connection here is provided by Theorem 2.2.2; both of these theorems are attempts to prove the converse of the statement that every convergent sequence is bounded. It is clear that there are bounded sequences which diverge, however, these two theorems show that there is more to be said. On the one hand, with an additional hypothesis we can conclude convergence, and on the other, with boundedness alone the sequence may diverge, but some part of it must converge.

The third main theorem of this section returns to the idea of a Cauchy sequence. We explored this in a previous section as a means of detecting divergence but it turns out that it is also capable of indicating convergence. The beauty of the theorem below is that it provides us a means of determining whether or not a sequence converges without the explicit knowledge of a limit. The two theorems presented above share this property to some degree.

Theorem 2.3.8 (Cauchy Criterion). *A sequence converges if and only if it is Cauchy.*

Proof. We showed one direction of the statement in Proposition 2.1.9 and now we need to verify that every Cauchy sequence is convergent. Let (a_n) be a Cauchy sequence. It was shown in Exercise 2.2.1 that every Cauchy sequence is bounded. Thus we can apply the Bolzano–Weierstrass Theorem to generate a subsequence (a_{n_j}) which converges. Denote the limit of this subsequence by a.

To show that the original sequence (a_n) also converges to a, let $\varepsilon > 0$. Since the original sequence is Cauchy, there exists an $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon/2$

whenever $n, m \geq N$. Furthermore, there exists a $J \in \mathbb{N}$ such that $|a_{n_j} - a| < \varepsilon/2$ for all $j \geq J$. At this point we have two ideas to work with, one which concerns the original sequence and one which concerns only the subsequence. To connect these ideas, we must take an extra bit of care. Choose $K \in \mathbb{N}$ such that $K \geq J$ and $n_K \geq N$. This second choice is permissible since the sequence (n_j) is an increasing sequence of natural numbers and will therefore eventually exceed the value that we have specified for N. These two conditions on the value of K guarantee us that $|a_n - a_{n_K}| < \varepsilon/2$ for all $n \geq N$ since $n_K \geq N$, and $|a_{n_K} - a| < \varepsilon/2$ since $K \geq J$. Using these two conditions with the triangle inequality, for $n \geq N$,

$$|a_n - a| = |a_n - a_{n_K} + a_{n_K} - a| \leq |a_n - a_{n_K}| + |a_{n_K} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence we conclude that $(a_n) \to a$. □

Each of these three theorems and the Nested Interval Property are direct consequences of the completeness axiom of $\mathbb{R}$ and each other. Indeed, consider the proofs of these four statements. The Nested Interval Property depended on the axiom of completeness and the structure of intervals while the Monotone Convergence Theorem came directly from the axiom and the behavior of monotone sequences. The Bolzano–Weierstrass Theorem was a result of the Nested Interval Property, which in turn supplied the machinery for the Cauchy Criterion. Exploring these implications indicates that while each result seems to be doing something very different with respect to the real number system, they are connected to each other. The connection is brought to light in a more obvious manner by considering the bigger picture, that is, each result is asserting the existence of a real number satisfying a specific set of hypotheses. And though the hypotheses of these five statements are drastically different, each result can be thought of in a visual sense as asserting that the real number line is a continuum.

It is also typical when considering mathematical implications to understand when statements are logically equivalent. As it turns out, all five of the statements discussed in the previous paragraph are equivalent. In other words, the axiom of completeness is not the only way to move from the ordered field $\mathbb{Q}$ to the complete, ordered field $\mathbb{R}$. We could have taken any of these other statements as our completeness axiom. For example, it is common to take the statement that all Cauchy sequences converge as the completeness axiom. The excellent text [30] takes this approach. Our choice of using the existence of suprema was simply the easiest to motivate early on. However, we will see in Sect. 2.5 that our choices change when considering completeness in an arbitrary metric space.

To close this section, we present a proof that the Monotone Convergence Theorem implies our axiom of completeness. The proof also depends on the Order Limit Theorem for sequences; however, the proof of that fact does not depend on the completeness property in $\mathbb{R}$. The reader is then encouraged to consider using the other statements under consideration to prove the axiom of completeness and/or any of the other statements.

Proof. We assume that every bounded, monotone sequence in $\mathbb{R}$ converges. We then assume that $A \subseteq \mathbb{R}$ is nonempty and bounded above. Our task is to then produce a point $s \in \mathbb{R}$ which satisfies the definition of supremum. In order to apply the Monotone Convergence Theorem, we begin by constructing two sequences. Let $a_1 \in A$ and let y_1 be an upper bound for A. If $a_1 = y_1$, then this number will satisfy the definition of supremum and we are done. Indeed, the point is certainly an upper bound, and since the point is in A, it is also less than or equal to any upper bound.

If this is not the case, then $M = y_1 - a_1 > 0$ and we consider the interval $[a_1, y_1]$. Let b_1 be the midpoint of this interval. If b_1 is an upper bound for A, we define $y_2 = b_1$ and set $a_2 = a_1$. If b_1 is not an upper bound for A, then we set $y_2 = y_1$ and choose a point $a_2 \in [b_1, y_1]$. If $a_2 = y_2$, then this point is the supremum of A and we are done. If not, then the interval $[a_2, y_2]$ has a positive length less than or equal to $M/2$.

Continuing in this manner, we construct two sequences (a_n) and (y_n). If at some point we have $a_n = y_n$, then the process terminates and the supremum of the set is this common value. If the process continues without terminating, the sequence (a_n) is an increasing sequence of elements of A and is bounded above, while (y_n) is a decreasing sequence of upper bounds for A and is hence bounded below. Moveover, $|y_n - a_n| \leq M/2^{n-1} \to 0$ as $n \to \infty$. By the Monotone Convergence Theorem, (a_n) and (y_n) both converge, and since $|y_n - a_n| \to 0$, they have the same limit; call it s. We claim now that s is the supremum of A.

For each $n \in \mathbb{N}$, we know that $a \leq y_n$ since each y_n is an upper bound for A. The Order Limit Theorem implies that $a \leq s$, showing that s is an upper bound. Finally, assume that x is some other upper bound for A. To show that $s \leq x$, we assume to the contrary that $x < s$. Since $(a_n) \to s$, we can find an $n \in \mathbb{N}$ such that $|a_n - s| < |s - x|$ for all $n \geq N$. In particular, the term a_N satisfies $|a_N - s| < |s - x|$, i.e. the distance between a_N and s is less than the distance between s and x. Our hypothesis concerning the order of these values now implies that $x < a_N \leq s$. But this is a contradiction since x is an upper bound for A. Thus $s \leq x$ and s is the supremum of A. ◻

The proof above is technical in nature and should remind you of the technique we used to prove the Bolzano–Weierstrass Theorem. It should also be easy to see that with some simple modifications, we could use a very similar argument to show that the Monotone Convergence Theorem implies the Nested Interval Property or that the Nested Interval Property implies the axiom of completeness. While we won't pursue this collection of statements further, we encourage the reader to attempt proving several of the other equivalences mentioned before the proof.

