

Chapter 2

Inference on Mean Residual Life-Overview

Statistical inference based on the remaining lifetimes would be intuitively more appealing than the popular hazard function defined as the risk of immediate failure, whose interpretation could be sometimes difficult to be grasped. For example, when an efficacy of a new drug is concerned, it would be more straightforward to explain it as “if one with the similar genetic and environmental background like you takes this drug, it is expected, on average, that it will prolong your remaining life years by 10 years” rather than simply saying “the average hazard reduction in the treatment group will be 25%.” Common summary measures for the remaining lifetimes have been the mean and median residual lifetimes. This chapter presents a brief overview of statistical inference on the mean residual life, because the main focus of this book will be on the quantile residual life function.

We first define the mean residual life function and discuss the asymptotic properties of one-sample nonparametric estimator. Various regression models are then reviewed such as the proportional mean residual life model (Oakes and Dasu, 1990), the expectancy regression model (Chen and Cheng, 2006; Chen, 2007), the proportional scaled mean residual life model (Liu and Ghosh,

2008), and a general family of transformation models under the additive mean residual life structure (Sun and Zhang, 2009).

2.1 Mean Residual Life Function

The usage of mean residual life dates back to the third century A.D. (Deevey, 1947; Chiang, 1968; Guess and Proschan, 1985). Assuming that T is a continuous random variable with survival function $S(t)$, the mean residual life function is defined as the expected value of the remaining lifetimes after a fixed time point t , i.e.

$$\begin{aligned} e(t) &= E(T - t | T > t) \\ &= \frac{\int_t^\infty S(v) dv}{S(t)} \\ &= \frac{\int_t^\infty v f(v) dv}{S(t)} - t, \end{aligned} \tag{2.1}$$

which exists for all t if and only if $e(0) = E(T)$ is finite (Oakes and Dasu, 2003). For example, for an exponential distribution with the probability density function $f(t) = \mu \exp(-\mu t)$, the mean residual life function is given as the mean of the distribution, i.e. $e(t) = 1/\mu$.

As succinctly summarized in McLain and Ghosh (2011), Hall and Wellner (1981, Proposition 2) provided a characterization theorem that gives the necessary and sufficient conditions for existence of the mean residual life function of a continuous nonnegative random variable:

- (a) $e(t) \geq 0$ for all $t \geq 0$, and continuous,
- (b) $e(t) + t$ is nondecreasing in t ,
- (c) if there exists a ω such that $e(\omega) = 0$, then $e(t) = 0$ for all $t \geq \omega$; otherwise, $\int_0^\infty e^{-1}(v) dv = \infty$.

The mean residual life function can be inverted to the survival function for more tractability by using the Inversion Formula (Gumbel, 1924; Cox, 1962, p. 128)

$$S(t) = \frac{e(0)}{e(t)} \exp \left\{ - \int_0^t \frac{dv}{e(v)} \right\}. \quad (2.2)$$

The variance formula for the mean residual life function, attributed by Hall and Wellner (1981) to Pyke (1965), can be derived as

$$\sigma(t) \equiv \text{Var}(T - t | T > t) = \frac{\int_t^\infty e^2(v) f(v) dv}{S(t)}.$$

This formula also shows that $\sigma(t)$ is finite for all t if and only if $\sigma(0) = \text{Var}(T)$ is finite. Hall and Wellner (1981) considered a family of the mean residual life function linear in t , i.e. $e(t) = at + b$ (Hall–Wellner family), which gives the survival function

$$S(t) = \left(\frac{b}{at + b} \right)_+^{1+1/a},$$

where the subscript $+$ implies that only the positive part of the expression in the parentheses is taken. Special cases of this family are a Pareto distribution, an exponential distribution, and a beta distribution for $a > 0$, $a = 0$, and $-1 < a < 0$, respectively (Oakes and Dasu, 2003).

