Hamiltonian Walks

2

2.1 The Icosian Game

The year 1857 saw the introduction of a two-person game called the *Icosian Game*. In this game, one player is to place some of 20 given pieces on the points of a playing board (in the shape of dodecahedron) as shown in Fig. 2.1 so that successive pieces are placed along the lines of the board. These pieces may be required to fulfill other conditions as well. The other player then has the responsibility to place the remaining pieces on the remaining points in such a way that every consecutive pair of pieces lie along a line of the board and that the twentieth piece lies along a line of the first piece. Sometimes this can be done, sometimes it cannot.

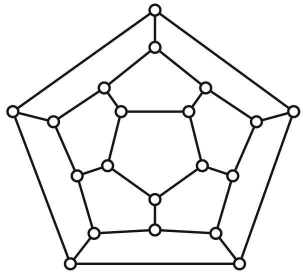
This game was the invention of William Rowan Hamilton. Before discussing this game in more detail, let us go back in history to learn some facts about Hamilton.

Hamilton was born in Dublin, Ireland in 1805. He was brought up by his uncle, who educated him. Very early on, it became clear that Hamilton was an extraordinarily talented individual. Indeed, at age 5, young William had mastered the languages Latin, Greek, and Hebrew. By the time he reached 12 years of age, he had become quite accomplished with mental arithmetic. At age 15, he had studied the work of Sir Isaac Newton and Pierre-Simon Laplace. That Hamilton discovered an error in Laplace's work on celestial mechanics brought him to the attention of the Astronomer Royal of Dublin. At the age of 18, Hamilton became a student at Trinity College Dublin. There he placed first in every examination in every subject. During his first year he was awarded "optime" in classics, which had not been awarded in 20 years. Later he was awarded "optime" in physics, an unheard of distinction to receive two such awards in different subjects. His education stopped at age 21 when he became Professor of Astronomy at Trinity College. With this came the title of Royal Astronomer of Ireland.

In 1832 Hamilton predicted that a ray of light passing through a biaxial crystal would be refracted into the shape of a cone. When this was experimentally confirmed, this resulted in a major scientific announcement. Hamilton was knighted for his discovery in 1835, thereby becoming *Sir* William Rowan Hamilton.

F. Fujie and P. Zhang, *Covering Walks in Graphs*, SpringerBriefs in Mathematics,
DOI 10.1007/978-1-4939-0305-4_2,

Fig. 2.1 The Icosian Game



In 1835 Hamilton observed that complex numbers could be represented as ordered pairs of real numbers. For the next 8 years, Hamilton attempted to extend his theory to ordered triples but he was never successful. In 1843, however, while walking across the Brougham Bridge on the Royal Canal in Dublin, Hamilton discovered a set of 4-dimensional numbers $a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$, where a, b, c, and d are real numbers, called *quaternions*. This was the first example of non-commutative algebra. He was so excited that he carved the formula he had discovered into the bridge. Today, on a plaque attached to the bridge, the following is written:

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$

As a consequence of doing this, Hamilton had essentially introduced the cross product and dot product for vector algebra.

In 1856 Hamilton discovered a non-commutative algebraic structure referred to as the *Icosian Calculus*. This discovery came from his failed attempts to find an algebra of ordered triples that would reflect the three Cartesian axes in the Euclidean 3-space just as ordered pairs reflect the two Cartesian axes in the Euclidean plane. The symbols Hamilton used in his Icosian Calculus represented moves between vertices on a dodecahedron. This led to Hamilton's invention of the Icosian Game, which he used as a means of illustrating and popularizing his mathematical discovery. The Icosian Game was introduced to the public in 1857 at a meeting of the British Association in Dublin.

How Hamilton thought of connecting his Icosian Calculus to traveling along the edges of a dodecahedron is unknown. The mathematician John Graves was one of Hamilton's best friends. In 1859 a friend of Graves manufactured a version of the Icosian Game in the form of a small table consisting of a game board with legs, which was sent to Hamilton. Graves put Hamilton in contact with John Jaques,

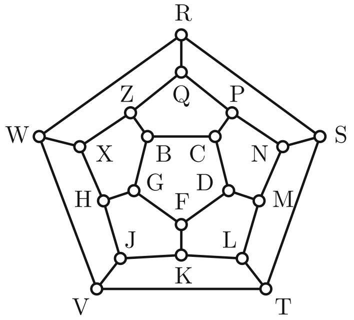


Fig. 2.2 The traveler version of the Icosian Game

whose company John Jaques and Son manufactured toys and games. Hamilton sold the rights to his game for 25 pounds to this manufacturer, which was later known as John Jaques of London and then Jaques of London. This company, still in existence after two centuries, is known for the chess sets it sells. It also invented the game of ping pong.

Two versions of Hamilton's game were manufactured by Jaques, one played on a flat board and another for a "traveler," played on an actual dodecahedron. The traveler version of the game was labeled as:

NEW PUZZLE
TRAVELLER'S DODECAHEDRON
or
A VOYAGE ROUND THE WORLD

Here the 20 vertices of the dodecahedron are labeled with the 20 consonants of the English alphabet (see Fig. 2.2), which stands for the following 20 cities:

B.	Brussels	N.	Naples
C.	Canton	P.	Paris
D.	Delhi	Q.	Quebec
F.	Frankfort	R.	Rome
G.	Geneva	S.	Stockholm
H.	Hanover	T.	Toholsk
J.	Jeddo	V.	Vienna
K.	Kashmere	W.	Washington
L.	London	X.	Xenia
M.	Moscow	Z.	Zanzibar

The goal of this game was to construct a round trip in which each of the 20 cities would be visited exactly once. Hamilton played a role in marketing the game. The preface to the instruction pamphlet, written by Hamilton, began as follows:

In this new Game (invented by

Sir WILLIAM ROWAN HAMILTON, LL.D., & c., of Dublin,

and by him named Icosian from a Greek word signifying 'twenty') a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed along the lines of the figure, and also to fulfill certain other conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in proposing as well as in resolving problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be cyclical, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind. Thus, if B C D F G are the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No. 6 in hole H, as above, it is also allowed (by the supposed conditions) to put No. 7 in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z. Other examples of Icosian Problems, with solutions of some of them, will be found in the following page.

2.2 Hamiltonian Graphs

Hamilton's Icosian Game gave rise to a much-studied class of graphs named for him. A path containing all the vertices of a graph G is a *Hamiltonian path* in G, while a cycle containing all the vertices of G is a *Hamiltonian cycle* in G. If G contains a Hamiltonian path, then G is connected; if G contains a Hamiltonian cycle, then G is 2-connected (that is, $G - v$ is connected for every $v \in V(G)$). A graph is *Hamiltonian* if it contains a Hamiltonian cycle and a graph is *traceable* if it contains a Hamiltonian path. While the graph of the dodecahedron (shown in Figs. 2.1 and 2.2) and the graph G_1 of Fig. 2.3 are both Hamiltonian, none of the graphs G_2, G_3, and G_4 of Fig. 2.3 are Hamiltonian. The graphs G_2 and G_3 have Hamiltonian paths, however. The graph G_4 does not contain a Hamiltonian path. Therefore, G_1, G_2, and G_3 are traceable but G_4 is not.

In 1855, the year before Hamilton invented the Icosian Game, Thomas Penyngton Kirkman studied questions as to whether all vertices of a polyhedron could be visited exactly once by moving along edges of a polyhedron and returning to the starting vertex. So even though Kirkman had considered this concept before Hamilton did, these paths, cycles, and graphs were named for Hamilton, not Kirkman.

Determining conditions under which a graph is Hamiltonian did not occur until 1952 when Gabriel Andrew Dirac [29] introduced a sufficient condition for a graph to be Hamiltonian in terms of the degrees of the vertices of a graph. Gabriel Dirac was the stepson of Paul Adrien Maurice Dirac, who was awarded a Nobel Prize in Physics in 1933. The smallest and largest degrees among the vertices of a graph G are the *minimum degree* $\delta(G)$ and *maximum degree* $\Delta(G)$, respectively, of G.

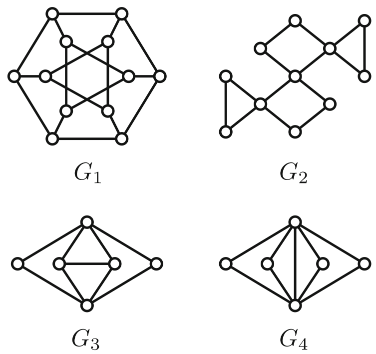


Fig. 2.3 Illustrating Hamiltonian and traceable graphs

Theorem 2.1 (Dirac's Theorem). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then G is Hamiltonian.*

Proof. Assume that this statement is false. Then for some integer $n \geq 3$, there is a non-Hamiltonian graph G of order n and maximum size for which $\deg v \geq n/2$ for each vertex v of G. Surely G is not complete, so G contains pairs of nonadjacent vertices. Let u, v be such a pair. Thus $G + uv$ is Hamiltonian and every Hamiltonian cycle of $G + uv$ necessarily contains the edge uv. This in turn implies that G contains a Hamiltonian $u - v$ path $(u = v_1, v_2, \ldots, v_n = v)$. Let $v_{n+1} = v_1$ and define $A = \{v_i : uv_{i+1} \in E(G)\}$ and $B = \{v_i : vv_i \in E(G)\}$. Since $v_n \notin A \cup B$, it follows that $|A \cup B| \leq n - 1$. Corresponding to each vertex adjacent to u is an element of A; that is, $|A| = \deg u$. Similarly, $|B| = \deg v$. Thus, $|A| + |B| \geq n$.

If there exists a vertex v_{i^*} belonging to $A \cap B$, then $2 \leq i^* \leq n - 2$ and $uv_{i^*+1}, vv_{i^*} \in E(G)$. However then,

$$(u, v_{i^*+1}, v_{i^*+2}, \ldots, v_n = v, v_{i^*}, v_{i^*-1}, \ldots, v_2, v_1 = u)$$

is a Hamiltonian cycle in G, producing a contradiction. Thus $A \cap B = \emptyset$ and so $n \leq |A| + |B| = |A \cup B| \leq n - 1$, which is clearly impossible. $\square$

In 1960 Oystein Ore [56] generalized Dirac's theorem. In fact, the proof of Theorem 2.1 given above also serves as a proof of Ore's theorem.

Theorem 2.2 (Ore's Theorem). *If G is a graph of order $n \geq 3$ such that* $\deg u + \deg v \geq n$ *for every pair u, v of nonadjacent vertices of G, then G is Hamiltonian.*

Suppose that G is a nontrivial graph and consider the graph $H = G \vee K_1$, the join of G and a vertex. This new graph H is certainly Hamiltonian if H satisfies the condition in either Theorems 2.1 or 2.2. Since G is traceable if and only if H is Hamiltonian, we obtain the following as a corollary.

Theorem 2.3. *Let G be a graph of order $n \geq 2$.*
(a) *If $\delta(G) \geq (n-1)/2$, then G is traceable.*
(b) *If $\deg u + \deg v \geq n-1$ for every pair u, v of nonadjacent vertices of G, then G is traceable.*

Following the publication of Ore's theorem was a succession of new sufficient conditions for a graph G to be Hamiltonian in terms of the degrees of the vertices of G, each more general than those that preceded it. The most general of these was based on Ore's theorem and is due to J. Adrian Bondy and Vašek Chvátal.

Let G be a graph of order n. If u_1 and v_1 are two nonadjacent vertices such that $\deg_G u_1 + \deg_G v_1 \geq n$, then join u_1 and v_1 by an edge producing the graph $G_1 = G + u_1v_1$. If, in G_1, there are two nonadjacent vertices u_2 and v_2 such that $\deg_{G_1} u_2 + \deg_{G_1} v_2 \geq n$, then join u_2 and v_2 by an edge producing the graph $G_2 = G_1 + u_2v_2$. This procedure is continued until no such pairs of nonadjacent vertices remain. This final graph is called the *closure* of G and is denoted by $CL(G)$. Adding the edges described above can occur in many different orders but the resulting graph is always the same graph, namely $CL(G)$. The primary importance of this concept lies in the following theorem [12].

