

# Explicit Bounds Concerning Non-trivial Zeros of the Riemann Zeta Function

Mehdi Hassani

*Dedicated to Professor Hari M. Srivastava*

**Abstract** In this paper, we get explicit upper and lower bounds for  $\gamma_n$ , where  $0 < \gamma_1 < \gamma_2 < \gamma_3 < \dots$  are consecutive ordinates of non-trivial zeros  $\rho = \beta + i\gamma$  of the Riemann zeta function. Meanwhile, we obtain the asymptotic relation  $\gamma_n \log^2 n - 2\pi n \log n \sim 2\pi n \log \log n$  as  $n \rightarrow \infty$ .

## 1 Introduction

The Riemann zeta function is defined for  $\text{Re}(s) > 1$  by  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$  and extended by analytic continuation to the complex plan with a simple pole at  $s = 1$  with residues 1. It is known [3, 7] that

$$N(T) := \sum_{\substack{0 < \gamma \leq T \\ \zeta(\beta + i\gamma) = 0}} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \quad (1)$$

As a consequence of (1) we get

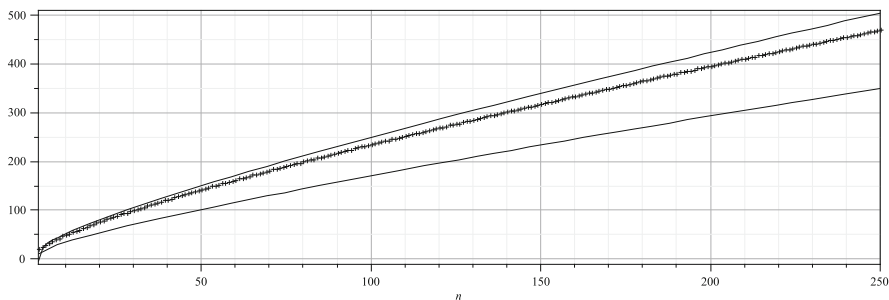
$$\sum_{\substack{0 < \gamma \leq T \\ \zeta(\beta + i\gamma) = 0}} \frac{1}{\gamma} = \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T + E(T),$$

with  $E(T) = O(1)$ . Recently, we obtained an explicit form of this approximate formula by proving that  $\frac{3}{50} < E(T) < \frac{109}{250}$  for any  $T \geq \gamma_1$  (see [1, Theorem 1]), where

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M. Hassani (✉)

Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran  
e-mail: [mehdi.hassani@znu.ac.ir](mailto:mehdi.hassani@znu.ac.ir)



**Fig. 1** Graph of the point set  $(n, \gamma_n)$  for  $2 \leq n \leq 250$  and functions  $\frac{2\pi n}{\log n} \left(1 + a \frac{\log \log n}{\log n}\right)$  with  $a = 3/4$  and  $a = 5/2$

$$\gamma_1 = \min\{\gamma > 0 : \zeta(\beta + i\gamma) = 0\} \cong 14.134725141734693790457251983562.$$

More generally, we set  $0 < \gamma_1 < \gamma_2 < \gamma_3 < \dots$  to be consecutive ordinates of the imaginary parts of non-trivial zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$ . Another consequence of (1) is

$$\gamma_n \sim \frac{2\pi n}{\log n}, \quad \text{as } n \rightarrow \infty.$$

Our intention in writing this note is to obtain explicit forms of this approximate formula. More precisely, we show the following.

**Theorem 1.1.** *For any integer  $n \geq 5$  we have*

$$\frac{2\pi n}{\log n} \left(1 + \frac{3}{4} \frac{\log \log n}{\log n}\right) \leq \gamma_n \leq \frac{2\pi n}{\log n} \left(1 + \frac{5}{2} \frac{\log \log n}{\log n}\right). \quad (2)$$

Figure 1 shows graph of the point set  $(n, \gamma_n)$  for  $2 \leq n \leq 250$ , and lower and upper bounds appeared in (2). We note that the left-hand side of (2) is valid for  $2 \leq n \leq 4$ , too.

One may obtain better bounds for  $\gamma_n$  by using numerical information, which we obtain during proof of Theorem 1.1. More precisely, by considering Tables 1 and 2, we have the following.

