Chapter 2
The LMI Approach in an Infinite-Dimensional Setting

Extended via the Lyapunov–Krasovskii method to linear time-delay systems (LTDS), the LMI approach has long been recognized as a powerful analysis tool of such systems. In the present chapter, this approach is further extended to the stability analysis of LTDSs evolving in a Hilbert space. The operator acting on the delayed state is supposed to be bounded. The system delay is unknown and time-varying, with an a priori given upper bound on the delay. Following [50], sufficient exponential stability conditions are derived in the form of linear operator inequalities, where the decision variables are operators in the Hilbert space. When applied to a heat equation and to a wave equation, these conditions are reduced to standard LMIs.

2.1 Historical Remarks and Notation

Time delay naturally appears in many control systems, and it is frequently a source of instability [59]. In the case of distributed parameter systems (DPS)s, even arbitrarily small delays in the feedback may destabilize the system [37, 79, 84, 101]. The stability issue of systems with delay is, therefore, of theoretical and practical value.

During the past decade, a considerable amount of attention has been paid to the stability of ODEs with uncertain constant or time-varying delays (see, e.g., [58, 73, 86, 103]). Special forms of Lyapunov–Krasovskii functionals have been used to derive simple finite-dimensional conditions in terms of LMIs [23]. These conditions are either delay-independent or delay-dependent.

The stability analysis of PDEs with delay is essentially more complicated. There are only a few works on Lyapunov-based technique for PDEs with delay. The second Lyapunov method was extended to abstract nonlinear time-delay systems in the Banach spaces in [126]. Stability and instability conditions for delay wave equations were established in [85]. Stability conditions and exponential bounds were derived

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in [127,128] for some scalar heat equations and wave equations with constant delays and with Dirichlet boundary conditions. In [50], the exponential stability of general DPSs was studied within the framework of LTDSs evolving in a Hilbert space.

It is the latter framework that is adopted in the present chapter. Provided the system delay is unknown and time-varying, sufficient *delay-dependent* exponential stability conditions are derived in the form of LOIs, where the decision variables are operators in the Hilbert space. Although available methods for solving LOIs are confined to finite-dimensional approximations [65], however, if exemplified with test heat and wave equations, the derived conditions prove to be reduced to standard finite-dimensional LMIs that are well known to guarantee the exponential stability of the first-order and, respectively, second-order delay-differential equations, describing the first modal dynamics of these test PDEs. Surprisingly, such a *reduction of infinite-dimensional LOIs to finite-dimensional LMIs is tight* in the sense that the stability of the latter delay-differential equations is necessary and sufficient for the stability of the PDEs in question, a feature presently not recognized for ODEs with delay.

The specific notation in the infinite-dimensional setting, used throughout, is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$, and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$, means that P is symmetric and positive definite, whereas $\lambda_{min}(P)$ $[\lambda_{max}(P)]$ denotes its minimum (maximum) eigenvalue.

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $|\cdot|$. Denote by $\mathcal{L}(\mathcal{H})$ bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. Given a linear operator $P : \mathcal{H} \to \mathcal{H}$ with a dense domain $\mathcal{D}(P) \subset \mathcal{H}$, the notation P^* stands for the adjoint operator. Such an operator P is strictly positive definite, that is, $P > 0$ iff it is self-adjoint in the sense that $P = P^*$ and there exists a constant $\beta > 0$ such that $\langle x, Px\rangle \geq \beta\langle x, x\rangle$ and for all $x \in \mathcal{D}(P)$, whereas $P \geq 0$ means that P is self-adjoint and nonnegative definite, that is, $\langle x, Px\rangle \geq 0$ for all $x \in \mathcal{D}(P)$.

If an infinitesimal operator A generates a strongly continuous semigroup $T(t)$ on the Hilbert space $\mathcal{H}$ (see, e.g., [35] for details), the domain of the operator A forms another Hilbert space $\mathcal{D}(A)$ with the graph inner product $(\cdot,\cdot)_{\mathcal{D}(A)}$ defined as follows: $(x, y)_{\mathcal{D}(A)} = \langle x, y\rangle + \langle Ax, Ay\rangle$, $x, y \in \mathcal{D}(A)$. Moreover, the induced norm $\|T(t)\|$ of the semigroup $T(t)$ satisfies the inequality $\|T(t)\| \leq \kappa e^{\sigma t}$ everywhere with some constant $\kappa > 0$ and growth bound σ.

The space of the continuous $\mathcal{H}$-valued functions $x : [a,b] \to \mathcal{H}$ with the induced norm $\|x\|_{C([a,b],\mathcal{H})} = \max_{s\in[a,b]} |x(s)|$ is denoted by $C([a,b],\mathcal{H})$. The space of the continuously differentiable $\mathcal{H}$-valued functions $x : [a,b] \to \mathcal{H}$, with the induced norm $\|x\|_{C^1([a,b],\mathcal{H})} = \max(\|x\|_{C([a,b],\mathcal{H})}, \|\dot{x}\|_{C([a,b],\mathcal{H})})$, is denoted by $C^1([a,b],\mathcal{H})$.

$L_2(a,b;\mathcal{H})$ is the Hilbert space of square-integrable $\mathcal{H}$-valued functions on (a,b) with the corresponding norm; $W^{l,2}([a,b],R)$ is the Sobolev space of absolutely continuous scalar functions on $[a,b]$ with square-integrable derivatives of the order $l \geq 1$.

The notation $x^t = x(t+\theta) \in L_2(-h,0;\mathcal{H})$ stands for $x(\cdot) \in L_2(a,b;\mathcal{H})$ and $t \in [a+h,b]$.

2.2 The Lyapunov–Krasovskii Method

For later use, the following instrumental results are extracted from the literature.

Lemma 1 (Wirtinger's inequality and its generalization [127]). *Let* $u \in W^{1,2}([a,b], R)$ *be a scalar function with* $u(a) = u(b) = 0$. *Then*

$$\int_a^b u^2(\xi)d\xi \leq \frac{(b-a)^2}{\pi^2} \int_a^b (u'(\xi))^2 d\xi, \tag{2.1}$$

$$max_{\xi \in [a,b]} z^2(\xi) \leq (b-a) \int_a^b (z'(\xi))^2 d\xi. \tag{2.2}$$

If, additionally, $u \in W^{2,2}([a,b], R)$*, then*

$$\int_a^b (u'(\xi))^2 d\xi \leq \frac{(b-a)^2}{\pi^2} \int_a^b (u''(\xi))^2 d\xi. \tag{2.3}$$

Lemma 2 (Jensen's inequality [58]). *Let* $\mathcal{H}$ *be a Hilbert space with the inner product* $\langle \cdot, \cdot \rangle$. *For any linear bounded operator* $R : \mathcal{H} \to \mathcal{H}$, $R > 0$, *scalar* $l > 0$, *and* $x \in L_2([a,b], \mathcal{H})$*, the following holds:*

$$l \int_0^l \langle x(s), Rx(s) \rangle ds \geq \left\langle \int_0^l x(s)ds, R \int_0^l x(s)ds \right\rangle. \tag{2.4}$$

Next, consider a linear infinite-dimensional system

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \quad t \geq t_0, \tag{2.5}$$

evolving in a Hilbert space $\mathcal{H}$, where $x(t) \in \mathcal{H}$ is the instantaneous state of the system. Let the following assumptions be satisfied:

A1 The operator A generates a strongly continuous semigroup $T(t)$ and the domain $\mathcal{D}(A)$ of the operator A is dense in $\mathcal{H}$.
A2 The linear operator A_1 is bounded in $\mathcal{H}$.
A3 The function $\tau(t)$ is piecewise-continuous of class C^1 on the closure of each continuity subinterval and it satisfies

$$\inf_t \tau(t) \geq 0, \ \sup_t \tau(t) \leq h \tag{2.6}$$

with some constant $h > 0$ for all $t \geq t_0$.