Exercises

Exercise 2.3.1. Show that the sequence defined recursively by $y_1 = 1$ and $y_{n+1} = (3y_n + 4)/4$ converges and find its limit.

Exercise 2.3.2. Show that the sequence defined recursively by $y_1 = 8$ and $y_{n+1} = (3y_n + 4)/4$ converges and find its limit.

Exercise 2.3.3. Show that the sequence defined recursively by $y_1 = 1$ and $y_{n+1} = 4 - 1/y_n$ converges and find its limit.

Exercise 2.3.4. Consider the sequence defined by

$$x_1 = \sin 1, \quad x_2 = \max\{\sin 1, \sin 2\}, \quad x_3 = \max\{\sin 1, \sin 2, \sin 3\}$$

and in general,

$$x_k = \max\{\sin 1, \sin 2, \ldots, \sin k\}.$$

Show that the sequence converges (you do not need to find the limit).

Exercise 2.3.5. Consider the sequence defined by the pattern

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots.$$

(a) Find a recursive definition for this sequence.
(b) Show that the sequence converges.
(c) Find the limit of this sequence.

Exercise 2.3.6. (a) Let (a_n) be an increasing sequence which is not bounded above. Show that the sequence diverges to ∞.
(b) Let (a_n) be an decreasing sequence which is not bounded below. Show that the sequence diverges to $-\infty$.

Exercise 2.3.7. (a) Let $0 < x < 1$. Show that the sequence $(x^n) \to 0$.
(b) Let $-1 < x < 0$. Show that the sequence $(x^n) \to 0$.

Exercise 2.3.8. A sequence (a_n) is called *additive* if $a_{m+n} = a_m + a_n$ for all $m, n \in \mathbb{N}$.

(a) Show that the sequence $(2n)_{n=1}^{\infty}$ is additive.
(b) Suppose that (a_n) is an additive sequence and show that the sequence $\left(\frac{a_n}{n}\right)$ converges. What is the limit?

Exercise 2.3.9. A sequence (a_n) is called *subadditive* if $a_{m+n} \leq a_m + a_n$ for all $m, n \in \mathbb{N}$.

(a) Show that the sequence $(2n + 1)_{n=1}^{\infty}$ is subadditive.
(b) Suppose that (a_n) is a subadditive sequence of positive real numbers. Show that the sequence $\left(\frac{a_n}{n}\right)$ has a convergent subsequence.
(c) Suppose that (a_n) is a subadditive sequence of positive real numbers. Extend the previous exercise by showing that the sequence $\left(\frac{a_n}{n}\right)$ is convergent.

Exercise 2.3.10. The following sequence of exercises explore the notion of the limit superior and inferior of a bounded sequence. Assume that (a_n) is a bounded, not necessarily convergent, sequence of real numbers.

(a) Define a new sequence (y_n) by $y_n = \sup\{a_k : k \geq n\}$. For the sake of clarity,

$$y_1 = \sup\{a_1, a_2, a_3, \ldots\}$$

and

$$y_2 = \sup\{a_2, a_3, a_4, \ldots\}$$

and so on. Show that the sequence (y_n) converges.
(b) The *limit superior* of the sequence (a_n) is defined to be the limit of the sequence (y_n) constructed in part (a) and is denoted by

$$\limsup a_n = \lim y_n.$$

Provide a definition for the *limit inferior*, denoted $\liminf a_n$. Does this quantity always exist? Briefly explain.
(c) Let (a_n) be the sequence $(1, -1, 1, -1, 1, -1, \ldots)$. Find $\lim\sup a_n$ and $\liminf a_n$.
(d) Show that $\liminf a_n \leq \limsup a_n$ for any bounded sequence (a_n).
(e) Suppose that $\liminf a_n = \limsup a_n$. Show that $\lim a_n$ exists and has the same value.
(f) Suppose that $\lim a_n$ exists. Show that $\liminf a_n$ and $\limsup a_n$ both exist and have the same value.
(g) Show that $\liminf(-a_n) = -\limsup a_n$.

Exercise 2.3.11. (a) Let (a_n) be a bounded sequence of real numbers. Show that there is a subsequence of (a_n) which converges to $\limsup a_n$.
(b) Can a similar statement be made for the $\liminf a_n$? Supply the details.
(c) Use either (a) or (b) to supply a short proof for the Bolzano–Weierstrass Theorem.
(d) Let (a_n) be a bounded, non-convergent sequence of real numbers. Show that (a_n) has at least two subsequences which converge to different values.

(e) Assume that (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show then that (a_n) must converge to a. (It's probably easiest to use parts (a) and (b); however, another proof can be given by assuming that (a_n) doesn't converge and using the Bolzano–Weierstrass Theorem.)

Exercise 2.3.12. Give an example of each of the following. If it is not possible to give an example, supply a proof explaining why.

(a) A convergent sequence which is not Cauchy.
(b) A Cauchy sequence which is not increasing.
(c) A bounded sequence which is not Cauchy.
(d) A divergent sequence with a subsequence that is Cauchy.

Exercise 2.3.13. Let (a_n) and (b_n) be Cauchy sequences.

(a) Use the definition of Cauchy to show that $(a_n + b_n)$ is Cauchy.
(b) Show that $(a_n b_n)$ is Cauchy.

2.4 Set Structures in $\mathbb{R}$ via Sequences

In Sect. 1.4 we discussed various set structures in $\mathbb{R}$ with open sets forming the basis of our discussion. Here we return to closed and compact sets and explore their characterizations via sequential criteria.

Definition 2.4.1. Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is called a *limit point of* A if there is a sequence (a_n) in A with $a_n \neq c$ for every $n \in \mathbb{N}$ and such that $(a_n) \to c$. A point $c \in A$ which is not a limit point is called an *isolated point*.

It is important to keep in mind that limit points may be outside the set A but isolated points must be in the set in question.

