Preface

Researchers in graph theory, including graduate students, are the primary audience for this book. Although it is assumed that the reader is acquainted with the basic concepts in graph theory, this book is self-contained in that all concepts and terminology needed for the topic are clearly presented and illustrated. This book can be used for a reading course, a seminar or a short course for graduate students who are interested in Eulerian and Hamiltonian properties of graphs as well as covering walks of graphs in general. In addition, this book contains the background needed to begin a research program on a variety of topics concerning covering walks in graphs and provides easy access to recent results and open problems in this area of research.

Many theorems involving walks in graphs can be traced back to problems that led to some of the best-known and most-studied concepts in graph theory. A walk in a graph G begins at some vertex u of G, proceeds to an edge $e = uu'$ incident with u, then proceeds to u' and next to an edge incident with u' (possibly e again). This continues until the procedure stops at some vertex v, producing a $u - v$ walk W. If W contains every edge of G, then W is an edge-covering walk, while if W contains every vertex of G, then it is a vertex-covering walk.

Graph theory is considered to have begun in 1736 when the great Swiss mathematician Leonhard Euler solved the famous Königsberg Bridge Problem, which asks whether it is possible to walk about the city of Königsberg (located in Prussia at the time) and cross each of its seven bridges in the city exactly once. Eventually it was seen that the Königsberg Bridge Problem could be expressed as a problem in graph theory, an area of mathematics that did not exist in 1736. This led to the concept of Eulerian graphs and later to the more general concept of edge-covering walks in graphs. This is the topic of Chap. 1.

After presenting Euler's characterization of Eulerian graphs as those connected graphs containing only even vertices, graphs containing Eulerian trails are described as those connected graphs containing exactly two odd vertices. Oswald Veblen's characterization of Eulerian graphs as those connected graphs that can be decomposed into cycles is presented. From this theorem, a number of results and conjectures emanated. One of the recent conjectures is the Eulerian Cycle Decomposition Conjecture. Many results obtained on this conjecture are presented.

Several results are presented that deal with connected graphs containing a specified number of odd vertices. While connected graphs with odd vertices do not have circuits containing each edge exactly once, they do contain closed walks containing each edge at least once. Determining the minimum number of edges in such a closed walk is the famous Chinese Postman Problem. This led to the recent study of irregular Eulerian walks in which no two edges are encountered the same number of times in the walk.

Some Eulerian graphs contain vertices u having the property that every trail with initial vertex u can be extended to an Eulerian circuit. Graphs with this property are described. The analogous result for connected graphs with two odd vertices is also presented.

Chapter 2 deals with graphs that possess closed vertex-covering walks. This concept emanates from the so-called Icosian Game of the Irish mathematician William Rowan Hamilton and his Around the World puzzle, which dealt with cycles in a dodecahedron containing every vertex. This led to the concepts of Hamiltonian cycles and Hamiltonian graphs. Although no characterization of Hamiltonian graphs has ever been found, many sufficient conditions for a graph to be Hamiltonian have been discovered, the first of which was a 1952 theorem of the Danish mathematician Gabriel Andrew Dirac, who proved that if the minimum degree of a graph is at least half of the order of the graph, then that graph is guaranteed to be Hamiltonian. This was extended somewhat by the Norwegian mathematician Oystein Ore, who showed that if the sum of the degrees of every two nonadjacent vertices in a graph is at least its order, then that graph is Hamiltonian. This led to the study of the closure of graphs and its connection with Hamiltonian graphs.

The best-known necessary condition for a Hamiltonian graph is one that states that every Hamiltonian graph G has the property that when a set of vertices is removed from G, then the number of components in the resulting graph never exceeds the number of vertices removed. This observation led to the well-studied concept of toughness in graphs and its relationship to Hamiltonian graphs. The famous Traveling Salesman Problem is also described.

Of a number of operations defined on a graph that result in new graphs, two of the most common are the line graph and powers of a graph. There have been numerous theorems dealing with those operations that result in Hamiltonian graphs and graphs with related properties.

Although many connected graphs do not contain Hamiltonian cycles, every connected graph contains a closed vertex-covering walk. The major interest here is the minimum length of such walks in a graph, which is the Hamiltonian number of the graph. Recent research showed that the Hamiltonian number of a graph G can be determined by computing the sum of the distances of consecutive terms in each cyclic ordering of the vertices of G and then finding the minimum of these sums. The maximum of these sums is the upper Hamiltonian number of G and results on this topic are presented as well in Chap. 2. Furthermore, several results on the set of all such numbers obtained in this manner are discussed.

In Chap. 3, the emphasis changes from closed vertex-covering walks in a connected graph G to the recent research topic of open vertex-covering walks in G. Such a walk of minimum length is referred to as a traceable walk and its length is the traceable number of G. Here too it is seen that the traceable number of G

can be obtained by first computing the sum of the distances of consecutive terms in each linear ordering of the vertices of G. The minimum value of such a sum is the traceable number of G; the maximum such sum is the upper traceable number of G. Special attention is given to these two parameters of trees. Also, comparisons between the Hamiltonian and traceable numbers are described as are comparisons between the upper Hamiltonian and upper traceable numbers of given graphs.

One of the differences between closed vertex-covering walks and open vertex-covering walks in a graph is that, for an open vertex-covering walk, its length depends on which vertex the walk begins (or ends). For this reason, for each vertex v in a graph, sequences of the vertices of the graph with v as their initial term and the resulting sums of distances of consecutive terms are considered. The minimum such sum is the traceable number of v. Related concepts such as the maximum vertex-traceable numbers of graphs, traceably singular graphs, and the total traceable numbers of graphs are described.

The numerous new areas of research presented in this book have led to a number of conjectures and open problems, which are described throughout the book.

Nagoya, Aichi, Japan Futaba Fujie
Kalamazoo, MI, USA Ping Zhang

http://www.springer.com/978-1-4939-0304-7

Covering Walks in Graphs
Fujie, F.; Zhang, P.
2014, XIV, 110 p. 37 illus., 11 illus. in color., Softcover
ISBN: 978-1-4939-0304-7