Let the initial conditions

$$x^{t_0} = \varphi(\theta),\ \theta \in [-h, 0],\ \phi \in W, \tag{2.7}$$

be given in the space

$$W = C([-h, 0], \mathcal{D}(A)) \cap C^1([-h, 0], \mathcal{H}). \tag{2.8}$$

Recall that a function $x(t) \in C([t_0 - h, t_0 + \eta], \mathcal{D}(A))$ is said to be a solution of the initial-value problem (2.5), (2.7) on $[t_0 - h, t_0 + \eta]$ if $x(t)$ is initialized with (2.7), it is absolutely continuous for $t \in [t_0, t_0 + \eta]$, and it satisfies (2.5) for almost all $t \in [t_0, t_0 + \eta]$.

The initial-value problem (2.5), (2.7) turns out to be well posed on the semiinfinite time interval $[t_0, \infty)$ and due to Lemma 3 given below, its solutions can be found as mild solutions, namely, as those of the integral equation

$$\begin{aligned} &x(t) = T(t - t_0)x(t_0) \\ &+ \int_{t_0}^{t} T(t - s)A_1 x(s - \tau(s))ds,\ t \geq t_0. \end{aligned} \tag{2.9}$$

The following result is in order.

Lemma 3 ([50]). *Under **A1–A3**, there exists a unique solution of the initial-value problem (2.5), (2.7) on* $[t_0, \infty)$. *This solution is also a unique solution of the integral initial-value problem (2.7), (2.9).*

The present aim is to derive exponential stability criteria for linear time-delay systems (2.5), (2.6), thus defined. The stability concept under study is based on the initial data norm

$$\|\phi\|_W = \sqrt{|A\phi(0)|^2 + \|\phi\|^2_{C^1([-h,0],\mathcal{H})}} \tag{2.10}$$

in space (2.8). Suppose $x(t, t_0, \phi)$ denotes a solution of (2.5), (2.7) at a time instant $t \geq t_0$.

Definition 1. System (2.5) is said to be exponentially stable with a decay rate $\delta > 0$ if there exists a constant $K \geq 1$ such that the following exponential estimate holds:

$$|x(t, t_0, \phi)|^2 \leq K e^{-2\delta(t-t_0)} \|\phi\|^2_W \quad \forall t \geq t_0. \tag{2.11}$$

Consider Lyapunov–Krasovskii functionals, which depend on x and $\dot{x}$ [73]. Given a continuous functional $V : \mathbb{R} \times W \times C([-h, 0], \mathcal{H}) \to \mathbb{R}$, its upper right-hand derivative along solutions $x^t(t_0, \phi),\ t \geq t_0$ of (2.5), (2.7) is defined as follows:

$$\begin{aligned} &\dot{V}(t,\phi,\dot{\phi}) = \limsup_{s \to 0^+} \tfrac{1}{s}[V(t+s, x^{t+s}(t,\phi), \dot{x}^{t+s}(t,\phi)) \\ &-V(t, \phi, \dot{\phi})]. \end{aligned}$$

Lemma 4. *Let **A1–A3** be in force and let there exist positive numbers* $\delta,\ \beta,\ \gamma$ *and a continuous functional*

$$V : \mathbb{R} \times W \times C([-h, 0], \mathcal{H}) \to \mathbb{R}$$

such that the function $\bar{V}(t) = V(t, x^t, \dot{x}^t)$ *is absolutely continuous for* x^t*, satisfying (2.5), and*

$$\beta|\phi(0)|^2 \leq V(t, \phi, \dot{\phi}) \leq \gamma\|\phi\|_W^2, \tag{2.12}$$

$$\dot{V}(t, \phi, \dot{\phi}) + 2\delta V(t, \phi, \dot{\phi}) \leq 0. \tag{2.13}$$

Then (2.5) is exponentially stable with the decay rate δ *and (2.11) holds with* $K = \frac{\gamma}{\beta}$.

Proof. As in the case of ODEs, from (2.13) with $\phi = x^t$, one derives

$$\tfrac{d}{dt} V(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t) \leq 0,$$

where $V(t_0, x^{t_0}, \dot{x}^{t_0}) = V(t_0, \phi, \dot{\phi})$. Hence, by the comparison principle argument [71], it follows that

$$\begin{aligned} &\beta|x(t)|^2 \leq V(t, x^t, \dot{x}^t) \leq V(t_0, \phi, \dot{\phi})e^{-2\delta(t-t_0)} \\ &\leq \gamma e^{-2\delta(t-t_0)}\|\phi\|_W^2. \end{aligned}$$

2.3 Exponential Stability in a Hilbert Space

In this section, the delay is assumed to be either slow-varying subject to

$$\dot{\tau} \leq d < 1 \tag{2.14}$$

or fast-varying with no restrictions on $\dot{\tau}$. Let **A1–A3** be in force.

Delay-dependent conditions are derived by using a simple Lyapunov–Krasovskii functional, inherited from [58]:

$$\begin{aligned} V(t, x^t, \dot{x}^t) &= \langle x(t), Px(t)\rangle + \int_{t-h}^{t} e^{2\delta(s-t)}\langle x(s), Sx(s)\rangle ds \\ &+h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\delta(s-t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds d\theta \\ &+ \int_{t-\tau(t)}^{t} e^{2\delta(s-t)}\langle x(s), Qx(s)\rangle ds, \end{aligned} \tag{2.15}$$

where $P : \mathcal{D}(A) \to \mathcal{H}$ is a linear positive definite operator and $R, Q, S \in \mathcal{L}(\mathcal{H})$ are nonnegative definite operators, satisfying the following inequalities:

$$\begin{aligned} &\beta\langle x, x\rangle \leq \langle x, Px\rangle \leq \gamma_P[\langle x, x\rangle + \langle Ax, Ax\rangle], \\ &\langle x, Qx\rangle \leq \gamma_Q\langle x, x\rangle, \ \langle x, Rx\rangle \leq \gamma_R\langle x, x\rangle, \\ &\langle x, Sx\rangle \leq \gamma_S\langle x, x\rangle, \quad \forall x \in D(A), \end{aligned} \tag{2.16}$$

for some positive constants $\beta, \gamma_P, \gamma_Q, \gamma_S, \gamma_R$. Thus, condition (2.12) of Lemma 4 is satisfied.

Note that the first inequality (2.16) allows one to use not only bounded operators P (like those considered in [35, p. 217]), but also unbounded operators P, which are upper-estimated by the unbounded operator A according to (2.16). In the case of ODEs where A is a matrix, the above upper bound is equivalent to the standard one with $A = 0$. For ODEs with delay, the Lyapunov functional of the form (2.15) was recently introduced in [60] for $\delta = 0$, whereas this functional with $S = 0$ was introduced earlier in [52] for $\delta = 0$ and in [118] for $\delta > 0$.