Example 2.4.2. Consider the set $A = \{1, 1/2, 1/3, 1/4, \ldots\}$. The point 0 is not in A but it is a limit point of A since the sequence $(1/n)$ is contained in A, no point of the sequence is equal to 0, and $(1/n) \to 0$. Is 1 a limit point? Keep in mind that to satisfy the definition we must build a sequence converging to 1, but we cannot use the point 1. It's obvious then that 1 is not a limit point since all other points of A are at least $\frac{1}{2}$ a unit away from 1 and hence there is no way for a sequence in A to converge to 1. This shows that 1 is an isolated point of A. Similar reasoning will show that no other point of A, or of $\mathbb{R}$ for that matter, is a limit point. From this we conclude that every point of A is isolated.

For another example, think about the set $A = (0, 1)$. If $a \in (0, 1)$, then we can find a sequence in A with the properties of Definition 2.4.1. In particular, the sequence $(a - a/n)_{n=2}$ is in A, no point of the sequence is equal to a and the sequence converges to a. Thus every point of A is a limit point of A. Are there any others? What about 0 and 1? For 0, the sequence $(1/n)$ will suffice and, for 1, we can choose the sequence $(n/(n + 1))$. Now consider a real number $a > 1$. Since all the points of A are less than one, it is clear by the Order Limit Theorem that there is no way to construct a sequence in A which converges to a. A similar argument shows that no real number $a < 0$ is a limit point of A. Does A have any isolated points?

In the last two examples, the set in question had limit points which are not elements of the set. The set $[0, 1]$ does not have this property. Reasoning just as with the open interval above, we can show that every point of the set is a limit point, but this time 0 and 1 are also in the set. Hence $[0, 1]$ contains all its limit points. The next theorem states that this is a way to characterize closed sets. On a related note, *every* element of $[0, 1]$ is also a limit point, but this is not necessary for general closed sets; the set $\{1, 2\}$ has no limit points, only isolated points, but is still closed.

Theorem 2.4.3. *A set $F \subseteq \mathbb{R}$ is closed if and only if it contains all its limit points.*

Proof. First suppose that F is closed, which means that F^c is open by definition. To show that this set contains all its limit points, let $c \in \mathbb{R}$ be a limit point of F, that is, suppose there is a sequence (a_n) in F with $a_n \neq c$ for all n and $(a_n) \to c$. Now we must show that $c \in F$. To make use of our assumption that F^c is open, we will assume towards a contradiction that $c \in F^c$. Hence there is an $\varepsilon > 0$ such that $B_\varepsilon(c) \subseteq F^c$. However, since $(a_n) \to c$, there is an $N \in \mathbb{N}$ such that $|a_N - c| < \varepsilon$, meaning that $a_N \in B_\varepsilon(c) \subseteq F^c$. But a_N is in F, a contradiction. Thus $c \in F$.

For the converse, suppose that F contains all its limit points. To show that F is closed, we must show that F^c is open. Here we will also work by contradiction and we assume that F^c is not open. Hence there is a point $c \in F^c$ such that for every $\varepsilon > 0$, $B_\varepsilon(c)$ is not a subset of F^c. We claim now that the point c must be a limit point of F, which will provide a contradiction since $c \notin F$.

To verify the claim, we construct a sequence in F which converges to c. To define this sequence, choose a_n to be a point in F which is in $B_{1/n}(c)$; this is possible since for every $\varepsilon > 0$, we know that $B_\varepsilon(c)$ is *not* a subset of F^c. It follows immediately that (a_n) is in F and $a_n \neq c$ for all n. Furthermore,

$$c - \frac{1}{n} < a_n < c + \frac{1}{n}$$

and hence converges to c by the Squeeze Theorem. This shows that c is a limit point of F and provides our contradiction. □

Thinking about terminology for a moment, when we use the word closed it is typically with respect to a set and operations. For example, we say that $\mathbb{R}$ is closed with respect to the operations of addition and multiplication. The theorem above demonstrates that a closed set is closed with respect to the operation of limits. Continuing this line of inquiry, we now define the closure of a set.

Definition 2.4.4. Let $A \subseteq \mathbb{R}$. The *closure of* A, denoted $\overline{A}$, is defined to be the set which consists of A together with all its limit points. Letting L denote the set of limit points of A, we can write $\overline{A} = A \cup L$.

Example 2.4.5. Let $A = \{1, 1/2, 1/3, \ldots\}$. From our discussion in the previous example we see that $L = \{0\}$ and $\overline{A} = \{0, 1, 1/2, 1/3, \ldots\}$. From that example we also see that $\overline{(0,1)} = \overline{[0,1]} = [0,1]$ with $L = [0,1]$ for both $(0,1)$ and $[0,1]$.

Exercise 2.1.8 shows that $\overline{\mathbb{Q}} = \overline{\mathbb{I}} = \mathbb{R}$ with $L = \mathbb{R}$ for both sets. Considering $\mathbb{N}$, we have $L = \emptyset$ and hence $\overline{\mathbb{N}} = \mathbb{N}$.

Notice that in all of these examples, both L and $\overline{A}$ are closed. The following theorem formalizes this.

Proposition 2.4.6. *Let $A \subseteq \mathbb{R}$. The set $\overline{A}$ is a closed set. Furthermore, A is closed if and only if $A = \overline{A}$.*

Proof. We will show that $\overline{A}$ is closed and leave the second statement as an exercise. Our goal is to show that $\overline{A}^c$ is an open set and hence we choose an arbitrary point $c \in \overline{A}^c$. In order to generate an ε–neighborhood for c, notice that the definition of closure implies $c \in (A \cup L)^c = A^c \cap L^c$, i.e. c is not in A and c is not a limit point of A. Using the negation of Exercise 2.4.1, the statement that $c \notin L$ means that there is an $\varepsilon > 0$, such that for every $a \in A$, either $a = c$ or $a \notin B_\varepsilon(c)$. Now, we know that $c \notin A$, and thus $a \neq c$, so it must be the case that no point of A is in $B_\varepsilon(c)$ for this particular value of ε.