**Theorem 1.2.** *Assume that we choose pairs  $\lambda$  and  $n_\lambda$  from Table 1, and also we choose pairs  $\eta$  and  $n_\eta$  from Table 2. Then, we have*

$$\frac{2\pi n}{\log n} \left(1 + \frac{\lambda}{2\pi} \frac{\log \log n}{\log n}\right) \leq \gamma_n \quad \text{and} \quad \gamma_n \leq \frac{2\pi n}{\log n} \left(1 + \frac{\eta}{2\pi} \frac{\log \log n}{\log n}\right),$$

respectively, for  $n \geq n_\lambda$  and for  $n \geq n_\eta$ .

**Table 1** Some values of  $\lambda$  and  $n_\lambda$  for which the inequality (11) is valid for  $n \geq n_\lambda$ 

$\lambda$	$n_\lambda \cong$	$\lambda$	$n_\lambda \cong$
$-2\pi$	3.9	$3\pi/2$	4984.5
$-\pi$	5.1	$5\pi/3$	392062.1
$-1$	7.8	$7\pi/4$	138610176.5
0	10.7	$9\pi/5$	2499273431483.9
1	16.7	$11\pi/6$	109511051064367600190250.3
$\pi$	97.1	5.795	876581819433015771165641491644046075.5

**Table 2** Some values of  $\eta$  and  $n_\eta$  for which the inequality (12) is valid for  $n \geq n_\eta$ 

$\eta$	$n_\eta \cong$	$\eta$	$n_\eta \cong$
$20\pi$	8.8	$6\pi$	1197.1
$10\pi$	11.7	$5\pi$	26245.8
$8\pi$	64.3	$4\pi$	80727920.5
$7\pi$	217.7	$3\pi$	74219923532062069835922351534787.7

On the other hand, we mention that the constants  $\frac{3}{4}$  and  $\frac{5}{2}$  in Theorem 1.1, as more as, the constants  $\frac{\lambda}{2\pi}$  and  $\frac{\eta}{2\pi}$  in Theorem 1.2, are not optimal. More precisely, if we let

$$R_n := \frac{\frac{\gamma_n}{\frac{2\pi n}{\log n}} - 1}{\frac{\log \log n}{\log n}}, \quad (3)$$

then Theorem 1.1 yields that  $\frac{3}{4} \leq R_n \leq \frac{5}{2}$  for any integer  $n \geq 5$ . But, the proof of above theorems includes an argument in its heart, which implies that  $\lim_{n \rightarrow \infty} R_n = 1$ . Indeed, we show the following.

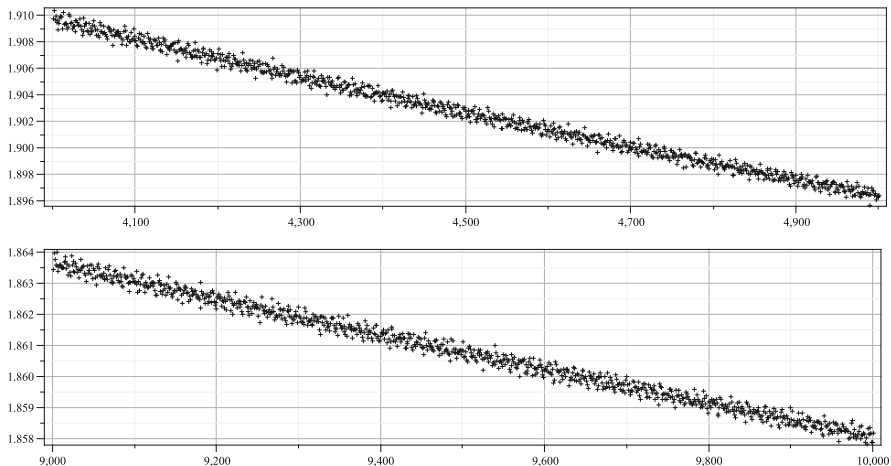
**Theorem 1.3.** *Let*

$$\Lambda_n = \frac{\gamma_n \log^2 n - 2\pi n \log n}{n \log \log n}. \quad (4)$$

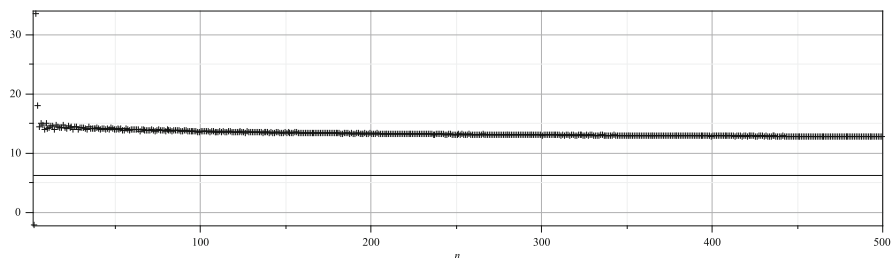
*Then, we have  $\lim_{n \rightarrow \infty} \Lambda_n = 2\pi$ .*