Viewed on solutions of (2.5), the Lyapunov–Krasovskii functional (2.15) is absolutely continuous as a function of t because the solutions are absolutely continuous in t. Differentiating V yields

$$\begin{aligned}
&\dot{V}(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t)\\
&\leq 2\langle x(t), P\dot{x}(t)\rangle + 2\delta\langle x(t), Px(t)\rangle + h^2\langle \dot{x}(t), R\dot{x}(t)\rangle\\
&-he^{-2\delta h}\textstyle\int_{t-h}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds + \langle x(t), (Q+S)x(t)\rangle\\
&-(1-\dot{\tau}(t))\langle x(t-\tau(t)), Qx(t-\tau(t))\rangle e^{-2\delta h}\\
&-\langle x(t-h), Sx(t-h)\rangle e^{-2\delta h}.
\end{aligned} \tag{2.17}$$

Following [60], the Jensen inequality (2.4), applied to the representation

$$\begin{aligned}
&-h\textstyle\int_{t-h}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds = -h\int_{t-h}^{t-\tau(t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds\\
&-h\textstyle\int_{t-\tau(t)}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds,
\end{aligned} \tag{2.18}$$

results in

$$\begin{aligned}
&\textstyle\int_{t-\tau(t)}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds\\
&\geq \frac{1}{h}\left\langle \int_{t-\tau(t)}^{t}\dot{x}(s)ds, R\int_{t-\tau(t)}^{t}\dot{x}(s)ds\right\rangle,\\
&\textstyle\int_{t-h}^{t-\tau(t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds\\
&\geq \frac{1}{h}\left\langle \int_{t-h}^{t-\tau(t)}\dot{x}(s)ds, R\int_{t-h}^{t-\tau(t)}\dot{x}(s)ds\right\rangle.
\end{aligned} \tag{2.19}$$

Then, taking into account (2.14) and following [55] lead to

$$\begin{aligned}
&\dot{V}(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t)\\
&\leq 2\langle x(t), P\dot{x}(t)\rangle + 2\delta\langle x(t), Px(t)\rangle + h^2\langle \dot{x}(t), R\dot{x}(t)\rangle\\
&-\Big[\langle x(t)-x(t-\tau(t)), R(x(t)-x(t-\tau(t)))\rangle\\
&+\langle x(t-\tau(t))-x(t-h), R(x(t-\tau(t))-x(t-h))\rangle\\
&+(1-d)\langle x(t-\tau(t)), Qx(t-\tau(t))\rangle\Big]e^{-2\delta h}\\
&+\langle x(t), (Q+S)x(t)\rangle - \langle x(t-h), Sx(t-h)\rangle e^{-2\delta h}.
\end{aligned} \tag{2.20}$$

Stability conditions are now derived in two forms. The first form will subsequently be applied to the wave equation and the second one to the heat equation. The first form is derived by substituting the right-hand side of (2.5) for $\dot{x}(t)$. Once condition (2.13) of Lemma 4 is given in terms of $\eta(t) = col\{x(t), x(t-h), x(t-\tau(t))\}$, thus yielding

$$\dot{V}(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t) \leq \langle \eta(t), \Phi_h \eta(t) \rangle \leq 0, \tag{2.21}$$

it is readily verified if the following LOI:

$$\Phi_h = \begin{bmatrix} \Phi_{11} & 0 & PA_1 \\ 0 & 0 & 0 \\ A_1^* P & 0 & 0 \end{bmatrix} + h^2 \begin{bmatrix} A^* RA & 0 & A^* RA_1 \\ 0 & 0 & 0 \\ A_1^* RA & 0 & A_1^* RA_1 \end{bmatrix}$$
$$-e^{-2\delta h} \begin{bmatrix} R & 0 & -R \\ 0 & (S+R) & -R \\ -R & -R & 2R+(1-d)Q \end{bmatrix} \leq 0 \tag{2.22}$$

holds provided that

$$\Phi_{11} = A^* P + PA + 2\delta P + Q + S. \tag{2.23}$$

The resulting inequality (2.22) is convex with respect to h; that is, given $h_0 > 0$, it becomes feasible for all $\bar{h} \in [0, h_0]$ whenever it is feasible for h_0. The convexity follows from the fact that $\Phi_{\bar{h}} \leq \Phi_{h_0}$ since h^2 and $-e^{-2\delta h}$ multiply the nonnegative definite operators. Summarizing, the following result is obtained.

Theorem 5. *Let **A1–A3** be in force. Given $\delta > 0$, let there exist linear operators $P > 0$ and $R \geq 0, S \geq 0, Q \geq 0$ subject to (2.16) such that the LOI (2.22) with notation (2.23) holds in the Hilbert space $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$. Then system (2.5) is exponentially stable with the decay rate δ for all slow-varying differentiable delays (2.14). The inequality (2.11) is satisfied with $K = \max\{\gamma_P, h(\gamma_Q + \gamma_S + h^2\gamma_R/2)\}/\beta$. Moreover, (2.5) is exponentially stable for all fast-varying delays $\tau(t)$ with no restrictions on $\dot{\tau}$ if the LOI (2.22) is feasible with $Q = 0$.*

The conditions of Theorem 5 are delay-dependent (h-dependent) even for $\delta \to 0$. Taking in the above derivations $S = R = 0$, one arrives at the following "quasi-delay-independent" conditions, which become delay-independent for $\delta \to 0$; in the case of ODEs, they are simplified to the well-known result.

Corollary 1 ([83]). *Let **A1–A3** be in force. Given $\delta > 0$, system (2.5) is exponentially stable with the decay rate δ for all differentiable slow-varying delays (2.14) if there exist linear operators $P > 0$ and $Q \geq 0$ subject to (2.16) such that the LOI*

$$\begin{bmatrix} (A+\delta)^* P + P(A+\delta) + Q & PA_1 \\ A_1^* P & -(1-d)Qe^{-2\delta h} \end{bmatrix} \leq 0 \tag{2.24}$$

holds in the Hilbert space $\mathcal{D}(A) \times \mathcal{D}(A)$. Moreover, the inequality (2.11) is satisfied with $K = \max\{\gamma_P, h\gamma_Q\}/\beta$.

Unlike the finite-dimensional case, the feasibility of the strict LOI (2.22) [(2.24)] for $h = 0$ ($\delta = 0$) does not necessarily imply the feasibility of (2.22) [(2.24)] for small enough h (δ) because h^2 (δ) is multiplied by the operator, which may be unbounded.

It may be difficult to verify the feasibility of (2.22) if the operator that multiplies h^2 (and depends on A) in Φ_h is unbounded. To avoid this, we will derive the second form of LOI by the descriptor method [48], where the right-hand sides of the expressions

$$\begin{aligned} 0 &= 2\langle x(t), P_2^*[Ax(t) + A_1 x(t-\tau(t)) - \dot{x}(t)]\rangle, \\ 0 &= 2\langle \dot{x}(t), P_3^*[Ax(t) + A_1 x(t-\tau(t)) - \dot{x}(t)]\rangle \end{aligned} \tag{2.25}$$

with some $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ are added into the right-hand side of (2.20). Setting $\eta_d(t) = col\{x(t), \dot{x}(t), x(t-h), x(t-\tau(t))\}$, we obtain

$$\dot{V}(t, x^t, \dot{x}^t) + 2\langle x(t), P\dot{x}(t)\rangle \le \langle \eta_d(t), \Phi_d \eta_d(t)\rangle \le 0,$$

if the LOI

$$\Phi_d = \begin{bmatrix} \Phi_{d11} & \Phi_{d12} & 0 & P_2^* A_1 + Re^{-2\delta h} \\ * & \Phi_{d22} & 0 & P_3^* A_1 \\ * & * & -(S+R)e^{-2\delta h} & Re^{-2\delta h} \\ * & * & * & -[2R + (1-d)Q]e^{-2\delta h} \end{bmatrix} \le 0 \tag{2.26}$$

holds, where

$$\begin{aligned} \Phi_{d11} &= A^* P_2 + P_2^* A + 2\delta P + Q + S - Re^{-2\delta h}, \\ \Phi_{d12} &= P - P_2^* + A^* P_3, \ \Phi_{d22} = -P_3 - P_3^* + h^2 R \end{aligned} \tag{2.27}$$

and $*$ denotes the symmetric terms of the operator matrix. Thus, the following result is obtained.