Theorem 2.4 (Bondy and Chvátal's Theorem). *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Proof. First, if a graph G is Hamiltonian, then surely $CL(G)$ is Hamiltonian. Suppose then that G is a graph of order $n \geq 3$ such that $CL(G)$ is Hamiltonian. Let $G, G_1, G_2, \ldots, G_{k-1}, G_k = CL(G)$ be a sequence of graphs produced during the process of obtaining $CL(G)$. In particular, $CL(G) = G_k = G_{k-1} + uv$, where u and v are nonadjacent vertices in G_{k-1} and $\deg_{G_{k-1}} u + \deg_{G_{k-1}} v \geq n$. Therefore, according to the proof of Theorem 2.1, G_{k-1} is Hamiltonian. Proceeding backwards, we see that $G_{k-2}, G_{k-3}, \ldots, G_2, G_1$ and finally G are Hamiltonian. □

By Theorem 2.4, a graph G of order at least 3 is Hamiltonian if $CL(G)$ is complete. A graph can be Hamiltonian without its closure being complete, however. For example, for the n-cycles C_n, which are clearly Hamiltonian, $CL(C_n) = C_n \neq K_n$ for $n \geq 5$.

The sufficient conditions presented for a graph to be Hamiltonian in Dirac's and Ore's theorems are just that, namely, they are sufficient only. That is, a graph can be Hamiltonian without satisfying either of these conditions. The fact that the cycles of order at least 5 are Hamiltonian is not a consequence of any of the theorems by Dirac, Ore, and Bondy and Chvátal. While Dirac's theorem requires every vertex of a graph G of order $n \geq 3$ to have degree at least $n/2$ in order to guarantee that G is Hamiltonian and Ore's theorem requires many vertices to have degree at least $n/2$, neither theorem can be applied if the maximum degree of G is less than $n/2$.

There are some sufficient conditions for an r-regular graph G to be Hamiltonian that do not require $r \geq |V(G)|/2$. One of these is due to Crispin Nash-Williams [52].

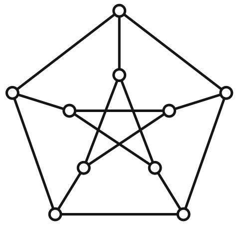


Fig. 2.4 The Petersen graph

Theorem 2.5 (Nash-Williams' Theorem). *Every r-regular graph of order $2r + 1 \geq 5$ is Hamiltonian.*

Proof. Let G be an r-regular graph of order $2r + 1$, where then r is a positive even integer. Since C_5 is the unique 2-regular graph of order 5, suppose that $r \geq 4$. By Theorem 2.3 (a), we may assume that G contains a Hamiltonian path, say $P = (v_0, v_1, \dots, v_{2r})$. Suppose, however, that G is not Hamiltonian. For $1 \leq i \leq 2r$, it follows that if $v_0 v_i \in E(G)$, then $v_{i-1}v_{2r} \notin E(G)$; for otherwise, $(v_0, v_i, v_{i+1}, \dots, v_{2r}, v_{i-1}, v_{i-2}, \dots, v_0)$ is a Hamiltonian cycle of G, which is impossible. Since $\deg_G v_0 = \deg_G v_{2r} = r$, it follows that exactly one of $v_0 v_i$ and $v_{i-1}v_{2r}$ belongs to $E(G)$ for $1 \leq i \leq 2r$. We consider the following two cases.

Case 1. $N(v_0) = \{v_1, v_2, \dots, v_r\}$. Then $N(v_{2r}) = \{v_r, v_{r+1}, \dots, v_{2r}\}$. If $G - v_r$ is disconnected, then $G - v_r$ is a subgraph of $2K_r$. However then, either $\delta(G) < r$ or $\deg_G v_r = 2r$, which contradicts the fact that G is r-regular. Thus, v_r is not a cut-vertex and so $v_{i^*}v_{j^*} \in E(G)$ for some i^* and j^* satisfying $1 \leq i^* \leq r - 1$ and $r + 1 \leq j^* \leq 2r - 1$. However, then, this produces a Hamiltonian cycle $(v_0, v_1, \dots, v_{i^*}, v_{j^*}, v_{j^*+1}, \dots, v_{2r}, v_{j^*-1}, v_{j^*-2}, \dots, v_{i^*+1}, v_0)$. Thus, this case never occurs.

Case 2. $N(v_0) \neq \{v_1, v_2, \dots, v_r\}$. Then there exists an integer i^* such that $v_0 v_{i^*} \notin E(G)$ and $v_0 v_{i^*+1} \in E(G)$. Thus, $v_{2r}v_{i^*-1} \in E(G)$ and $v_{2r}v_{i^*} \notin E(G)$ and so we have a $2r$-cycle $C = (v_0, v_1, \dots, v_{i^*-1}, v_{2r}, v_{2r-1}, \dots, v_{i^*+1}, v_0)$. Renaming the vertices, let us write $C = (u_1, u_2, \dots, u_{2r}, u_1)$. Since $\deg v_{i^*} = r$ and G is not Hamiltonian, it follows that either $N(v_{i^*}) = \{u_1, u_3, \dots, u_{2r-1}\}$ or $N(v_{i^*}) = \{u_2, u_4, \dots, u_{2r}\}$, say the former. Then for each u_{2i} $(1 \leq i \leq r)$, the cycle obtained from C by replacing u_{2i} by v_{i^*} is also a $2r$-cycle. This implies that $N(v_{i^*}) = N(u_{2i})$ for $1 \leq i \leq r$. It then follows that $\deg u_1 \geq r + 1$, which is again a contradiction. □

If G is r-regular and $|V(G)| \geq 2r + 2$, then G may or may not be connected. (Consider, for example, $G = 2K_{r+1}$.) Jackson [43] has shown that if G is 2-connected and its order is at most $3r$, then G is guaranteed to be Hamiltonian. In fact, Zhu, Liu, and Yu [72] showed that $3r$ can be replaced by $3r + 1$ by excluding the *Petersen graph* (see Fig. 2.4).

Theorem 2.6. *Every 2-connected r-regular graph of order at most $3r + 1$ is Hamiltonian unless it is the Petersen graph.*

The following is therefore a consequence of the above theorem.

Theorem 2.7. *Let r be a positive integer. If G is an r-regular graph of order $2r+2$, then either G is Hamiltonian or $G = 2K_{r+1}$.*

Proof. Let G be an r-regular graph of order $2r+2$. If $r = 1$, then clearly $G = 2K_2$. For $r \geq 2$, it suffices to show by Theorem 2.6 that if G is not 2-connected, then $G = 2K_{r+1}$. Since G is disconnected if and only if $G = 2K_{r+1}$, we may assume that G is connected and has a cut-vertex v. Clearly G is not complete. Also, G being r-regular implies that each component of $G - v$ contains at least r vertices. Thus, $G - v$ consists of exactly two components, say G_1 and G_2, whose orders are r and $r + 1$, respectively. However then, the vertices in G_1 have degree r in G only if $N(v) = V(G_1)$ and $G_1 = K_r$, which then implies that G is already disconnected even without deleting v. □

Another sufficient condition for a graph to be Hamiltonian is due to Chvátal and Paul Erdős [27]. Before presenting their result, let us state a few lemmas.

For a graph G that is not a forest, the length of a longest cycle in G is called the *circumference* $\mathrm{cir}(G)$ of G. Therefore, G is Hamiltonian if and only if $\mathrm{cir}(G) = |V(G)| \geq 3$.

Lemma 2.1. *If G is a graph with $\delta(G) \geq 2$, then G contains cycles and* $\mathrm{cir}(G) \geq \delta(G) + 1$.

Proof. That G is not a forest is immediate. Suppose that $P = (v_0, v_1, v_2, \ldots, v_\ell)$ is a longest path in G. Since P cannot be extended any further, the neighborhood of v_0 must be a subset of $V(P)\backslash\{v_0\}$. Thus, there exists an integer ℓ' satisfying $\delta(G) \leq \deg v_0 \leq \ell' \leq \ell$ such that $v_{\ell'}$ belongs to P and $v_0 v_{\ell'} \in E(G)$. Thus, $(v_0, v_1, \ldots, v_{\ell'-1}, v_{\ell'}, v_0)$ is a cycle in G whose length is at least $\delta(G) + 1$. □

For a graph G, consider a subset $S \subseteq V(G)$. The set S is *independent* if no two vertices in S are adjacent in G. The *independence number* $\alpha(G)$ of G is the maximum number of vertices in an independent set of vertices of G. If G is not complete and $G - S$ is disconnected, then the set S is called a *vertex-cut* of G. The cardinality of a minimum vertex-cut of G is the *connectivity* of G, denoted by $\kappa(G)$. When G is a complete graph, its connectivity is defined as $|V(G)| - 1$. Observe that $1 \leq \alpha(G) \leq |V(G)|$ while $0 \leq \kappa(G) \leq |V(G)| - 1$. If k is a positive integer satisfying $k \leq \kappa(G)$, then G is said to be *k-connected*. In other words, for a k-connected graph G, deleting $k - 1$ arbitrary vertices from G does not disconnect the graph. See [24, p. 92], for example, for the proof of the following well-known result.

Theorem 2.8. *For every graph G, $\kappa(G) \le \delta(G)$.*

Theorem 2.9 (Chvátal-Erdős' Theorem). *Let G be a k-connected graph of order at least 3. If $k \ge \alpha(G)$, then G is Hamiltonian.*

Proof. Suppose that G is a k-connected graph containing more than two vertices, where $k \ge \alpha(G)$, and assume, to the contrary, that G is not Hamiltonian. Since $\alpha(G) = 1$ if and only if G is a complete graph, which is Hamiltonian, we may assume that $\alpha(G) \ge 2$. Lemma 2.1 and Theorem 2.8 then imply that $3 \le k + 1 \le \mathrm{cir}(G) \le |V(G)| - 1$. Suppose that $C = (v_1, v_2, \ldots, v_\ell, v_{\ell+1} = v_1)$ is a cycle in G whose length is $\ell = \mathrm{cir}(G)$ and let H be a component in $G - V(C)$.

Consider the subsets S and S' of $V(C)$ such that, for $1 \le i \le \ell$, $v_i \in S$ if and only if v_i is adjacent to a vertex in H if and only if $v_{i+1} \in S'$. Then S is nonempty since G is connected. Note also that if S contains two distinct vertices, say u and v, then G contains a $u - v$ path of length at least 2 each of whose internal vertices is a vertex in H. Therefore, by the fact that C is a longest cycle in G, no two consecutive vertices on C belong to S, that is, $S \cap S' = \emptyset$. Thus, S is a vertex-cut of G and so $|S'| = |S| \ge k$.

We now verify that S' is an independent set. If this is not the case, then there are integers i and j satisfying $1 \le i < j \le \ell$ such that $v_{i+1}, v_{j+1} \in S'$ and $v_{i+1}v_{j+1} \in E(G)$. Then G contains the $v_i - v_j$ path P, where

$$P = \begin{cases} (v_1, v_\ell, v_{\ell-1}, \ldots, v_{j+1}, v_2, v_3, \ldots, v_j) & \text{if } i = 1 \\ (v_i, v_{i-1}, \ldots, v_1, v_{i+1}, v_{i+2}, \ldots, v_\ell) & \text{if } j = \ell \\ (v_i, v_{i-1}, \ldots, v_1, v_\ell, v_{\ell-1}, \ldots, v_{j+1}, v_{i+1}, v_{i+2}, \ldots, v_j) & \text{otherwise.} \end{cases}$$

In each case, $V(P) = V(C)$. On the other hand, since both v_i and v_j belong to S, there is also a $v_i - v_j$ path Q of length at least 2 in G such that $V(C) \cap V(Q) = \{v_i, v_j\}$. However, this is impossible since P and Q form a cycle in G whose length is at least $\ell + 1 = \mathrm{cir}(G) + 1$. Thus, S' is independent, as claimed. Furthermore, for an arbitrary vertex $x \in V(H)$, the set $S' \cup \{x\}$ is independent. However then, $k + 1 \le |S' \cup \{x\}| \le \alpha(G)$, which contradicts our original assumption that $k \ge \alpha(G)$. □

Therefore, a graph of order at least 3 must be Hamiltonian provided its connectivity is at least as large as its independence number. The complete bipartite graph $K_{n,n+1}$ $(n \ge 1)$ shows that the bound is sharp as $\kappa(K_{n,n+1}) = n = \alpha(K_{n,n+1}) - 1$. Note that $K_{n,n+1}$ is traceable although it is not Hamiltonian.