**Corollary 1.1.** *For any real  $\varepsilon \in (0, 1)$ , there exists positive integer  $n_\varepsilon$  such that for  $n \geq n_\varepsilon$  we have*

$$\frac{2\pi n}{\log n} \left( 1 + (1 - \varepsilon) \frac{\log \log n}{\log n} \right) \leq \gamma_n \leq \frac{2\pi n}{\log n} \left( 1 + (1 + \varepsilon) \frac{\log \log n}{\log n} \right).$$



**Fig. 2** Graph of the pointset  $(n, R_n)$  for  $4000 \leq n \leq 5000$  and  $9000 \leq n \leq 10000$ , where  $R_n$  is defined by (3)



**Fig. 3** Graph of the point set  $(n, A_n)$  for  $2 \leq n \leq 500$ , where  $A_n$  is defined by (4), and horizontal line at height  $2\pi$

*Remark 1.1.* Figure 2 pictures some values of  $R_n$  for several values of  $n$ . As our computations show, one may have the inequality  $R_n > 1$  for  $n \geq 3$ . This means that one may have the validity of the left-hand side of (2) with 1 instead of  $\frac{3}{4}$ , for any integer  $n \geq 3$ . This conjecture is pictured in Fig. 3 in another point of view, where we plot values of  $A_n$  for  $2 \leq n \leq 500$  and horizontal line at height  $2\pi$ . Also, it seems that there exists a positive integer  $m \cong 250$  such that  $R_{n+m} \geq R_n$  for any integer  $n \geq 3$ .

*Remark 1.2.* The truth of Corollary 1.1 asserts that as  $n \rightarrow \infty$  we have

$$\gamma_n = \frac{2\pi n}{\log n} \left( 1 + (1 + o(1)) \frac{\log \log n}{\log n} \right).$$

One may ask for such asymptotic expansions with more precise terms.

In the next two sections, we prove our results. To generate figures which appeared on present paper, as more as, during proofs, we will do several computations running over the numbers  $\gamma_n$ , all of which have been done by using Maple software and are based on the tables of zeros of the Riemann zeta function due to Odlyzko [4].

## 2 Lambert $W$ Function, the Key of Proof

The main idea to get explicit results similar to (2) is applying an explicit version of the Riemann–von Mangoldt formula (1). This can be found in the following result due to Rosser, which is Theorem 19 of [6].

**Proposition 2.1.** *For any  $T \geq 2$  we have  $|N(T) - F(T)| \leq R(T)$ , with*

$$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} \quad \text{and} \quad R(T) = \frac{137}{1000} \log T + \frac{443}{1000} \log \log T + \frac{397}{250}.$$

For our purpose, we need to modify the truth of above proposition as follows. For the whole text, we set

$$\ell = \frac{14}{25} \quad \text{and} \quad u = \frac{11}{50}.$$

**Lemma 2.1.** *Let*

$$L(T) = \frac{1}{2\pi} T \log T - \ell T \quad \text{and} \quad U(T) = \frac{1}{2\pi} T \log T - uT. \quad (5)$$

*Then, for  $T \geq \gamma_1 - 10^{-5}$  we have*

$$L(T) \leq N(T) \leq U(T). \quad (6)$$

*Moreover,  $U(T)$  and  $L(T)$  are strictly increasing for  $T \geq e^{2\pi u-1} \cong 1.465653$  and  $T \geq e^{2\pi \ell-1} \cong 12.411008$ , respectively.*

*Proof.* We consider Proposition 2.1 to write  $F(T) - R(T) \leq N(T) \leq F(T) + R(T)$  for  $T \geq 2$ . On the other hand, for  $T \geq \gamma_1 - 10^{-5}$  we have  $F(T) + R(T) \leq U(T)$  and  $L(T) \leq F(T) - R(T)$ . This proves both sides of (6). Monotonicity of the functions  $U(T)$  and  $L(T)$  is straightforward.  $\square$

The following lemma brings lower and upper bounds for  $N(T)$  to bounds for  $\gamma_n$  in terms of inverses of mentioned bounds for  $N(T)$ .