We can conclude that $B_\varepsilon(c) \subseteq \overline{A}^c$ if we can also show that no point of L is in this ε-neighborhood. To this end, suppose that $d \in L$ and $d \in B_\varepsilon(c)$. The fact that the ε-neighborhood is open then implies that there is a $\delta > 0$ such that $B_\delta(d) \subseteq B_\varepsilon(c)$. However, since d is a limit point of A, Exercise 2.4.1 implies the existence of a point $a \in A$ with $a \neq d$ and $a \in B_\delta(d) \subseteq B_\varepsilon(c)$. But this is impossible since we know that $B_\varepsilon(c)$ contains no point of A. Thus $B_\varepsilon(c)$ also contains no limit points of A and we have $B_\varepsilon(c) \subseteq \overline{A}^c$. Therefore $\overline{A}^c$ is open and $\overline{A}$ is closed. □

Recall from the previous example that the closure of both the rationals and irrationals is $\mathbb{R}$. This is a restatement of the fact that both these sets are dense in $\mathbb{R}$.

Proposition 2.4.7. *Let S be a subset of $\mathbb{R}$. Then S is dense in $\mathbb{R}$ if and only if $\overline{S} = \mathbb{R}$.*

Proof. First suppose that S is dense in $\mathbb{R}$. In order to show that $\overline{S} = \mathbb{R}$, we must show that $\mathbb{R} \subseteq \overline{S}$; the reverse inclusion is trivially true. In order to verify the desired inclusion, it suffices to show that every point of $\mathbb{R}$ is a limit point of S. To this end, let $t \in \mathbb{R}$. Applying the density condition, for each $n \in \mathbb{N}$, choose $s_n \in S$ such that

$$t - \frac{1}{n} < s_n < t.$$

It is clear that $s_n \neq t$ for all n and the Squeeze Theorem guarantees us that $(s_n) \to t$. Thus every point of $\mathbb{R}$ is a limit point of S.

For the converse suppose that $\overline{S} = \mathbb{R}$. To show that S is dense in $\mathbb{R}$, let $r, t \in \mathbb{R}$ with $r < t$. To produce a point $s \in S$ with $r < s < t$, let $r_1 = (r + t)/2$, the midpoint of r and t. If $r_1 \in S$, then we are done. If $r_1 \notin S$, the fact that $\overline{S} = \mathbb{R}$ implies that there is a sequence (s_n) in S with $s_n \neq r_1$ for all n and $(s_n) \to r_1$. If we then choose $\varepsilon = \frac{1}{2}(r_1 - r) = \frac{1}{2}(t - r_1)$, the definition of convergence implies that there is an $N \in \mathbb{N}$ such that $|r_1 - s_N| < \varepsilon$. This immediately implies that

$$r < r_1 - \varepsilon < s_N < r_1 + \varepsilon < t$$

as desired. □

We turn our attention now to compact sets which were originally defined as closed and bounded subsets of $\mathbb{R}$. With the sequential characterization of closed sets in mind, we know that any compact set must contain its limit points. In order to understand the bounded condition one only need to consider the Bolzano Weierstrass Theorem which states that any bounded sequence must have a convergent subsequence. Thus if we take a sequence (a_n) in a compact set K, then we know that (a_n) has a convergent subsequence. The fact that K is closed then implies that this subsequential limit must be in K. This motivates the following definition which turns out to be equivalent to compactness.

Definition 2.4.8. Let $K \subseteq \mathbb{R}$. We say that K is *sequentially compact* if every sequence in K has a subsequence which converges to a point in K.

Example 2.4.9. From the comments before the definition, we know that any compact set will be sequentially compact which provides an abundance of examples. For an example of a set which is not sequentially compact, take the open interval $(0, 1)$. We must now identify a sequence (x_n) in $(0, 1)$ for which no subsequence converges to a point in $(0, 1)$. The easiest way to do this is to think about limit points. Certainly 0 is a limit point, and the sequence $(1/n)$ is in $(0, 1)$, but no subsequence converges to a point in $(0, 1)$. Indeed, every subsequence converges to 0 which is not in this interval.

The following theorem relates the notion of compactness and sequential compactness in $\mathbb{R}$ though the proof is postponed to the end of the section where we include a third characterization of compact subsets of $\mathbb{R}$.

Theorem 2.4.10 (Heine–Borel Theorem). *A set $K \subseteq \mathbb{R}$ is compact if and only if it is sequentially compact.*

Compactness has been characterized in terms of closed sets and sequences, but it turns out that it is also possible to do so in terms of open sets (our first characterization in terms of closed sets was really about open sets too).

Definition 2.4.11. Let $K \subseteq \mathbb{R}$. We say a collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ is an *open cover* for K if $K \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$; the collection Λ may be an uncountable set. For an open cover $\{O_\lambda : \lambda \in \Lambda\}$ of K, we say that $\{O_{\lambda_1}, O_{\lambda_2}, \dots, O_{\lambda_N}\}$ is a *finite subcover* if $K \subseteq \bigcup_{n=1}^{N} O_{\lambda_n}$. In this case we say that the open cover *admits* a finite subcover for K.

Example 2.4.12. The definition above is somewhat more general than sequential compactness and can be harder to digest at first glance. An open cover is a set consisting of sets and the union of these sets contains the set in question. The collection $\{(-n, n) : n \in \mathbb{N}\}$ is an open cover for $\mathbb{R}$ since

$$\mathbb{R} \subseteq \bigcup_{n=1}^{\infty} (-n, n).$$

For the interval $(0, 1)$, we can take $\{(0, 1)\}$ as an open cover, though this is rather boring. Other open covers include $\{(1/n, 1) : n \geq 2\}$, $\{(0, 1 - 1/n) : n \geq 2\}$, and $\{(1/n, 1 - 1/n) : n \geq 3\}$.

What we are truly interested in are sets whose open covers *always* admit at least one finite subcover. For an example where this does not happen, we again consider $(0, 1)$ and the open cover $\{(1/n, 1) : n \geq 2\}$. To see that there is no finite subcover, suppose that $N \in \mathbb{N}$ is fixed and represents the largest natural number index in a particular finite subcover. But then the union of these finitely many open intervals is $(1/N, 1)$ which does not cover all of $(0, 1)$. Hence there is no finite subcover for this particular open cover.

Theorem 2.4.13 (Heine–Borel Theorem). *Let $K \subseteq \mathbb{R}$. The following are equivalent.*

(a) K is compact.
(b) K is sequentially compact.
(c) Every open cover for K admits a finite subcover.

When proving a string of equivalent statements as we have above it is common place to show the string of implications (a)⇒(b)⇒(c)⇒(a). Our proof will hinge on our original definition for compactness and we will show that both (b) and (c) are logically equivalent to (a).

Proof. $(a) \Rightarrow (b)$. The proof for this direction follows immediately from the comments which motivated the definition of sequential compactness though there is one tiny detail to consider which arises from the requirement that "$a_n \neq c$ for all $n \in \mathbb{N}$" in the definition of a limit point.