If G is a nontrivial k-connected graph with $k \ge \alpha(G) - 1$, then consider the graph $H = G \vee K_1$, the join of G and a vertex. One can verify that H is $(k + 1)$-connected and $\alpha(H) = \alpha(G)$. Hence, Theorem 2.9 guarantees that H contains a Hamiltonian cycle, which in turn implies that G contains a Hamiltonian path.

Theorem 2.10 ([27]). *If G is a nontrivial k-connected graph, where $k \ge \alpha(G) - 1$, then G is traceable.*

2.3 The Toughness of a Graph

While a number of sufficient conditions have been derived for a graph to be Hamiltonian, there is one well-known and useful necessary condition. We have already stated that every Hamiltonian graph is 2-connected, that is, no Hamiltonian graph contains a cut-vertex. Stated in another manner, no Hamiltonian graph G contains a vertex v such that $G - v$ contains two or more components. In fact, every Hamiltonian graph satisfies an even more general condition. The number of components in a graph G is denoted by $k(G)$.

Theorem 2.11. *Let G be a Hamiltonian graph. Then $k(G - S) \leq |S|$ for every nonempty proper subset S of $V(G)$.*

Proof. Let S be a nonempty proper subset of $V(G)$. Suppose that $k(G - S) = k \geq 2$ and that $G_1, G_2, \ldots, G_k$ are the k components of $G - S$. Therefore, each vertex in G_i can only be adjacent to vertices in S or to other vertices in G_i. Let $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a Hamiltonian cycle in G, where $n = |V(G)|$, and let $i_j = \max\{i : v_i \in V(G_j), 1 \leq i \leq n\}$ for $1 \leq j \leq k$. Thus, the set $\{v_{i_1+1}, v_{i_2+1}, \ldots, v_{i_k+1}\}$ is a subset of S containing k distinct vertices. It then follows that $|S| \geq k = k(G - S)$. □

Necessary conditions are typically most useful when stated in their contrapositive forms.

Theorem 2.12. *If G is a graph containing a nonempty proper subset S of $V(G)$ such that $k(G - S) > |S|$, then G is not Hamiltonian.*

As a consequence of Theorem 2.11, if G is Hamiltonian, then $|S|/k(G - S) \geq 1$ for every nonempty proper subset S of $V(G)$. This observation led Chvátal to introduce a new concept in 1973.

For a positive real number t, a noncomplete graph G is *t-tough* if

$$\frac{|S|}{k(G - S)} \geq t$$

for every vertex-cut S of G. The *toughness* $t(G)$ of G is the maximum real number t for which G is t-tough. For a complete graph K_n, its toughness is taken as $t(K_n) = (n - 1)/2$.

By our earlier observations, every Hamiltonian graph is 1-tough. The converse is not true, however. For example, the graph G of Fig. 2.5 is 1-tough but is not Hamiltonian. In addition, it is well known that the Petersen graph P is not Hamiltonian; yet P is not only 1-tough, it is $(4/3)$-tough. In 1973, Chvátal [26] made the following conjecture.

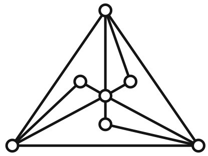


Fig. 2.5 A non-Hamiltonian 1-tough graph

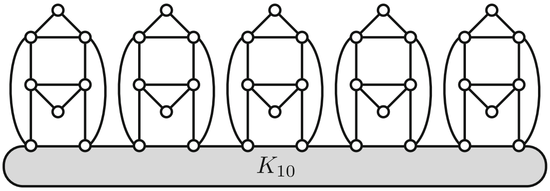


Fig. 2.6 Constructing the Bauer-Broersma-Veldman graph

Chvátal's Toughness Conjecture. *There exists a real number t_0 such that every t_0-tough graph is Hamiltonian.*

In 1985, Enomoto, Jackson, Katernis, and Saito [31] proved that every 2-tough graph contains a 2-factor (a spanning 2-regular subgraph) and that there is no real number $t < 2$ for which every t-tough graph contains a 2-factor. This added credence to the following.

The 2-Tough Conjecture. *Every 2-tough graph is Hamiltonian.*

However, in 2000, Bauer, Broersma, and Veldman [9] showed that the 2-Tough Conjecture is false by constructing the so-called *Bauer-Broersma-Veldman graph*, a graph that is 2-tough but not Hamiltonian. This graph is formed by taking the join of K_2 and the graph of order 40 shown in Fig. 2.6. In fact, Bauer, Broersma, and Veldman established the following.

Theorem 2.13 ([9]). *For every real number $t < 9/4$, there is a t-tough nontraceable graph.*

Hamiltonian-Connected Graphs

A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u - v$ path for every two distinct vertices u and v of G. Thus, every Hamiltonian-connected graph of order at least 3 is Hamiltonian (but the converse is of course false as cycles show). In fact, every edge in a Hamiltonian-connected graph belongs to a Hamiltonian cycle in that graph.

A number of conditions for a graph to be Hamiltonian-connected are known that are similar to those for a graph to be Hamiltonian. The following results are analogous to Theorems 2.1, 2.2, 2.12, and 2.9, respectively.

Theorem 2.14 ([58]). *If G is a graph of order $n \geq 4$ such that* $\deg u + \deg v \geq n+1$ *for every pair u, v of nonadjacent vertices of G, then G is Hamiltonian-connected. Consequently, $\delta(G) \geq (n+1)/2$ implies that G is Hamiltonian-connected.*

Theorem 2.15. *If G is a graph containing a nonempty proper subset S of $V(G)$ such that $k(G-S) > |S| - 1$, then G is not Hamiltonian-connected.*

Theorem 2.16 ([27]). *If G is a k-connected graph satisfying $k \geq \alpha(G) + 1$, then G is Hamiltonian-connected.*

2.4 The Traveling Salesman Problem

One of the best known problems concerning Hamiltonian cycles is of a more applied nature.

The Traveling Salesman Problem. *A salesman wishes to make a round trip that visits certain cities once each. He knows the distance between each pair of cities. What is the minimum total distance of such a round trip?*

This problem can be expressed in terms of weighted graphs. In particular, let G be a weighted complete graph whose vertices are the cities and where each edge uv is assigned a weight equal to the distance between u and v. The *weight* $w(C)$ of a Hamiltonian cycle C in G is the sum of the weights of the edges of C. Finding a solution to the Traveling Salesman Problem then consists of determining the minimum weight of a Hamiltonian cycle in G.

If the number n of cities involved is large, then the number of Hamiltonian cycles in G that need to be investigated is quite large. We can consider a Hamiltonian cycle as beginning at any vertex v. Then the remaining $n-1$ vertices can follow v in any of $(n-1)!$ orders. This produces $(n-1)!$ Hamiltonian cycles whose weights we need to compute. In fact, we need *only* consider $(n-1)!/2$ Hamiltonian cycles as we would obtain the same sum if the order in which the vertices appear in a cycle were reversed.

Even though the Traveling Salesman Problem is an extremely difficult problem in general, there are instances where this problem has been solved for a large number of cities. In 1998 Applegate, Bixby, Chvátal, and Cook [5] solved a Traveling Salesman Problem for the 13,509 largest cities in the United States (those whose population exceeded 500 at that time). They also solved a Traveling Salesman Problem for 15,113 German cities in 2001 and for 24,978 Swedish cities in 2004. Their ultimate goal was to solve the Traveling Salesman Problem for every registered city or town in the world plus a few research bases in Antartica (1,904,711 locations in all). In 2006, the four wrote a book titled *The Traveling Salesman Problem: A Computational Study* [6], in which they describe the history of the Traveling Salesman Problem as well as the method they used to solve a range of largescale problems. In 2012 Cook [28] wrote a book titled *In Pursuit of the Traveling Salesman* for a more general audience.

2.5 Line Graphs and Powers of Graphs

There are two operations on graphs in which much attention has been focused regarding Hamiltonian properties of the resulting graphs.

Line Graphs

The *line graph* $L(G)$ of a nonempty graph G is that graph whose vertices correspond to the edges of G where two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. The line graph of a graph G is therefore Hamiltonian provided the m edges of G can be listed as $e_1, e_2, \dots, e_m, e_{m+1} = e_1$ in such a way that e_i and e_{i+1} are adjacent for $i = 1, 2, \dots, m$. As a consequence of this observation, we have the following.

Theorem 2.17. *The line graph of an Eulerian graph is Hamiltonian.*

Each of the three graphs G_1, G_2, G_3 in Fig. 2.7 is neither Eulerian nor Hamiltonian, while $L(G_1)$ is Hamiltonian but not Eulerian, $L(G_2)$ is Eulerian but not Hamiltonian, and $L(G_3)$ is neither Eulerian nor Hamitonian. The graph G_1 shows that the converse of Theorem 2.17 is not true. In fact, Harary and Nash-Williams [39] characterized those graphs whose line graphs are Hamiltonian. A circuit C in a graph G is called a *dominating circuit* if every edge of G is incident with at least one vertex of C.

Theorem 2.18. *Let G be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if either G is a star of size at least 3 or G contains a dominating circuit.*

Proof. Let m be the size of G. If G is the star $K_{1,m}$, then $L(G) = K_m$, which is Hamiltonian for $m \geq 3$. Suppose then that G contains a dominating circuit $C = (v_1, v_2, \dots, v_\ell, v_1)$. It suffices to show that there exists an ordering $s : e_1, e_2, \dots, e_m$ of the m edges of G such that e_i and e_{i+1} are adjacent edges of G, for $1 \leq i \leq m-1$, as are e_1 and e_m, since such an ordering s corresponds to a Hamiltonian cycle of $L(G)$. Begin s by selecting, in any order, all edges of G incident with v_1 that are not edges of C, followed by the edge v_1v_2. At each successive vertex v_i, $2 \leq i \leq \ell-1$, select, in any order, all edges of G incident with v_i that are neither edges of C nor previously selected edges, followed by the edge v_iv_{i+1}. This process terminates with the edge $v_{\ell-1}v_\ell$. The ordering s is completed by adding the edge $v_\ell v_1$. Since C is a dominating circuit of G, every edge of G appears exactly once in s. Furthermore, consecutive edges of s as well as the first and last edges of s are adjacent in G.

