Preface

Like many mathematical textbooks, this project evolved from a set of course notes. These notes were developed and refined over a 5-year period and represent my interpretation of a first course in analysis. Much of the material comes from the standard undergraduate canon including an axiomatic or constructive exploration of the real number system, numerical sequences and series, limits and continuity, differentiation, sequences and series of functions, and some introductory form of integration. In addition I have included material on measure theory and the Lebesgue integral, and a brief invitation to functional analysis. Although these advanced topics are not typically introduced until graduate study, I find that with the proper presentation they are perfectly manageable for undergraduates wishing to delve deeper into the subject matter. My primary reason for the inclusion of this advanced material is to prepare students for graduate study, and to also help them experience the evolution of mathematics. While each chapter covers a topic central to a beginning course in real analysis, the last section in each chapter introduces a topic from functional analysis which is derived from the core chapter content in a natural way; these sections are developed alongside a concrete topic in order to provide grounding. This takes the shape of a particular idea, complete normed linear spaces, eigenvalues, invariant subspaces, or focuses on a particular example or class of examples, ℓ^p spaces, continuous function spaces, and L^p spaces over general measure spaces. Taken as a whole, these sections provide a nice connection to linear algebra and point to a thriving area of mathematical research. The Epilogue then provides a brief road map to further points of interest in the realm of functional analysis.

When I began this project, I first mapped out the sections within each chapter. This led to nine chapters which built on each other, but seemed to lack a sense of harmony. As a reworking, I began to think about the overall story line of the text, of each chapter, and then of each section. The goal was to tie these story lines together in a cohesive manner that conveys the story of real analysis. This activity led to five core beliefs which have influenced every word written here.

- An introductory text should be written for students. There are a great many analysis texts available on the market. Some classical texts are too difficult for today's audience (which has changed drastically in the last few decades) and many of the newer expositions are too watered down. Others seem to strike a balance between rigorous exposition and intuitive reasoning, but fall short when it comes to adequate exercises and examples. In thinking about all of this, my goal in writing this text was to provide a thorough treatment of the necessary subject matter without sacrificing rigor, while shaping the presentation so that every student required to take a course in real analysis can succeed. Examples are frequent, discussion of new ideas motivates definitions, theorems, and proofs before formal statements and arguments are given, and the exercise sets are detailed and thorough. The discussion portions of the text are meant for the newcomer and there will inevitably be instances where an instructor thinks I'm belaboring a point. And while that may be the case, these explanations have arisen out of conversations with students about the material, both in the classroom and in the office hours. Undergraduates think differently about new material than a professional mathematician and my audience is comprised of undergraduates.
- A first course in analysis should, to some degree, focus on providing a theoretical treatment of the material from the first-year calculus sequence. I see this course as a means of coming full circle. Students have experienced calculus, their intuition has been developed, and they have been exposed to the many applications of differentiation and integration. To complete this experience, it is necessary then that they should understand why the things they have taken on faith are in fact true. To do this requires a thorough study of the concept of a limit, and it is necessary that they understand why it is the underlying construct for every topic in analysis. This can take many forms and I have chosen to focus on sequential limits. So, while I do strive to provide a concrete foundation for the differential and integral calculus, I do not believe that a text should be confined to developing this material in the same manner or order as it is presented in the first-year calculus sequence.
- Examples and exercises are the keys to understanding theory. The sections are written so as to provide most of the details for the ideas under consideration, but the reader is encouraged time and time again to pick up a pencil and check a calculation or complete the details of an argument. That being said, core ideas are spelled out in detail and major results are proved in full generality. It is my strong belief that core results should not be left as exercises as these are typically too difficult for many students at the beginner level.
- Students deserve to see how their mathematics courses relate to each other; this in turn leads to an understanding of how mathematics evolves as a discipline. Mathematics is at its most powerful when it incorporates knowledge from its various branches, and I find analysis to be particularly rich in this area as it depends so heavily on the algebra, topology, and geometry of the underlying domain space. The study of operators on a Hilbert space may be the most ideal example of such a lush and vibrant area of inquiry.

- The story of analysis is not over, so a text should not attempt to wrap it up in a tidy package. This again is where the introduction of functional analysis comes into play. The functional topics included are a generalization of the basic analysis topics they accompany and these sections are designed to produce more questions than they necessarily answer. This is the beauty of mathematics and any academic discipline. There will always be questions to ask and my point here is to help the students see that new questions are being asked not only about new topics but about old topics as well. When taken as a whole, these culminating sections provide a basic introduction to very abstract material.

As for prerequisites, the text assumes that students have experienced calculus up through sequences and series, which typically comes after differentiation and integration, and that they have had a formal course in proof writing, including basic logic and set theory, proof techniques (induction, proof by contradiction, and proof by contrapositive), basic function theory, cardinality, and some exposure to the nature of quantifiers. There are several instances where basic topics from linear algebra come into play, but these are only present in the culminating sections of each chapter and do not affect the core material of the chapters; these can be easily supplemented to the student who has not had a formal linear algebra course.