$(b) \Rightarrow (a)$. Here we assume that K is sequentially compact and we must show that K is closed and bounded. First suppose to the contrary that K is not bounded. For a contradiction we will construct a sequence such that every subsequence is unbounded and hence not convergent. For $n \in \mathbb{N}$, choose $x_n \in K$ such that $|x_n| > n$. Now let (x_{n_k}) be a subsequence of (x_n) and let $M > 0$. Then there is an $N \in \mathbb{N}$ such that $N > M$. Furthermore, since (n_k) is an increasing sequence of natural numbers, there is a $k \in \mathbb{N}$ such that $n_k > N > M$ which implies that $|x_{n_k}| > n_k > N > M$. Hence (x_{n_k}) is unbounded and cannot possibly converge. But this contradicts the fact that K is sequentially compact and thus it must be the case that K is bounded.

To show that K is also closed, we use the sequential characterization of closed sets. Let $x \in \mathbb{R}$ be a limit point of K. Then we know there is a sequence (x_n) in K with $x_n \neq x$ for all n and $(x_n) \to x$. The sequential compactness then implies that there is a subsequence (x_{n_k}) which converges to a point $y \in K$. By

Proposition 2.2.11 we conclude that $x = y$ and thus $x \in K$ showing that K is closed.

$(a) \Rightarrow (c)$. We again assume that K is closed and bounded, and let $\mathscr{O} = \{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. Our goal is to show that there is a finite subcollection of $\mathscr{O}$ whose union contains K. First notice that $\inf(K)$ and $\sup(K)$ both exist and are elements of K by Exercise 2.4.8; call these a and b, respectively. It follows that

$$K \subseteq [a, b];$$

modifying our previous goal, we will now seek to show that there is a finite subcollection of $\mathscr{O}$ whose union contains $[a, b]$, from which our desired conclusion will immediately follow. To this end, we define a set

$$E = \left\{ x \in [a, b] : K \cap [a, x] \subseteq \bigcup_{i=1}^{N} O_{\lambda_i} \text{ for some } \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subseteq \Lambda \right\}.$$

The set E is certainly not empty; it should be clear that $a \in E$ since a is in some $O_\lambda \in \mathscr{O}$. Notice also that E is bounded above since $E \subseteq [a, b]$; set $c = \sup(E)$. As a first observation, we show that $c \in K$. For a contradiction, suppose $c \notin K$. Then, since K^c is open, there is a $\delta > 0$ such that $B_\delta(c) \subseteq K^c$. By definition of c and our assumption that $c \notin K$, we have $a < c < b$ and hence it must be the case that $B_\delta(c) \subseteq (a, b)$ since $a, b \in K$. However, by Lemma 1.2.10, there is a $y \in E$ with $c - \delta < y$. Using the fact that $B_\delta(c) \subseteq K^c$, it follows that $K \cap [a, y] = K \cap [a, c + \delta/2]$. But this implies that $c + \delta/2 \in E$ which contradicts the fact that c is the supremum of E. Thus it must be the case that $c \in K$.

Next we will show that $b = c$. Using the fact that $c \in K$, we immediately find that there is an open set $O_\alpha \in \mathscr{O}$ with $c \in O_\alpha$. With this open set specified, we know that there is an $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq O_\alpha$, and appealing to Lemma 1.2.10 again, there must be a point $d \in E$ with $d \in (c - \varepsilon, c)$. This is the crux. The fact that $d \in E$ implies that there is a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subseteq \Lambda$ such that

$$K \cap [a, d] \subseteq \bigcup_{i=1}^{N} O_{\lambda_i}.$$

From this and the fact that $d \in (c - \varepsilon, c) \subseteq \mathscr{O}_\alpha$, it is clear that

$$K \cap [a, c] \subseteq O_\alpha \cup \left(\bigcup_{i=1}^{N} O_{\lambda_i} \right);$$

this last union is also a finite subcover. However, O_α contains all points in the interval $(c, c + \varepsilon)$, which shows that

$$K \cap [a, c+\varepsilon] \subseteq O_\alpha \cup \left(\bigcup_{i=1}^{N} O_{\lambda_i} \right).$$

If $c < b$, then there are points of $[a, b]$ in the interval $(c, c+\varepsilon)$ and the above set inclusion then contradicts the fact that $c = \sup(E)$. Thus it must be the case that $c = b$ (c cannot be greater than b) and the above finite subcover satisfies

$$K = K \cap [a, b] = K \cap [a, c] \subseteq O_\alpha \cup \left(\bigcup_{i=1}^{N} O_{\lambda_i} \right),$$

as desired.

$(c) \Rightarrow (a)$. The proof here is also left as an exercise with a hint provided. □

Exercises

Exercise 2.4.1. Let $A \subseteq \mathbb{R}$ with $c \in \mathbb{R}$. Show that c is a limit point of A if and only if for every $\varepsilon > 0$, there is a point $a \in A$ such that $a \neq c$ and $a \in B_\varepsilon(c)$.

Exercise 2.4.2. Let $O \subseteq \mathbb{R}$ be an open set with $a \in O$. If (a_n) converges to a, show that at most finitely many terms of this sequence are not in O.

Exercise 2.4.3. Let $O \subseteq \mathbb{R}$ be an open set. Prove that O has no isolated points.

Exercise 2.4.4. (a) Give an example of a bounded set A for which $\sup(A)$ is not a limit point of A.
(b) Give an example of an open set which contains all the rational numbers, but which is not all of $\mathbb{R}$.

Exercise 2.4.5. Let $A \subseteq \mathbb{R}$ and let L be the set of limit points of A. Show that L is a closed set.

Exercise 2.4.6. Let $A \subseteq \mathbb{R}$. Show that A is closed if and only if $A = \overline{A}$.

Exercise 2.4.7. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Use the definition of closure to show that $\sup(A) \in \overline{A}$.

Exercise 2.4.8. Let $K \subseteq \mathbb{R}$ be a compact set. Show that $\sup K$ and $\inf K$ are elements of K.

Exercise 2.4.9. A set $A \subseteq \mathbb{R}$ is called a G_δ (pronounced "G-delta") set if it can be written as a countable intersection of open sets.

(a) Explain why a G_δ set is not necessarily an open set.
(b) Explain why every open set is a G_δ set.
(c) Show that the set $\{0\}$ is a G_δ set.
(d) Show that the interval $(0, 1]$ is a G_δ set.
(e) Is $[0, 1]$ a G_δ set?