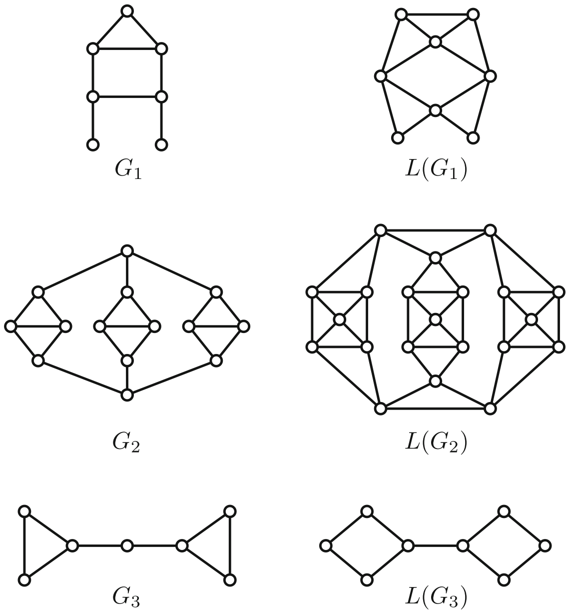


Fig. 2.7 Graphs and their line graphs

Conversely, suppose that G is not a star but $L(G)$ is Hamiltonian. We show that G contains a dominating circuit. Since $L(G)$ is Hamiltonian, there is an ordering $s : e_1, e_2, \ldots, e_m$ of the m edges of G such that e_i and e_{i+1} are adjacent edges of G for $1 \le i \le m-1$, as are e_1 and e_m. For $1 \le i \le m-1$, let v_i be the vertex of G incident with both e_i and e_{i+1}. (Note that $1 \le i \ne j \le m-1$ does not necessarily imply that $v_i \ne v_j$.) Since G is not a star, there is a smallest integer i_1 exceeding 1 such that $v_{i_1} \ne v_1$. Thus, the edges $e_1, e_2, \ldots, e_{i_1-1}$ are incident with v_1 and $e_{i_1} = v_1 v_{i_1}$. Next, let i_2 (if it exists) be the smallest integer exceeding i_1 such that $v_{i_2} \ne v_{i_1}$. Then the edges $e_{i_1}, e_{i_1+1}, \ldots, e_{i_2-1}$ are incident with v_{i_1} and $e_{i_2} = v_{i_1} v_{i_2}$. Continuing in this fashion, we finally arrive at a vertex v_{i_ℓ} such that $e_{i_\ell} = v_{i_{\ell-1}} v_{i_\ell}$, where $v_{i_\ell} = v_{m-1}$. Note that (i) v_1 is incident with the edges $e_1, e_2, \ldots, e_{i_1}$, (ii) v_{i_j} $(1 \le j \le \ell - 1)$ is incident with the edges $e_{i_j}, e_{i_j+1}, \ldots, e_{i_{j+1}}$, and (iii) v_{i_ℓ} is incident with the edges $e_{i_\ell}, e_{i_\ell+1}, \ldots, e_m$. Since each edge of G appears exactly once in s and $1 < i_1 < i_2 < \cdots < i_\ell \le m-1$, we obtain a trail

$$T = (v_1, v_{i_1}, v_{i_2}, \ldots, v_{i_\ell} = v_{m-1}) = (e_{i_1}, e_{i_2}, \ldots, e_{i_\ell})$$

in G with the properties that every edge of G is incident with a vertex in T and neither e_1 nor e_m belongs to T. Thus, T itself is a dominating circuit if $v_1 = v_{m-1}$. If not, let v_0 be the vertex of G incident with both e_1 and e_m. Now, if $v_0 \notin \{v_1, v_{m-1}\}$, then $(e_{i_1}, e_{i_2}, \ldots, e_{i_\ell}, e_m, e_1)$ is a dominating circuit. Otherwise, $v_0 = v_1$ or $v_0 = v_{m-1}$. If $v_0 = v_1$, then $e_m = v_1 v_{m-1}$ and so $(e_{i_1}, e_{i_2}, \ldots, e_{i_\ell}, e_m)$ is a dominating circuit. Similarly, if $v_0 = v_{m-1}$, then $(e_{i_1}, e_{i_2}, \ldots, e_{i_\ell}, e_1)$ is a dominating circuit. □

It is an immediate corollary of Theorem 2.18 that $L(G)$ is Hamiltonian whenever G is a graph that is either Eulerian or Hamiltonian. In fact, $L(G)$ is Hamiltonian if G contains an Eulerian spanning subgraph. Another consequence of Theorem 2.18 is that if $L(G)$ is Hamiltonian, then every bridge in G must be a pendant edge. Indeed, if G contains a bridge $e = uv$ where neither u nor v is an end-vertex, then the vertex in $L(G)$ corresponding to e is a cut-vertex. (See the graphs G_1 and G_3 in Fig. 2.7, for example.)

For nearly every connected graph, successively taking the line graphs results in Hamiltonian graphs. For a nonempty graph G, we write $L^0(G)$ for G and $L^1(G)$ for $L(G)$. By $L^2(G)$, we mean $L(L(G))$. More generally, for a positive integer k, the graph $L^k(G)$ is defined as $L(L^{k-1}(G))$.

If G is connected and r-regular, then $L(G)$ is a connected $2(r-1)$-regular graph, that is, $L(G)$ is Eulerian. In this case, $L(G)$ is Hamiltonian if r is even but the same cannot be guaranteed when r is odd. (Consider, for example, the graph G_2 in Fig. 2.7 and the complete graph K_4. Both are 3-regular and $L(G_2)$ is not Hamiltonian while $L(K_4) = K_{2,2,2}$ is.) Taking the line graph again, however, $L^2(G)$ is a $2(2r-3)$-regular graph that is both Eulerian and Hamiltonian for every integer $r \geq 2$. That is, $L^2(G)$ is always both Eulerian and Hamiltonian when G is a connected regular graph containing three or more vertices. For those connected graphs that are not regular, we have the following result due to Chartrand and Wall [17].

Theorem 2.19. *If G is a connected graph with $\delta(G) \geq 3$, then $L^2(G)$ is Hamiltonian.*

Proof. Let v be a vertex of G. Then the edges incident with v in G give rise to a subgraph G_v in $L(G)$ which is isomorphic to a complete graph whose order equals $\deg v$ (≥ 3). Let H_v be a Hamiltonian cycle in G_v and define the spanning subgraph H of $L(G)$ by $V(H) = V(L(G))$ and $E(H) = \bigcup_{v \in V(G)} E(H_v)$. Then H is connected and certainly has a cycle decomposition. Thus, H has an Eulerian circuit, which is a dominating circuit of $L(G)$. The desired result now follows by Theorem 2.18. $\square$

The graphs G_2 and G_3 in Fig. 2.7 show that Theorem 2.19 cannot be improved in general, as neither $L(G_2)$ nor $L^2(G_3)$ is Hamiltonian.

Note that $L(P_n) = P_{n-1}$ for each integer $n \geq 2$. Thus $L^{n-1}(P_n)$ is trivial and $L^k(P_n)$ is not defined for $k \geq n$. Also, $L(C_n) = C_n$ for every $n \geq 3$ and $L(K_{1,3}) = C_3$. Therefore, $L^k(C_n) = C_n$ for $k \geq 0$; while $L^k(K_{1,3}) = C_3$ for $k \geq 1$. If, however, G is a connected graph that is none of a path, cycle, and the star $K_{1,3}$ (called a *claw*), then we eventually arrive at some positive integer k such that $\deg v \geq 3$ for every vertex v of $L^k(G)$. The following is due to Chartrand [15].

Theorem 2.20. *If G is a connected graph that is not a path, then there exists a positive integer K such that $L^k(G)$ is Hamiltonian for every integer $k \geq K$.*

Powers of Graphs

Another operation on graphs is the k-th power of a graph for various positive integers k, a topic discussed in Sect. 1.2. Recall for a connected graph G and a positive integer k that the k-th power G^k of G is the graph with $V(G^k) = V(G)$ and $E(G^k) = \{uv : 1 \leq d_G(u, v) \leq k\}$.

If G is connected, then G^k is complete if and only if $k \geq \text{diam}(G)$. Thus, it suffices to consider G^k only when $1 \leq k < \text{diam}(G)$. Since G is a spanning subgraph of G^k for every positive integer k, the graph G^k is certainly Hamiltonian if G itself is. For a connected graph G of order $n \geq 3$, there is a smallest positive integer k such that G^k is Hamiltonian. That G^3 is Hamiltonian for every connected graph of order 3 or more is a consequence of a result of Sekanina [61]. Recall that a graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u - v$ path for every two distinct vertices u and v of G.

Theorem 2.21. *The cube of every connected graph is Hamiltonian-connected.*

Proof. If H is a spanning subgraph of G and H^3 is Hamiltonian-connected, then G^3 is also Hamiltonian-connected. Hence, it suffices to prove that the cube of every tree is Hamiltonian-connected. We proceed by induction on n, the order of the tree. Since the result is obvious for those graphs having diameter at most 3, assume for every tree of order less than n that its cube is Hamiltonian-connected for some $n \geq 5$. Let T be a tree of order n. For two arbitrary distinct vertices $u, v \in V(T)$, let $(u = v_1, v_2, \ldots, v_{d+1} = v)$ be the unique $u - v$ path in T, where $d = d_T(u, v)$. Also, let T_1 and T_2 be the two components of $T - v_1v_2$, where v_i belongs to T_i for $i = 1, 2$. Thus, for each tree T_i, either T_i is trivial or T_i^3 is Hamiltonian-connected. We consider the following two cases.

Case 1. $v = v_2$, that is, $uv \in E(T)$. For each $i = 1, 2$, let $w_i \in N_{T_i}(v_i)$ if T_i is nontrivial and let $w_i = v_i$ otherwise. (Note that at most one of T_1 and T_2 is trivial.) Then $d_T(w_1, w_2) \leq 3$ and so $w_1w_2 \in E(T^3)$. If we let $P^{(i)}$ be a Hamiltonian $v_i - w_i$ path in T_i^3 (which may be trivial) for $i = 1, 2$, then $P^{(1)}$ and $P^{(2)}$ with the edge w_1w_2 form a Hamiltonian $u - v$ path in T^3.

Case 2. $v \neq v_2$. Then T_2^3 contains a Hamiltonian $v_2 - v$ path P. Let $P^{(1)}$ be a Hamiltonian $v_1 - w_1$ path in T_1^3, as described in Case 1. Since $d_T(w_1, v_2) \leq 2$, the paths $P^{(1)}$ and P with the edge w_1v_2 form a Hamiltonian $u - v$ path in T^3. □

The following is therefore immediate by the previous result.

Theorem 2.22. *The cube of every connected graph of order at least* 3 *is Hamiltonian.*

The graph of order 7 in Fig. 2.5 is the square of the tree obtained by subdividing each edge of $K_{1,3}$ exactly once. We have already seen that this graph is not Hamiltonian. Consequently, even though the cube of every connected graph of order at least 3 is Hamiltonian, such is not the case for the square. On the other hand, in the 1960s, Plummer and Nash-Williams independently conjectured that the square of every 2-connected graph is Hamiltonian. In 1974, this conjecture was verified by Fleischner [33].

Theorem 2.23. *The square of every* 2*-connected graph is Hamiltonian.*

2.6 Hamiltonian Walks and Cyclic Orderings

Let G be a nontrivial connected graph. By a *Hamiltonian walk* in G is meant a closed spanning walk of minimum length in G. Thus, while an Eulerian walk is a closed *edge-covering* walk, not necessarily of minimum length, a Hamiltonian walk is a closed *vertex-covering* walk of *minimum* length. The length of a Hamiltonian walk in G is called the *Hamiltonian number* of G and is denoted by $h(G)$. Therefore, $h(G) \geq |V(G)|$ and $h(G) = |V(G)|$ if and only if G is either Hamiltonian or K_2.

For a connected graph G, recall that $e(G)$ denotes the minimum length of an Eulerian walk in G. We saw in Sect. 1.4 that $|E(G)| \leq e(G) \leq 2|E(G)|$. Therefore, if G is a nontrivial connected graph, then

$$|V(G)| \leq h(G) \leq e(G) \leq 2|E(G)|. \tag{2.1}$$

That the upper bound $2|E(G)|$ for $h(G)$ cannot be improved is shown in the next result due to Goodman and Hedetniemi [37].

Theorem 2.24. *If T is a tree of order $n \geq 2$, then $h(T) = 2(n-1)$.*

Proof. Since the size of a tree of order n is $n-1$, it suffices to show by (2.1) that $h(T) \geq 2(n-1)$. Let W be a Hamiltonian walk in T and consider an edge $uv \in E(T)$. We may assume that u precedes v on W. Since uv is a bridge, it lies on W. We may therefore assume that W begins with u and is immediately followed by v. Since W terminates at u, the vertex u appears a second time on W and this occurrence of u is immediately preceded by v. Thus the edge uv appears at least twice on W. Hence $h(T) \geq 2(n-1)$ and therefore $h(T) = 2(n-1)$. □

The proof of Theorem 2.24 in fact shows that every bridge in a connected graph G must appear at least twice on any Hamiltonian walk in G. Since a Hamiltonian walk in a spanning tree T of G is also a Hamiltonian walk in G, it follows that $h(G) \leq h(T)$. Thus we have the following.

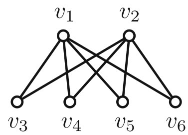


Fig. 2.8 Illustrating cyclic orderings of the vertices in a graph

Theorem 2.25. *If G is a nontrivial connected graph of order n, then*

$$n \leq h(G) \leq 2(n-1).$$

In [23] Chartrand, Thomas, Saenpholphat, and Zhang described an alternative way to compute the Hamiltonian number of a graph. If a graph G of order n is Hamiltonian, then each Hamiltonian walk in G is a Hamiltonian cycle C in G, say $C = (v_1, v_2, \ldots, v_n, v_1)$, and so $h(G) = n$. Since the edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$ belong to G, it follows that there is a *cyclic ordering* $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of the vertices of G such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$.