Theorem 6. *Let* ***A1–A3*** *be in force. Given* $\delta > 0$, *let there exist* $P > 0$ *and* $R \ge 0$, $S \ge 0$, $Q \ge 0$ *subject to (2.16) and indefinite operators* $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ *such that the LOI (2.26) with notations given in (2.27) holds in the Hilbert space* $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$. *Then system (2.5) is exponentially stable with the decay rate* δ *for all differentiable slow-varying delays (2.14). The inequality (2.11) is satisfied with* $K = \max\{\gamma_P, h(\gamma_Q + \gamma_S + h^2\gamma_R/2)\}/\beta$. *Moreover, (2.5) is exponentially stable for all fast-varying delays with no restrictions on the delay derivative if the LOI (2.26) is feasible with* $Q = 0$.

Consider now system (2.5) with A and A_1 from the uncertain time-invariant polytope

$$\Omega = \textstyle\sum_{j=1}^{M} f_j \Omega_j \text{ for some } 0 \le f_j \le 1, \ \sum_{j=1}^{M} f_j = 1, \tag{2.28}$$

where $\Omega_j = \left[A^{(j)} \ A_1^{(j)} \right]$, $A_1^{(j)} \in \mathcal{L}(\mathcal{H})$ and the operators $A^{(j)}$ have a common domain, which is dense in $\mathcal{H}$ and $A = \sum_{j=1}^{M} f_j A^{(j)}$ generates a strongly continuous semigroup for all f_j, satisfying (2.28). Applying conditions of Theorem 5 to the uncertain system, one concludes that (2.5) is exponentially stable under **A3**, provided LOI (2.22) is feasible. Since LOI (2.22) is affine in A and A_1, by applying the same arguments as for LMIs (see [23]), we may conclude that (2.22) is feasible if the following LOIs:

$$\begin{bmatrix} \Phi_{11} & 0 & PA_1^{(j)} \\ 0 & 0 & 0 \\ A_1^{(j)*}P & 0 & 0 \end{bmatrix} + h^2 \begin{bmatrix} A^{(j)*}RA^{(j)} & 0 & A^{(j)*}RA_1^{(j)} \\ 0 & 0 & 0 \\ A_1^{(j)*}RA^{(j)} & 0 & A_1^{(j)*}RA_1^{(j)} \end{bmatrix}$$
$$-e^{-2\delta h} \begin{bmatrix} R & 0 & -R \\ 0 & (S^{(j)} + R) & -R \\ -R & -R & 2R+(1-d)Q^{(j)} \end{bmatrix} \le 0,$$
$$\Phi_{11} = A^{(j)*}P + PA^{(j)} + 2\delta P + Q^{(j)} + S^{(j)}, \ j = 1, \dots, M,$$

in the vertices are feasible for the same $P > 0$, $R > 0$ and for different $Q^{(j)}, S^{(j)}$.

Similarly, if we apply Theorem 6, system (2.5) proves to be exponentially stable provided that LOIs (2.26) in M vertices are feasible for the same P_2, P_3 and for different $Q^{(j)} \ge 0, S^{(j)} > 0, R^{(j)} > 0, P^{(j)} > 0, \ j = 1, \dots, M$.

2.4 Exponential Stability of a Delay Heat Equation

Consider the heat equation

$$z_t(\xi, t) = az_{\xi\xi}(\xi, t) - a_0 z(\xi, t) - a_1 z(\xi, t - \tau(t)), \tag{2.29}$$

where $t \ge t_0$, $0 \le \xi \le l$, with the constant parameters $a > 0, a_0$, and a_1, with the time-varying delay $\tau(t)$, satisfying (2.6), and with the Dirichlet boundary condition

$$z(0, t) = z(l, t) = 0, \ t \ge t_0. \tag{2.30}$$

The boundary-value problem (2.29), (2.30) describes the propagation of heat in a homogeneous one-dimensional rod with a fixed temperature at the ends in the case of the delayed (possibly, due to actuation) heat exchange with the surroundings. Here a and a_i, $i = 0, 1$, stand for the heat conduction coefficient and for the coefficients of the heat exchange with the surroundings, respectively, $z(\xi, t)$ is the value of the temperature field of the plant at time moment t and location ξ along the rod. In the sequel, the state dependence on time t and spatial variable ξ is suppressed whenever possible.

Stability issues of the boundary-value problem (2.29), (2.30) are studied in the Hilbert space $\mathcal{H} = L_2(0, l)$ so that it is represented as the differential equation (2.5) in the Hilbert space $L_2(0, l)$ with the infinitesimal operator $A = a\frac{\partial^2}{\partial \xi^2} - a_0$, possessing the dense domain

$$\mathcal{D}\left(\frac{\partial^2}{\partial \xi^2}\right) = \{z \in W^{2,2}([0, l], \mathbb{R}) : z(0) = z(l) = 0\}, \tag{2.31}$$

and with the bounded operator $A_1 = -a_1$ of the multiplication by the constant $-a_1$. The infinitesimal operator A generates an exponentially stable semigroup (see, e.g., [35] for details).

Simple delay-independent conditions, based on LOI (2.24), are derived first. For this purpose, let's consider the Lyapunov–Krasovskii functional of the form

$$\begin{array}{l} V(t, z^t) = p \int_0^l z^2(\xi, t) d\xi \\ +q \int_{t-\tau(t)}^t \int_0^l e^{2\delta(s-t)} z^2(\xi, s) d\xi ds \end{array} \tag{2.32}$$

with some positive constants p and q. Then the operators P and Q in (2.24) take the form $P = p$, $Q = q$ of the bounded operators of the multiplication by positive constants p and q, respectively. Integrating by parts and taking into account (2.30) yield

$$\begin{array}{l} \langle x, (A^* P + PA)x \rangle = 2a \int_0^l p z z_{\xi\xi} d\xi - 2a_0 \int_0^l p z^2 d\xi \\ = -2 \left[a \int_0^l p z_\xi^2 d\xi + a_0 \int_0^l p z^2 d\xi \right] \le -2 \left(\frac{\pi^2}{l^2} a + a_0 \right) \int_0^l p z^2 d\xi \end{array} \tag{2.33}$$

for $x \in \mathcal{D}(A)$, where the last inequality follows from Wirtinger's inequality (2.1). We thus obtain that (2.24) is satisfied if

$$\Psi_\delta \triangleq \begin{bmatrix} q - 2\left(\frac{\pi^2}{l^2} a + a_0 - \delta\right) p & -a_1 p \\ -a_1 p & -(1-d) q e^{-2\delta h} \end{bmatrix} < 0. \tag{2.34}$$

Since $\Psi_\delta = \Psi_0 + diag\{2\delta p, (1-d)q(1-e^{-2\delta h})\}$, it follows from $\Psi_0 < 0$ that $\Psi_\delta < 0$ for sufficiently small δ. The following result is concluded.