To the Instructor

This text is designed for use over the course of two semesters or quarters. I currently teach the majority of the material over two 10-week terms covering roughly Chaps. 1–5 and 6–9 in the two respective sessions. As mentioned earlier, the text is written so that each chapter culminates with the introduction of a topic from functional analysis so that the students can see how the topics evolve from the content of that particular chapter. However, I will be the first to admit that the natural order will be strenuous for most students. To prepare the students for the challenge I walk a smoother path.

I. *First Term*: Chapters 1–5 excluding the last section in each chapter;
II. *Second Term*:
   a. Chapters 6–7 excluding the last section in each chapter;
   b. Sections 1.5, 2.5, 3.4, 6.5, 7.5;
   c. Chapters 8–9;
   d. Sections 4.5 and 5.6 (if time allows).

For a particularly strong group or as an independent reading project, the natural order may be more manageable. And also please keep in mind that the culminating sections in each chapter are only meant for enticement; in fact, they are actually meant to raise just as many questions as they answer. The material in the last sections is not relevant to the core material in subsequent chapters, so doing things a little out of order should cause no confusion. However, be aware that I find the material in Sect. 7.5 to be the driving force behind introducing the Lebesgue integral at this level. I have found that the material can be introduced along a variety of paths, each of which has its own set of advantages; this provides some room for creating a course that suits a variety of needs depending on the level of the students and the time span of the course.

There are several places where I have varied from tradition. Some of these are simply personal preferences, but all have a pedagogical intent. A few of these are detailed below.

- I do not include a formal chapter on topology of the real number system. All of the topics typically outlined in such a chapter appear throughout the text as necessary. I find that incorporating this material into the text at various places allows for a smoother understanding rather than singling out the study of set structures in an isolated chapter. In Chap. 1 there is an introductory section with a discussion of open, closed, and compact sets. This initial discussion centers around open sets; closed sets are then defined as complements of open sets. The idea of a limit point is introduced in the second chapter where we revisit the notion of set structures via sequential criteria. Compact sets are therefore introduced in the first chapter as closed and bounded subsets of the real line and the characterizations in terms of sequential compactness and open cover criteria are given as equivalent notions. This is very atypical, but the closed and bounded criteria is easier to digest at first, though I do feel that this may detract from the importance of the Heine–Borel theorem.
- My basic treatment of most of the constructs of analysis is presented via a sequential approach. On the whole, this provides a unifying theme to the text and places sequences at the center of all our work. This is particularly relevant to our discussion of functional limits where the sequential definition is presented before the more traditional $\varepsilon - \delta$ definition. When discussing integrals, the sequential approach is presented as a secondary characterization.
- The last two chapters of the text provide an introduction to measure theory and Lebesgue integration. This material is not typical of an undergraduate text and these chapters are more dense than the earlier chapters, but I have taken care to present the material in a manner that is appropriate for the beginner.

As a final comment, the exercise sets are designed to be completed in full meaning that many of the problems are referenced later in the text. This does not mean that every student has to do every problem for submission. Rather, I find it helpful to create homework sets by selecting a few problems from a section without worrying too much at the future. Then, if I need a particular problem later in the course, I do not hesitate to assign it with a later homework set. This gives the student the opportunity to revisit early topics and I find repetition key at this point in their development. However, I do make it clear to students that the homework sets represent the minimal amount of preparation for exams and encourage them to investigate all the problems in each section.

To the Student

Be aware that what you are about to undertake represents several hundred years worth of mathematical inquiry and the topics are difficult. I don't say this to deter you, but my point is that you should expect to devote a significant amount of time to mastering the ideas covered here. There are a few keys to doing this. First, read the text carefully. Second, read the text often. Third, reread the text carefully. Mathematics takes time to sink in and allowing yourself ample time is necessary to fully digest the material. When you read, have a paper and pencil handy. You need to be involved with the ideas and sketching out the details of an argument or checking a calculation is the best way to invest yourself in the subject matter. I have devoted much time to writing clear and concise proofs, to developing and motivating ideas before stating definitions and theorems, and before giving proofs; I have spent many hours crafting problem sets that emphasize the key techniques used in this area of mathematics. All of this is to say that the text is designed *for you*. Your instructor has already learned the material and this text is not designed as a guide for him/her. It is designed as a guide *for you*.

Also, while this text is aimed at the beginner, there will inevitably be a moment when you need some topic clarified. In this moment, ask for help. I have strived to convey my thoughts clearly but this doesn't always translate as effectively as an in-person clarification. This is the nature of writing. I have one chance to express a thought in an effective manner, but that may not be the best explanation for your particular point of confusion. So, while I have been meticulous about details, it may be the case that I have inadvertently glossed over a point that causes you strife. In this moment, ask for help.