In general, for a connected graph G of order $n \geq 2$ and a cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of the vertices of G, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}). \tag{2.2}$$

Since $d(v_i, v_{i+1}) \geq 1$ for $i = 1, 2, \ldots, n$, it follows that $d(s) \geq n$. To illustrate this concept, consider the graph $G = K_{2,4}$ shown in Fig. 2.8. The distance between every two vertices of G is either 1 or 2. In every cyclic ordering of the vertices of G, there are either two pairs or four pairs of consecutive vertices with distance 2. Consider, for example, the two cyclic orderings $s_1 : v_1, v_3, v_2, v_4, v_5, v_6, v_1$ and $s_2 : v_1, v_2, v_3, v_4, v_5, v_6, v_1$. Then $d(s_1) = 8$ and $d(s_2) = 10$.

We define the number $h^*(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of the vertices of G. Then $h^*(G) \geq n$ for each connected graph G of order $n \geq 3$ and $h^*(G) = n$ if and only if G is Hamiltonian. In the graph $G = K_{2,4}$ of Fig. 2.8, for every cyclic ordering s of $V(G)$, either $d(s) = 8$ or $d(s) = 10$. Thus $h^*(G) = 8$.

The interest in the parameter $h^*(G)$ lies in the following theorem.

Theorem 2.26. *For every connected graph G, $h^*(G) = h(G)$.*

Proof. The result clearly holds if the order of G is 2 and so we assume that $|V(G)| = n \geq 3$. For a cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of $V(G)$ with $d(s) = h^*(G)$, let $P^{(i)}$ be a $v_i - v_{i+1}$ geodesic in G for $1 \leq i \leq n$. Then the walk obtained by proceeding along the paths $P^{(1)}, P^{(2)}, \ldots, P^{(n)}$ in the given order is a closed spanning walk of G whose length equals $h^*(G)$. Therefore, $h(G) \leq h^*(G)$.

Next, let $W = (x_0, x_1, \ldots, x_\ell)$ be a Hamiltonian walk in G. Hence, $L(W) = h(G) \geq n$. Let $v_i = x_{i-1}$ for $i = 1, 2$. For $3 \leq i \leq n$, let $v_i = x_{j_i}$, where j_i is the smallest positive integer such that $x_{j_i} \notin \{v_1, v_2, \ldots, v_{i-1}\}$. Then $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ is a cyclic ordering of $V(G)$. For $1 \leq i \leq n$, let W_i be the $v_i - v_{i+1}$ subwalk of W. Then

$$h^*(G) \leq d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}) \leq \sum_{i=1}^{n} L(W_i) = L(W) = h(G),$$

which completes the proof. □

By Theorems 2.25 and 2.26, $h^*(T) = h(T) = 2(n-1)$ for every tree T of order n. In fact, trees are the only graphs G satisfying $h(G) = 2(|V(G)| - 1)$.

Theorem 2.27 ([23]). *Let G be a nontrivial connected graph of order n. Then $h(G) = 2(n-1)$ if and only if G is a tree.*

Proof. By Theorem 2.24, it remains to show that $h(G) < 2(n-1)$ if G contains cycles. We proceed by induction on n. If $n = 3$, then $G = K_3$ and the result is immediate. Assume for an integer $n \geq 4$ that $h(H) < 2(n-2)$ for each connected graph H of order $n-1$ that is not a tree. Let G be a connected graph of order n that is not a tree. Since $h(C_n) = n < 2(n-1)$, we may assume that $G \neq C_n$.

We claim that G contains a vertex u such that $G - u$ is connected but not a tree. If G contains cut-vertices, then there is a vertex u in an end-block of G with the desired property. Thus we may assume that G is 2-connected. Since $G \neq C_n$, it follows that G contains a cycle C whose length is less than n. Thus, there is a vertex u that is not a cut-vertex and $G - u$ still contains C. Since $h(G-u) < 2(n-2)$, there is a cyclic ordering $s_0 : v_1, v_2, \ldots, v_{n-1}, v_1$ of the vertices of $G - u$ such that $d(s_0) = h(G-u) < 2(n-2)$. Suppose that u is adjacent to the vertex v_i and consider the cyclic ordering s of $V(G)$ defined by $s : v_1, v_2, \ldots, v_i, u, v_{i+1}, \ldots, v_{n-1}, v_1$. Since $d(u, v_i) = 1$, it follows that $d(u, v_{i+1}) \leq d(u, v_i) + d(v_i, v_{i+1}) = 1 + d(v_i, v_{i+1})$. Hence

$$\begin{aligned} d(s) &= d(s_0) - d(v_i, v_{i+1}) + d(v_i, u) + d(u, v_{i+1}) \\ &\leq d(s_0) - d(v_i, v_{i+1}) + 1 + (1 + d(v_i, v_{i+1})) \\ &< 2(n-2) + 2 = 2(n-1). \end{aligned}$$

Therefore, $h(G) \leq d(s) < 2(n-1)$. □

In Theorem 1.23, we saw for a connected graph G of size $m \geq 1$ that $e(G) = 2m$ if and only if G is a tree. It then follows by Theorem 2.24 that if G is a tree of order n, then $h(G) = e(G) = 2(n-1)$. It was shown in [37] that there are connected graphs G that are not trees and yet $h(G) = e(G)$. In order to describe a class of graphs with this property, it is useful to introduce a new term. A cycle C in a connected graph G is a *cut-cycle* of G if $G - E(C)$ is disconnected.

For an Eulerian walk W of a connected graph G, let M be the multigraph obtained from G by replacing each edge uv of G by i parallel edges, where i equals the number of times the edge uv is encountered on W. In this case, M is said to be *induced* by W. In other words, M is the multigraph induced by $E(W)$. Consequently, M is Eulerian and $V(M) = V(G)$.

Theorem 2.28. *If G is a connected graph such that $h(G) = e(G)$, then every cycle of G is a cut-cycle.*

Proof. If G is a tree, then $h(G) = e(G)$ and the result follows vacuously. Otherwise, we may assume, to the contrary, that $h(G) = e(G)$ and G contains a cycle C that is not a cut-cycle. Therefore, $G - E(C)$ is a connected spanning subgraph of G. Suppose that C is an ℓ-cycle. Let W be an Eulerian walk of G with $L(W) = e(G)$ and let M be the multigraph induced by $E(W)$. Then M is an Eulerian multigraph and $V(M) = V(G)$. Since C is not a cut-cycle, $M - E(C)$ is an Eulerian spanning submultigraph of M. An Eulerian circuit in $M - E(C)$ gives rise to a closed spanning walk in $G - E(C)$ and so in G. Hence $h(G) \le |E(M) - E(C)| = e(G) - \ell < e(G)$, which is a contradiction. □

Theorem 2.28 was strengthened in 1974 by Goodman and Hedetniemi [37].

Theorem 2.29. *If a connected graph G contains a cycle such that more than half of its edges can be removed without disconnecting G, then $h(G) < e(G)$.*

Proof. Let W be an Eulerian walk of G with $L(W) = e(G)$ and let M be the Eulerian multigraph induced by W. Let C be a cycle of G such that $E(C)$ can be partitioned into E_1 and E_2 with $|E_1| > |E_2|$ and $G - E_1$ is connected. Certainly, C is a cycle in M. Also, $M - E_1$ is a connected spanning submiltigraph of M, since $G - E_1 \subseteq M - E_1$ and $G - E_1$ is connected. For each edge $e = uv$ in E_2, we add an additional edge joining u and v in $M - E_1$. This produces an Eulerian multigraph M' whose vertex set is $V(G)$. An Eulerian circuit in M' gives rise to a closed spanning walk in $G - E_1$ and so in G. Hence $h(G) \le |E(M)| - |E_1| + |E_2| = |E(M)| - (|E_1| - |E_2|) < |E(M)| = e(G)$. □

The converse of Theorem 2.29 is false. For example, consider $G = K_{2,3}$, where $h(G) = 6$ and $e(G) = 8$. In this case, the removal of more than half of the edges of every cycle results in a subgraph of $K_1 + P_4$, which is disconnected. On the other hand, if G is an Eulerian graph, then the converse is true, as the next result shows [37]. In order to present a proof of this result, we first make some preliminary observations. We saw that if W is an Eulerian walk of minimum length in a graph G, then each edge of G appears at most twice in W. It was shown in [37] that this is also the case for a Hamiltonian walk in a graph.

Theorem 2.30. *Every edge in a connected graph G appears at most twice in a Hamiltonian walk in G.*

Proof. Suppose that there exists some edge uv of a connected graph G that appears at least three times in a Hamiltonian walk W in G. We may assume that W has one of the following two forms

$$W' = (u, v, W_1, u, v, W_2, u, v, W_3) \text{ and } W'' = (u, v, W_1, u, v, W_2, v, u, W_3),$$

where W_1, W_2, W_3 are (possibly empty) subwalks in W. Let $\overleftarrow{W}_i$ denote the reverse of the subwalk W_i. If $W = W'$, then the walk $(u, \overleftarrow{W}_1, v, W_2, u, v, W_3)$ is a closed spanning walk of G which is shorter than W. This contradicts the defining property of W. Similarly, if $W = W''$, then the walk $(u, \overleftarrow{W}_1, v, W_2, v, u, W_3)$ is a closed spanning walk of G which is shorter than W, another contradiction. □

The following theorem deals with Hamiltonian walks in Eulerian graphs.

Theorem 2.31. *Let G be an Eulerian graph. Then $h(G) < e(G)$ if and only if G contains a cycle such that more than half of its edges can be removed without disconnecting G.*

Proof. By Theorem 2.29, we only show that an Eulerian graph G with $h(G) < e(G)$ has a cycle the removal of more than half of whose edges from G does not disconnect G. Let G be an Eulerian graph and consider a Hamiltonian walk W of G. By Theorem 2.30, we have a partition $\{E_0, E_1, E_2\}$ of $E(G)$ such that $e \in E_i$ if and only if e appears i times in W for $0 \le i \le 2$. Therefore, E_0 is not an edge-cut of G. Also, $h(G) = L(W) = |E_1| + 2|E_2|$ and $e(G) = |E(G)| = |E_0| + |E_1| + |E_2|$. Thus, $h(G) < e(G)$ implies that $|E_0| > |E_2|$.

For a vertex $v \in V(G)$, let E_v be the set of the edges incident with v in G. Of course, $\deg_G v = |E_v|$ is even since G is Eulerian. Also, since the multigraph M induced by $E(W)$ is Eulerian whose vertex set equals $V(G)$, it follows that $\deg_M v = |E_v \cap E_1| + 2|E_v \cap E_2|$ is also even, which in turn implies that $|E_v \cap (E_0 \cup E_2)|$ is even. Thus, the graph G' induced by $E_0 \cup E_2$ is a nonempty spanning subgraph of G in which every nontrivial component is Eulerian. Thus, G' has a cycle decomposition according to Veblen's Theorem and so G' (and G as well) contains a cycle C such that $|E(C) \cap E_0| > |E(C) \cap E_2|$. Now, $G - (E(C) \cap E_0)$ must be connected since E_0 is not an edge-cut of G. □

One of the best known sufficient conditions for a graph to be Hamiltonian is that due to Ore (Theorem 2.2). This theorem can be stated in terms of the Hamiltonian number of a graph as follows.

Theorem 2.32. *If G is a graph of order $n \ge 3$ such that $\deg u + \deg v \ge n$ whenever $uv \notin E(G)$, then $h(G) = n$.*

Jean-Claude Bermond [10] generalized this result by showing that if G is a connected graph of order n for which the minimum degree sum σ of every two nonadjacent vertices of G satisfies $2 \leq \sigma \leq n$, then $h(G)$ is no more than $2n - \sigma$.

Theorem 2.33 (Bermond's Theorem). *Let G be a connected graph G of order $n \geq 3$. If $\deg u + \deg v \geq \sigma$ for every pair u, v of nonadjacent vertices of G and $2 \leq \sigma \leq n$, then $h(G) \leq 2n - \sigma$.*

Among the results obtained by Goodman and Hedetniemi is the following [37].