**Lemma 2.2.** *Assume that  $L(T)$  and  $U(T)$  are defined as in (5), and denote by  $L^{-1}(T)$  and  $U^{-1}(T)$  their inverses, respectively. Then, for any integer  $n \geq 1$  we have*

$$U^{-1}(n) \leq \gamma_n \leq L^{-1}(n). \quad (7)$$

*Proof.* Assume that  $n \geq 1$  is any arbitrary integer and  $\delta \in (0, 1)$  is any arbitrary real. We have  $N(\gamma_n) = n$ . Thus, we get  $N(\gamma_n + \delta) \geq n$  and  $N(\gamma_n - \delta) \leq n - 1$ . Therefore, we obtain

$$1 + N(\gamma_n - \delta) \leq n \leq N(\gamma_n + \delta). \quad (8)$$

Right-hand sides of (6) and (8) give  $\gamma_n + \delta \geq U^{-1}(n)$ . Thus, we get  $\gamma_n \geq U^{-1}(n)$ . Similarly, left-hand sides of (6) and (8) give  $L(\gamma_n - \delta) \leq N(\gamma_n - \delta) \leq n - 1 < n$ . So, we get  $\gamma_n - \delta \leq L^{-1}(n)$ , and this implies validity of  $\gamma_n \leq L^{-1}(n)$ .  $\square$

In order to use inequalities (7), we need formulas for the inverses of the functions  $L(T)$  and  $U(T)$ . This may be done in terms of the Lambert  $W$  function  $W(x)$ , which is defined by the relation  $W(x)e^{W(x)} = x$  for  $x \in [-e^{-1}, +\infty)$ . The Lambert  $W$  function has the asymptotic expansion  $W(x) = \log x + O(\log \log x)$  as  $x \rightarrow \infty$ , (see [5, p. 111]). The following lemma summarizes what we need about the inverses of the functions  $L(T)$  and  $U(T)$ .

**Lemma 2.3.** *Assume that  $a$  and  $b$  are some positive real numbers, and let*

$$f(T) = \frac{1}{a}T \log T - bT.$$

*We denote the inverse function of  $f$  by  $f^{-1}$ . Then, for  $T \geq e^{ab-1}$  the function  $f$  is strictly increasing and we have*

$$f^{-1}(T) = \frac{aT}{W(ae^{-ab}T)}. \quad (9)$$

*In particular, as  $T \rightarrow +\infty$ , we obtain  $f^{-1}(T) \sim aT / \log T$ .*

*Proof.* Assume that  $T > 0$ . Then, by definition of the Lambert  $W$  function, we imply that  $f(e^{W(ae^{-ab}T)+ab}) = T$  or equivalently  $f^{-1}(T) = e^{W(ae^{-ab}T)+ab}$ . Definition of the Lambert  $W$  function also gives that  $aT = W(ae^{-ab}T)e^{W(ae^{-ab}T)+ab}$ . Thus, we obtain (9). The asymptotic relation comes from  $W(ae^{-ab}T) \sim \log T$ , which is valid as  $T \rightarrow +\infty$ .  $\square$

Finally, to get our desired explicit results, we need some explicit bounds for the Lambert  $W$  function. The following proposition, which is Theorem 2.8 of [2], offers such sharp bounds.

**Proposition 2.2.** *Assume that  $\alpha > 0$  is real, and let*

$$\omega_\alpha(x) := \log x - \log \log x + \alpha \frac{\log \log x}{\log x}.$$

Then, for every  $x \geq e$  we have

$$\omega_{\frac{1}{2}}(x) \leq W(x) \leq \omega_{\frac{e}{e-1}}(x), \quad (10)$$

with equality only for  $x = e$ .

### 3 Proof of Results

#### 3.1 Proof of the Left-Hand Side of (2)

We let  $c_u = 2\pi e^{-2\pi u}$ . By applying the validity of Lemma 2.3, considering the left-hand side of (7), and considering the right-hand side of (10), we obtain

$$\gamma_n \geq U^{-1}(n) = \frac{2\pi n}{W(c_u n)} \geq \frac{2\pi n}{\omega_{\frac{e}{e-1}}(c_u n)},$$

for  $c_u n \geq e$  or equivalently for  $n \geq \frac{e}{c_u} \approx 1.7$ . Moreover, by computation, for any integer  $n \geq 1$  we get

$$\gamma_n \geq \frac{2\pi n}{\omega_{\frac{e}{e-1}}(c_u n)} := g(n),$$

say. We let

$$h(n) := \frac{g(n) - \frac{2\pi n}{\log n}}{\frac{n \log \log n}{\log^2 n}}.$$

Now, we note that the function  $h : (e, +\infty) \rightarrow (-\infty, 2\pi)$  defined by  $h(n)$  is continuous and strictly increasing. Moreover, we have

$$\lim_{n \rightarrow e^+} h(n) = -\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} h(n) = 2\pi.$$

