Chapter 2
Mathematical Background

Abstract To follow all the upcoming discussion of this book clearly and accurately, this chapter describes the mathematical language used throughout. That is, it reviews all the necessary mathematical concepts so that no other sources are needed to grasp all the topics covered in the book. Simply stated, the present chapter makes this book completely self-contained. The concepts covered in the present chapter, divided into four sections, primarily include fundamental areas of discrete mathematics. First, Sect. 2.1 reviews some of the most basic concepts from set theory. Then, Sect. 2.2 give the essentials concerning relations, after which Sect. 2.3 concentrates its attention on their crucially important special cases—functions. Finally, Sect. 2.4 reviews a number of concepts from graph theory, used throughout the remainder of the book.

Keywords Discrete mathematics • Sets • Relations • Functions • Graphs • Trees

Although this book is self-contained in the sense that no other sources are needed to grasp all the presented results, the reader is expected to have at least basic knowledge regarding mathematical background of formal language theory. Of course, a deeper knowledge of mathematics is more than welcome because this introductory chapter describes the necessary mathematical terminology rather briefly. For a good treatment of mathematical foundations underlying formal language theory, consult [2].

The present four-section chapter reviews rudimentary concepts concerning sets (Sect. 2.1), relations (Sect. 2.2), functions (Sect. 2.3), and graphs (Sect. 2.4). For readers familiar with these concepts, this chapter can be skimmed and treated as a reference for notation and definitions.

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2.1 Sets and Sequences

A *set* Q is a collection of differentiable elements taken from some prescribed *universe* $\mathbb{U}$ without any structure other than membership. To indicate that x is an element (a *member*) of Q, we write $x \in Q$. The statement that x is not in Q is written as $x \notin Q$. If Q has a finite number of members, then Q is a *finite set*; otherwise, Q is an *infinite set*. The set that has no members is the *empty set*, denoted by $\emptyset$. The *cardinality* of Q, denoted by $\mathrm{card}(Q)$, is, for finite sets, the number of members of Q. If Q is infinite, then we assume that $\mathrm{card}(Q) > k$ for every integer k. Note that $\mathrm{card}(\emptyset) = 0$.

Sets can specified by enclosing some description of their elements in curly brackets; for example, the set Q of three consecutive integers, $1, 2$ and 3, is denoted by

$$Q = \{1, 2, 3\}$$

Ellipses can be used whenever the meaning is clear. Thus, $\{a, b, \ldots, z\}$ stands for all the lower-case letters of the English alphabet. When the need arises, we use a more explicit notation, in which a set Q is specified by a property σ so Q contains all elements satisfying σ. This specification has the following format

$$Q = \{x \mid \sigma(x)\}$$

Let $\mathbb{N}_0$ denote the set of all nonnegative integers. Then, for example, the set of all even nonnegative integers can be defined as $\mathbb{N}_0^{even} = \{i \mid i \in \mathbb{N}_0, i \text{ is even}\}$.

The usual set operations are *union* ($\cup$), *intersection* ($\cap$), and *difference* ($-$). Let Q_1 and Q_2 be two sets. Then,

$$\begin{aligned}
Q_1 \cup Q_2 &= \{x \mid x \in Q_1 \text{ or } x \in Q_2\} \\
Q_1 \cap Q_2 &= \{x \mid x \in Q_1 \text{ and } x \in Q_2\} \\
Q_1 - Q_2 &= \{x \mid x \in Q_1 \text{ and } x \notin Q_2\}
\end{aligned}$$

For n sets, $Q_1, Q_2, \ldots, Q_n$, where $n \geq 1$, instead of $Q_1 \cup Q_2 \cup \cdots \cup Q_n$ and $Q_1 \cap Q_2 \cap \cdots \cap Q_n$, we usually write $\bigcup_{1 \leq i \leq n} Q_i$ and $\bigcap_{1 \leq i \leq n} Q_i$, respectively. If there are infinitely many sets, we omit the upper bound n.