Exercise 2.4.10. A set $A \subseteq \mathbb{R}$ is called an F_σ (pronounced "F-sigma") set if it can be written as a countable union of closed sets.

(a) Explain why an F_σ set is not necessarily a closed set.
(b) Explain why every closed set is a F_σ set.
(c) Show that the interval $(0, 1]$ is an F_σ set.
(d) Is $(0, 1)$ an F_σ set?
(e) Show that $\mathbb{Q}$ is an F_σ set.

Exercise 2.4.11. (a) Let $A \subseteq \mathbb{R}$. Show that A is a G_δ set if and only if A^c is an F_σ set.
(b) Show that $\mathbb{I}$ is a G_δ set.
(c) Is the set $(-\infty, 0) \cap (0, \infty)$ a G_δ set, an F_σ set, or both?

Exercise 2.4.12. Follow the outline below to complete the proof of the Heine–Borel Theorem.

(a) Verify that $(a) \Rightarrow (b)$.
(b) Verify that $(c) \Rightarrow (a)$. One possible approach is to argue by contrapositive: if K is not closed or not bounded, then there is an open cover which does not admit a finite subcover. If K is not closed, construct an open cover which does not admit a finite subcover, and likewise if K is not bounded.

2.5 Complete Spaces

As with our study of $\mathbb{R}$, the foundation of analysis in any setting is the investigation of convergent sequences. The following definitions are natural extensions of the notions we have encountered thus far.

Definition 2.5.1. Let (X, d) be a metric space. A sequence (x_n) in X *converges to* $x \in X$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for every $n \geq N$. Likewise the sequence (x_n) is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for every $n, m \geq N$.

The definition of convergence here is closely related to the convergence of a sequence of real numbers. If (x_n) is a sequence in a metric space (X, d), then (x_n) converges to $x \in X$ if the sequence of real numbers $d(x_n, x)$ converges to 0 as $n \to \infty$. When working in a normed linear space, the metric of choice is the metric induced by the norm as stated in Proposition 1.5.10. In this case, the statement above is interpreted by saying that (x_n) converges to x in X if $\|x_n - x\| \to 0$ as $n \to \infty$. Similar comparisons can be made for Cauchy sequences in a metric space and Cauchy sequences of real numbers.

The first result of this section exhibits a familiar connection between these two ideas. It is also a good idea to compare the proof in this more generatl setting to the proof given for sequences in $\mathbb{R}$ in Proposition 2.1.9.

Theorem 2.5.2. *In any metric space, a convergent sequence is Cauchy.*

Proof. Let (X, d) be a metric space and suppose (x_n) is a sequence in X converging to $x \in X$. For $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon/2$. For $n, m \geq N$ observe that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and thus we conclude that (x_n) is a Cauchy sequence. □

In $\mathbb{R}$, the converse of this theorem also holds, the Cauchy Criterion, and provides a means of determining convergence in situations when no limit candidate can be exhibited explicitly. Unfortunately, in a general metric space it is not always true that Cauchy sequences converge. In order to single out the exceptional metric spaces with this property, we have the following definition.

Definition 2.5.3. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges to a point in X.

Recall that we asserted the completeness of $\mathbb{R}$ via the existence of suprema. However, we then explored various other equivalent statements: the Nested Interval Property, the Monotone Convergence Theorem, the Bolzano–Weierstrass Theorem, and the Cauchy Criterion. Several of these statements depend on the fact that there is an order on $\mathbb{R}$. It should be clear, consider $\mathbb{R}^2$ for example, that general metric spaces do not always afford an order, and thus we are restricted in our approach to completeness in these spaces. While there are other ways to classify completeness in general metric spaces, the use of Cauchy sequences is the most standard. Complete metric spaces which have the additional structure of a norm or an inner product are named for two of the most prominent analysts of the early twentieth century, Stefan Banach and David Hilbert.

Definition 2.5.4. A complete metric space in which the metric is induced by a norm as in Proposition 1.5.10 is called a *complete normed linear space* or, more commonly, a *Banach space*. A complete normed linear space whose norm is induced by an inner product as in Proposition 1.5.15 is called a *Hilbert space*.

We now present two examples which exhibit a standard technique for demonstrating completeness.

Example 2.5.5. The space $\mathbb{R}^2$ with the Euclidean metric is a Hilbert space. We know that this space is an inner product space, so we simply need to show that the metric induced by the inner product is complete. Before beginning, if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are elements of $\mathbb{R}^2$, recall that metric distance between x and y is given by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

To show that the space is complete, let (x_n) be a Cauchy sequence in $\mathbb{R}^2$. It is then our task to show that this sequence converges to some $x \in \mathbb{R}^2$. We begin by identifying x. For notation, we will write x_n as the ordered pair (x_{n1}, x_{n2}). Since (x_n) is Cauchy, for $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that

$$d(x_n, x_m) = \sqrt{(x_{n1} - x_{m1})^2 + (x_{n2} - x_{m2})^2} < \varepsilon$$

for all $n, m \geq N$. Using a simple estimate we see that

$$|x_{n1} - x_{m1}| \leq \sqrt{(x_{n1} - x_{m1})^2 + (x_{n2} - x_{m2})^2} < \varepsilon$$

for $n, m \geq N$ and thus the sequence $(x_{n1})_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. The Cauchy Criterion now guarantees the existence of $x_1 \in \mathbb{R}$ such that $(x_{n1}) \to x_1$ as $n \to \infty$. Similarly we can find $x_2 \in \mathbb{R}$ such that $(x_{n2}) \to x_2$ as $n \to \infty$. Let $x = (x_1, x_2) \in \mathbb{R}^2$.

To complete the proof, we show that (x_n) converges to x in the Euclidean metric. By the Algebraic Limit Theorem, $(x_{n1} - x_1)^2 + (x_{n2} - x_2)^2 \to 0$ as $n \to \infty$. Thus

$$d(x_n, x) = \sqrt{(x_{n1} - x_1)^2 + (x_{n2} - x_2)^2} \to 0,$$

completing the proof.

Our second example deals with a new space which will reappear in a different guise later on.