2.2 One- and Two-sample Cases

For a random sample T_1, T_2, \dots, T_n from a distribution with the cumulative distribution function $F(t)$, and hence the survival function $S(t) = 1 - F(t)$, without censoring the natural one-sample nonparametric estimator for the mean residual life function at age t would be the sample mean of the residual lifetimes of the observations that exceed t , i.e.

$$\hat{e}(t) = \frac{\sum_{i=1}^n (T_i - t) I(T_i > t)}{\sum_{i=1}^n I(T_i > t)},$$

where $I(T_i > t) = 1$ if $T_i > t$ and 0 otherwise, so the denominator is the total number of observations that exceed t . Yang (1978) and Csörgö, Csörgö and Horváth (1986) showed that the process

$$Z_n(t) = \sqrt{n}[\hat{e}(t) - e(t)]$$

converges to a Gaussian process $Z(t)$ with 0 mean and covariance function

$$\text{Cov}[Z(s), Z(t)] = \frac{\sigma(\max(s, t))}{S(\min(s, t))}.$$

Yang (1978) noted that for $t > 0$, $\hat{e}(t)$ become slightly biased as $E[\hat{e}(t)] = e(t)[1 - F^n(t)]$, which, however, converges to $e(t)$ asymptotically (see also Gertsbakh and Kordonskiy, 1969).

Hall and Wellner (1981) and Bhattacharjee (1982) studied thoroughly on the mean residual life function, deriving necessary and sufficient conditions for an arbitrary function to be a mean residual life function. As mentioned above, Hall and Wellner (1981) also characterized a family of the mean residual life functions that are linear in age t . Guess and Proschan (1985) extensively reviewed both theory and application aspects of the mean residual life function. As they stated, for any distribution with a finite mean, the mean residual life function completely determines the distribution via the Inversion Formula as the probability density function, the moment generating function, or the characteristic function does. Bryson and Siddiqui (1969) defined various criteria for aging such as increasing failure rate (IFR) class, new better than used (NBU) class, decreasing mean residual life (DMRL) class, and new better than used in expectation (NBUE) class. Hollander and Proschan (1975) derived statistics to test the null hypothesis that the underlying failure distribution is exponential against the alternative hypothesis that it has a monotone mean residual life function. Chen *et al.* (1983) extended the Hollander–Proschan tests to censored survival data. Nair and Nair (1989) introduced the concept of the bivariate mean residual life function. Other related work in reliability theory also includes Watson and Wells (1961), Mute (1977), Bartholomew (1973), and Morrison and Schmittlein (1980). Berger, Boos, and Guess (1988) proposed a nonparametric test statistic to compare the mean residual life functions based on two independent samples.

2.3 Regression on Mean Residual Life

Analogous to Cox's proportional hazards model (Cox, 1972), Oakes and Dasu (1990) proposed the proportional mean residual life model

$$e_1(t; \theta) = \theta e_0(t), \quad (2.3)$$

where $\theta = \exp(\beta'z)$, β being a vector of regression coefficients and z a vector of covariates. Here $e_0(t)$ and $e_1(t; \theta)$ are the mean residual life function for the baseline and one adjusted for z , respectively. It seems that early literature on regression modeling for the mean residual life function has been revolving around the proportional mean residual life model. Therefore, we first review briefly existing statistical inference procedures under the proportional mean residual life model.

Applying the Inversion Formula in (2.2) twice under the model (2.3), we have

$$\begin{aligned} S_1(t; \theta) &= \frac{e_1(0; \theta)}{e_1(t; \theta)} \exp \left\{ - \int^t \frac{dv}{e_1(v; \theta)} \right\} \\ &= \frac{e_0(0)}{e_0(t)} \exp \left\{ - \frac{1}{\theta} \int^t \frac{dv}{e_0(v)} \right\} \\ &= S_0(t) \left(\frac{\int_t^\infty S_0(v) dv}{\mu_0} \right)^{1/\theta-1}, \end{aligned} \quad (2.4)$$

where $\mu_0 = e_0(0)$ is the mean of the distribution at the origin.

Taking the first derivative of the middle line in (2.4) gives the probability density function

$$f_1(t; \theta) = -dS_1(t; \theta)/dt = \frac{\mu_0}{e_0^2(t)} \left\{ e_0'(t) + \frac{1}{\theta} \right\} \exp \left\{ - \frac{1}{\theta} \int_0^t \frac{dv}{e_0(v)} \right\}. \quad (2.5)$$

Once the baseline survival function is specified in (2.4) or the baseline mean residual life function is specified in (2.5), the maximum likelihood estimation method under the proportional mean residual life model can be readily applied to estimate the parameters

for the baseline distribution and the proportionality parameter θ simultaneously, for both uncensored and censored survival data. Specifically, denoting C_i ($i = 1, \dots, n$) to be the random censoring time, $X_i = \min(T_i, C_i)$, and $\delta_i = I(T_i < C_i)$, the log-likelihood function will be given as

$$l(\theta; x_i) = \sum_{i=1}^n [\delta_i \log\{f_1(x_i; \theta)\} + (1 - \delta_i) \log\{S_1(x_i; \theta)\}].$$