Theorem 7. *Given $\delta > 0$, let the LMI (2.34) hold for some scalars $p > 0$ and $q > 0$. Then the Dirichlet boundary-value problem (2.29), (2.30) is exponentially stable with the decay rate δ for all differentiable slow-varying delays subject to (2.6) and (2.14), and the inequality*

$$\int_0^l z^2(\xi, t) d\xi \le K e^{-2\delta(t-t_0)} \max_{s \in [t_0 - h, t_0]} \int_0^l z^2(\xi, s) d\xi \tag{2.35}$$

is satisfied for all $t \ge t_0$ with $K = 1 + hq/p$. If (2.34) holds for $\delta = 0$, then inequality (2.35) is satisfied with $K = 1 + hq/p$ and a sufficiently small δ.

By the Schur complements' formula, LMI (2.34) with $\delta = 0$ is feasible iff $q^2 - 2\left(\frac{\pi^2}{l^2}a + a_0\right)pq + a_1^2 p^2/(1-d) < 0$ for some $p > 0$ and $q > 0$. The left part of the latter inequality achieves its minimum at $q = \left(\frac{\pi^2}{l^2}a + a_0\right)p$ and, thus, the inequality holds iff

$$\frac{\pi^2}{l^2}a + a_0 > 0, \quad a_1^2 < \left(\frac{\pi^2}{l^2}a + a_0\right)^2 (1-d). \tag{2.36}$$

Next, we'll apply Theorem 6 to derive delay-dependent conditions. For simplicity, for this investigation, we've chosen the case where $l = \pi$. The Lyapunov–Krasovskii functional V is thus specified in the form

$$\begin{aligned}V(t, z^t, z_s^t) &= (p_1 - p_3 a)\int_0^\pi z^2(\xi, t)d\xi + p_3 a \int_0^\pi z_\xi^2(\xi, t)d\xi \\ &+ \int_0^\pi \Big[r \int_{-h}^0 \int_{t+\theta}^t e^{2\delta(s-t)} z_s^2(\xi, s)ds d\theta \\ &+ s\int_{t-h}^t e^{2\delta(s-t)} z^2(\xi, s)ds + q\int_{t-\tau(t)}^t e^{2\delta(s-t)} z^2(\xi, s)ds\Big]d\xi,\end{aligned}$$

with some constants $p_1 > 0$, $p_3 > 0$, $s > 0$, $r > 0$, and $q \geq 0$. Then the operators in (2.15) are as follows: $P = -p_3\left(a\frac{\partial^2}{\partial\xi^2} + a\right) + p_1$, $R = r$, $Q = q$, $S = s$. Furthermore, $P_2 = p_2$ and $P_3 = p_3$ are chosen, where $p_2 > 0$ and

$$p_2 - \delta p_3 \geq 0. \tag{2.37}$$

Here P is an unbounded operator and the other operators are bounded in $L_2(0, \pi)$. It should be noted that the above choice of P, depending on the slack variable P_3, is different from that of the ODEs (where these matrices are independent). Thus, the slack variable allows one to construct an appropriate Lyapunov–Krasovskii functional.

Integrating by parts and utilizing Wirtinger's inequality (2.1), we find that

$$\begin{aligned}\langle x, Px\rangle &= \int_0^\pi \left[-p_3 a z_{\xi\xi} z - p_3 a z^2 + p_1 z^2\right] d\xi \\ &= \int_0^\pi \left[a p_3 [z_\xi^2 - z^2] + p_1 z^2\right] d\xi \geq p_1 \int_0^\pi z^2 d\xi > 0\end{aligned}$$

for $x \in D(A)$. Moreover, (2.16) is satisfied because the inequality

$$\langle x, Px\rangle \leq \int_0^\pi [a p_3 (z_{\xi\xi})^2 + (a p_3 + p_1) z^2] d\xi \leq \gamma_P [|Ax|^2 + |x|^2]$$

holds for some $\gamma_P > 0$ by virtue of the generalized Wirtinger inequality (2.3). We thus obtain that

$$\begin{aligned}
&\langle \dot{x}, (P - P_2^* + A^* P_3)x \rangle = \langle \dot{x}, (p_1 - p_2 - (a + a_0)p_3)x \rangle; \\
&\langle x, A^* P_2 x \rangle + \langle x, P_2^* A x \rangle + 2\delta \langle x, Px \rangle \\
&= 2a(p_2 - \delta p_3) \int_0^\pi z_{\xi\xi} z d\xi + 2[-a_0 p_2 + \delta(p_1 - a p_3)] \int_0^\pi z^2 d\xi \\
&= -2a(p_2 - \delta p_3) \int_0^\pi z_\xi^2 d\xi + 2[-a_0 p_2 + \delta(p_1 - a p_3)] \int_0^\pi z^2 d\xi \\
&\le [-2(a + a_0)p_2 + 2\delta p_1] \int_0^\pi z^2 d\xi,
\end{aligned}$$

where the latter inequality follows from (2.37) and Wirtinger's inequality (2.1). Therefore, (2.26) holds if

$$\begin{aligned}
&\begin{bmatrix} \phi_{11} & \phi_{12} & 0 & \phi_{14} \\ * & -2p_3 + h^2 r & 0 & -p_3 a_1 \\ * & * & -(s + r)e^{-2\delta h} & re^{-2\delta h} \\ * & * & * & \phi_{44} \end{bmatrix} < 0, \\
&\phi_{11} = -2(a + a_0)p_2 + 2\delta p_1 + q + s - re^{-2\delta h}, \\
&\phi_{12} = p_1 - p_2 - (a + a_0)p_3, \\
&\phi_{14} = -p_2 a_1 + re^{-2\delta h}, \ \phi_{44} = -[2r + (1 - d)q]e^{-2\delta h}.
\end{aligned} \tag{2.38}$$

We have thus proved the following result.

Theorem 8. *Given $\delta > 0$, let there exist scalars $p_1 > 0, p_2 > 0, p_3 > 0, s > 0, r > 0$, and $q \ge 0$ such that LMIs (2.37) and (2.38) hold. Then the boundary-value problem (2.29), (2.30), where $l = \pi$, is exponentially stable with the decay rate δ for all differentiable slow-varying delays subject to (2.6) and (2.14), and the inequality*

$$\begin{aligned}
&p_1 \int_0^\pi z^2(\xi, t) d\xi \le e^{-2\delta(t - t_0)} \Big\{ a p_3 \int_0^\pi z_\xi^2(\xi, t_0) d\xi \\
&+ \max[p_1 - p_3 a + hq + hs, h^3 r/2] \\
&\times \max_{s \in [t_0 - h, t_0]} \int_0^\pi [z^2(\xi, s) + z_t^2(\xi, s)] \, d\xi \Big\}
\end{aligned} \tag{2.39}$$

is satisfied for all $t \ge t_0$. Moreover, (2.29), (2.30) is exponentially stable with the decay rate δ for all fast-varying delays (2.6) with no restrictions on the derivative of the delay if (2.37), (2.38) are feasible with $q = 0$. If (2.38) holds for $\delta = 0$, then (2.29), (2.30) is exponentially stable with a sufficiently small decay rate.