Example 2.5.6. Let ℓ^∞ denote the space of all bounded sequences of real numbers, i.e.

$$\ell^\infty = \{(a_n)_{n=1} = (a_1, a_2, a_3 \ldots) : \sup_n |a_n| < \infty\}.$$

If we define addition and scalar multiplication by

$$(a_n) + (b_n) = (a_n + b_n) \quad \text{and} \quad c(a_n) = (ca_n),$$

then this collection of sequences forms a real vector space. Furthermore, if (a_n) is a bounded sequence, the rule

$$\|(a_n)\|_\infty = \sup_n |a_n|$$

defines a norm on ℓ^∞, referred to commonly as the *sup-norm*. We leave this as an exercise and prove the completeness.

Proposition 2.5.7. *The space ℓ^∞ is complete with respect to the sup-norm.*

Proof. To begin, assume we have a Cauchy sequence in ℓ^∞. The proof will be broken down into several steps; the first, and most lengthy, will be to identify a limit candidate. Since the elements in this set are themselves sequences, we will need to take some care with notation. Let $\Big((a_{nk})_{k=1}\Big)_{n=1}$ denote this Cauchy sequence. Writing this out with an array (we borrow heavily from the standard matrix notation where the first index represents the row of the entry and the second represents the column), we have

$$\begin{array}{ccl}
(a_{1k}) & = & a_{11},\ a_{12},\ a_{13},\ \ldots,\ a_{1k},\ \ldots \\
(a_{2k}) & = & a_{21},\ a_{22},\ a_{23},\ \ldots,\ a_{2k},\ \ldots \\
(a_{3k}) & = & a_{31},\ a_{32},\ a_{33},\ \ldots,\ a_{3k},\ \ldots \\
\vdots & & \vdots \qquad\qquad \vdots \\
(a_{nk}) & = & a_{n1},\ a_{n2},\ a_{n3},\ \ldots,\ a_{nk},\ \ldots \\
\vdots & & \vdots \qquad\qquad \vdots \qquad\quad \vdots
\end{array}$$

For the sake of clarity, applying the sup-norm to one of these sequences is expressed by

$$\|(a_{nk})_{k=1}\|_\infty = \sup_k |a_{nk}|;$$

the point is to be aware of the fact that the norm is taken by considering the supremum as the second subscript varies, i.e. along the rows in the array above.

Now, since this sequence is Cauchy, given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\|(a_{nk}) - (a_{mk})\|_\infty = \sup_k\{|a_{nk} - a_{mk}|\} < \varepsilon$$

whenever $n, m \geq N$. Further, notice that for each $k \in \mathbb{N}$ it is true that

$$|a_{nk} - a_{mk}| \leq \|(a_{nk}) - (a_{mk})\|_\infty.$$

Thus $|a_{nk} - a_{mk}| < \varepsilon$ for all $n, m \geq N$, and hence the sequence $(a_{nk})_{n=1}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed $k \in \mathbb{N}$; in other words, for each k there is an $a_k \in \mathbb{R}$ such that $(a_{nk})_{n=1}$ converges to a_k as $n \to \infty$. Adding this new information to our array gives

$$\begin{array}{ccl}
(a_{1k}) & = & a_{11},\ a_{12},\ a_{13},\ \ldots,\ a_{1k},\ \ldots \\
(a_{2k}) & = & a_{21},\ a_{22},\ a_{23},\ \ldots,\ a_{2k},\ \ldots \\
(a_{3k}) & = & a_{31},\ a_{32},\ a_{33},\ \ldots,\ a_{3k},\ \ldots \\
\vdots & & \vdots \qquad\qquad \vdots \\
(a_{nk}) & = & a_{n1},\ a_{n2},\ a_{n3},\ \ldots,\ a_{nk},\ \ldots \\
\downarrow ? & & \ \downarrow \quad\ \downarrow \quad\ \downarrow \qquad\quad \downarrow \\
(a_k) & = & a_1,\ \ a_2,\ \ a_3,\ \ \ldots,\ a_k,\ \ldots
\end{array}$$

We will be finished if we can show that $(a_k)_{k=1}$ is in ℓ^∞ and that the sequence $\Big((a_{nk})_{k=1}\Big)_{n=1}$ (the left-hand column of our array) converges to the sequence $(a_k)_{k=1}$ (the lower left-hand entry) as $n \to \infty$. Notice here that when we speak of convergence we mean convergence in the metric induced by the sup-norm.

First, we will show that (a_k) is in ℓ^∞. To see this, we appeal to the fact that Cauchy sequences are bounded, a fact you will show in Exercise 2.5.1. Thus there is an $M > 0$ such that $\|(a_{nk})_{k=1}\|_\infty \leq M$ for every $n \in \mathbb{N}$. This immediately implies that $|a_{nk}| \leq M$ for every $n, k \in \mathbb{N}$. The Order Limit Theorem now implies that $|a_k| \leq M$ for all k and hence (a_k) is a bounded sequence.

Finally, we must show that $\|(a_{nk}) - (a_k)\|_\infty \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Using the original hypothesis that our sequence is Cauchy, we can find $N \in \mathbb{N}$ such that

$$\|(a_{nk}) - (a_{mk})\|_\infty = \sup_k\{|a_{nk} - a_{mk}|\} < \varepsilon/4$$

for $n, m \geq N$. As before, this immediately implies that

$$|a_{nk} - a_{mk}| \leq \|(a_{nk}) - (a_{mk})\|_\infty < \varepsilon/4$$

for all $n, m \geq N$ and all $k \in \mathbb{N}$. The proof of the Cauchy Criterion now immediately shows that

$$|a_{nk} - a_k| < \varepsilon/2$$

for all $n \geq N$ and all $k \in \mathbb{N}$. To conclude, the previous inequality ensures us that

$$\|(a_{nk}) - (a_k)\|_\infty = \sup_k\{|a_{nk} - a_k|\} \leq \varepsilon/2 < \varepsilon$$

for all $n \geq N$. Thus the space ℓ^∞ equipped with the sup-norm is a complete normed linear space. $\square$

Before moving on, notice that the technique used to identify the limit candidate in both of these examples was exactly the same. In the case of $\mathbb{R}^2$ it was easy to see that the potential limit was in the space while in ℓ^∞ we had to work a little harder. The final convergence argument for these two examples differed but that was to be expected due to the nature of the norms being considered.

Definition 2.5.8. Let X be a vector space and let $W \subseteq X$. We call W a *subspace* of X if W is a vector space with respect to the operations on X.