For the uncensored case where the baseline distribution is completely known, Dasu (1991) and Oakes and Dasu (2003) showed that the score function per observation takes the form of

$$U(\theta) = \frac{d \log f(x; \theta)}{d\theta} = \frac{1}{\theta^2} \int_0^x \frac{dv}{e_0(v)} - \frac{1}{\theta\{1 + \theta e'_0(x)\}},$$

and hence the Fisher information is given by

$$I(\theta) = \frac{1}{\theta^2} \int_0^\infty \frac{S_1^2(v; \theta)}{e_1^2(v; \theta) f_1(v; \theta)} dv.$$

Therefore from the large sample theory, the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$, where $\hat{\theta}$ is the maximum likelihood estimator of θ , follows a normal distribution with mean 0 and variance $I^{-1}(\theta)$.

For the semiparametric inference under the proportional mean residual life model, Oakes and Dasu (2003) proposed an estimator for θ for a binary covariate case, but its asymptotic behavior was not proved. Maguluri and Zhang (1994) modified the partial likelihood-based inference (Cox, 1972, 1975) by using the fact that for any stationary renewal process (Karlin and Taylor, 1975) the mean residual life function can be expressed as the reciprocal of the hazard function of the residual life distribution, but mainly for the uncensored case. Chen and Cheng (2005) and Chen *et al.* (2005) employed the counting process theory to develop a new inference procedure for censored survival data under the proportional mean residual life model. Their approach mimics the Cox partial score function, resulting in a closed form of the baseline mean residual life estimator and hence a regression coefficient estimator that resembles the maximum partial likelihood estimator.

Zhao and Qin (2006) applied an empirical likelihood ratio to construct the confidence regions for the regression parameters. Chan *et al.* (2012) considered inference on the proportional mean residual life model for right-censored and length-biased data. Chen and Wang (2013) proposed an estimation procedure via the augmented inverse probability weighting and kernel smoothing techniques under the proportional mean residual life model with missing at random.

For other regression models, Chen and Cheng (2006) and Chen (2007) proposed expectancy regression model where the mean residual life functions are additive, i.e.

$$e(t; z) = e_0(t) + \beta'z.$$

However, estimation under this model is subject to the constraint that $e(t; z) \geq 0$ for all z and $t \geq 0$, which is difficult to be satisfied, as noted in McLain and Ghosh (2011). Zhang *et al.* (2010) developed goodness-of-fit tests for this model. Liu and Ghosh (2008) proposed a proportional scaled mean residual life model that satisfies the first two characterization conditions (a) and (b),

$$e(t; z) = e_0\{t \exp(-\beta'z)\} \exp(\beta'z).$$

This model can be shown to be equivalent to the accelerated failure time model, but interpretation of β is not straightforward in terms of the mean residual life function. Sun and Zhang (2009) proposed the general family of semiparametric transformation models,

$$e(t; z) = g\{e_0(t) + \beta'z\},$$

for a transformation function $g(\cdot)$, which includes the proportional and additive mean residual life models as special cases, but they noted that the characterization condition (b) is still difficult to be met in general. Sun *et al.* (2011) extended this model to the case with time-dependent covariates under right censoring. McLain and Ghosh (2011) interestingly proposed two semiparametric estimators to estimate directly the mean residual life function adjusted for covariates. By using existing smoothing techniques, they estimated the adjusted survival functions

first and plug them into the middle line in (2.1) to ensure that their estimators produce the mean residual life estimates that satisfy all of the characterization conditions (a)–(c). Their methods were applied to both proportional and additive mean residual life models.

In this chapter, a brief overview of statistical inference on the mean residual life function was presented. Major advantages of using the mean residual life function would be that it can be uniquely defined as long as the mean and variance of the distribution at the origin are defined, and that the proper statistical methods are well established in the literature and are expected to work nicely especially when the distribution of interest is symmetric. However, as first mentioned by Schmittlein and Morrison (1981), event-time data can be easily skewed, which might introduce biases in inference based on the center of the distribution. Parallel to the development of statistical methods in the topic of the mean residual life, there also has been vigorous research recently on the median or quantile residual life function. Starting from the next chapter, this book will be devoted to review recent developments in statistical inference on the quantile residual life function and to introduce new approaches where it is appropriate.

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Jeong, J.-H.

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