Theorem 2.34. *Let G be a connected graph having blocks $B_1, B_2, \ldots, B_k$. Then the union of the edges in a Hamiltonian walk for each of the blocks B_i forms a Hamiltonian walk for G and, conversely, the edges in a Hamiltonian walk of G that belong to B_i form a Hamiltonian walk in B_i.*

Theorem 2.34 implies that the study of Hamiltonian walks can be restricted to 2-connected graphs. For k-connected graphs ($k \geq 2$) of a specified diameter, the following appears in [37]. The *diameter* of a connected graph G is the largest distance between two vertices of G and is denoted by $\mathrm{diam}(G)$.

Theorem 2.35. *If G is a k-connected graph of order n having diameter d, then*

$$h(G) \leq 2(n-1) - 2\lfloor k/2 \rfloor (d-1).$$

The *clique number* of a graph G is the maximum order among the complete subgraphs of G. In [59] an upper bound was established for $h(G)$ in terms of the order and clique number of a connected graph G.

Theorem 2.36. *If G is a nontrivial connected graph of order n having clique number ω, then $h(G) \leq 2n - \omega$. Furthermore, for each integer ω with $2 \leq \omega \leq n$, there exists a connected graph G of order n having clique number ω such that $h(G) = 2n - \omega$.*

By Theorem 2.27, trees of order n are the only connected graphs of order n with Hamiltonian number $2(n-1)$. All connected graphs of order n with Hamiltonian number $2n - 3$ or $2n - 4$ are characterized in [59]. A connected graph with exactly one cycle is called a *unicyclic graph*. Therefore, a unicyclic graph is a graph obtained from a tree by joining two nonadjacent vertices. In other words, G is unicyclic if G itself is a cycle or G contains exactly one block that is a cycle and each of the remaining block is K_2.

Theorem 2.37. *Let G be a connected graph of order $n \geq 3$. Then $h(G) = 2n - 3$ if and only if G is a unicyclic graph whose unique cycle is a triangle.*

Let $\mathscr{G}_1$ be the set of connected graphs G of order $n \geq 5$ with cut-vertices such that G contains exactly two blocks that are K_3 and each of the remaining blocks of G is K_2. Also, let $\mathscr{G}_2$ be the set of connected graphs G of order $n \geq 5$ with cut-vertices such that G contains exactly one block that is one of the graphs in the set

$$\{K_4\} \cup \{K_{2,n'-2}, K_{1,1,n'-2} : 4 \leq n' \leq n-1\}$$

and each of the remaining blocks of G is K_2.

Theorem 2.38. *Let G be a connected graph of order n. Then $h(G) = 2n-4$ if and only if* (a) *$n \geq 4$ and $G \in \{K_4, K_{2,n-2}, K_{1,1,n-2}\}$ or* (b) *$n \geq 5$ and $G \in \mathscr{G}_1 \cup \mathscr{G}_2$.*

We have seen that if T is a nontrivial tree of order n, then $h(T) = 2(n-1)$, which is clearly even. With the aid of the alternative definition of the Hamiltonian number of a graph in terms of $d(s)$ defined in (2.2), we can extend this fact to all connected bipartite graphs.

Theorem 2.39 ([35]). *If G is a nontrivial connected bipartite graph, then $d(s)$ is even for every cyclic ordering s of $V(G)$.*

Proof. For an arbitrary cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of $V(G)$, where $n = |V(G)|$, consider the set $\{i_1, i_2, \ldots, i_k\}$ of integers with $1 = i_1 < i_2 < \cdots < i_k = n+2$ (where the subscripts of the vertices are expressed as integers modulo n) such that (i) v_{i_j} and $v_{i_{j+1}}$ belong to different partite sets ($1 \leq j \leq k-2$) and (ii) the set $S_j = \{v_i : i_j \leq i < i_{j+1}\}$ is contained in a partite set ($1 \leq j \leq k-1$). Since $v_1 = v_{n+1}$ belongs to both S_1 and S_{k-1}, it follows that k must be even and the partite sets of G are $S_1 \cup S_3 \cup \cdots \cup S_{k-1}$ and $S_2 \cup S_4 \cup \cdots \cup S_{k-2}$. Therefore, $d(v_i, v_{i+1})$ is odd if and only if $i = i_j - 1$ ($2 \leq j \leq k-1$), that is, exactly $k-2$ of the n summands in $d(s)$ are odd. □

Alternatively, we may consider Theorem 2.39 as follows. For a nontrivial connected graph G of order n, suppose that $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ is a cyclic ordering of $V(G)$. If $P^{(i)}$ is a $v_i - v_{i+1}$ geodesic for $1 \leq i \leq n$, then the walk W obtained by traversing the n paths $P^{(1)}, P^{(2)}, \ldots, P^{(n)}$ in this order is a closed walk in which every vertex of G appears at least once. Furthermore, the length of W equals $d(s)$. Since the length of a $u - v$ walk in a bipartite graph is even if and only if u and v belong to the same partite set, every closed walk in a bipartite graph has even length.

Theorem 2.40. *If G is a connected bipartite graph, then $h(G)$ is even.*

Hamiltonian walks in maximal planar graphs were studied by Asano, Nishizeki, and Watanabe [7, 8]. In [7], it was shown that if G is a maximal planar graph of order $n \leq 10$, then G is Hamiltonian and so $h(G) = n$. For a maximal planar graph of order $n \geq 11$, an upper bound was established in terms of n.

Theorem 2.41. *If G is a maximal planar graph of order $n \geq 11$, then $h(G) \leq 1.5(n-3)$.*

As indicated in [8], the problem of finding a Hamiltonian walk in a given graph is NP-complete. This problem is a generalized Hamiltonian cycle problem and is a special case of the Traveling Salesman Problem. With the aid of the techniques of divide-and-conquer and augmentation, an approximation algorithm for this problem on maximal planar graphs was presented in [8]. This algorithm finds in $O(n^2)$ time, a closed spanning walk of length at most $3(n-3)/2$ in a given arbitrary maximal planar graph of order $n \geq 9$. More recent results include the following by Kawarabayashi and Ozeki [44].

Theorem 2.42. *Let G be a 3-connected planar graph. Then $h(G) \leq 4(n-1)/3$, where $|V(G)| = n$.*

2.7 The Upper Hamiltonian Number of a Graph

In Sect. 2.6, we saw for the graph $G = K_{2,4}$ (shown in Fig. 2.8) that $d(s) = 8 = h(G)$ or $d(s) = 10$ for *every* cyclic ordering s of $V(G)$.

For a connected graph G in general, the *upper Hamiltonian number* $h^+(G)$ is defined as

$$h^+(G) = \max\{d(s)\},$$

where the maximum is taken over all cyclic orderings s of the vertices of G. This concept was introduced in [23]. Thus, $h(G) = 8$ while $h^+(G) = 10$ for $G = K_{2,4}$. In fact, Theorem 2.39 implies that both $h(G)$ and $h^+(G)$ are even when G is bipartite.

Obviously, $h^+(G) \geq h(G)$ for every connected graph G in general, while the two parameters are equal when G is complete. As another example, let us consider the hypercubes Q_n. Note that $Q_1 = K_2$ and so $h(Q_1) = h^+(Q_1) = 2$. For $n \geq 2$, the graph Q_n is Hamiltonian and so $h(Q_n) = 2^n$ for each $n \geq 1$. The upper Hamiltonian number of Q_n was obtained in [23].

Theorem 2.43. *For each integer $n \geq 2$, $h^+(Q_n) = 2^{n-1}(2n-1)$.*

Proof. First, we show that $h^+(Q_n) \leq 2^{n-1}(2n-1)$. Let s be an arbitrary cyclic ordering of $V(Q_n)$ with $d(s) = h^+(Q_n)$. Since $\operatorname{diam}(Q_n) = n$ and each vertex $v \in V(Q_n)$ has exactly one vertex $v' \in V(Q_n)$ such that $d(v, v') = n$, at most 2^{n-1} terms in $d(s)$ are equal to n. Thus, $h^+(Q_n) = d(s) \leq 2^{n-1}n + 2^{n-1}(n-1) = 2^{n-1}(2n-1)$.

To verify that $h^+(Q_n) \geq 2^{n-1}(2n-1)$, note that the result is straightforward to verify for $Q_2 = C_4$ and so we may assume that $n \geq 3$. Let $G = Q_n$. Then G consists of two disjoint copies G_1 and G_2 of Q_{n-1}, where corresponding vertices of G_1 and G_2 are adjacent. For each vertex v of G, there is a unique vertex v' of G

such that $d(v, v') = n = \mathrm{diam}(Q_n)$. Necessarily, exactly one of v and v' belongs to G_1. Let $(v_1, v_2, \ldots, v_{2^{n-1}}, v_{2^{n-1}+1} = v_1)$ be a Hamiltonian cycle in G_1 and consider the cyclic ordering $s : v_1, v_1', v_2, v_2', \ldots, v_{2^{n-1}}, v_{2^{n-1}}', v_1$ of $V(G)$. By the triangle inequality, $d(v_{i+1}, v_i') \geq d(v_i, v_i') - d(v_i, v_{i+1}) = n - 1$ for $1 \leq i \leq 2^{n-1}$. Hence, $h^+(Q_n) \geq d(s) = 2^{n-1}n + 2^{n-1}(n-1) = 2^{n-1}(2n-1)$. □

The upper Hamiltonian numbers of trees and cycles have been calculated in [23, 47].

Theorem 2.44. *If T is a nontrivial tree of order n, then*

$$2(n-1) = h(T) \leq h^+(T) \leq \left\lfloor n^2/2 \right\rfloor.$$

Furthermore, $h^+(T) = 2(n-1)$ if and only if T is a star and $h^+(T) = \lfloor n^2/2 \rfloor$ if and only if T is a path.

Theorem 2.45. *For each integer $n \geq 3$, $h(C_n) = n$ and*

$$h^+(C_n) = (n-2)\lfloor (n-1)/2 \rfloor + 2\lceil (n-1)/2 \rceil.$$

Theorems 2.43–2.45 show, not surprisingly, that $h^+(G)$ can be considerably larger than $h(G)$. There are, however, only two graphs G of a fixed order for which $h(G) = h^+(G)$, a fact established in [23]. Two vertices u and v are *antipodal vertices* in a connected graph G if $d(u, v) = \mathrm{diam}(G)$.

Theorem 2.46. *Let G be a nontrivial connected graph. Then $h(G) = h^+(G)$ if and only if G is either complete or a star.*

Proof. Let G be a connected graph of order $n \geq 2$. For every cyclic ordering s of $V(G)$, observe that $d(s) = n$ if G is complete while $d(s) = 2(n-1)$ if G is a star. In other words, $h(K_n) = h^+(K_n) = n$ and $h(K_{1,n-1}) = h^+(K_{1,n-1}) = 2(n-1)$.

For the converse, suppose that G is a connected graph of order n and $G \neq K_n, K_{1,n-1}$. Thus, $n \geq 4$ and $d = \mathrm{diam}(G) \geq 2$. We may also assume by Theorems 2.44 and 2.45 that G is neither a path nor a cycle. We now consider two cases, according to whether $d \geq 3$ or $d = 2$.

Case 1. $d \geq 3$. Let $P = (v_1, v_2, \ldots, v_{d+1})$ be a $v_1 - v_{d+1}$ geodesic, where v_1 and v_{d+1} are antipodal vertices in G. Since G itself is not a path, the set $U = V(G) - V(P)$ is not empty. Write $U = \{u_1, u_2, \ldots, u_{n-d-1}\}$ and define cyclic orderings s and s' of $V(G)$ by

$$s : v_1, v_2, v_3, v_4, \ldots, v_{d+1}, u_1, u_2, \ldots, u_{n-d-1}, v_1 \tag{2.3}$$

$$s' : v_1, v_3, v_2, v_4, \ldots, v_{d+1}, u_1, u_2, \ldots, u_{n-d-1}, v_1. \tag{2.4}$$

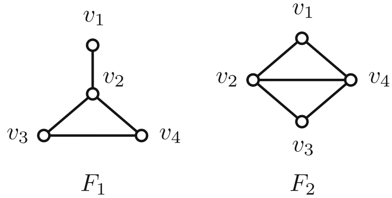


Fig. 2.9 Induced subgraphs F_1 and F_2 of G

Observe then that $h^+(G) \geq d(s') = d(s) + 2 \geq h(G) + 2$.