Notice that the definition of subspace subtly requires W to be nonempty by vector space axiom (d). Also, the definition above is stated in the most general form; however, the operational axioms of vector space, (b), (c), and (g)–(j), must hold no matter what collection W we are considering since $W \subseteq X$ and X is a vector space. Thus we really only need to check that all of the correct vectors are in

W. The following proposition captures this thought and we leave the formal proof to the reader.

Proposition 2.5.9. *Let X be a vector space and let $W \subseteq X$. If W is not empty, then W is a subspace of X if and only if W is closed with respect to the addition and scalar multiplication of X.*

One immediate consequence here is that the additive identity in a subspace W is the same as the additive identity in the larger space X. This often provides a quick means of showing a set is not a subspace.

Example 2.5.10. Let $X = \mathbb{R}^2$. If $x \in \mathbb{R}^2$, the set $W = \{cx : c \in \mathbb{R}\}$ is a subspace of W. However, the set $W = \{(x_1, x_2) : 2x_1 + 3x_2 = 1\}$ is not a subspace since it does not contain the zero vector.

For the polynomial spaces P_n (see Example 1.5.3), it should be apparent that P_n is a subspace of P_{n+1} for all $n \in \mathbb{N}$.

To close this section, we consider several subspaces of ℓ^∞ and explore an example of a space which is not complete.

Example 2.5.11. Let c denote the collection of all sequences of real numbers which converge and let c_0 denote the collection of all sequence of real numbers which converge to zero. In Exercise 2.5.9 you are asked to show that $c_o \subseteq c \subseteq \ell^\infty$. When equipped with the sup-norm of the previous example, these two spaces are complete normed linear spaces.

Example 2.5.12. Define p^∞ to be the collection of sequences of real numbers with at most finitely many nonzero terms. Any sequence of this form must be in ℓ^∞ since the supremum of a finite number of points always exists; hence, we conclude that $p^\infty \subseteq \ell^\infty$. To confirm that this collection is a subspace of ℓ^∞ we need to verify that it is closed with respect to the addition and scalar multiplication; it is certainly not empty. To this end suppose that (a_k) is a sequence with n nonzero terms and (b_k) is a sequence with m nonzero terms. Then the sum $(a_k) + (b_k)$ has at most $n + m$ nonzero terms, and, for $c \in \mathbb{R}$, $c(a_k)$ has at most n nonzero terms. Thus both new elements are in p^∞.

We will now show that this space is not complete with respect to the sup-norm. To do this, we must produce a sequence of elements in p^∞ which is Cauchy but has no limit in p^∞. Take the sequence of sequences $\left((a_{nk})_{k=1}\right)_{n=1}$ defined by

$$a_{nk} = \begin{cases} 1/k, & k \leq n; \\ 0, & k > n. \end{cases}$$

Writing this out in an array form to better see what each sequence looks like, we have

$$\begin{array}{lllllll}
(a_{1k}) = 1, & 0, & 0, & 0, \ldots, & 0, & 0, \ldots \\
(a_{2k}) = 1, & 1/2, & 0, & 0, \ldots, & 0, & 0, \ldots \\
(a_{3k}) = 1, & 1/2, & 1/3, & 0, \ldots, & 0, & 0, \ldots \\
\vdots & \vdots & \vdots & & & & \\
(a_{nk}) = 1, & 1/2, & 1/3, & \ldots, & 1/n, & 0, \ldots \\
\vdots & \vdots & \vdots & & \vdots & &
\end{array}$$

To see that this sequence is Cauchy, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then, for a fixed $k \in \mathbb{N}$, (we assume $n > m$ to simplify the calculation)

$$|a_{nk} - a_{mk}| = \begin{cases} 0, & k \le m < n; \\ 1/k, & m < k \le n; \\ 0, & m < n < k. \end{cases}$$

If we now assume that $n > m \ge N$, it follows that $1/(m+1) \le 1/N < \varepsilon$ and thus

$$\|(a_{nk}) - (a_{mk})\|_\infty = \sup_k\{|a_{nk} - a_{mk}|\} = 1/(m+1) < \varepsilon;$$

hence we have a Cauchy sequence. We also note that this sequence converges in the sup-norm to the sequence $(a_k) = (1/k)_{k=1}$ since

$$\|(a_{nk}) - (a_k)\|_\infty = \sup_k\{|a_{nk} - a_k|\} = 1/(n+1).$$

This convergence is in the larger space ℓ^∞ and hence (a_k) is the *only* possible limit for our chosen Cauchy sequence. However, this limit is not in p^∞ since it has countably many nonzero terms and we therefore conclude that p^∞ is not complete.

Exercises

Exercise 2.5.1. Let $(X, \|\cdot\|)$ be a normed linear space. We say that a sequence (x_n) is *bounded* if there exists an $M > 0$ such that $\|x_n\| \le M$ for every $n \in \mathbb{N}$.

(a) Show that a convergent sequence is bounded.
(b) Show that a Cauchy sequence is bounded. Keep in mind that we are not assuming our space is complete, and hence you cannot simply reference part (a).

Exercise 2.5.2. Let (X, d) be a metric space and suppose (x_n) is a convergent sequence in X. Show that the limit is unique.

Exercise 2.5.3 (Algebraic Limit Theorem for Normed Linear Spaces). Suppose $(X, \|\cdot\|)$ is a normed linear space and let (x_n) and (y_n) be sequences in X with limits x and y, respectively.

(a) Show that $(x_n + y_n)$ converges to $x + y$.
(b) If $c \in \mathbb{R}$, show that (cx_n) converges to cx.

Exercise 2.5.4. Let $(X, \|\cdot\|)$ be a normed linear space and let (x_n) be a sequence in X converging to $x \in X$. Use Exercise 1.5.3 to show that the sequence of real numbers $(\|x_n\|)$ converges to $\|x\|$.

Exercise 2.5.5. Let (X, d) be a metric space. Show that X is complete if and only if every Cauchy sequence has a convergent subsequence (with limit x in X).

Exercise 2.5.6. Show that ℓ^∞ is a vector space and that the rule $\|\cdot\|_\infty$ given in Example 2.5.6 defines a norm.

Exercise 2.5.7. Use the parallelogram equality to show that the sup-norm does not arise as an inner product as in Proposition 1.5.15.

Exercise 2.5.8. Let X be a vector space and let $W \subseteq X$ be a subspace. Show that $\mathbf{0}_W = \mathbf{0}_X$.

Exercise 2.5.9. (a) Show that $p^\infty \subseteq c_0 \subseteq c \subseteq \ell^\infty$.
(b) Show that each space in the chain above is a subspace of its superset.
(c) Show that the spaces c and c_0 are complete normed linear spaces with respect to the sup-norm.

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