If you haven't already picked up on the fact that I am a believer that repetition is a key pedagogical technique, let me repeat. I believe in repetition. The only way to learn is to do and not just once. You should commit the definitions and major statements to memory. As a student I did this by writing out the statements and reading them several times a day. This way it doesn't feel like memorization but you have to find what works for you. For problems, you should work as many exercises as possible. In each section there are multiple problems that get at the same point or technique, and this variety is intended to be repetitive so as to emphasize the importance of a concept or technique.

You should also be aware of the fact that this is a writing course. Your homework sets should look more like short papers than like a calculus assignment. The proofs given in the text then have two purposes. They are included so as to demonstrate the validity of claims made. But they are also present to act as examples of good mathematical writing as you attempt to write your own proofs. Below are a few things to keep in mind.

- The words "show," "verify," "explain," "discuss," etc. all have the same meaning in this context. That is, they are all asking you to prove a specified statement. This may sometimes be just a sentence, but a proof nonetheless.

- Developing a proof is actually a two-step process. Don't expect to read a problem and immediately sit down and write out a proof. First, you need to sketch out the ideas and determine a logical path that takes you from hypothesis to conclusion. I recommend a notebook, almost like a journal, to keep your sketch work in. The second phase of this is then the writing process. You want to translate the ideas from the sketching phase into a clear argument and this presents its own set of challenges. This technique is demonstrated throughout the text as I have included much of the "scratch work" necessary for building a proof before presenting a formal argument. This has added substantial length to the text, but is present to aid you in understanding how mathematicians work. This is analogous to how writers, artists, musicians, film makers, and other creative individuals construct masterpieces.
- Your proofs should consist only of complete sentences with proper punctuation and grammar. And every problem, even if it is simply a computational exercise, should be written in this fashion (there are only a handful of computational exercises in the text). For some problems, e.g. those asking for examples with a specified property, a short explanation is all that is necessary, but this should always be included. As a simple rule of thumb, always supply proof. If the directions of a problem are unclear, ask your instructor.
- Do not abuse symbology, i.e. your proof should have more words than symbols. In fact, you should only use symbols when absolutely necessary. This is not a strict rule and you should consult with your instructor for their particular preference here. For example, I find it perfectly acceptable to write "$x \in A$" or "x in A," but I would be put off by the use of "$\forall \varepsilon > 0, \exists \delta > 0$" as a substitute for the phrase "for every $\varepsilon > 0$, there is a $\delta > 0$" in a formal proof.
- Read a section clearly and carefully before beginning a problem set. Often there are hints in the section for problems. In particular, proof techniques in a section are likely relevant to the problems in that section. Also, mathematics builds on itself, as does this text. Referencing previously completed homework exercises will likely occur and this is expected. To do this effectively, you have to remember things you have previously encountered. In addition, the problems are designed to be completed using *only* ideas from the current section or previous sections. This is for your own development even though there may be a later theorem which makes the argument extremely simple.
- The problems are not easy (though they vary in difficulty level) and you want to allow yourself plenty of time.
- Reread your work before submitting it for assessment. This will help you catch typos/errors and will also help you to solidify your arguments. If upon rereading you do not understand your argument, then there is a high probability that your instructor will not understand your argument. This is an encouragement to work early and often.

Finally, let me take a moment to indicate what makes an analysis course slightly different from your previous mathematics courses. In most areas of study, there are certain obstacles which make that particular discipline challenging. From my perspective, analysis presents two such obstacles. First, you have been trained to understand equality exceptionally well. However, analysis is about inequality. One reason for this is that it is often hard to tell when two quantities which are allowed to vary are equal, and the task is made easier by contenting ourselves with understanding when one quantity is less than another. For example, we will see in the first chapter that a real number a is equal to 0 if and only if $|a| < \varepsilon$ for every $\varepsilon > 0$. This should make intuitive sense as 0 is the unique real number which "separates" the positive and negative real numbers, and the statement above demonstrates that inequality can often tell us something about equality. Also, many of the constructs of analysis will deal exclusively with inequality, sequence convergence for example, as these ideas are really about being close to a fixed value rather than being equal to that fixed value.

The second obstacle will appear in the nature of our proofs. As we will deal primarily with functions, we will most often be working with at least two quantities simultaneously, a domain point and a range point. The goal then will be to *force* some outcome in the range by *controlling* the corresponding domain quantity. This will require us to make very explicit choices with respect to domain values and therein lies the challenge. The nature of such arguments may seem circular at first and the key is practice.

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With all of that, I hope that you, the reader, will find this text as engaging to read as it has been for me to write. And while many folks have contributed their efforts to ensure that the text is error free, I take complete responsibility for any remaining errors, oversights, or shortcomings.

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