Case 2. $d = 2$. Since G is not a star, G contains cycles. Let g be the *girth* (the length of a smallest cycle) in G. So $g \geq 3$. Assume first that $g = 3$. Since G is connected and not complete, there exists a set $V \subseteq V(G)$ such that the subgraph induced by V in G is isomorphic to one of the graphs F_1 and F_2 in Fig. 2.9.

If $n = 4$, then the cyclic orderings $s : v_1, v_2, v_3, v_4, v_1$ and $s' : v_1, v_3, v_2, v_4, v_1$ show that $h^+(G) \geq h(G) + 1$. For $n \geq 5$, the set $U = V(G) - V$ is nonempty. Then define the cyclic orderings s and s' of $V(G)$ as described in (2.3) and (2.4) with $d = 3$, respectively, and verify that $d(s') = d(s) + 1$. Thus, $h^+(G) \geq h(G) + 1$.

If $g \geq 4$, then let $C = (v_1, v_2, \ldots, v_g, v_1)$ be an induced cycle of G and let $U = V(G) - V(C) = \{u_1, u_2, \ldots, u_{n-g}\}$, which is nonempty since $G \neq C_n$. Again, by considering the cyclic orderings s and s' of $V(G)$ as described in (2.3) and (2.4) with $d = g - 1$, respectively, we see that $h^+(G) \geq h(G) + 2$. $\square$

The proof of Theorem 2.46 suggests that if G is a graph with $h^+(G) - h(G) = 1$, then $|V(G)| \geq 4$, $\text{diam}(G) = 2$, and G must contain a triangle. In order to obtain a complete characterization of those graphs G for which the difference between $h(G)$ and $h^+(G)$ is exactly 1, the following is useful. Note that $G \vee H$ denotes the join of vertex-disjoint graphs G and H (while $G + H$ is the union of G and H).

Lemma 2.2. *For a graph G, let $G_1 = K_1 \vee G$ and $G_2 = K_1 \vee \overline{G}$. Then $h^+(G_1) - h(G_1) = h^+(G_2) - h(G_2)$.*

Proof. For a graph G of order $n - 1$ (≥ 1), construct each of G_1 and G_2 by adding a new vertex and joining it to every vertex of G and $\overline{G}$, respectively. Let $V = V(G_1) = V(G_2)$. Since $G_1 = G_2 = K_2$ if $n = 2$, we may assume that $n \geq 3$. For every two distinct vertices $u, v \in V(G)$, we have $d_{G_1}(u, v) + d_{G_2}(u, v) = 3$. Therefore, $d_{G_1}(s) + d_{G_2}(s) = 3n - 2$ for every cyclic ordering s of V. Let s_1 and s_2 be cyclic orderings of $V(G_1) = V(G_2)$ such that $d_{G_1}(s_1) = h(G_1)$ and $d_{G_2}(s_2) = h^+(G_2)$. Then $3n - 2 = d_{G_1}(s_1) + d_{G_2}(s_1) \leq h(G_1) + h^+(G_2) \leq d_{G_1}(s_2) + d_{G_2}(s_2) = 3n - 2$, implying that $h(G_1) + h^+(G_2) = 3n - 2$. Similarly, $h(G_2) + h^+(G_1) = 3n - 2$. Therefore, $h^+(G_1) - h(G_1) = h^+(G_2) - h(G_2)$. $\square$

For a set $U \subseteq V(G)$, where say $|U| = \ell$, an ordering $v_1, v_2, \ldots, v_\ell$ of the ℓ vertices in U is called a *linear ordering* of U.

Theorem 2.47. *Let G be a nontrivial connected graph of order n. Then $h^+(G) - h(G) = 1$ if and only if $n \geq 4$ and $G = K_1 \vee H$, where*

$$H \in \{K_{1,\ldots,1,2}, \overline{K_{1,\ldots,1,2}}, K_{1,n-2}, \overline{K_{1,n-2}}\}.$$

Proof. For $n \geq 4$, let $H_1 = \overline{K_{1,n-2}}$, $H_2 = K_{1,\ldots,1,2}$, $H_3 = \overline{H_2}$, and $H_4 = \overline{H_1}$. Then it is straightforward to verify that

$$h(K_1 \vee H_i) = h^+(K_1 \vee H_i) - 1 = \begin{cases} n + 2 - i & \text{if } i = 1, 2 \\ 2n - i & \text{if } i = 3, 4. \end{cases}$$

For the converse, suppose that G is a connected graph of order n and $h^+(G) - h(G) = 1$. Then $n \geq 4$ since G is neither complete nor a star by Theorem 2.46. Furthermore, as we saw in the proof of Theorem 2.46, there is neither P_4 nor C_4 as an induced subgraph in G. We may therefore assume that $\Delta(G) = n - 1$ and G contains triangles. That is, $G = K_1 \vee H$ for some graph H of order $n - 1$ that is neither complete nor empty. For $n = 4$, therefore, $H \in \{K_{1,2}, \overline{K_{1,2}}\}$.

Now assume that $n \geq 5$. We next show that none of $2K_2$, P_4, and C_4 is an induced subgraph in H. We have already seen that neither P_4 nor C_4 can be an induced subgraph in G, that is, neither is contained in H as an induced subgraph. Also, $2K_2 = \overline{C}_4$ cannot be an induced subgraph in H by Lemma 2.2. For $n = 5$, therefore, $H \in \{K_{1,1,2}, \overline{K_{1,1,2}}, K_{1,3}, \overline{K_{1,3}}\}$ or $H \in \{H_0, \overline{H}_0\}$, where $H_0 = K_1 + P_3$. One can quickly verify that $h(K_1 \vee H_0) = 6 = h^+(K_1 \vee H_0) - 2$ and so $h^+(K_1 \vee H_0) - h(K_1 \vee H_0) = h^+(K_1 \vee \overline{H}_0) - h(K_1 \vee \overline{H}_0) = 2$ by Lemma 2.2.

Finally, assume that $n \geq 6$. We next show that $\deg_H v \in \{0, 1, n-3, n-2\}$ for every $v \in V(H)$. Assume, to the contrary, that v_1 is a vertex in H with $2 \leq \deg_H v_1 \leq n - 4$. Then let v_2, v_3, v_4, v_5 be vertices in H such that v_2 and v_3 are adjacent to v_1 while v_4 and v_5 are not. Let v_0 be the vertex in G that is adjacent to every vertex in H. Then by considering two orderings $s_1 : v_2, v_1, v_3, v_4, v_0, v_5, v_2$ and $s_2 : v_2, v_0, v_3, v_4, v_1, v_5, v_2$ (and by inserting some fixed linear ordering of $V(G) - \{v_0, v_1, \ldots, v_5\}$ between v_5 and v_2 in each of s_1 and s_2 in case $n \geq 7$), we see that $h^+(G) - h(G) \geq 2$. This verifies the claim. Furthermore, $\Delta(H) \in \{1, n-3, n-2\}$ since H is nonempty. If $\Delta(H) = 1$, then $H = \overline{K_{1,\ldots,1,2}}$ since $2K_2$ cannot be an induced subgraph in H. Thus, we now consider the following two cases. Let $V(H) = \{v_1, v_2, \ldots, v_{n-1}\}$ and $\deg_H v_1 = \Delta(H)$.

Case 1. $\Delta(H) = n - 3$. Then suppose that $v_1v_2 \notin E(H)$. If $\deg_H v_2 \geq 1$, say $v_2v_3 \in E(H)$, then we may assume that $v_3v_4 \notin E(H)$ since $\deg_H v_3 \leq n - 3$. However, this implies that the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ is either C_4 or P_4, which cannot occur. Hence, $\deg_H v_2 = 0$. If $H \neq \overline{K_{1,n-2}}$, then $H = K_1 + K_{1,n-3}$ since $\deg_H v \in \{0, 1, n-3, n-2\}$ for every $v \in V(H)$. To see that this cannot occur, observe that $\overline{H}$ is traceable and $K_1 \vee \overline{H}$ is Hamiltonian while $d_{K_1 \vee \overline{H}}(s) \geq n + 2$ for any cyclic ordering of $V(K_1 \vee \overline{H})$ whose first three terms are v_3, v_1, v_4. Thus,

$h^+(K_1 \vee H) - h(K_1 \vee H) = h^+(K_1 \vee \overline{H}) - h(K_1 \vee \overline{H}) \geq 2$ by Lemma 2.2. Therefore, $H = \overline{K_{1,n-2}}$ is the only possibility in this case.

Case 2. $\Delta(H) = n-2$. Then $\delta(H) \in \{1, n-3\}$ since H is not complete. If there are two or more vertices having degree $n-2$ in H, then $\delta(H) = n-3$. Furthermore, $H = K_{1,\ldots,1,2}$ since C_4 cannot occur as an induced subgraph in H. On the other hand, if v_1 is the only vertex whose degree in H equals $n-2$, then the number of end-vertices in H is either 1 or $n-2$. If the former occurs, then $\overline{H} = K_1 + K_{1,n-3}$. However, this is impossible by Case 1 and Lemma 2.2. Therefore, $H = K_{1,n-2}$. □

Observe that if $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ is any cyclic ordering of the vertices of a connected graph, then for each vertex v_i $(1 \leq i \leq n)$, both $d(v_{i-1}, v_i) \leq e(v_i)$ and $d(v_i, v_{i+1}) \leq e(v_i)$, where $e(v_i)$ denotes the *eccentricity* of v_i (the distance from v_i to a vertex farthest from v_i). Therefore, if G is a connected graph of order $n \geq 3$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$h^+(G) \leq \sum_{i=1}^{n} e(v_i).$$

Since the eccentricity of a vertex in G is at most the diameter of G, we have the following upper bound for $h^+(G)$ in terms of the order and diameter of G.

Theorem 2.48. *If G is a nontrivial connected graph of order n and diameter d, then $h^+(G) \leq nd$.*

The upper bound in Theorem 2.48 has been shown to be sharp in [23]. A sharp lower bound for the upper Hamiltonian number of a connected graph G, also in terms of the order and diameter of G, was obtained by Král, Tong, and Zhu in [47].

Theorem 2.49. *If G is a nontrivial connected graph of order n and diameter d, then*

$$h^+(G) \geq n + \lceil d^2/2 \rceil - 1.$$

Furthermore, for each pair n, d of integers satisfying $1 \leq d \leq n-1$, there is a connected graph G of order n and diameter d with $h^+(G) = n + \lceil d^2/2 \rceil - 1$.

2.8 The Hamiltonian Spectrum of a Graph

For a connected graph G, the *Hamiltonian spectrum* $\mathscr{H}(G)$ of G is defined in [47] as

$$\mathscr{H}(G) = \{d(s) : \ s \text{ is a cyclic ordering of the vertices of } G\}.$$

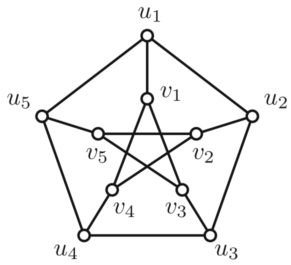


Fig. 2.10 Illustrating the Hamiltonian spectrum of a graph

Of course, this implies that $h(G), h^+(G) \in \mathscr{H}(G)$ and, in general,

$$\mathscr{H}(G) \subseteq \{k : k = h(G), h(G)+1, \dots, h^+(G)\}. \tag{2.5}$$

The following is therefore an immediate consequence of Theorem 2.46.