It is of interest to note that under $l = \pi$, the same LMIs (2.34) and (2.38) guarantee the exponential stability of the scalar ODE

$$\dot{y}(t) + (a + a_0)y(t) + a_1 y(t - \tau(t)) = 0. \tag{2.40}$$

System (2.40) corresponds to the first modal dynamics, corresponding to $k = 1$ in the modal representation

$$\dot{y}_k(t) + (ak^2 + a_0)y_k(t) + a_1 y_k(t - \tau(t)) = 0, \ k = 1, 2, \ldots, \tag{2.41}$$

of the Dirichlet boundary-value problem (2.29), (2.30) with $l = \pi$, projected on the eigenfunctions of the operator $\frac{\partial^2}{\partial \xi^2}$ (this operator has eigenvalues $-k^2$; see, e.g., [135]). The stability of (2.29), (2.30) implies the stability of (2.41). Thus, the reduction of the infinite-dimensional LOI of Corollary 1 (Theorem 2) to the finite-dimensional LMI of Theorem 7 (Theorem 8) is tight, since the stability of (2.40) is necessary for the stability of (2.29), (2.30).

The above is consistent with the frequency-domain analysis in the case of constant delays, where the characteristic equations of (2.29), (2.30) are given by

$$\lambda_k + ak^2 + a_0 + a_1 e^{-\lambda_k \tau} = 0, \quad k = 1, 2, \ldots \tag{2.42}$$

(see, e.g., [135]). The exponential stability of (2.29), (2.30) is shown in [64] to be determined by (2.42) with $k = 1$, namely, by the stability of the ODE (2.40).

Remark 2. Consider the Dirichlet boundary-value problem (2.29), (2.30) with the uncertain coefficients from the uncertain time-invariant polytope Ω given by (2.28) with $\Omega_j = \left[a^{(j)} \ a_0^{(j)} \ a_1^{(j)} \right]$. Here $M = 2^k$ and k is the number of uncertain parameters, and it may take values from the finite set $\{1, 2, 3\}$. The uncertain infinitesimal operator $A = \sum_{j=1}^{M} f_j a^{(j)} \frac{\partial^2}{\partial \xi^2} - a_0^{(j)}$ with the dense domain (2.31) generates a strongly continuous semigroup, whereas the uncertain operator $A_1 = \sum_{j=1}^{M} f_j a^{(j)}$ is bounded. By applying Theorem 8, one concludes that similar to the Hilbert space-valued dynamics, the boundary problem (2.29), (2.30) is exponentially stable if (2.37) holds and LMIs (2.38) in the vertices are feasible for the same p_2, p_3 and for different $q^{(j)} \geq 0, s^{(j)} > 0, r^{(j)} > 0, p^{(j)} > 0,\ j = 1, \ldots, M$. By Theorem 7, (2.29), (2.30) is exponentially stable if LMIs (2.34) in the vertices are feasible for the same $p > 0$ and for different $q^{(j)},\ j = 1, \ldots, M$.

Example 3. To exemplify the above theoretical results, consider the controlled heat equation

$$z_t(\xi, t) = z_{\xi\xi}(\xi, t) + rz(\xi, t) + u, \ \ z(0, t) = z(l, t) = 0, \tag{2.43}$$

where $\xi \in (0, l),\ t > 0$, and where r is an uncertain parameter satisfying $|r| \leq \beta$ with given β. It was shown in [102] that for $l = 1$, the state feedback $u = -\gamma z(\xi, t)$ with $\gamma > \left(\frac{\beta}{2\pi}\right)^2$ exponentially stabilizes (2.43). By verifying the LMI (2.34), we conclude that the closed-loop system is exponentially stable if there exists $p > 0$ such that $-2(\pi^2 - r + \gamma)p < 0$ for all $|r| \leq \beta$, that is, if $\gamma > \beta - \pi^2$. Since $\beta - \pi^2 \leq \left(\frac{\beta}{2\pi}\right)^2$, the developed method guarantees the exponential stabilization of (2.43) via a lower gain, which becomes essentially lower for large β.

Noticing that a time delay often appears in the feedback, let's also consider the case where $l = \pi$, $\beta = 0.1$, and the delayed feedback $u = -z(\xi, t - \tau(t))$ is applied with an uncertain delay satisfying **A3**. This is a polytopic system reached by choosing $r = \pm 0.1$. Applying Theorem 8 with $\delta = 0$ and Remark 2 to the resulting closed-loop system establishes the feasibility of LMI (2.38) in the two vertices corresponding to $r = \pm 0.1$. Given a particular upper bound d in (2.14), we can compute the maximum delay value h_{max} of h in (2.6), for which the closed-loop system remains exponentially stable, by using the MATLAB® LMI toolbox; for example, the toolbox yields $h_{max} = 2.04$ for $d = 0.5$ and $h_{max} = 1.34$ for unknown d. As noted before, these results are inherited from the exponential stability of the ODE $\dot{y} = (-1 + r)y(t) - y(t - \tau(t))$ with $|r| \leq 0.1$.

2.5 Exponential Stability of a Delay Wave Equation

Consider the wave equation

$$\begin{aligned} z_{tt}(\xi, t) &= a z_{\xi\xi} - \mu_0 z_t(\xi, t) - \mu_1 z_t(\xi, t - \tau(t)) \\ &-a_0 z(\xi, t) - a_1 z(\xi, t - \tau(t)), \ t \geq t_0, \ 0 \leq \xi \leq \pi \end{aligned} \tag{2.44}$$

with the Dirichlet boundary condition (2.30), where $l = \pi$, and with the constant parameters $a > 0$, $\mu_0 > 0, \mu_1, a_0$, and a_1, with the time-varying delay $\tau(t)$, satisfying (2.6). The boundary-value problem (2.30), (2.44) describes the oscillations of a homogeneous string with fixed ends in the case of the delayed (possibly, due to actuation) stiffness restoration and dissipation. Here a stands for the elasticity coefficient, μ_0 and μ_1 stand for the dissipation coefficients, a_0, a_1 stand for the restoring stiffness coefficients, and the state vector $x = col\{z, z_t\}$ consists of the deflection $z(\xi, t)$ of the string and its velocity $z_t(\xi, t)$ at time moment t and location ξ along the string.

Let's introduce the operators

$$A = \begin{bmatrix} 0 & 1 \\ a\frac{\partial^2}{\partial \xi^2} - a_0 & -\mu_0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ -a_1 & -\mu_1 \end{bmatrix}, \tag{2.45}$$

where A_1 is a bounded operator of multiplication by the constant matrix and where the domain $\mathcal{D}\left(\frac{\partial^2}{\partial \xi^2}\right)$ of the double-differentiation operator is determined by (2.31). Then the boundary-value problem (2.30), (2.44) can be represented as the differential equation (2.5) in the Hilbert space $\mathcal{H} = L_2(0, \pi) \times L_2(0, \pi)$ with the infinitesimal operator A, possessing the domain $\mathcal{D}(A) = \mathcal{D}\left(\frac{\partial^2}{\partial \xi^2}\right) \times L_2(0, \pi)$ and generating an exponentially stable semigroup (see, e.g., [35] for details).