Another basic operation is the *complementation* of Q, which is denoted by $\overline{Q}$ and defined as

$$\overline{Q} = \{x \mid x \in \mathbb{U} \text{ and } x \notin Q\}$$

A set P is said to be a *subset* of Q if every element of P is also an element of Q; we write this as $P \subseteq Q$. We also say that Q is a *superset* of P. If $P \subseteq Q$ and $Q - P \neq \emptyset$, we say that P is a *proper subset* of Q, written as $P \subset Q$. Similarly, we also say that Q is a *proper superset* of P.

Let Q_1 and Q_2 be two sets. If $Q_1 \cap Q_2 = \emptyset$, then Q_1 and Q_2 are *disjoint*. If $Q_1 \subseteq Q_2$ and $Q_2 \subseteq Q_1$, then Q_1 and Q_2 are *identical*, written as $Q_1 = Q_2$; otherwise, they are *non-identical*, written as $Q_1 \neq Q_2$.

Let $Q_1, Q_2, \ldots, Q_n$ be n sets, for some $n \geq 1$. If $Q_i \cap Q_j = \emptyset$ for all $i = 1, 2, \ldots, n$ and all $j = 1, 2, \ldots, n$ such that $i \neq j$, then we say that all Q_i are *pairwise disjoint*.

The *power set* of Q, denoted by 2^Q, is the set of all subsets of Q. In symbols,

$$2^Q = \{U \mid U \subseteq Q\}$$

Sets whose members are sets are customarily called *families of sets*, rather than sets of sets. For instance, 2^Q is a family of sets. As obvious, all notions and operations from sets apply to families of sets as well.

For a finite set $Q \subseteq \mathbb{N}_0$, let $\max(Q)$ denote the smallest integer m such that m is greater or equal to all members of Q. Similarly, $\min(Q)$ denotes the greatest integer n such that n is lesser or equal to all members of Q.

A *sequence* is a list of elements. Contrary to a set, a sequence can contain an element more than once and the elements appear in a certain order. Elements in sequences are usually separated by a comma. As sets, sequences can be either *finite* or *infinite*. Finite sequences are also called *tuples*. More specifically, sequences of two, three, four, five, six, and seven elements are called *pairs*, *triples*, *quadruples*, *quintuples*, *sextuples*, and *septuples*, respectively.

2.2 Relations

Let Q_1 and Q_2 be two sets. The *Cartesian product* of Q_1 and Q_2, denoted by $Q_1 \times Q_2$, is a set of pairs defined as

$$Q_1 \times Q_2 = \{(x_1, x_2) \mid x_1 \in Q_1 \text{ and } x_2 \in Q_2\}$$

A *binary relation* ρ from Q_1 to Q_2 is any subset of their Cartesian product. That is,

$$\rho \subseteq Q_1 \times Q_2$$

The *inverse* of ρ, denoted by ρ^{-1}, is defined as

$$\rho^{-1} = \{(y, x) \mid (x, y) \in \rho\}$$

Instead of $(x, y) \in \rho$, we often write $x\rho y$; in other words, $(x, y) \in \rho$ and $x\rho y$ are used interchangeably. If $Q_1 = Q_2$, then we say that ρ is a *relation on* Q_1 or *relation over* Q_1. As relations are sets, all common operations over sets apply to relations as well. If ρ is a finite set, then ρ is a *finite relation*; otherwise, ρ is an *infinite relation*.

For every $k \geq 0$, the *kth power* of ρ, denoted by ρ^k, is recursively defined as follows:

(1) $x\rho^0 y$ if and only if $x = y$
(2) for $k \geq 1, x\rho^k y$ if and only if $x\rho z$ and $z\rho^{k-1} y$, for some $z \in Q$

The *transitive closure* of ρ, denoted by ρ^+, is defined as $x\rho^+ y$ if and only if $x\rho^k y$, for some $k \geq 1$. The *reflexive-transitive closure* of ρ, denoted by ρ^*, is defined as $x\rho^* y$ if and only if $x\rho^k y$, for some $k \geq 0$.