Theorem 2.50. *The Hamiltonian spectrum of a connected graph G consists of a single number if and only if G is either a complete graph or a star.*

As another illustration, consider the Petersen graph P in Fig. 2.10. Since P is a non-Hamiltonian graph of order 10, it follows that $h(P) \geq 11$. On the other hand, $h^+(P) \leq 20$ by Theorem 2.48. Therefore, $11 \leq h(P) < h^+(P) \leq 20$. In fact, $h(P) = 11$ and $h^+(P) = 20$. Consider the sequences s_i $(1 \leq i \leq 10)$ given by

$$\begin{aligned}
s_1 &: u_1, u_2, u_3, u_4, u_5, v_5, v_2, v_4, v_3, v_1, u_1\\
s_2 &: u_1, u_2, u_3, u_4, u_5, v_5, v_2, v_3, v_4, v_1, u_1\\
s_3 &: u_1, u_2, u_3, u_5, u_4, v_4, v_2, v_3, v_5, v_1, u_1\\
s_4 &: u_1, u_3, u_5, u_2, u_4, v_4, v_2, v_5, v_3, v_1, u_1\\
s_5 &: u_1, u_3, u_5, u_2, u_4, v_3, v_5, v_2, v_4, v_1, u_1\\
s_6 &: u_1, u_3, u_5, u_2, u_4, v_5, v_2, v_4, v_3, v_1, u_1\\
s_7 &: u_1, u_3, u_5, u_2, u_4, v_3, v_5, v_4, v_2, v_1, u_1\\
s_8 &: u_1, u_3, u_5, u_2, v_2, u_4, v_3, v_4, v_5, v_1, u_1\\
s_9 &: u_1, u_3, u_5, u_2, u_4, v_2, v_3, v_4, v_5, v_1, u_1\\
s_{10} &: u_1, u_3, u_5, u_2, u_4, v_1, v_2, v_3, v_4, v_5, u_1.
\end{aligned}$$

Since $d(s_i) = 10 + i$ for $1 \leq i \leq 10$, it follows that $\mathscr{H}(P) = \{11, 12, \dots, 20\}$, that is, equality holds in (2.5). On the other hand, Theorem 2.39 implies that the Hamiltonian spectrum of a connected bipartite graph consists only of even integers, that is, equality in (2.5) does not hold in general.

The Hamiltonian spectrum of an n-cycle was determined in [47] for each integer $n \geq 3$. Recall that $h^+(C_n) = (n-2)\lfloor (n-1)/2 \rfloor + 2\lceil (n-1)/2 \rceil$.

Theorem 2.51. *Let $n \geq 3$ be an integer.*
(a) *If n is even, then $\mathscr{H}(C_n) = \{n, n+2, \dots, h^+(C_n)-2, h^+(C_n)\}$.*

(b) *If n is odd, then*

$$\mathscr{H}(C_n) = \{n, n+2, \ldots, 2n-5, 2n-3\} \cup \{2n-2, 2n-1, \ldots, h^+(C_n)-2\} \cup \{h^+(C_n)\}.$$

The Hamiltonian spectrum of a tree was determined by Liu [50]. In order to present this result, we introduce some additional definitions. For a vertex v of a connected graph G, the *total distance* $\mathrm{td}(v)$ of v is the sum of the distances from v to all vertices of G. The minimum total distance over all vertices of G is the *median number* of G and is denoted by $\mathrm{med}(G)$.

Theorem 2.52. *For a nontrivial tree T of order n,*

$$\mathscr{H}(T) = \{2k : k = n-1, n, n+1, \ldots, \mathrm{med}(T)\}.$$

The following is a consequence of Theorem 2.52.

Theorem 2.53. *The upper Hamiltonian number of a nontrivial tree T equals* $2\,\mathrm{med}(T)$.

According to Theorems 2.44 and 2.53 (or Theorem 2.39), the upper Hamiltonian number of a tree of order n is an even integer between $2(n-1)$ and $\lfloor n^2/2 \rfloor$. In fact, for each integer $n \geq 3$, every even integer between $2(n-1)$ and $\lfloor n^2/2 \rfloor$ is the upper Hamiltonian number of some tree of order n. In order to show this, we first present some preliminary results. A vertex of a connected graph G whose total distance equals the median number of G is a *median vertex* of G. The subgraph of G induced by its median vertices of G is the *median* of G. The following two lemmas will be useful to us, the first of which is an easy observation and the second of which was established by Truszczyński [65].

Lemma 2.3. *No end-vertex of a tree T of order at least* 3 *is a median vertex of T.*

Lemma 2.4. *The median of every connected graph G lies in a single block of G.*

It therefore follows by Lemma 2.4 that the median of a tree is either K_1 or K_2.

Theorem 2.54. *For each pair n, k of integers satisfying $1 \leq n-1 \leq k \leq \lfloor n^2/4 \rfloor$, there exists a tree T of order n such that $h^+(T) = 2k$.*

Proof. By Theorem 2.44, the result holds when $k \in \{n-1, \lfloor n^2/4 \rfloor\}$. Thus, let $n \geq 5$ be a fixed integer and suppose that k is an integer satisfying $n+1 \leq k \leq \lfloor n^2/4 \rfloor$ and there exists a tree T_k of order n with $h^+(T_k) = 2k$. We show that there exists a tree T of order n with $h^+(T) = 2(k-1)$.

Let x be a median vertex of T_k and select a vertex y farthest from x. Thus, y is an end-vertex in T_k while x is not by Lemma 2.3. Also, $\mathrm{td}_{T_k}(x) = \mathrm{med}(T_k) = k$ by Theorem 2.53. Now consider the $y - x$ geodesic $P = (y = v_0, v_1, v_2, \ldots, v_{e(x)} = x)$, where $e(x)$ is the eccentricity of x. Note that $e(x) \geq 2$ since T_k is not a star. Let T be the tree obtained from T_k by deleting the edge v_0v_1 and adding the edge v_0v_2. Then y is an end-vertex in T while v_1 may or may not. We claim that $\mathrm{med}(T) = k - 1$. For each vertex $v \in V(T) - \{y\}$, observe that

$$\mathrm{td}_T(v) = \begin{cases} \mathrm{td}_{T_k}(v) + 1 & \text{if } v \in V(T') \\ \mathrm{td}_{T_k}(v) - 1 & \text{otherwise,} \end{cases}$$

where T' is the component of $T - v_1v_2$ containing v_1. Since $\mathrm{td}_T(y) > \mathrm{med}(T)$ again by Lemma 2.3, it follows that $\mathrm{med}(T) = \mathrm{td}_T(x) = k - 1$ and so $h^+(T) = 2(k-1)$ by Theorem 2.53. □

As we have seen earlier, $d(s)$ and $d(s')$ are of the same parity for every two cyclic orderings s and s' of $V(G)$ if G is either complete or bipartite. In fact, these are the only two classes of connected graphs with this property.

Theorem 2.55 ([35]). *A nontrivial connected graph G has the property that $d(s)$ and $d(s')$ are of the same parity for every two cyclic orderings s and s' of $V(G)$ if and only if G is complete or bipartite.*

Proof. By the discussion above, we may assume that G is neither complete nor bipartite. We consider two cases.

Case 1. G contains a triangle. Let $G' = K_\omega$ be a largest clique in G, where then $\omega \geq 3$. Since G is not complete and G is connected, there is a vertex in $V(G) - V(G')$ that is adjacent to some but not all vertices of G'. Thus, there is a triangle (v_1, v_2, v_3, v_1) and a vertex $v_4 \in V(G) - \{v_1, v_2, v_3\}$ such that $v_2v_4 \notin E(G)$ and $v_3v_4 \in E(G)$. For a fixed linear ordering s of $V(G) - \{v_2, v_3, v_4\}$ whose terminal vertex is v_1, let s_1 be the ordering v_1, v_2, v_3, v_4 followed by s. Similarly, let s_2 be the ordering v_1, v_3, v_2, v_4 followed by s. Then both s_1 and s_2 are cyclic orderings of $V(G)$ and $d(s_2) - d(s_1) = 1$. Hence, $d(s_1)$ and $d(s_2)$ are of opposite parity.

Case 2. G is triangle-free. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a shortest odd cycle in G. Thus, $\ell \geq 5$ and C is an induced subgraph of G. We consider two subcases.

Subcase 2.1. $\ell = 5$. If G itself is a cycle, that is, if $n = \ell = 5$, then let $s_1 : v_1, v_3, v_2, v_4, v_5, v_1$ and $s_2 : v_1, v_3, v_4, v_2, v_5, v_1$ be two cyclic orderings of the vertices of G. Hence, $d(s_1) = 7$ and $d(s_2) = 8$. If $n \geq 6$, then let s be a fixed linear ordering of the vertices of $V(G) - \{v_2, v_3, v_4, v_5\}$ whose terminal vertex is v_1. Now consider $s_1' : v_1, v_3, v_2, v_4, v_5$ and $s_2' : v_1, v_4, v_2, v_3, v_5$. For $i = 1, 2$, let s_i be the ordering s_i' followed by s. Then s_i is a cyclic ordering of $V(G)$ and $d(s_2) - d(s_1) = 1$.

Subcase 2.2. $\ell \geq 7$. Let s be a fixed linear ordering of the set $V(G)-\{v_2,\ldots,v_{\ell-1}\}$ whose terminal vertex is v_1. Let $\ell^* = (\ell+1)/2$ and consider

$$s_1' : v_1, v_{\ell^*}, v_2, v_3, \ldots, v_{\ell^*-1}, v_{\ell^*+1}, v_{\ell^*+2}, \ldots, v_{\ell-1}$$

$$s_2' : v_1, v_{\ell^*+1}, v_2, v_3, \ldots, v_{\ell^*}, v_{\ell^*+2}, v_{\ell^*+3}, \ldots, v_{\ell-1}.$$

For $i = 1, 2$, let s_i be the ordering s_i' followed by s. Then both s_1 and s_2 are cyclic orderings of $V(G)$ and $d(s_2) - d(s_1) = 1$. □

The following is an immediate consequence of the proof of Theorem 2.55.

Theorem 2.56. *If G is a nontrivial connected graph that is neither complete nor bipartite, then there are cyclic orderings s and s' of $V(G)$ such that $d(s)-d(s') = 1$. In other words, $\mathscr{H}(G)$ contains two consecutive integers.*

We have seen that the Hamiltonian spectrum of a graph G consists of a single element if and only if G is either complete or a star. Suppose now that G is a graph for which $\mathscr{H}(G)$ contains exactly two elements. If G is not bipartite, then it follows by Theorem 2.56 that $h^+(G) - h(G) = 1$. Such graphs have been completely characterized in Theorem 2.47.

A tree T is a *double star* if it contains exactly two vertices that are not end-vertices. Necessarily, these two vertices are adjacent in T. If their degrees are r and s $(r, s \geq 2)$, respectively, then we write $T = S_{r,s}$. For those graphs G that are bipartite and $|\mathscr{H}(G)| = 2$, we have the following.

Theorem 2.57 ([35]). *Let G be a nontrivial connected bipartite graph of order n. Then $|\mathscr{H}(G)| = 2$ if and only if $n \geq 4$ and G is either $S_{2,n-2}$ or $K_{2,n-2}$.*

Combining Theorems 2.56 and 2.57, we have the following.

Theorem 2.58. *Let G be a nontrivial connected graph of order n. Then $|\mathscr{H}(G)| = 2$ if and only if $n \geq 4$ and either*
(a) *$G \in \{S_{2,n-2}, K_{2,n-2}\}$ or*
(b) *$G = K_1 \vee H$, where $H \in \{K_{1,\ldots,1,2}, \overline{K_{1,\ldots,1,2}}, K_{1,n-2}, \overline{K_{1,n-2}}\}$.*
Furthermore, the two integers in $\mathscr{H}(G)$ are of the same parity if and only if (a) *occurs.*

Theorem 2.59 ([35]). *If G is a connected graph of order n such that $h^+(G) - h(G) = 2$, then exactly one of the following* (a)–(c) *occurs:*
(a) *$n \geq 4$ and $G \in \{S_{2,n-2}, K_{2,n-2}\}$.*
(b) *$n \geq 5$ and $G = H_1 \vee H_2$, where H_1 is complete and*
  i. *$n \geq 6$ and $H_2 = \overline{K_3}$ or*
  ii. *$n \geq 5$ and $H_2 = K_{2,2}$ or*
  iii. *$n \geq 5$ and $H_2 = K_1 + K_\ell$, where $2 \leq \ell \leq n-3$.*
(c) *$n \geq 5$ and G is neither bipartite nor Hamiltonian.*

http://www.springer.com/978-1-4939-0304-7

Covering Walks in Graphs
Fujie, F.; Zhang, P.
2014, XIV, 110 p. 37 illus., 11 illus. in color., Softcover
ISBN: 978-1-4939-0304-7