Stability issues of the boundary-value problem (2.30), (2.44) are further studied in the Hilbert space $\mathcal{H} = L_2(0, \pi) \times L_2(0, \pi)$. First, quasi-delay-independent conditions are derived by choosing V in the form

$$
\begin{array}{l}
V(t, v^t) = ap_3 \int_0^\pi z_\xi^2(\xi, t) d\xi + \int_0^\pi v^T(\xi, t) P_0 v(\xi, t) d\xi \\
+ \int_{t-\tau(t)}^t \int_0^\pi v^T(\xi, s) e^{2\delta(s-t)} Q v(\xi, s) d\xi ds, \\
P_0 = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}, \ P_w \stackrel{\Delta}{=} P_0 + diag\{ap_3, 0\} > 0, \ Q \geq 0,
\end{array} \tag{2.46}
$$

where $v^T(\xi, t) = [z(\xi, t) \ z_t(\xi, t)]$. Then the operators P (unbounded) and Q (bounded) in (2.24) are given by

$$
P = diag\left\{-ap_3 \frac{\partial^2}{\partial \xi^2}, 0\right\} + P_0, \quad Q \geq 0. \tag{2.47}
$$

Now, integrating by parts and taking into account the inequality $p_3 > 0$ (extracted from $P_w > 0$) and Wirtinger's inequality (2.1) yield

$$
\begin{array}{l}
\langle x, Px \rangle = \int_0^\pi \left[-ap_3 z_{\xi\xi} z + z^T P_0 z\right] d\xi = a \int_0^\pi p_3 (z_\xi)^2 d\xi \\
+ \langle x, P_0 x \rangle \geq \langle x, P_w x \rangle \geq \lambda_{min}(P_w) |x|^2 > 0
\end{array} \tag{2.48}
$$

for all $x \in \mathcal{D}(A) \times L_2(0, \pi)$. Moreover, by the generalized Wirtinger inequality (2.3), the following

$$
\langle x, Px \rangle \leq \int_0^\pi ap_3 (z_{\xi\xi})^2 d\xi + \langle x, P_0 x \rangle \leq \gamma_P (|Ax|^2 + |x|^2) \tag{2.49}
$$

holds with some constant $\gamma_P > 0$, and (2.16) is thus satisfied.

Finally, integrating by parts and applying Wirtinger's inequality (2.1) under condition (2.37) result in

$$
\begin{array}{l}
\langle x, P(A + \delta) x \rangle + \langle x, (A^* + \delta) P x \rangle = \int_0^\pi [z \ z_t] \\
\times \left\{ \begin{bmatrix} p_1 - ap_3 \frac{\partial^2}{\partial \xi^2} & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} \delta & 1 \\ a \frac{\partial^2}{\partial \xi^2} - a_0 & -\mu_0 + \delta \end{bmatrix} \right. \\
\left. + \begin{bmatrix} \delta & a \frac{\partial^2}{\partial \xi^2} - a_0 \\ 1 & -\mu_0 + \delta \end{bmatrix} \begin{bmatrix} p_1 - ap_3 \frac{\partial^2}{\partial \xi^2} & p_2 \\ p_2 & p_3 \end{bmatrix} \right\} \begin{bmatrix} z \\ z_t \end{bmatrix} d\xi \\
= -2a(p_2 - p_3 \delta) \int_0^\pi (z_\xi)^2 d\xi - \int_0^\pi [z \ z_t] \\
\times \begin{bmatrix} -2p_2 a_0 + 2p_1 \delta & p_1 - (\mu_0 - 2\delta) p_2 - p_3 a_0 \\ p_1 - (\mu_0 - 2\delta) p_2 - p_3 a_0 & 2p_2 - 2(\mu_0 - \delta) p_3 \end{bmatrix} \\
\times \begin{bmatrix} z \\ z_t \end{bmatrix} d\xi \leq \int_0^\pi [z \ z_t] (P_w C_\delta + C_\delta^T P_w) \begin{bmatrix} z \\ z_t \end{bmatrix} d\xi,
\end{array} \tag{2.50}
$$

where

$$C_\delta = \begin{bmatrix} \delta & 1 \\ -a - a_0 & -\mu_0 + \delta \end{bmatrix}. \tag{2.51}$$

Therefore, (2.24) is feasible if the following LMI

$$\Omega_\delta \stackrel{\Delta}{=} \begin{bmatrix} C_\delta^T P_w + P_w C_\delta + Q & P_w A_1 \\ A_1^T P_w & -(1-d)e^{-2\delta h} Q \end{bmatrix} \le 0 \tag{2.52}$$

is feasible. Since the LMI (2.52) ensures that the (1,1)-term of its left-hand side satisfies

$$-2p_2(a_0 + a) + 2(p_1 + ap_3)\delta + q_1 \le 0,$$

whereas $q_1 + 2\delta(p_1 + ap_3) \ge 0$, it follows that $p_2 \ge 0$. Hence, (2.37) does not follow from (2.52).

However, it follows from $\Omega_0 < 0$ that $p_2 > 0$, and (2.37) is thus validated for sufficiently small δ. Moreover, since $\Omega_\delta = \Omega_0 + diag\{2\delta P_w, (1-d)Q(1-e^{-2\delta h})\}$, the inequality $\Omega_0 < 0$ ensures that $\Omega_\delta < 0$ for sufficiently small δ. We have thus proved the following result.

Theorem 9. *Given $\delta > 0$, let the LMIs (2.37) and (2.52) hold for some symmetric $2 \times 2-$matrices $P_w > 0$ and $Q \ge 0$, where p_2 and p_3 are, respectively, the (1,2)- and (2,2)-terms of P_w. Then the Dirichlet boundary-value problem (2.30), (2.44), specified with $l = \pi$, is exponentially stable with the decay rate δ for all differentiable slow-varying delays subject to (2.6), (2.14), and the inequality*

$$\begin{aligned} &\lambda_{min}(P_w) \int_0^\pi [z^2(\xi,t) + z_t^2(\xi,t)]d\xi \le e^{-2\delta(t-t_0)} \\ &\times \left\{ K_w \max_{s\in[t_0-h,t_0]} \int_0^\pi [z^2(\xi,s) + z_t^2(\xi,s)]d\xi + ap_3 \int_0^\pi z_\xi^2(\xi,t_0)d\xi \right\} \end{aligned} \tag{2.53}$$

is satisfied, with

$$K_w = \lambda_{max}(P_w - diag\{ap_3, 0\}) + h\lambda_{max}(Q) \tag{2.54}$$

for all $t \ge t_0$. If $\Omega_0 < 0$ holds for $\delta = 0$, then (2.30), (2.44) is exponentially stable with a sufficiently small decay rate.

In a particular case, the above theorem admits further specification.

Corollary 2. *Once specified with $l = \pi$, $a = 1$, and $a_0 = a_1 = 0$, the Dirichlet boundary-value problem (2.30), (2.44) is exponentially stable for all bounded differentiable slow-varying delays subject to (2.6), (2.14) if*

$$\mu_1^2 < (1-d)\mu_0^2. \tag{2.55}$$

Proof. Since the delay appears only in z_t, the decision variables of the LMI (2.52) can be chosen in the form $P_w = \begin{bmatrix} 1 & 2\delta \\ 2\delta & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & \mu_0 \end{bmatrix}$, and thus $P_w A_1 = \begin{bmatrix} 0 & -2\mu_1\delta \\ 0 & -\mu_1 \end{bmatrix}$. Deleting from Ω_δ the column and the row consisting of zero elements yields the matrix

$$\begin{bmatrix} -2\delta & 4\delta^2 - 2\mu_0\delta & -2\mu_1\delta \\ * & -\mu_0 + 6\delta & -\mu_1 \\ * & * & -(1-d)e^{-2\delta h}\mu_0 \end{bmatrix}.$$

Applying the Schur complements' formula to the last column and the last row of this matrix, we may conclude that (2.52) is feasible if the following LMI:

$$\begin{bmatrix} -2\delta + O(\delta^2) & O(\delta) \\ O(\delta) & -\mu_0 + \frac{\mu_1^2}{(1-d)\mu_0} e^{2\delta h} + 6\delta \end{bmatrix} < 0 \tag{2.56}$$

holds with $|O(\delta^k)| \leq c\delta^k$ $(k = 1, 2)$ for some constant $c > 0$ and for all sufficiently small δ. Subject to (2.55), the (2,2)-term of the left-hand side of (2.56) is negative for sufficiently small δ. Then, applying the Schur complements' formula to the second column and the second row of the matrix in (2.56), we obtain the expression of the form $-2\delta + O(\delta^2)$, which is negative for small δ. Therefore, (2.56) and, hence, (2.52) are feasible for small δ. The proof is completed.