2.3 Functions

A *function* ψ from Q_1 to Q_2 is a relation from Q_1 to Q_2 such that for every $x \in Q_1$,

$$\operatorname{card}\big(\{y \mid y \in Q_2 \text{ and } (x, y) \in \psi\}\big) \leq 1$$

If for every $y \in Q_2, \operatorname{card}(\{x \mid x \in Q_1 \text{ and } (x, y) \in \psi\}) \leq 1$, ψ is an *injection*. If for every $y \in Q_2, \operatorname{card}(\{x \mid x \in Q_1 \text{ and } (x, y) \in \psi\}) \geq 1$, ψ is a *surjection*. If ψ is both an injection and a surjection, ψ represents a *bijection*.

The *domain* of ψ, denoted by domain(ψ), and the *range* of ψ, denoted by range(ψ), are defined as

$$\operatorname{domain}(\psi) = \big\{x \mid x \in Q_1 \text{ and } (x, y) \in \psi, \text{ for some } y \in Q_2\big\}$$

and

$$\operatorname{range}(\psi) = \big\{y \mid y \in Q_2 \text{ and } (x, y) \in \psi, \text{ for some } x \in Q_1\big\}$$

If domain(ψ) $= Q_1$, ψ is *total*; otherwise, ψ is *(strictly) partial*. Instead of $(x, y) \in \psi$, we usually write $\psi(x) = y$.

Let P and Q be two sets. A *unary operation* over P is a function f from P to Q. We say that P is *closed under a unary operation* f if for each $a \in P$, $f(a) \in P$.

A *binary operation* over P is a function $\odot$ from $(P \times P)$ to Q. We say that P is *closed under a binary operation* $\odot$ if for each $a, b \in P$, $a \odot b \in P$ (we use the infix notation $a \odot b$ instead of $\odot(a, b)$). A set that is closed under an operation is said to satisfy a *closure property*.

2.4 Graphs

In this section, we review the basics of graph theory. For more information, please consult [1, 3, 4].

A *directed graph* is a pair, $G = (V, \rho)$, where V is a finite set and ρ is a relation over V. For brevity, by a *graph*, we automatically mean a directed graph throughout the book. Members of V are called *nodes* and pairs in ρ are called *edges*. If $e = (a, b)$ and $e \in \rho$, then e *leaves* a and *enters* b; at this point, a is a *direct predecessor* of b and b is a *direct descendant* of a. A sequence of nodes, $a_0, a_1, \ldots, a_n$, where $n \geq 1$, forms a *walk* from a_0 to a_n if $(a_{i-1}, a_i) \in \rho$ for all $i = 1, \ldots, n$. If, in addition, $a_0 = a_n$, then $a_0, a_1, \ldots, a_n$ is a *cycle*. If there is no cycle in G, then G is an *acyclic graph*.

Let Q be a nonempty set. An *ordered labeled tree* or, briefly, a *tree* is an acyclic graph, $T = (V, \rho)$, satisfying the following four conditions:

(1) there is exactly one special node, called the *root* such that no edge enters it;
(2) for each node $a \in V$ other than the root, there is exactly one walk from the root to a;
(3) every node is labeled with a member of Q (there is a total function from V to Q);
(4) each node $a \in V$ has its direct descendants $b_1, b_2, \ldots, b_n$ ordered from the left to the right, so b_1 is the leftmost descendant of a and b_n is the rightmost descendant of a.

For brevity, we often refer to nodes by their labels.

Let $T = (V, \rho)$ be a tree and $a \in V$ be a node. If no edge leaves a, then a is a *leaf*. The *frontier* of T is the sequence of leaves of T, ordered from the left to the right. A tree, $T' = (V', \rho')$, is a *subtree* of T if it satisfies the following three conditions

(1) $\emptyset \subset V' \subseteq V$;
(2) $\rho' = (V' \times V') \cap \rho$;
(3) in T, no node in $V - V'$ is a direct descendant of a node in V'.

Let $r \in V'$ be the root of T'. Then, we say that T' is *rooted* at r.

A *multigraph* is a triple $G = (V, \rho, Q)$, where V and Q are two finite sets and $\rho \subseteq V \times Q \times V$ is a ternary relation. Informally, it is a graph where there can be more than one edge between two nodes, each being labeled with a symbol from Q.

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