Therefore, for any real  $\lambda \in (-\infty, 2\pi)$ , there exists unique  $n_\lambda \in (e, +\infty)$  such that  $h(n) \geq \lambda$  for  $n \geq n_\lambda$  with equality only for  $n = n_\lambda$ . Hence, for  $n \geq n_\lambda$  we obtain

$$\gamma_n \geq \frac{2\pi n}{\log n} + \lambda \frac{n \log \log n}{\log^2 n}. \quad (11)$$

In Table 1 we list some values of  $\lambda$  and related values of  $n_\lambda$ . We use information of this table choosing  $\lambda = \frac{3\pi}{2}$ , from which we obtain the inequality

$$\gamma_n \geq \frac{2\pi n}{\log n} + \frac{3\pi n \log \log n}{2 \log^2 n},$$

for  $n \geq 4985$ . By computation, we confirm validity of it for  $2 \leq n \leq 4984$ , too. This completes the proof of left-hand side of (2).

### 3.2 Proof of the Right-Hand Side of (2)

Let  $c_\ell = 2\pi e^{-2\pi\ell}$ . We use the validity of Lemma 2.3, the right-hand side of (7), and the left-hand side of (10) to get

$$\gamma_n \leq L^{-1}(n) = \frac{2\pi n}{W(c_\ell n)} \leq \frac{2\pi n}{\omega_{1/2}(c_\ell n)},$$

for  $c_\ell n \geq e$  or equivalently for  $n \geq \frac{e}{c_\ell} \approx 14.6$ . As more as, by computation, for any integer  $n \geq 8$ , we obtain

$$\gamma_n \leq \frac{2\pi n}{\omega_{1/2}(c_\ell n)} := v(n),$$

say. We set

$$z(n) := \frac{v(n) - \frac{2\pi n}{\log n}}{\frac{n \log \log n}{\log^2 n}}.$$

Also, we let

$$y_1 := \lim_{n \rightarrow 1/c_\ell +} \frac{1}{z(n)} = \frac{\log(-\log c_\ell)}{2\pi \log c_\ell} \approx -0.049167.$$

We note that the function  $y : (1/c_\ell, +\infty) \rightarrow (y_1, 1/(2\pi))$  defined by  $y(n) = 1/z(n)$  is continuous and strictly increasing. Thus, there exists unique  $n_0 > 1/c_\ell$  such that  $y(n_0) = 0$ . By computation, we observe that  $n_0 \approx 7.745051$ . Now, we note that the function  $z : (n_0, +\infty) \rightarrow (2\pi, +\infty)$  defined by  $z(n)$  is continuous and strictly decreasing. Moreover, we have

$$\lim_{n \rightarrow n_0^+} z(n) = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} z(n) = 2\pi.$$

Therefore, for any  $\eta \in (2\pi, +\infty)$ , there exists unique  $n_\eta \in (n_0, +\infty)$  such that  $z(n) \leq \eta$  for  $n \geq n_\eta$  with equality only for  $n = n_\eta$ , and consequently, for  $n \geq n_\eta$  we get



$$\gamma_n \leq \frac{2\pi n}{\log n} + \eta \frac{n \log \log n}{\log^2 n}. \quad (12)$$

Table 2 includes some values of  $\eta$  and related values of  $n_\eta$ . Considering our computational tools, we choose  $\eta = 5\pi$  from this table, from which for  $n \geq 26246$  we obtain the inequality

$$\gamma_n \leq \frac{2\pi n}{\log n} + \frac{5\pi n \log \log n}{\log^2 n}.$$

By computation, we confirm validity of it for  $5 \leq n \leq 26245$ , too. This completes the proof of right-hand side of (2).

### 3.3 Proof of Theorem 1.3

We note that inequalities (11) and (12) imply

$$\liminf_{n \rightarrow \infty} \Lambda_n \geq 2\pi \quad \text{and} \quad \limsup_{n \rightarrow \infty} \Lambda_n \leq 2\pi,$$

respectively. This gives assertion of Theorem 1.3.

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Analytic Number Theory, Approximation Theory, and  
Special Functions

In Honor of Hari M. Srivastava

Milovanović, G.V.; Rassias, M.T. (Eds.)

2014, XI, 880 p. 24 illus. in color., Hardcover

ISBN: 978-1-4939-0257-6