In the constant-delay case $d = 0$, the condition $0 \leq \mu_1 < \mu_0$, which ensures the exponential stability of the wave equation with a mixed Dirichlet–Neumann boundary condition and with $a = 1$, $a_0 = a_1 = 0$, was carried out in [85], where it was also shown that if $\mu_1 \geq \mu_0$, there exists a sequence of arbitrary small delays that destabilize the system.

In order to derive delay-dependent stability conditions for (2.44), (2.30) with $\mu_1 = 0$, we utilize the conditions of Theorem 5. Since the delay appears only in z, V is chosen as follows:

$$\begin{aligned} V &= ap_3 \int_0^\pi z_\xi^2(\xi,t)d\xi + \int_0^\pi [z(\xi,t)\ z_t(\xi,t)] P_0 \\ &\times \begin{bmatrix} z(\xi,t) \\ z_t(\xi,t) \end{bmatrix} d\xi + \int_0^\pi \Big[hr \int_{-h}^0 \int_{t+\theta}^t z_t^2(\xi,s) e^{2\delta(s-t)} ds d\theta \\ &+ s \int_{t-h}^t z^2(\xi,s) e^{2\delta(s-t)} ds + q \int_{t-\tau}^t z^2(\xi,s) e^{2\delta(s-t)} ds \Big] d\xi, \\ P_0 &= \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix},\ P_w = \begin{bmatrix} ap_3 + p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0, \end{aligned}$$

where $r > 0,\ s > 0,\ q \geq 0$. Then the operators P, Q, R in (2.22) are given by

$$
\begin{array}{l}
P = diag\left\{-ap_3 \frac{\partial^2}{\partial \xi^2}, 0\right\} + P_0 > 0,\ Q = diag\{r, 0\} \geq 0, \\
R = diag\{r, 0\} \geq 0,\ S = diag\{s, 0\} \geq 0.
\end{array}
$$

Since $h^2 A^* R A = diag\{0, h^2 r\},\ h^2 A^* R A_1 = 0,\ h^2 A_1^* R A_1 = 0$, it follows from (2.50) that (2.22) is feasible if the (1,2) and (2,2) terms p_{02}, p_{03} of P_w meet the condition

$$
p_{02} - \delta p_{03} \geq 0 \tag{2.57}
$$

and the following LMI:

$$
\begin{bmatrix}
\phi_w & 0 & P_w \begin{bmatrix} 0 \\ -a_1 \end{bmatrix} + \begin{bmatrix} re^{-2\delta h} \\ 0 \end{bmatrix} \\
* & -(s + r)e^{-2\delta h} & re^{-2\delta h} \\
* & * & -(2r + (1 - d)q)e^{-2\delta h}
\end{bmatrix} < 0 \tag{2.58}
$$

is satisfied provided that $\phi_w = C_\delta^T P_w + P_w C_\delta + diag\{q + s - re^{-2\delta h}, h^2 r\}$. The following result is thus obtained.

Theorem 10. *Given $\delta > 0$, let there exist a 2×2-matrix $P_w > 0$ and scalars $q \geq 0, r > 0, s > 0$ such that LMIs (2.57) and (2.58) hold, with p_{02} and p_{03} as, respectively, the (1,2)- and (2,2)-terms of P_w. Then the delay wave equation (2.44) with $\mu_1 = 0$ and with the Dirichlet boundary condition (2.30), corresponding to $l = \pi$, is exponentially stable with the decay rate δ for all differentiable slow-varying delays subject to (2.6), (2.14), and the inequality (2.53) is satisfied with*

$$
K_w = \lambda_{max}(P_w - diag\{ap_3, 0\}) + \max\{hq + hs, h^3 r/2\} \tag{2.59}
$$

for all $t \geq t_0$. Moreover, if the LMIs (2.37) and (2.58) are feasible with $q = 0$, then the boundary-value problem (2.30), (2.44) is exponentially stable with the decay rate δ for all fast-varying delays (2.6) with no restrictions on the delay derivative. If the LMI (2.58) holds for $\delta = 0$, then the boundary-value problem (2.30), (2.44) is exponentially stable with a sufficiently small decay rate.

As in the case of the delay heat equation, the LMIs (2.52) and (2.58) ensure the exponential stability of the delay ODE $\dot{\bar{z}}(t) = C_0 \bar{z}(t) + A_1 \bar{z}(t - \tau(t)),\quad \bar{z}(t) \in \mathbb{R}^2$ or, equivalently, the scalar delay ODE

$$
\begin{array}{l}
\ddot{y}(t) + \mu_0 \dot{y}(t) + \mu_1 \dot{y}(t - \tau(t)) \\
+(a + a_0) y(t) + a_1 y(t - \tau(t)) = 0.
\end{array} \tag{2.60}
$$

Provided that $l = \pi$, the ODE (2.60) governs the first modal dynamics of the modal representation

$$\begin{aligned}&\ddot{y}_k(t) + \mu_0 \dot{y}_k(t) + \mu_1 \dot{y}_k(t - \tau(t))\\&+(ak^2 + a_0) y_k(t) + a_1 y_k(t - \tau(t)) = 0,\ k = 1, 2, \ldots,\end{aligned} \tag{2.61}$$

of the Dirichlet boundary-value problem (2.30), (2.44) on the eigenfunctions of the operator $\frac{\partial^2}{\partial \xi^2}$. Hence, the results of Theorems 9 and 10 are tight in the sense that the stability of ODE (2.60) is necessary for the stability of (2.30), (2.44).

One can also derive "mixed" stability conditions for the wave equation (2.44) with $\mu_1 \neq 0$: delay-dependent (with respect to delay in z)/delay-independent (with respect to delay in z_t). This is similar to stability analysis of neutral systems, where the delay in the state derivative is treated in the delay-independent manner [86].

Similar to Remark 2, the exponential stability of the uncertain wave equation with the coefficients from the uncertain polytope and with the Dirichlet boundary conditions can be verified by solving the LMIs of Theorems 9 and 10 in the vertices of the polytope.

Example 4. To this end, consider the controlled wave equation

$$z_{tt}(\xi, t) = 0.1 z_{\xi\xi}(\xi, t) - 2 z_t(\xi, t) + u, \tag{2.62}$$

with the boundary condition (2.30) and $l = \pi$. By applying Theorem 10, we establish the open-loop system (2.62) with $u = 0$ to be exponentially stable with the decay rate $\delta = 0.05$. Involving a delayed feedback $u = -z(\xi, t - \tau(t))$ and then verifying the conditions of Theorem 10 show that the closed-loop system is exponentially stable with a greater decay rate $\delta = 0.8$ for all $0 \leq \tau(t) \leq 0.31$.

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